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Funnel Control With Saturation: Linear MIMO Systems

Norman Hopfe, Achim Ilchmann, and Eugene P. Ryan

***Abstract*—Tracking—by the system output—of a reference signal (assumed bounded with essentially bounded derivative) is considered in the context of linear m-input, m-output systems subject to input saturation. The system is assumed to have strict relative degree one with stable zero dynamics. Prespecified is a performance funnel, within which the tracking error is required to evolve: transient and asymptotic behavior of the tracking error is influenced through choice of parameter values which define the funnel. The control structure is a saturating error feedback with a gain function designed to evolve so as to preclude contact with the funnel boundary. A feasibility condition (formulated in terms of the system data, the saturation bounds, the funnel data, bounds on the reference signal, and the initial state) is presented under which the tracking objective is achieved, whilst maintaining boundedness of all signals.**

***Index Terms*—Input saturation, linear systems, output feedback, tracking, transient behavior.**

I. Introduction

In the early 1980s, a novel feature was introduced in adaptive control, namely, adaptive strategies which do not require identification of the particular system being controlled. Pioneering contributions to the area include [1], [5], [6], [8], [11] (see, also, the survey [2] and references therein). The prototypical example of a system class—rather than a single system—is that of linear m-input, m-output systems with relative degree one, positive high-frequency gain and stable zero dynamics, i.e., minimum phase. The simple output feedback $u(t) = -k(t)\,y(t)$ stabilizes each system belonging to the above class provided $k(\cdot)$ is appropriately generated: e.g., by the differential equation $\dot{k}(t) = \|y(t)\|^2$ or variants thereof. The two major drawbacks of the latter strategy (and its variants) are (i) albeit bounded, the gain $k(t)$ is monotonically increasing and (ii) whilst asymptotic performance is guaranteed, transient behavior is not generally taken into account, an exception being the contribution [7]. A fundamentally different approach—so called 'funnel control'—was introduced in [3] in the context of output tracking: this control ensures prespecified transient behavior of the tracking error, has a non-monotone gain, is simpler than the above adaptive controller (insofar as the gain is not dynamically generated), and does not invoke any internal model. It has been successfully applied in experiments controlling the speed of electric devices [4]; see [2] for further applications.

The present technical note adopts the funnel control viewpoint. In common with its precursor [3], we restrict attention to systems with strict relative degree one: however, in contrast with [3], here the presence of explicit input constraints is a distinguishing feature of the underlying system class. A feasibility relationship involving the system

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N. Hopfe and A. Ilchmann are with the Institute of Mathematics, Technical University Ilmenau, Ilmenau 98693, Germany (e-mail: norman.hopfe@tu-ilmenau.de; achim.ilchmann@tu-ilmenau.de).

E. P. Ryan is with the Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, U.K. (e-mail: epr@maths.bath.ac.uk).

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data, funnel data, reference signal data and the saturation bound is derived under which the efficacy of funnel control in the presence of input saturation is established. However, there is a price to pay: sufficient a priori plant information is required in order to check the feasibility condition.

For motivation, consider the simple scalar linear system

$$\dot{y} = ay + bu, \quad a \in \mathbb{R}, \quad b > 0, \quad y(0) = y^0.$$

The control objective is tracking, of a (suitably regular) reference signal r, with prescribed transient and asymptotic behavior: precisely, for some given function $\psi : [0,\infty) \to [\lambda,\infty)$, $\lambda > 0$, the tracking error is required to be bounded by ψ in the sense that $|y(t) - r(t)| < \psi(t)$ for all $t \geq 0$. For example, if $|y^0 - r(0)| < 1$ and $\psi(t) = \max\{1 - \Lambda t, \lambda\}$, with $\Lambda > 0$ and $\lambda \in (0,1)$, then attainment of the tracking objective implies that a prescribed tracking accuracy, quantified by $\lambda > 0$, is achieved in prescribed time $t^* = (1-\lambda)/\Lambda$: specifically, $|y(t) - r(t)| < \lambda$ for all $t \geq t^*$. In the general case, if ψ is globally Lipschitz and bounded away from zero, and the reference signal r is a bounded absolutely continuous function with essentially bounded derivative, then it is known (see [3]) that the tracking objective is achieved by the strategy

$$\left.\begin{aligned} u(t) &= -k(t)[y(t) - r(t)], \\ k(t) &= [\psi(t) - |y(t) - r(t)|]^{-1} \end{aligned}\right\}. \tag{1.1}$$

if, and only if, the following holds: $|y^0 - r(0)| < \psi(0)$. Moreover, the gain k, and hence the control u, is bounded. Next, reconsider the above scalar system, with the same control objective, but now with saturation in the input channel

