Funnel control for nonlinear functional differential-algebraic systems*

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***Abstract*— We consider output regulation for a class of nonlinear functional differential-algebraic systems. Funnel control, that is a static nonlinear proportional output error feedback, is applied to achieve tracking of a reference signal by the output signal with prescribed transient behavior.**

***Index Terms*— Differential-algebraic equations, nonlinear systems, functional differential equations, funnel control.**

I. INTRODUCTION

Differential-algebraic equations (DAEs) are an appropriate tool to model systems coming from applications such as multibody dynamics [1] and electrical networks [2]. The dynamics and constraints of the system are modeled as a set of differential and algebraic equations. If the internal dynamics of the system are autonomous and the input does affect at most the first derivative of the output (roughly speaking, the largest relative degree part is one), then the DAE model may be written in the form

$$\begin{aligned} \Gamma\big(y(t)\big)\,\dot y(t) &= f_1\big(y(t)\big) + f_2\big(d_1(t), x(t)\big) \\ &\quad + f_3\big(d_2(t), x(t)\big)\,u(t), \\ \dot x(t) &= f_4\big(x(t), y(t), d_3(t)\big). \end{aligned} \tag{1.1}$$

The functions $u : \mathbb{R} \to \mathbb{R}^m$ and $y : \mathbb{R} \to \mathbb{R}^m$ are called *input* and *output* of the system, resp., and d_1, d_2, and d_3 are bounded disturbances. The second equation in (1.1) represents the internal dynamics, governed by the *state* $x : \mathbb{R} \to \mathbb{R}^q$. It is possible, that there are also algebraic variables in the system which depend on x and y and their derivatives, but these do not affect the input-output behavior of the system and hence we omit them in the model (1.1). The differentiable functions f_1, f_2 and f_4 are vector valued, f_3 is scalar valued and Γ is matrix valued; for more details see Section II.

If the internal dynamics of (1.1) are input-to-state stable (ISS) [3], then system (1.1) can be rewritten, by the choice of an appropriate operator T (which depends on f_4, d_3 and the initial value $x(0)$) explained in [4, Sect. 2.3], as a nonlinear functional differential-algebraic multi-input, multi-output systems of the form

$$\begin{aligned} \Gamma\big(y(t)\big)\,\dot y(t) &= f_1\big(y(t)\big) + f_2\big(d_1(t), (Ty)(t)\big) \\ &\quad + f_3\big(d_2(t), (Ty)(t)\big)\,u(t). \end{aligned} \tag{1.2}$$

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In the present paper we consider DAE systems (1.2) which may arise from different models or applications, i.e., T is not necessarily a solution operator as in the motivation above but some causal operator with a bounded input, bounded output property, see Section II for details.

We consider output regulation for systems (1.2). It is the aim of the present paper to prove that the application of the *funnel controller*

$$\begin{aligned} u(t) &= -k(t)\,e(t), \quad \text{where} \quad e(t) = y(t) - y_{\mathrm{ref}}(t), \\ k(t) &= \frac{\hat k}{1 - \varphi(t)^2 \|e(t)\|^2}\,. \end{aligned} \tag{1.3}$$

to the system (1.2) achieves tracking of the reference signal y_{ref} by the output signal y within the pre-specified performance funnel

$$\mathcal{F}_\varphi := \{\ (t, e) \in \mathbb{R}_{\ge 0} \times \mathbb{R}^m \ \mid \varphi(t)\|e\| < 1\ \}. \tag{1.4}$$

The concept of funnel control as a simple strategy for output regulation has been developed in [5] for ODEs, see also the survey [6] and the references therein. Funnel control for linear DAE systems has been investigated in the recent papers [7], [8], [9], [10]. In the present paper we study funnel control for nonlinear DAE systems. This generalizes the results for nonlinear ODE systems obtained in [5], [11] and the results for linear DAE systems obtained in [9].

The paper is organized as follows: We introduce the class of systems (1.2) considered in the present paper and, in particular, the class of operators T allowed in (1.2) in Section II. Two preliminary results for the proof of our main result - Theorem 4.1 - are provided in Section III. In Section IV the concept of funnel control is introduced and it is proved that the funnel controller achieves tracking with prescribed transient behavior.

II. SYSTEM CLASS

We study nonlinear functional DAEs (1.2), where T is a causal operator and d_1, d_2 are extraneous disturbances. We extend a class of operators which has been introduced in [11].

Definition 2.1 (Operator class $\mathcal{T}_{m,q}$): For $t \ge 0$, $w \in \mathcal{C}([0,t];\mathbb{R}^m)$, $\tau > t$ and $\delta > 0$, define the following set of extensions of w:

$$\mathcal{C}(w; t, \tau, \delta) := \left\{ v \in \mathcal{C}([0,\tau];\mathbb{R}^m) \;\middle|\; \begin{array}{l} v|_{[0,t]} = w \ \wedge\ \forall\, s \in [t,\tau] : \\ \|v(s) - w(s)\| \le \delta \end{array} \right\}.$$

An operator $T : \mathcal{C}(\mathbb{R}_{\ge 0};\mathbb{R}^m) \to \mathcal{C}^1(\mathbb{R}_{\ge 0};\mathbb{R}^q)$ is said to be of class $\mathcal{T}_{m,q}$ if, and only if,

(i) T is a causal operator,

(ii) $\forall\, t \geq 0 \;\forall\, w \in \mathcal{C}([0,t];\mathbb{R}^m) \;\exists\, \tau > t \;\exists\, \delta > 0$
$\exists\, c_0 > 0 \;\forall\, u, v \in \mathcal{C}(w; t, \tau, \delta)$:

$$\max_{s\in[t,\tau]} \|(Tu)(s) - (Tv)(s)\| \leq c_0 \max_{s\in[t,\tau]} \|u(s) - v(s)\|,$$

(iii) $\forall\, c_1 > 0 \;\exists\, c_2 > 0 \;\forall\, v \in \mathcal{C}(\mathbb{R}_{\geq 0};\mathbb{R}^m)$:

$$\sup_{s\in\mathbb{R}_{\geq 0}} \|v(s)\| \leq c_1 \implies \sup_{t\in\mathbb{R}_{\geq 0}} \|(Tv)(t)\| \leq c_2,$$

(iv) $\exists\, h \in \mathcal{C}(\mathbb{R}^m \times \mathbb{R}^q;\mathbb{R}^q) \;\exists\, \tilde{T} : \mathcal{C}(\mathbb{R}_{\geq 0};\mathbb{R}^m) \to \mathcal{C}(\mathbb{R}_{\geq 0};\mathbb{R}^q)$ with Properties (i)–(iii) $\forall\, v \in \mathcal{C}(\mathbb{R}_{\geq 0};\mathbb{R}^m) \;\forall\, t \geq 0$:

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(Tv)(t) = h\big(v(t), (\tilde{T}v)(t)\big).$$

For a motivation of the properties of the operators within the class $\mathcal{T}_{m,q}$ see [11]. Compared to [11], we have added Property (iv) which is needed for the case where equation (1.2) has parts with relative degree smaller or equal to zero - a differentiation of these parts is required for the solvability of the closed-loop system (1.2), (1.3). Assumption (iv) is not very restrictive since usually T is an integral operator or a solution operator of a differential equation.

