Distance to singularity for DAEs

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Disclaimer: DAEs and matrix pencils

- Consider for quadratic $E, A \in \mathbb{C}^{n \times n}$ the differential-algebraic equation
$$\frac{d}{dt} Ex(t) = Ax(t), \quad x(0) = x_0.$$

- For $E = I_n$ it is well known that the solutions can be described by the eigenvalues and generalized eigenvectors of A.

- Similarly the solutions of the DAE can be described by the eigenvalues and generalized eigenvectors of the matrix pencil $sE - A$.

Recap: Regular matrix pencils

- Recall that $sE - A$ is called regular if and only if

$$\det(sE - A) \in \mathbb{C}[s] \setminus \{0\}$$

  otherwise it is singular.

- Regularity is equivalent to the existence of invertible $S, T \in \mathbb{C}^{n \times n}$ such that

$$S(sE - A)T = s \begin{pmatrix} I_r & 0 \\ 0 & N \end{pmatrix} - \begin{pmatrix} J & 0 \\ 0 & I_{n-r} \end{pmatrix}$$

  with J and N in Jordan canonical form and N nilpotent. The representation above is called Weierstraß canonical form.

- the eigenvalues of J are the finite eigenvalues of $sE - A$
- the Jordan chains of N at 0 correspond to the chains at ∞.

Motivation

- Consider the following example for $sE - A$

$$s\begin{pmatrix} 0.028 & 0.0233 & 0.02 & 0.0175 \\ -0.335 & -0.28 & -0.24 & -0.2097 \\ 0.84 & 0.701 & 0.6 & 0.5237 \\ -0.56 & -0.4667 & -0.399 & -0.348 \end{pmatrix} - \begin{pmatrix} 0.061 & 0.04 & 0.03 & 0.024 \\ -0.6 & -0.399 & -0.3 & -0.24 \\ 1.35 & 0.9 & 0.676 & 0.54 \\ -0.84 & -0.56 & -0.42 & -0.335 \end{pmatrix}$$

  MATLAB's `eig(E,A)` yields

$$\lambda_{1,2} = 0.0165 \pm 0.0154i, \quad \lambda_{3,4} = -0.0165 \pm 0.0171i.$$

- There is an invertible H such that

$$sH^{-1}EH - H^{-1}AH = s\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 10^{-5} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

  This is almost in Weierstraß canonical form, therefore the only eigenvalue is 0 with Jordan chain of length 4.

Explanation

- A reason why this pencil behaves numerically so bad might be that

$$\det(sE - A) = \det \begin{pmatrix} s & 1 & & \\ 0 & s10^{-5} & 1 & 0 \\ 0 & 0 & s & 1 \\ 0 & 0 & 0 & s \end{pmatrix} = 10^{-5}s^4$$

  which is close to the zero polynomial.

- It is likely that $sE - A$ behaves like a singular pencil.

Distance to Singularity

- For regular $sE - A$, $E, A \in \mathbb{C}^{n\times n}$, we want to measure how close it is to a singular pencil
- A natural approach goes back to Byers, He, Mehrmann '98

$$\delta(E,A) := \min\{\|\Delta E\|_F + \|\Delta A\|_F \mid s(E+\Delta E)-(A+\Delta A) \text{ is singular}\}$$

with the Frobenius norm $\|(a_{ij})_{i,j=1}^n\|_F := \sqrt{\sum_{i,j=1}^n |a_{ij}|^2}$.

- Let's look at some examples

$$s\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \quad \rightarrow \quad \delta(E,A) = 1$$

- Can we compute $\delta(E,A)$ or at least find bounds?

Upper bound for $\delta(E, A)$ from BHM'98

For $sE - A$ regular, using the Weiertraß canonical form we see that

$$\mathrm{rk}\,[A, -E] = \mathrm{rk}\,[SAT, -SET] = \mathrm{rk}\left[\begin{pmatrix} J & 0 \\ 0 & I_{n-r} \end{pmatrix}, \begin{pmatrix} I_r & 0 \\ 0 & N \end{pmatrix}\right] \geq n$$

Sufficient condition

Given that $\mathrm{rk}\,[A, -E] < n$ then $sE - A$ is singular.

This implies

$$\begin{aligned} \delta(E, A) &\leq \min\{\|\Delta E\|_F + \|\Delta A\|_F \mid \mathrm{rk}\,[A + \Delta A, -E + \Delta E] < n\} \\ &= \sigma_{min}([A, -E]) \end{aligned}$$

There are examples of singular pencils with $\mathrm{rk}\,[A, -E] = n$.

Subspace structure of a pencil (recall Henrik's talk)

We can associate with $sE - A$ the subspace $\ker[A, -E]$ with this the eigen structure decomposes into the following four parts

- Chains for $\lambda \in \mathbb{C}$ of the form
$\begin{pmatrix} x_\ell \\ \lambda x_\ell + x_{\ell-1} \end{pmatrix}, \dots, \begin{pmatrix} x_1 \\ \lambda x_1 \end{pmatrix} \in \ker[A, -E]$
- Chains at ∞ of the form
$\begin{pmatrix} 0 \\ x_\ell \end{pmatrix}, \begin{pmatrix} x_\ell \\ x_{\ell-1} \end{pmatrix}, \dots, \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \in \ker[A, -E]$
- Singular chains of the form
$\begin{pmatrix} 0 \\ x_1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \dots, \begin{pmatrix} x_k \\ 0 \end{pmatrix} \in \ker[A, -E]$
- Multi shifts of the form
$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}, \dots, \begin{pmatrix} x_{k-1} \\ x_k \end{pmatrix} \in \ker[A, -E]$

Gantmacher matrices

We associate with $sE - A$ the Gantmacher matrix

$$W_k(E,A) := \begin{bmatrix} E & 0 & \dots & 0 \\ A & E & & \vdots \\ 0 & \ddots & \ddots & \\ \vdots & & & E \\ 0 & \dots & & A \end{bmatrix} \in \mathbb{C}^{kn \times (k+1)n}$$

Satz (Gantmacher)

The pencil $sE - A$ with $E, A \in \mathbb{C}^{n \times n}$ is regular if and only if $W_n(E,A)$ has full rank.