$$\dot{y} = ay + b\,\mathrm{sat}_{\hat{u}}(u), \;\; a \in \mathbb{R}, \;\; b, \hat{u} > 0, \quad y(0) = y^0 \tag{1.2}$$

where $\mathrm{sat}_{\hat{u}}$ is the saturation function given by $\mathrm{sat}_{\hat{u}} = \hat{u}\,\mathrm{sgn}(u)$ if $|u| > \hat{u}$ and $\mathrm{sat}_{\hat{u}}(u) = u$ otherwise. Again, $|y^0 - r(0)| < \psi(0)$ is a necessary condition for attainment of the objective. However, a moment's reflection confirms that the latter condition is not sufficient: the question of feasibility of the tracking objective in the presence of input saturation is delicate and inevitably involves addressing the interplay between the plant data (a, b, y^0), the reference signal r, the function ψ and the saturation bound $\hat{u}$. For example, if $a > 0$, then it is readily seen that $b\hat{u} \geq a|y^0|$ is a necessary condition for feasibility. Moreover, the saturation level $\hat{u}$ should, for feasibility, also be commensurate with the magnitude of the reference signal r and its derivative $\dot{r}$. To illustrate the interplay between $\hat{u}$ and the function ψ, consider the case wherein $a = 0$, $r(\cdot) = 0$ and $\psi(t) = \max\{1 - \Lambda t, \lambda\}$ with $\Lambda > 0$ and $\lambda \in (0,1)$ (and so ψ is globally Lipschitz, with Lipschitz constant Λ). Assume feasibility of the tracking objective. Then

$$\begin{aligned} 1 - \lambda &= \psi(0) - \psi(t^*) < \psi(0) - y(t^*) \\ &= 1 - y^0 + y^0 - y(t^*) \\ &\leq 1 - y^0 + t^* b\hat{u}, \quad t^* := \frac{(1-\lambda)}{\Lambda} \end{aligned}$$

and since this must hold for all $|y^0| < 1$, we may conclude that $1 - \lambda \leq t^* b\hat{u}$. Therefore, $b\hat{u} \geq \lambda$ is a necessary condition for feasibility.

The purpose of the present technical note is to extend the above ideas to a more general context of m-input u, m-output y, n-dimensional linear systems (A, B, C) subject to input saturation. Two scenarios are investigated: in Scenario A, the saturation constraint is *Euclidean* in the sense that, for some $\hat{u} > 0$ the input u is required to satisfy the constraint

$$\|u(t)\| \leq \hat{u} \quad \forall t \geq 0 \tag{1.3}$$

where $\|\cdot\|$ denotes the Euclidean norm (the induced matrix norm is similarly denoted). In Scenario B, the saturation constraint is imposed *componentwise* in the sense that, for some $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_m)$, $\hat{u}_i > 0$, the input $u = (u_1, \ldots, u_m)$ is required to satisfy

$$|u_i(t)| \leq \hat{u}_i \quad \forall\, t \geq 0\, \forall\, i = 1, \ldots, m. \tag{1.4}$$

Restricting momentarily to the single-input, single-output case (in which case (1.3) and (1.4) are equivalent), prespecified is a performance funnel $\mathcal{F}(\psi) = \{(t,\xi) | \, |\xi| < \psi(t)\}$ determined by some suitable function ψ. The *control objective* is output tracking: determine a feedback structure which ensures that, for a given reference signal $r \in W^{1,\infty}(\mathbb{R}_+)$ (the space of bounded locally absolutely continuous functions $r : \mathbb{R}_+ := [0,\infty) \to \mathbb{R}$ with essentially bounded derivative $\dot{r}$), the output tracking error $e = y - r$ evolves within the funnel (i.e., $\mathrm{graph}(e) \subset \mathcal{F}(\psi)$): transient and asymptotic behavior of the tracking error is influenced through choice of ψ. The proposed control structure is a saturating error feedback of the form $u(t) = -\mathrm{sat}_{\hat{u}}(k(t)e(t))$ wherein the gain function $k : t \mapsto 1/(\psi(t) - |e(t)|)$ evolves so as to preclude contact with the funnel boundary. A feasibility condition (formulated in terms of the plant data (A, B, C) and $\hat{u}$, the funnel data ψ, the reference signal r, and the initial state x^0) is presented under which the tracking objective is achieved, whilst maintaining boundedness of the state x and gain function k.

