Laplacians on infinite graphs

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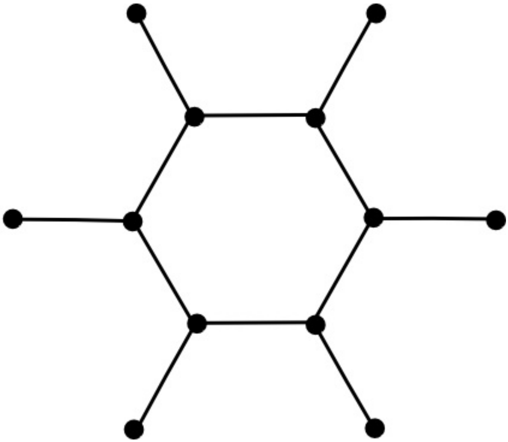
Quantum graph model



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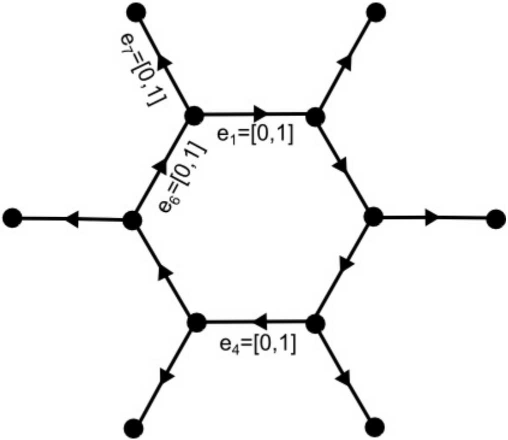

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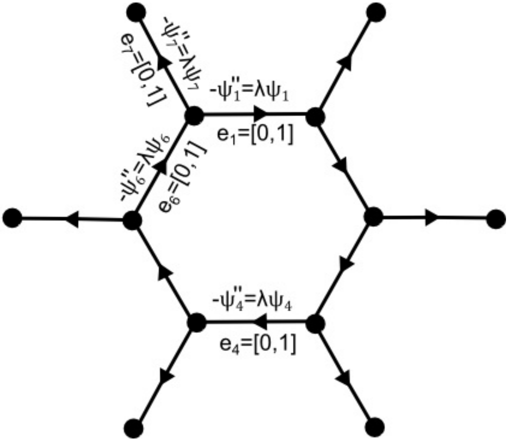

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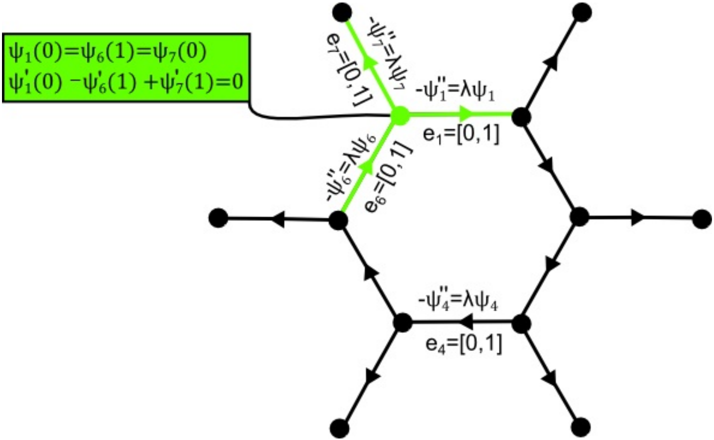

Quantum graph model

- Given: $E(G)$ – set of edges,
  $V(G)$ – set of vertices,
  $\ell : E(G) \to (0, \infty]$ length function,
  $-\frac{d^2}{dx^2}$ on each edge
- Task: Characterize self-adjoint Laplacians and their properties
- Today: $E(G)$ is infinite and $\inf_{e \in E(G)} \ell(e) = 0$

Minimal and maximal graph Laplacian

For $E(G) = \{e_n | n \in \mathbb{N}\}$ consider

- $S_n\psi_n := -\psi_n''$, $\operatorname{dom} S_n = H_0^2(0, \ell(e_n))$ in $L^2(0, \ell(e_n))$
- underlying Hilbert space $\mathcal{H} := \bigoplus_{n=0}^{\infty} L^2(0, \ell(e_n))$

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$$S := \bigoplus_{n=0}^{\infty} S_n, \quad \operatorname{dom} S := \bigoplus_{n=0}^{\infty} H_0^2(0, \ell(e_n)), \quad S(\psi_n)_{n\in\mathbb{N}} := (-\psi_n'')_{n\in\mathbb{N}}$$

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Boundary triplets

Definition *(Kochubei '75; Derkach, Malamud '91)*

Given closed densely defined symmetric linear operator S in Hilbert space $\mathcal{H}$, the triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is called boundary triplet for S^*, iff