Property (iii) of the operators in $\mathcal{T}_{m,q}$ is a bounded-input, bounded-output assumption and is the counterpart to the assumption of asymptotically stable zero dynamics used for linear systems e.g. in [6], [9].

Remark 2.2: Linear ODE minimum-phase systems with positive definite high-frequency gain matrix can be written in the form

$$\dot{y}(t) = p(t) + (Ty)(t) + Bu(t), \tag{2.1}$$

where $p \in \mathcal{C}^1(\mathbb{R}_{\geq 0};\mathbb{R}^m)$, $B \in \mathbb{R}^{m\times m}$ satisfies $B = B^\top > 0$ and

$$(Ty)(t) := A_1 y(t) + A_2 \int_0^t e^{A_4(t-s)} A_3 y(s)\,\mathrm{d}s\,, \tag{2.2}$$

where $A_1 \in \mathbb{R}^{m\times m}$, $A_2 \in \mathbb{R}^{m\times p}$, $A_3 \in \mathbb{R}^{p\times m}$, $A_4 \in \mathbb{R}^{p\times p}$, defines an operator T which satisfies properties (i) and (ii) in Definition 2.1. Property (iii) is satisfied if A_4 has spectrum in the open left complex half plane. For property (iv) to hold, the equality

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}(Ty)(t) &= A_1\dot{y}(t) + A_2A_3y(t)\\ &+ A_2A_4\int_0^t e^{A_4(t-s)}A_3y(s)\,\mathrm{d}s = h\big(y(t), (\tilde{T}y)(t)\big)\end{aligned}$$

needs to be satisfied for some $h \in \mathcal{C}(\mathbb{R}^m \times \mathbb{R}^q;\mathbb{R}^q)$ and $\tilde{T} : \mathcal{C}(\mathbb{R}_{\geq 0};\mathbb{R}^m) \to \mathcal{C}(\mathbb{R}_{\geq 0};\mathbb{R}^q)$ with properties (i)–(iii) and for all $y \in \mathcal{C}(\mathbb{R}_{\geq 0};\mathbb{R}^m)$ and all $t \geq 0$. Hence, property (iv) holds if, and only if, $A_1 = 0$. Therefore, the term $A_1y(t)$ in the definition of T has to be delegated into the function f_1 in the formulation (1.2). Then T belongs to $\mathcal{T}_{m,m}$.

The above observation for linear ODE systems implies that the initial points of Ty are uniquely determined by $y(0)$. In the general case, by causality of $T \in \mathcal{T}_{m,q}$ there exists $j : \mathbb{R}^m \to \mathbb{R}^q$ such that

$$\forall\, v \in \mathcal{C}(\mathbb{R}_{\geq 0};\mathbb{R}^m) : \; (Tv)(0) = j\big(v(0)\big). \tag{2.3}$$

Definition 2.3 (System class $\Sigma_{m,p,q,r}$): The functional differential-algebraic equation (1.2) is said to define a *system of class* $\Sigma_{m,p,q,r}$, and we write $(\Gamma, f_1, f_2, f_3, T, d_1, d_2) \in \Sigma_{m,p,q,r}$, if, and only if,

(i) $\exists\, R \in \mathbb{R}^{m\times r} \;\exists\, G \in \mathcal{C}(\mathbb{R}^m;\mathbb{R}^{r\times r}) \;\forall\, y \in \mathbb{R}^m$: $G(y) > 0 \;\wedge\; \Gamma(y) = RG(y)R^\top$,

(ii) $f_1 \in \mathcal{C}^1(\mathbb{R}^m;\mathbb{R}^m)$ and, for any basis matrix K of $\ker R^\top$, it holds that $K^\top f_1' K$ is bounded,

(iii) $f_2 \in \mathcal{C}^1(\mathbb{R}^p \times \mathbb{R}^q;\mathbb{R}^m)$,

(iv) $f_3 \in \mathcal{C}^1(\mathbb{R}^p \times \mathbb{R}^q;\mathbb{R}) \;\wedge\; \exists\, \alpha > 0 \;\forall\, (d, v) \in \mathbb{R}^p \times \mathbb{R}^q$: $f_3(d, v) \geq \alpha$,

(v) $T \in \mathcal{T}_{m,q}$,

(vi) $d_1, d_2 \in \mathcal{C}^1(\mathbb{R}_{\geq 0};\mathbb{R}^p)$ are bounded.

Linear ODE systems of the form (2.1) belong to $\Sigma_{m,m,m,m}$ with $\Gamma = B^{-1}$. Furthermore, the system class $\Sigma_{m,p,q,r}$ encompasses even singular DAE systems (descriptor systems). In (ii), the assumption on the derivative of f_1 is essential for the solvability of the closed-loop system (1.2), (1.3). More precise, we will require that $\hat{k}$ in (1.3) is larger than the infimum norm of $K^\top f_1' K$ multiplied with $\|(K^\top K)^{-1}\|$ and divided by α from (iv) in order to guarantee invertibility of $\alpha\hat{k}I - \|(K^\top K)^{-1}\| K^\top f_1' K$. The latter is crucial for the explicit solution of the hidden algebraic constraint on the output error in the closed-loop system (1.2), (1.3), i.e., it guarantees that this system is index-1, cf. [12], [13].

III. Preliminary results

We show three lemmata which are important for the proof of our main result.