- Advantage: the condition depends directly on E, A, no canonical form has to be computed.

Reformulation of Gantmacher

We can make a connection with singular chains.

Theorem

The pencil $sE - A$ has singular chains if and only if

$$\ker W_n(E, A) \neq \{0\}.$$

- **Proof:** Let $\begin{pmatrix} 0 \\ x_1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \dots, \begin{pmatrix} x_{k-1} \\ 0 \end{pmatrix} \in \ker[A, -E]$ be a singular chain
- By assumption $x_1 \in \ker E$ and $x_k \in \ker A$ such that

$$((-1)^{i+1} x_i)_{i=1}^k \in \ker W_k(E, A).$$

Lower bound for $\delta(E, A)$ from BHM'98

For $s(E + \Delta E) - (A + \Delta A)$ there exists $k \in \mathbb{N}$, $k \leq n$ with $\ker W_k(E, A) \neq \{0\}$ for some $x \in \mathbb{C}^{nk}$ with $\|x\|_2 = 1$

$$0 = W_k(E + \Delta E, A + \Delta A)x = W_k(E, A)x + W_k(\Delta E, \Delta A)x$$

Taking norms implies with $\|x\|_2 = 1$

$$\begin{aligned} \sigma_{min}(W_k(E, A)) \leq \|W_k(E, A)x\|_2 &\leq \|W_k(\Delta E, \Delta A)\|_F \\ &\leq \sqrt{k}(\|\Delta E\|_F + \|\Delta A\|_F) \end{aligned}$$

This holds for all such ΔE, ΔA hence

$$\delta(E, A) \geq \frac{\sigma_{min}(W_k(E, A))}{\sqrt{k}}.$$

The gap between subspaces

- A matrix pencil $sE - A$ corresponds to the subspace $\ker[A, -E]$.
- The natural measure for distance between subspaces is the gap.

Definition

The gap between subspaces $\mathcal{L}, \mathcal{M} \subseteq \mathbb{C}^n$ is with $S_{\mathcal{M}} := \{x \in \mathcal{M} \mid \|x\| = 1\}$ given by

$$\theta(\mathcal{L}, \mathcal{M}) := \max \left\{ \sup_{x \in S_{\mathcal{M}}} d(x, \mathcal{L}), \sup_{x \in S_{\mathcal{L}}} d(x, \mathcal{M}) \right\}$$

- The basic property we use is that

$$\theta(\mathcal{L}, \mathcal{M}) < 1 \quad \Longrightarrow \quad \dim \mathcal{L} = \dim \mathcal{M}.$$

Estimating the gap between to subspaces

- It was shown in Berger, Trunk, Winkler '16 that for $sE - A$ with $\mathrm{rk}\,[A, -E] = n$ there exist matrices $F, G \in \mathbb{C}^{n\times n}$ with

$$\ker[A, -E] = \mathrm{ran} \begin{bmatrix} F \\ G \end{bmatrix}.$$

Theorem

Let $\mathcal{L} = \mathrm{ran}\, A$ and $\widetilde{\mathcal{L}} = \mathrm{ran}\, B$ be subspaces with $A, B \in \mathbb{C}^{2n\times d}$, $d \le 2n$, then the following estimate holds

$$\theta(\mathcal{L}, \widetilde{\mathcal{L}}) \le \max\left\{ \frac{\min\{\|A\|, \|B - A\|\}}{\sigma^*_{min}(A)}, \frac{\min\{\|B\|, \|B - A\|\}}{\sigma^*_{min}(B)} \right\}.$$

An estimate of the gap between Gantmacher matrices

Using the self-similarity of the Gantmacher matrices we can prove a technical result.

Theorem

For $\Theta = \operatorname{ran} \begin{bmatrix} F \\ G \end{bmatrix}$, $\widetilde{\Theta} = \operatorname{ran} \begin{bmatrix} \widetilde{F} \\ \widetilde{G} \end{bmatrix}$ with G and $\widetilde{G}$ invertible we have

$$\theta(\operatorname{ran} W_k(F, G), \operatorname{ran} W_k(\widetilde{F}, \widetilde{G})) \le \max\{F_k, \widetilde{F}_k\}\theta(\Theta, \widetilde{\Theta})$$

with $\kappa := \frac{\|F\|}{\sigma^*_{min}(G)}$, $F_1 := \sqrt{2\kappa^2 + 2\kappa + 1}$ and

$$F_k := F_1\sqrt{F_1^{2(k-1)} + F_1^{2(k-2)} + \ldots + 1}.$$

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with $\widetilde{\kappa} := \frac{\|\widetilde{F}\|}{\sigma^*_{min}(\widetilde{G})}$, $\widetilde{F}_1 := \sqrt{2\widetilde{\kappa}^2 + 2\widetilde{\kappa} + 1}$ and

$$\widetilde{F}_k := \widetilde{F}_1 \sqrt{\widetilde{F}_1^{2(k-1)} + \widetilde{F}_1^{2(k-2)} + \ldots + 1}.$$

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