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- Subspaces $\Theta \subseteq \mathcal{G} \times \mathcal{G}$ describe extensions of S:

$$\mathrm{dom}\, S_\Theta := \left\{\psi \in \mathrm{dom}\, S^* \mid (\Gamma_0\psi, \Gamma_1\psi) \in \Theta\right\},$$
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- For S_n^* a boundary triplet is given by $\mathcal{G}_n := \mathbb{C}^2$,

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Theorem *(Kostenko, Malamud, Neidhardt '10)*

For $\sup_{e\in E(G)} \ell(e) < \infty$ we define a new boundary triplet $\widetilde{\Pi}_n := \{\mathbb{C}^2, \widetilde{\Gamma}_0^{(n)}, \widetilde{\Gamma}_1^{(n)}\}$ for S_n^* via

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Point interactions on infinite graphs

$$C(G) := \left\{ \psi \in \bigoplus_{n=0}^{\infty} H^2(0, \ell(e_n)) \;\middle|\; \psi \text{ is continuous on } G \right\}$$

For $\psi = (\psi_n)_{n\in\mathbb{N}} \in C(G)$ define

$$\psi(v) := \begin{cases} \psi_n(0), & \text{for } e_n = \overrightarrow{vw}, \\ \psi_n(\ell(e_n)), & \text{for } e_n = \overrightarrow{wv}, \end{cases}$$

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Given a set of real numbers $\{c(v)\}_{v\in V(G)}$, the domain of the point interaction S_c is given by

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Given a set of real numbers $\{0\}_{v\in V(G)}$, the domain of the Kirchhoff Laplacian S_{KH} is given by

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Properties of point interactions

- Using results on unbounded discrete Laplacians obtained in [LK11], [GHKLW13], we can characterize the point interactions:

Theorem

Given G connected and real $\{c(v)\}_{v\in V(G)}$, then S_c is

- Self-adjoint if all infinite paths have infinite length, and $c(v) \geq 0$ for all $v \in V(G)$ or $\sup_{v\in V(G)} \left|\frac{c(v)}{\sum_{w\sim v} \ell(vw)}\right| < \infty$.

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Weyl-type asymptotics for the discrete spectrum

Theorem

Assume $L := \sum_{e \in E(G)} \ell(e) < \infty$, then all semi-bounded self-adjoint extensions $\widetilde{S}$ of S with purely discrete spectrum $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ satisfy

$$\lim_{\lambda \to \infty} \frac{N_{\widetilde{S}}(\lambda)}{\sqrt{\lambda}} = \frac{L}{\pi},$$

with $N_{\widetilde{S}}(\lambda) := |\{n \in \mathbb{N} : \lambda_n \leq \lambda\}|$.

- Ideas: Dirichlet-Neumann-Bracketing based on [CW05], Krein-von Neumann and Friedrichs extension are decoupled cf. [MN12]

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Weyl-type asymptotics for bounded perturbations

Theorem

Assume $L := \sum_{e \in E(G)} \ell(e) < \infty$ and $\widetilde{S}$ lower semi-bounded self-adjoint extension of S.
Then for all bounded $Q = Q^* \in \bigoplus_{n=0}^{\infty} L^2(0, \ell(e_n))$, the set $\sigma(\widetilde{S} + Q)$ is purely discrete with

$$\lim_{\lambda \to \infty} \frac{N_{\widetilde{S}+Q}(\lambda)}{\sqrt{\lambda}} = \frac{L}{\pi}.$$

References

CW05 S. Currie, B.A. Watson, *Dirichlet Neumann bracketing for boundary value problems on graphs*, Electronic J. Differential Equations 2005 (93), 1-11 (2005).

DM91 V.A. Derkach, M.M. Malamud: *Generalised Resolvents and the boundary value problems for Hermitian operators with gaps*, J. Func. Anal. 95, 1-95 (1991).

GHKLW13 A. Georgakopoulos, S. Haeseler, M. Keller, D. Lenz, R.K. Wojciechowski: *Graphs of finite measure*, Preprint 2013, arXiv:1309.3501.

K75 A. Kochubei, *Extensions of symmetric operators and binary relations*, Math. Notes **17** (1975), 25–28.

KM10 A. Kostenko, M.M. Malamud: *1-D Schroedinger operators with local point interactions on a discrete set*, J. Differential Equations 249, 259-304 (2010).

KL11 M. Keller, D. Lenz: *Dirichlet forms and stochastic completeness of graphs and subgraphs*, Preprint 2011, arXiv:0904.2985v2.

LSV14 D. Lenz, C. Schubert, I. Veselić: *Unbounded quantum graphs with unbounded boundary conditions*, Mathematische Nachrichten 287(8-9), 962-979 (2014).

MN12 M.M. Malamud, H. Neidhardt: *Sturm-Liouville boundary value problems with operator potentials and unitary equivalence*, J. Differential Equations 252, 5875-5922 (2012).