Lemma 3.1: Let $D \in \mathcal{C}(\mathbb{R}^m;\mathbb{R}^{l\times l})$ be such that $D(y) > 0$ for all $y \in \mathbb{R}^m$ and let $K \subseteq \mathbb{R}^m$ be compact. Then

$$\exists\, \beta > 0 \;\forall\, y \in K : \; \min \sigma\big(D(y) + D(y)^\top\big) \geq \beta.$$

Proof: Since $D + D^\top$ is pointwise symmetric and positive definite, there exists a pointwise eigenvalue decomposition of the form

$$\begin{aligned}D(y) + D(y)^\top &= V(y)J(y)V(y)^\top,\\ J(y) &= \operatorname{diag}\left(\lambda_1(y), \ldots, \lambda_l(y)\right),\end{aligned}$$

where $V(y)$ is orthogonal and $\lambda_i(y) > 0$ for all $i = 1, \ldots, l$ and all $y \in \mathbb{R}^m$. Seeking a contradiction, we assume that

$$\forall\, \beta > 0 \;\exists\, y \in K \;\exists\, i \in \{1, \ldots, l\} : \; \lambda_i(y) < \beta.$$

Let $\beta > 0$ be arbitrary and choose $y \in K$ and $i \in \{1, \ldots, l\}$ such that $\lambda_i(y) < \beta$. Define $x(y) := V(y)e_i$ and observe that

$$\begin{aligned}\big(D(y) + D(y)^\top\big)x(y) &= V(y)J(y)e_i\\ &= \lambda_i(y)V(y)e_i = \lambda_i(y)x(y).\end{aligned}$$

This implies that

$$x(y)^\top\big(D(y) + D(y)^\top\big)x(y) = \lambda_i(y) < \beta,$$

and, since $x(y) \in \mathcal{S}_m^1 = \{\; x \in \mathbb{R}^m \;|\; \|x\| = 1 \;\}$, we have

$$\forall\, \beta > 0 \;\exists\, y \in K \;\exists\, x \in \mathcal{S}_m^1 : \; x^\top\big(D(y) + D(y)^\top\big)x < \beta.$$

We may hence choose sequences $(y_n) \in K^{\mathbb{N}}$ and $(x_n) \in (\mathcal{S}_m^1)^{\mathbb{N}}$ such that

$$\lim_{n\to\infty} x_n^\top \big(D(y_n) + D(y_n)^\top\big)x_n = 0.$$

However, this contradicts the fact that the continuous map

$$\mathbb{R}^l \times \mathbb{R}^m \ni (x,y) \mapsto x^\top \big(D(y) + D(y)^\top\big)x$$

has a positive minimum on the compact set $\mathcal{S}_m^1 \times K$. ■

Lemma 3.2: Let $D \in \mathbb{R}^{l\times l}$ be positive definite. Then

$$\forall\, z \in \mathbb{R}^l:\ z^\top D^{-1} z \geq \frac{\sigma_{\min}(D+D^\top)}{2\sigma_{\max}(DD^\top)}\|z\|^2,$$

where $\sigma_{\min}(D+D^\top)$ denotes the smallest eigenvalue of $D+D^\top$ and $\sigma_{\max}(DD^\top)$ denotes the largest eigenvalue of $DD^\top$, which are both real and positive.

Proof:

$$\begin{aligned} z^\top D^{-1} z &= \frac{1}{2} z^\top D^{-\top}(D+D^\top)D^{-1}z \\ &\geq \frac{1}{2}\sigma_{\min}(D+D^\top)\|D^{-1}z\|^2 \\ &= \frac{1}{2}\sigma_{\min}(D+D^\top) z^\top (DD^\top)^{-1} z \\ &\geq \frac{\sigma_{\min}(D+D^\top)}{2\sigma_{\max}(DD^\top)}\|z\|^2. \end{aligned}$$

■

Lemma 3.3: Let $K \in \mathbb{R}^{m\times r}$ with $\operatorname{rk} K = r$ and $M \in \mathbb{R}^{m\times m}$ such that $M = M^\top \geq 0$. Then

$$\left\|(K^\top K + K^\top M K)^{-1}\right\| \leq \left\|(K^\top K)^{-1}\right\|.$$

Proof: It follows from [14, Prop. 8.6.6] that

$$0 < (K^\top K + K^\top M K)^{-1} \leq (K^\top K)^{-1},$$

and this implies the assertion of the lemma. ■

IV. Funnel control

In this section we prove the main result of the paper: the funnel controller (1.3) achieves tracking of a reference trajectory by the output signal with prescribed transient behavior. Let

$$\mathcal{B}^1(\mathbb{R}_{\geq 0};\mathbb{R}^m) := \big\{\eta \in \mathcal{C}^1(\mathbb{R}_{\geq 0};\mathbb{R}^m) \,|\, \eta,\, \dot\eta \text{ are bounded}\big\}$$

and associate, for any function φ belonging to

$$\Phi := \left\{ \varphi \in \mathcal{B}^1(\mathbb{R}_{\geq 0};\mathbb{R}) \;\middle|\; \begin{array}{l} \varphi(0)=0,\ \varphi(s)>0 \text{ for all } s>0 \\ \text{and } \liminf_{s\to\infty}\varphi(s)>0 \end{array} \right\}$$

the performance funnel $\mathcal{F}_\varphi$, see (1.4) and Figure 1.

The control objective is feedback control so that the tracking error $e = y - y_{\rm ref}$, where $y_{\rm ref}$ is the reference signal, evolves within $\mathcal{F}_\varphi$ and all variables are bounded. More specific, the transient behavior is supposed to satisfy

$$\forall\, t > 0:\ \|e(t)\| < 1/\varphi(t).$$

The bounded-input, bounded-output property of the operator $T \in \mathcal{T}_{m,q}$ can be exploited for an inherent high-gain property of the system (1.2) and hence to maintain error evolution within the funnel: by the design of the controller (1.3), the gain $k(t)$ increases if the norm of the error $\|e(t)\|$ approaches the funnel boundary $1/\varphi(t)$. The control design (1.3) has two advantages: k is non-monotone and (1.3) is a simple static time-varying proportional output feedback.

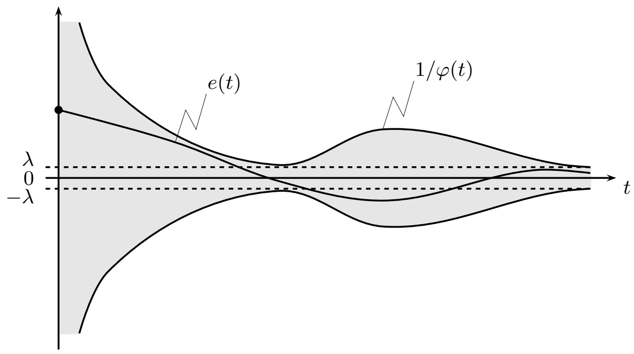

Fig. 1: Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $1/\varphi(t)$ for $t > 0$ and a pole at $t = 0$.