In the highly specialized context of the motivating scalar system (1.2), the main result of the technical note translates into the following: if

$$\left.\begin{aligned} |y^0 - r(0)| < \psi(0) \quad \text{and} \\ b\hat{u} \geq |a|\,[\|\psi\|_\infty + \|r\|_\infty] + \|\dot{r}\|_\infty + \Lambda, \end{aligned}\right\} \tag{1.5}$$

where $\|\cdot\|_\infty$ denotes the L^∞-norm, then the simple control strategy (wherein $e(t) = y(t) - r(t)$)

$$u(t) = -\mathrm{sat}_{\hat{u}}(k(t)e(t)), \quad k(t) = \Big[\psi(t) - |e(t)|\Big]^{-1}$$

ensures attainment of the tracking objective (and, moreover, the gain function k is bounded). Furthermore, if the first inequality in (1.5) is replaced by $|y^0 - r(0)| < \hat{u}[1 + \hat{u}]^{-1}\psi(0)$, then input saturation does not occur and so the control strategy coincides with (1.1).

II. The System Class

Consider the m-input, m-output linear system

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x^0 \in \mathbb{R}^n, \\ y(t) &= Cx(t), \end{aligned}\right\} \tag{2.1}$$

with $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ and assume that the *minimum-phase* condition holds

$$s \in \mathbb{C}, \quad \mathrm{Re}\, s \geq 0 \Longrightarrow \det \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} \neq 0. \tag{2.2}$$

We assume that the system has strict relative degree one

$$\det(CB) \neq 0. \tag{2.3}$$

It is immediate that, if (2.3) holds, then, for any $V \in \mathbb{R}^{n\times(n-m)}$ such that

$$\operatorname{im} V = \ker C, \quad N := (V^\top V)^{-1} V^\top [I_n - B(CB)^{-1}C] \quad (2.4)$$

the similarity transformation $S := (C^\top, N^\top)^\top$ has inverse $S^{-1} = \left(B(CB)^{-1},\, V\right)$ and takes system (2.1) into the form

$$\left.\begin{aligned} \dot y(t) &= A_1 y(t) + A_2 z(t) + CBu(t), \quad & y(0) = Cx^0 \\ \dot z(t) &= A_3 y(t) + A_4 z(t), & z(0) = Nx^0 \end{aligned}\right\} \quad (2.5)$$

where

$$\left.\begin{aligned} A_1 &:= CAB(CB)^{-1}, \quad & A_2 := CAV, \\ A_3 &:= NAB(CB)^{-1}, & A_4 := NAV. \end{aligned}\right\}. \quad (2.6)$$

Moreover, if (2.2) holds, then A_4 is a *Hurwitz* matrix, that is, $\operatorname{spec} A_4 \subset \{s \in \mathbb{C} \mid \operatorname{Re} s < 0\}$, in which case, there exist positive constants $\alpha,\ \beta > 0$ such that

$$\|\exp(A_4 t)\| \le \beta e^{-\alpha t} \quad \forall\, t \ge 0. \quad (2.7)$$

Now for any solution (y, z, u) of (2.5) on some interval $[0,\omega) \subset \mathbb{R}_+$ we have

$$z(t) = e^{A_4 t} N x^0 + \int_0^t e^{A_4(t-s)} A_3 y(s)\,\mathrm{d}s \quad \forall t \in [0,\omega)$$

and so, by (2.7)

$$\|z(t)\| \le \beta \|Nx^0\| + \frac{\beta}{\alpha}\|A_3\|\,\|y\|_{L^\infty(0,t)} \quad \forall t \in [0,\omega). \quad (2.8)$$

III. Performance Funnels

A central ingredient of our approach is the concept of a performance funnel within which the tracking error is $e = y - r$, where r is a reference signal, is required to evolve. First, we introduce the family of functions, parameterized by $\Lambda \ge 0$ and $\lambda > 0$

$$\mathcal{G}(\Lambda,\lambda) := \Big\{\psi : \mathbb{R}_+ \to [\lambda,\infty) \mid \psi \text{ bounded and Lipschitz with Lipschitz constant } \Lambda\Big\}. \quad (3.1)$$

The performance funnel takes one of two forms, depending on the nature of the input saturation constraint. In Scenario A (i.e., Euclidean saturation (1.3)), for $\Lambda \ge 0$, $\lambda > 0$ and $\psi \in \mathcal{G}(\Lambda,\lambda)$, the funnel is given by

$$\mathcal{F}(\psi) := \{(t,\eta) \in \mathbb{R}_+ \times \mathbb{R}^m \mid \|\eta\| < \psi(t)\}. \quad (3.2)$$

In Scenario B (i.e., componentwise saturation (1.4)), given $\Lambda_i \ge 0$ and $\lambda_i > 0$, $i = 1,\ldots,m$, the funnel is a Cartesian product of m component funnels

$$\mathcal{F}(\psi) := \mathop{\times}_{i=1}^{m} \Big\{(t,\eta) \in \mathbb{R}_+ \times \mathbb{R} \Big| \, |\eta| < \psi_i(t)\Big\} \quad (3.3)$$

determined by the family $\psi = (\psi_1,\ldots,\psi_m)$ of functions $\psi_i \in \mathcal{G}(\Lambda_i,\lambda_i)$, $i = 1,\ldots,m$. In each scenario, the control objective is a feedback structure which—given a reference signal $r \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R}^m)$ and under appropriate feasibility conditions—ensures that the closed-loop system has unique global bounded solution $x : \mathbb{R}_+ \to \mathbb{R}^n$ and the tracking error $e = y - r$ evolves within the corresponding performance funnel.