Theorem 4.1 (Funnel control): Let the system $(\Gamma, f_1, f_2, f_3, T, d_1, d_2) \in \Sigma_{m,p,q,r}$ be given, let $\varphi \in \Phi$ define a performance funnel $\mathcal{F}_\varphi$, and use the notation from Definition 2.3. Furthermore, let $y_{\rm ref} \in \mathcal{B}^1(\mathbb{R}_{\geq 0};\mathbb{R}^m)$ be any reference trajectory, K be a basis matrix of $\ker R^\top$, and assume that the initial gain satisfies

$$\hat k > \alpha^{-1}\|(K^\top K)^{-1}\| \sup_{y\in\mathbb{R}^m} \|K^\top f_1'(y) K\|. \tag{4.1}$$

Then, using the function j from (2.3), for any initial value

$$y^0 \in \left\{ w \in \mathbb{R}^m \;\middle|\; \begin{array}{l} K^\top\Big(f_1(w) + f_2\big(d_1(0), j(w)\big) \\ -\hat k\, f_3\big(d_2(0), j(w)\big)\big(w - y_{\rm ref}(0)\big)\Big) = 0 \end{array} \right\},$$

the application of the funnel controller (1.3) to (1.2), $y(0) = y^0$, yields a closed-loop initial-value problem that has a solution and every solution can be extended to a global solution. Furthermore, for every global solution y,

(i) the corresponding tracking error $e = y - y_{\rm ref}$ evolves uniformly within the performance funnel $\mathcal{F}_\varphi$; more precisely,

$$\exists\, \varepsilon > 0\ \forall\, t > 0:\ \|e(t)\| \leq \varphi(t)^{-1} - \varepsilon\,. \tag{4.2}$$

(ii) the corresponding gain function k given by (1.3) is bounded:

$$\forall\, t_0 > 0:\ \sup_{t\geq t_0}|k(t)| \leq \frac{\hat k}{1-(1-\varepsilon\lambda_{t_0})^2},$$

where $\lambda_{t_0} := \inf_{t\geq t_0}\varphi(t) > 0$ for all $t_0 > 0$.

Proof: We proceed in several steps.

Step 1: We show existence of a local solution of the closed-loop system (1.2), (1.3). Set

$$\tilde{\mathcal{D}} \;:=\; \big\{\ (t,e,k) \in \mathbb{R}_{\geq 0}\times\mathbb{R}^{m+1} \ \big|\ \varphi(t)\|e\| < 1\ \big\}.$$

The closed-loop system (1.2), (1.3) may be written in the form

$$\begin{aligned}\Gamma\big(y(t)\big)\,\dot e(t) &= f_1\big(y(t)\big) - k(t) f_3\big(d_2(t),(Ty)(t)\big)\,e(t)\\ &\quad + f_2\big(d_1(t),(Ty)(t)\big) - \Gamma\big(y(t)\big)\,\dot y_{\rm ref}(t),\\ k(t) &= \hat k\,(1-\varphi(t)^2\|e(t)\|^2)^{-1},\end{aligned} \tag{4.3}$$

where $y = e + y_{\rm ref}$. If $\Gamma\big(y^0 - y_{\rm ref}(0)\big)$ were invertible, then the solution theory of functional differential equations (see [11, Thm. B.1]) would guarantee the existence of a local solution with $(t, e(t), k(t)) \in \tilde{\mathcal D}$ and $(t, e(t)) \in \mathcal F_\varphi$ at initial data

$$\begin{pmatrix} e\\ k\end{pmatrix}(0) = \begin{pmatrix} y^0 - y_{\rm ref}(0)\\ \hat k\end{pmatrix}. \tag{4.4}$$

In the present case, we need to decompose equation (4.3) into an ODE part and an algebraic constraint. By differentiating the algebraic constraint we may obtain an ODE in all system variables, the solution of which satisfies the algebraic constraint. In this sense, equation (4.3) is an index-1 DAE, cf. [12], [13].

Step 2: We will now rewrite (4.3) as an explicit functional differential equation. Observe that, by the singular value decomposition, there exists an orthogonal matrix $U \in \mathbb R^{m\times m}$ such that $UR = [\tilde R^\top, 0]^\top$, where $\tilde R \in \mathbb R^{l\times r}$ has full row rank. This implies that, for all $w \in \mathbb R^m$,

$$U\Gamma(w)U^\top = \left[\begin{smallmatrix} D(w) & 0\\ 0 & 0\end{smallmatrix}\right],\ D(w) = \tilde R G(w)\tilde R^\top > 0.$$

We introduce new variables $e_1 := [I_l, 0]Ue$ and $e_2 := [0, I_{m-l}]Ue$. Then (4.3) may be written, by the use of $\|e(t)\|^2 = \|Ue(t)\|^2 = \|e_1(t)\|^2 + \|e_2(t)\|^2$, as the system

$$\begin{aligned}\dot e_1(t) &= [D\big(y(t)\big)^{-1}, 0]U\Big(f_1\big(y(t)\big) + f_2\big(d_1(t),(Ty)(t)\big)\Big)\\ &\quad - k(t) f_3\big(d_2(t),(Ty)(t)\big)\,D\big(y(t)\big)^{-1}\,e_1(t)\\ &\quad - [I_l, 0]U\,\dot y_{\rm ref}(t),\\ 0 &= [0, I_{m-l}]U\Big(f_1\big(y(t)\big) + f_2\big(d_1(t),(Ty)(t)\big)\Big)\\ &\quad - k(t) f_3\big(d_2(t),(Ty)(t)\big)\,e_2(t),\\ k(t) &= \hat k\,\big(1-\varphi(t)^2(\|e_1(t)\|^2+\|e_2(t)\|^2)\big)^{-1}.\end{aligned} \tag{4.5}$$

Note that, since D is continuous, the map $w \mapsto D(w)^{-1}$ is continuous as well. Now, differentiation of the second equation in (4.5), and using

$$\begin{aligned}F_{21}(w) &:= [0, I_{m-l}]Uf_1'(w)U^\top[I_l, 0]^\top,\\ F_{22}(w) &:= [0, I_{m-l}]Uf_1'(w)U^\top[0, I_{m-l}]^\top, \quad w \in \mathbb R^m,\end{aligned}$$

yields

$$\begin{aligned}0 &= F_{21}\big(y(t)\big)\,\dot e_1(t) + F_{22}\big(y(t)\big)\,\dot e_2(t)\\ &\quad + [0, I_{m-l}]Uf_2'\big(d_1(t),(Ty)(t)\big)\begin{pmatrix}\dot d_1(t)\\ h\big(y(t),(\tilde Ty)(t)\big)\end{pmatrix}\\ &\quad - k(t)e_2(t) f_3'\big(d_2(t),(Ty)(t)\big)\begin{pmatrix}\dot d_2(t)\\ h\big(y(t),(\tilde Ty)(t)\big)\end{pmatrix}\\ &\quad - f_3\big(d_2(t),(Ty)(t)\big)\Big(\dot k(t)e_2(t) + k(t)\dot e_2(t)\Big).\end{aligned} \tag{4.6}$$

Observe that the derivative of k is given by

$$\begin{aligned}\dot k(t) &= 2k(t)\,\big(1-\varphi(t)^2(\|e_1(t)\|^2+\|e_2(t)\|^2)\big)^{-1}\\ &\quad\times\big(\varphi(t)\dot\varphi(t)(\|e_1(t)\|^2+\|e_2(t)\|^2)\\ &\quad + \varphi(t)^2(e_1(t)^\top\dot e_1(t) + e_2(t)^\top\dot e_2(t))\big).\end{aligned} \tag{4.7}$$

Introduce the set

$$\mathcal D := \Big\{(t,k,e_1,e_2) \in \mathbb R_{\ge 0}\times[\hat k,\infty)\times\mathbb R^l\times\mathbb R^{m-l}\ \Big|\ \varphi(t)^2(\|e_1\|^2+\|e_2\|^2) < 1\Big\}$$

and define

$$\begin{aligned}\xi : \mathbb R_{\ge0}\times\mathbb R^l\times\mathbb R^{m-l} &\to \mathbb R^m,\\ (t,e_1,e_2) &\mapsto U^\top\begin{pmatrix}e_1\\ e_2\end{pmatrix} + y_{\rm ref}(t),\\ \Theta_1 : \mathcal C(\mathbb R_{\ge0};\mathbb R^l)\times\mathcal C(\mathbb R_{\ge0};\mathbb R^l) &\to \mathcal C^1(\mathbb R_{\ge0};\mathbb R^q),\\ (e_1,e_2) &\mapsto T(\xi(\cdot,e_1(\cdot),e_2(\cdot))),\\ \Theta_2 : \mathcal C(\mathbb R_{\ge0};\mathbb R^l)\times\mathcal C(\mathbb R_{\ge0};\mathbb R^l) &\to \mathcal C^1(\mathbb R_{\ge0};\mathbb R^q),\\ (e_1,e_2) &\mapsto \tilde T(\xi(\cdot,e_1(\cdot),e_2(\cdot))),\end{aligned}$$

$$\begin{aligned}&g_1 : \mathcal D\times\mathbb R^q \to \mathbb R^l, \qquad (t,k,e_1,e_2,\eta)\mapsto\\ &[D\big(\xi(t,e_1,e_2)\big)^{-1}, 0]U\Big(f_1\big(\xi(t,e_1,e_2)\big) + f_2\big(d_1(t),\eta\big)\Big)\\ &- kf_3\big(d_2(t),\eta\big)D\big(\xi(t,e_1,e_2)\big)^{-1}e_1(t) - [I_l,0]U\dot y_{\rm ref}(t).\end{aligned}$$

Now, the first equation in (4.5) can be written as

$$\dot e_1(t) = g_1\big(t,k(t),e_1(t),e_2(t),\Theta(e_1,e_2)(t)\big).$$

Further define

$$\begin{aligned}&M : \mathcal D\times\mathbb R^q \to \mathbf{Gl}_{m-l}(\mathbb R),\ (t,k,e_1,e_2,\eta)\mapsto\\ &F_{22}\big(\xi(t,e_1,e_2)\big) - k\,f_3\big(d_2(t),\eta\big)\Big(I_{m-l} +\\ &2\varphi(t)^2\big(1-\varphi(t)(\|e_1\|^2+\|e_2\|^2)\big)^{-1}e_2e_2^\top\Big)\end{aligned}$$

and

$$\begin{aligned}&g_2 : \mathcal D\times\mathbb R^l\times\mathbb R^q\times\mathbb R^q \to \mathbb R^{m-l},\\ &(t,k,e_1,e_2,\tilde e_1,\eta_1,\eta_2)\mapsto F_{21}\big(\xi(t,e_1,e_2)\big)\tilde e_1\\ &+ [0, I_{m-l}]Uf_2'\big(d_1(t),\eta_1\big)\begin{pmatrix}\dot d_1(t)\\ h\big(\xi(t,e_1,e_2),\eta_2\big)\end{pmatrix}\\ &- k\,e_2\,f_3'\big(d_2(t),\eta_1\big)\begin{pmatrix}\dot d_2(t)\\ h\big(\xi(t,e_1,e_2),\eta_2\big)\end{pmatrix}\\ &- 2k\,f_3\big(d_2(t),\eta_1\big)\,\big(1-\varphi(t)^2(\|e_1\|^2+\|e_2\|^2)\big)^{-1}\\ &\quad\times\big(\varphi(t)\dot\varphi(t)(\|e_1\|^2+\|e_2\|^2) + \varphi(t)^2e_1^\top\tilde e_1\big)e_2.\end{aligned}$$

If M is well defined, then inserting $\dot k$ from (4.7) into (4.6) and rearranging according to $\dot e_2$ gives

$$\begin{aligned}&M\big(t,k(t),e_1(t),e_2(t),\Theta_1(e_1,e_2)(t)\big)\,\dot e_2(t) =\\ &g_2\big(t,k(t),e_1(t),e_2(t),\dot e_1(t),\Theta_1(e_1,e_2)(t),\Theta_2(e_1,e_2)(t)\big).\end{aligned}$$

Now we show that M is well defined, i.e., that M is invertible everywhere on its domain. To this end, let K be the basis matrix of $\ker R^\top$ as in the statement of the theorem.

Since $\tilde{K} := U^\top [0, I_{m-l}]^\top$ is another basis matrix of $\ker R^\top$, there exists $T \in \mathbf{Gl}_{m-l}(\mathbb{R})$ such that $\tilde{K} = KT$. We may now write

$$I_{m-l} = \tilde{K}^\top \tilde{K} = T^\top K^\top K T,$$
$$\tilde{e}(e_1, e_2) := U^\top (e_1, e_2^\top)^\top, \ e_1 \in \mathbb{R}^l, e_2 \in \mathbb{R}^{m-l},$$
$$e_2 = \tilde{K}^\top \tilde{e}(e_1, e_2) = T^\top K^\top \tilde{e}(e_1, e_2).$$

The matrix-valued function

$$E : \tilde{\mathcal{D}} \to \mathbb{R}^{m \times m}, \ (t, e, k) \mapsto 2\varphi(t)^2 \left(1 - \varphi(t)^2 \|e\|^2\right)^{-1} ee^\top$$

is symmetric and positive semi-definite everywhere. Then we have that

$$M(t, k, e_1, e_2, \eta) = T^\top K^\top f_1'\big(\xi(t, e_1, e_2)\big) KT$$
$$- k f_3\big(d_2(t), \eta\big) T^\top \Big(K^\top K + K^\top E\big(t, \tilde{e}(e_1, e_2), k\big) K\Big) T$$

and clearly M is invertible everywhere if, and only if, $T^{-\top} M T^{-1}$ is invertible everywhere. By Lemma 3.3 and (4.1) we obtain, for all $(t, k, e_1, e_2, \eta) \in \mathcal{D} \times \mathbb{R}^q$, that

$$\|k^{-1} f_3\big(d_2(t), \eta\big)^{-1} \big(K^\top K + K^\top E(t, \tilde{e}(e_1, e_2), k) K\big)^{-1}$$
$$\times K^\top f_1'\big(\xi(t, e_1, e_2)\big) K\|$$
$$\le \hat{k}^{-1} \alpha^{-1} \|(K^\top K)^{-1}\| \|K^\top f_1'\big(\xi(t, e_1, e_2)\big) K\| < 1.$$

This implies that M is invertible everywhere. Now, with

$$\tilde{g}_2 : \mathcal{D} \times \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^{m-l}, \ (t, k, e_1, e_2, \eta_1, \eta_2) \mapsto$$
$$M(t, k, e_1, e_2, \eta_1)^{-1} g_2\big(t, k, e_1, e_2, g_1(t, k, e_1, e_2, \eta_1), \eta_1, \eta_2\big),$$

and

$$g_3 : \mathcal{D} \times \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}, \ (t, k, e_1, e_2, \eta_1, \eta_2) \mapsto$$
$$2k \left(1 - \varphi(t)^2 (\|e_1\|^2 + \|e_2\|^2)\right)^{-1} \Big(\varphi(t)\dot{\varphi}(t)(\|e_1\|^2 + \|e_2\|^2)$$
$$+ \varphi(t)^2 \big(e_1^\top g_1(t, k, e_1, e_2, \eta_1) + e_2^\top \tilde{g}_2(t, k, e_1, e_2, \eta_1, \eta_2)\big)\Big)$$

we obtain the system

$$\begin{aligned} \dot{e}_1(t) &= g_1\big(t, k(t), e_1(t), e_2(t), \Theta_1(e_1, e_2)(t)\big) \\ \dot{e}_2(t) &= \tilde{g}_2\big(t, k(t), e_1(t), e_2(t), \Theta_1(e_1, e_2)(t), \Theta_2(e_1, e_2)(t)\big) \\ \dot{k}(t) &= g_3\big(t, k(t), e_1(t), e_2(t), \Theta_1(e_1, e_2)(t), \Theta_2(e_1, e_2)(t)\big) \end{aligned} \tag{4.8}$$

with initial data

$$(k, e_1, e_2)(0) = \Big(\hat{k}, U(y^0 - y_{\rm ref}(0))\Big) =: \zeta \tag{4.9}$$

Step 3: We show existence of a maximal local solution of (4.8), (4.9) which evolves in $\mathcal{D}$ and leaves every compact subset of $\mathcal{D}$. We may write (4.8), (4.9) for appropriate $F : \mathcal{D} \times \mathbb{R}^{2q} \to \mathbb{R}^{m+1}$ in the form

$$\dot{z}(t) = F\big(t, z(t), (Sz)(t)\big), \quad z(0) = \zeta, \tag{4.10}$$

where $Sz = \big(\Theta_1(e_1, e_2)^\top, \Theta_2(e_1, e_2)^\top\big)^\top$ and $S : \mathcal{C}(\mathbb{R}_{\ge 0}; \mathbb{R}^{m+1}) \to \mathcal{C}^1(\mathbb{R}_{\ge 0}; \mathbb{R}^{2q})$ is an operator with the properties as in [11, Def. 2.1] (note that in [11] only operators with domain $\mathcal{C}(\mathbb{R}_{\ge 0}; \mathbb{R})$ are considered, but the generalization to domain $\mathcal{C}(\mathbb{R}_{\ge 0}; \mathbb{R}^{m+1})$ is straightforward), which follows by invoking that $y_{\rm ref}$ and $\dot{y}_{\rm ref}$ are bounded.

It is clear that g_1, $\tilde{g}_2$ and g_3 are continuous and hence F is continuous.

Then [11, Thm. B.1] is applicable to the system (4.10) (note that in [11] a domain $\mathcal{D} \subseteq \mathbb{R}_{\ge 0} \times \mathbb{R}$ is considered, but the generalization to the higher dimensional case is only a technicality) and we may conclude that

(a) there exists a solution of (4.10), i.e., a function $z \in \mathcal{C}([0, \rho); \mathbb{R}^{m+1})$ for some $\rho \in (0, \infty]$ such that z is locally absolutely continuous, $z(0) = \zeta$, $(t, z(t)) \in \mathcal{D}$ for all $t \in [0, \rho)$ and (4.10) holds for almost all $t \in [0, \rho)$,
(b) every solution can be extended to a maximal solution $z \in \mathcal{C}([0, \omega); \mathbb{R}^{m+1})$, i.e., z has no proper right extension that is also a solution,
(c) if $z \in \mathcal{C}([0, \rho); \mathbb{R}^{m+1})$ is a maximal solution, then the closure of $\operatorname{graph} z$ is not a compact subset of $\mathcal{D}$.

Property (c) follows since F is locally essentially bounded, as it is continuous. Let $z \in \mathcal{C}([0, \omega); \mathbb{R}^{m+1})$ be a maximal solution of (4.10) and observe that z is continuously differentiable since F is continuous. It is clear that z is a maximal solution of (4.8), (4.9) which leaves every compact subset of $\mathcal{D}$.

Step 4: We show that there exists a maximal solution of (4.3), (4.4) which evolves in $\tilde{\mathcal{D}}$ and leaves every compact subset of $\tilde{\mathcal{D}}$. The solution $(e_1, e_2, k) \in \mathcal{C}([0, \omega); \mathbb{R}^{m+1})$ of (4.8) in particular satisfies (4.6). Integration gives, for all $t \in [0, \omega)$,

$$\begin{aligned} &[0, I_{m-l}] U \Big(f_1\big(y(t)\big) + f_2\big(d_1(t), (Ty)(t)\big)\Big) \\ &- k(t) f_3\big(d_2(t), (Ty)(t)\big)\, e_2(t) \\ &- [0, I_{m-l}] U \Big(f_1\big(y^0\big) + f_2\big(d_1(0), j(y^0)\big)\Big) \\ &+ \hat{k} f_3\big(d_2(0), j(y^0)\big) [0, I_{m-l}] U \big(y^0 - y_{\rm ref}(0)\big) = 0, \end{aligned}$$

where $y = e + y_{\rm ref}$ and j is from (2.3). Since $\ker [0, I_{m-l}] U = \ker K$ it follows from the choice of y^0 that (e_1, e_2, k) satisfies the second equation in (4.5), and hence all equations in (4.5) are satisfied on $[0, \omega)$. This leads to a maximal solution $(e, k) \in \mathcal{C}^1([0, \omega); \mathbb{R}^{n+1})$ of (4.3) with $\operatorname{graph}(e, k) \subseteq \tilde{\mathcal{D}}$. Note that the solution is maximal, since the existence of a right extension would lead to a right extension of z, a contradiction. Furthermore, by (c) we have

$$\text{the closure of } \operatorname{graph}(e, k) \text{ is not a compact subset of } \tilde{\mathcal{D}}. \tag{4.11}$$

Step 5: We show that k is bounded. Seeking a contradiction, assume that $k(t) \to \infty$ for $t \to \omega$.

Step 5a: We show that $e_2(t) \to 0$ for $t \to \omega$. Seeking a contradiction, assume that there exist $\kappa > 0$ and a sequence $(t_n) \subseteq \mathbb{R}_{\ge 0}$ with $t_n \nearrow \omega$ such that $\|e_2(t_n)\| \ge \kappa$ for all $n \in \mathbb{N}$. Let $H(t) := [0, I_{m-l}] U f_2\big(d_1(t), (Ty)(t)\big)$, $t \ge 0$, and $\tilde{f}_1(y) := [0, I_{m-l}] U f_1(y)$, $y \in \mathbb{R}^m$. Then, from (4.5) we obtain, for all $t \geq 0$,

$$\|H(t)\| = \|\tilde{f}_1\big(y(t)\big) - k(t) f_3\big(d_2(t), (Ty)(t)\big) e_2(t)\| \\ \geq \Big| \|\tilde{f}_1\big(y(t)\big)\| - k(t) f_3\big(d_2(t), (Ty)(t)\big) \|e_2(t)\| \Big| .$$

Since y is bounded and f_1 is continuous, there exists $\gamma > 0$ such that $\sup_{t\geq 0} \|\tilde{f}_1\big(y(t)\big)\| \leq \gamma$. Since $k(t) \to \infty$, $\|e_2(t_n)\| \geq \kappa$ and $f_3\big(d_2(t_n), (Ty)(t_n)\big) \geq \alpha$, we find that for $n \in \mathbb{N}$ large enough

$$\|\tilde{f}_1\big(y(t_n)\big)\| < k(t_n) f_3\big(d_2(t_n), (Ty)(t_n)\big) \|e_2(t_n)\|$$

and hence

$$\|H(t_n)\| \geq \alpha\kappa k(t_n) - \beta \to \infty \quad \text{for } n \to \infty.$$

However, this contradicts the fact that H is bounded, as d_1 and Ty are bounded (the latter follows from boundedness of y and Property (iii) in Definition 2.1).

Step 5b: Now, if $l = 0$ then $e = e_2$ and we have $\lim_{t\to\omega} \|e(t)\| = 0$, which implies, by boundedness of φ, $\lim_{t\to\omega} \varphi(t)^2 \|e(t)\|^2 = 0$, hence $\lim_{t\to\omega} k(t) = \hat{k}$, a contradiction. Hence, in the following we assume that $l > 0$.

Let $\delta \in (0, \omega)$ be arbitrary but fixed and $\lambda := \inf_{t\in(0,\omega)} \varphi(t)^{-1} > 0$. Since $\dot{\varphi}$ is bounded and $\liminf_{t\to\infty} \varphi(t) > 0$ we find that $\frac{\mathrm{d}}{\mathrm{d}t} \varphi|_{[\delta,\infty)}(\cdot)^{-1}$ is bounded and hence there exists a Lipschitz bound $L > 0$ of $\varphi|_{[\delta,\infty)}(\cdot)^{-1}$. Furthermore, observe that by continuity of $D(\cdot)$, $D(\cdot)^{-1}$ is continuous as well and since y, Ty, d_1 and $\dot{y}_{\rm ref}$ are bounded, the number

$$\mu := \sup_{t\in[0,\omega)} \Big\| [D\big(y(t)\big)^{-1}, 0] U \Big(f_1\big(y(t)\big) \\ + f_2\big(d_1(t), (Ty)(t)\big)\Big) - [I_l, 0] U\, \dot{y}_{\rm ref}(t) \Big\|$$

is well defined. Moreover, Lemma 3.1 implies

$$\exists\, \beta_1 > 0\ \forall\, t \in [0,\omega): \ \min \sigma\big(D(y(t)) + D(y(t))^\top\big) \geq \beta_1.$$

Continuity of D and boundedness of y give

$$\exists\, \beta_2 > 0\ \forall\, t \in [0,\omega): \ \max \sigma\big(D(y(t)) D(y(t))^\top\big) \leq \beta_2.$$

We may now conclude from Lemma 3.2 that

$$\forall\, t \in [0,\omega)\ \forall\, z \in \mathbb{R}^l: \ z^\top D\big(y(t)\big)^{-1} z \geq \frac{\beta_1}{2\beta_2} \|z\|^2. \quad (4.12)$$

Define

$$\nu := \frac{\lambda^2 \hat{k} \alpha \beta_1}{8\beta_2}.$$

Now, choose $\varepsilon > 0$ small enough so that

$$\varepsilon \leq \min \left\{ \frac{\lambda}{2}, \min_{t\in[0,\delta]} (\varphi(t)^{-1} - \|e_1(t)\|) \right\}$$

and

$$L \leq -\mu + \frac{\nu}{\varepsilon}. \quad (4.13)$$

We show that

$$\forall\, t \in (0,\omega): \ \varphi(t)^{-1} - \|e_1(t)\| \geq \varepsilon. \quad (4.14)$$

By definition of ε this holds on $(0,\delta]$. Seeking a contradiction suppose that

$$\exists\, t_1 \in [\delta,\omega): \ \varphi(t_1)^{-1} - \|e_1(t_1)\| < \varepsilon.$$

Then for

$$t_0 := \max \left\{ \, t \in [\delta, t_1) \ \middle| \ \varphi(t)^{-1} - \|e_1(t)\| = \varepsilon \, \right\}$$

we have for all $t \in [t_0, t_1]$ that

$$\varphi(t)^{-1} - \|e_1(t)\| \leq \varepsilon \quad \text{and} \\ \|e_1(t)\| \geq \varphi(t)^{-1} - \varepsilon \geq \lambda - \varepsilon \geq \frac{\lambda}{2}$$

and

$$k(t) = \frac{\hat{k}}{1 - \varphi(t)^2 \|e(t)\|^2} \geq \frac{\hat{k}}{1 - \varphi(t)^2 \|e_1(t)\|^2} \\ = \frac{\hat{k}}{(1 - \varphi(t)\|e_1(t)\|)(1 + \varphi(t)\|e_1(t)\|)} \geq \frac{\hat{k}}{2\varepsilon\varphi(t)} \geq \frac{\lambda\hat{k}}{2\varepsilon}.$$

Now we have, for all $t \in [t_0, t_1]$,

$$\frac{1}{2} \tfrac{\mathrm{d}}{\mathrm{d}t} \|e_1(t)\|^2 = e_1(t)^\top \dot{e}_1(t) \\ \overset{(4.5)}{\leq} \mu \|e_1(t)\| - \alpha k(t) e_1(t)^\top D\big(y(t)\big)^{-1} e_1(t)^\top \\ \overset{(4.12)}{\leq} \mu \|e_1(t)\| - \frac{\lambda \hat{k} \alpha \beta_1}{4\varepsilon\beta_2} \|e_1(t)\|^2$$

This yields that

$$\frac{1}{2} \tfrac{\mathrm{d}}{\mathrm{d}t} \|e_1(t)\|^2 \leq \left(\mu - \frac{\nu}{\varepsilon}\right) \|e_1(t)\| \overset{(4.13)}{\leq} -L \|e_1(t)\|.$$

Therefore, using

$$\frac{1}{2} \tfrac{\mathrm{d}}{\mathrm{d}t} \|e_1(t)\|^2 = \|e_1(t)\| \tfrac{\mathrm{d}}{\mathrm{d}t} \|e_1(t)\|,$$

we find that

$$\|e_1(t_1)\| - \|e_1(t_0)\| = \int_{t_0}^{t_1} \frac{1}{2} \|e_1(t)\|^{-1} \tfrac{\mathrm{d}}{\mathrm{d}t} \|e_1(t)\|^2 \, \mathrm{d}t \\ \leq -L(t_1 - t_0) \leq -|\varphi(t_1)^{-1} - \varphi(t_0)^{-1}| \\ \leq \varphi(t_1)^{-1} - \varphi(t_0)^{-1},$$

and hence

$$\varepsilon = \varphi(t_0)^{-1} - \|e_1(t_0)\| \leq \varphi(t_1)^{-1} - \|e_1(t_1)\| < \varepsilon,$$

a contradiction.

Therefore, (4.14) holds and as $e_2(t) \to 0$ there exists $\tilde{t} \in [0,\omega)$ such that $\|e_2(t)\| \leq \varepsilon$ for all $t \in [\tilde{t}, \omega)$. Then, invoking $\varepsilon \leq \frac{\lambda}{2}$, we obtain for all $t \in [\tilde{t}, \omega)$

$$\|e(t)\|^2 = \|e_1(t)\|^2 + \|e_2(t)\|^2 \leq (\varphi(t)^{-1} - \varepsilon)^2 + \varepsilon^2 \\ \leq \varphi(t)^{-2} - 2\varepsilon\lambda + 2\varepsilon^2 \leq \varphi(t)^{-2} - 2\varepsilon^2.$$

This implies boundedness of k, a contradiction.

Step 6: We show that $\omega = \infty$. First note that by Step 2 and Step 3 we have that $(e,k) : [0,\omega) \to \mathbb{R}^{m+1}$ is bounded. Further noting that boundedness of k is equivalent to (4.2) (for $t \in [0,\omega)$), the assumption $\omega < \infty$ implies existence of a compact subset $\mathcal{K} \subseteq \tilde{\mathcal{D}}$ such that $\operatorname{graph}(x_1, e, x_3, k) \subseteq \mathcal{K}$.

This contradicts (4.11).

Step 7: It remains to show (ii). This follows from

$$\forall\, t > 0: \; k(t) = \hat{k} + k(t)\varphi(t)^2\|e(t)\|^2$$
$$\overset{(4.2)}{\leq} \hat{k} + k(t)\varphi(t)^2(\varphi(t)^{-1} - \varepsilon)^2 = \hat{k} + k(t)(1 - \varphi(t)\varepsilon)^2.$$

This completes the proof of the theorem. ■

We like to stress again that the condition (4.1) in Theorem 4.1 is sufficient for the closed-loop system to be index-1. It is an open problem as to whether (4.1) is also necessary for the index-1 property, although it seems that this is the case.

Remark 4.2: It is only a technicality to extend the proof of Theorem 4.1 to incorporate disturbances and the operator dependency in Γ, i.e., on the left hand side of (1.2) we have

$$\Gamma\big(y(t), (Ty)(t), d_4(t)\big)\, \dot{y}(t).$$

Furthermore, one could allow for f_3 to be matrix valued, incorporate disturbances in f_1 and replace $u(t)$ by $f_5\big(u(t) + d_5(t)\big)$ for some appropriate function f_5 and disturbance d_5.

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