A variety of funnels are possible. Consider, for example, Scenario A with given $\Lambda > 0$ and $\lambda > 0$. Choosing $a,\ b > 0$ such that $a > \lambda$ and $ab \le \Lambda$, then the function $t \mapsto \psi(t) := \lambda \max\{ae^{-bt},\lambda\}$ is in $\mathcal{G}(\Lambda,\lambda)$ and evolution within the associated funnel ensures a prescribed exponential decay in the transient phase $[0,T]$, $T = \ln(a/\lambda)/b$, and tracking accuracy $\lambda > 0$ thereafter. We remark that, as in Fig. 1, the funnel boundary need not be monotone. For Scenario B, an example is given in Section V of a funnel with a component having non-monotone boundary.

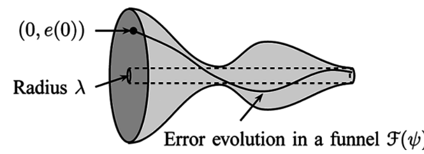


Fig. 1. Scenario A: prescribed performance funnel $\mathcal{F}(\psi)$.

IV. Main Results

The main contributions lie in the following two theorems, with proofs in the Appendix. In the precursor [3], the efficacy of *unconstrained* funnel control was established for the class of systems (2.1) which are minimum phase, relative degree one and have positive high-frequency gain. We consider the same class of systems, but now subject to input saturation.

A. Euclidean Saturation Constraint

First, we consider the case wherein the input is subject to the constraint (1.3) for some $\widehat{u} > 0$. With the input constraint parameter $\widehat{u}$, we associate the saturation function (the m-dimensional analogue of the scalar function in Section I)

$$\begin{aligned} \operatorname{sat}_{\hat u} : \mathbb{R}^m &\to \{w \in \mathbb{R}^m \mid \|w\| \le \widehat{u}\}, \\ v &\mapsto \operatorname{sat}_{\hat u}(v) := \begin{cases} \widehat{u}\|v\|^{-1}v, & \|v\| > \widehat{u} \\ v, & \text{otherwise.} \end{cases} \end{aligned} \quad (4.1)$$

In addition to the hypotheses of its precursor [3], the presence of input saturation in the present technical note necessitates an additional assumption on the system, namely, the feasibility assumption (4.4) of Theorem 4.1 below.

Theorem 4.1: Assume that system (2.1) is minimum phase, i.e., (2.2) holds, has strict relative degree one and positive-definite high-frequency gain, i.e.,

$$\exists \gamma > 0 \quad \forall v \in \mathbb{R}^m : \quad v^\top CBv \ge \gamma\|v\|^2. \quad (4.2)$$

Let $\Lambda \ge 0$, $\lambda > 0$ and $\psi \in \mathcal{G}(\lambda,\Lambda)$ define the performance funnel $\mathcal{F}(\psi)$ as in (3.2). Assume that the initial data $x^0 \in \mathbb{R}^n$ and reference signal $r \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R}^m)$ are such that

$$\|Cx^0 - r(0)\| < \psi(0). \quad (4.3)$$

Adopting the notation of (2.4)–(2.7), assume that $\widehat{u} > 0$ is such that the feasibility assumption

$$\gamma\widehat{u} - (L + \Lambda) =: \Delta > 0 \quad (4.4)$$

holds, with

$$L := \left[\|A_1\| + \|A_2\|\|A_3\|\frac{\beta}{\alpha}\right]\left[\|\psi\|_\infty + \|r\|_\infty\right] + \beta\|A_2\|\|Nx^0\| + \|\dot r\|_\infty. \quad (4.5)$$

Then application of the feedback strategy

$$\left.\begin{aligned} u(t) &= -\operatorname{sat}_{\hat u}(k(t)e(t)), \\ k(t) &= [\psi(t) - \|e(t)\|]^{-1}, \quad e(t) = Cx(t) - r(t) \end{aligned}\right\} \quad (4.6)$$

to (2.1) yields a closed-loop initial-value problem with the following properties.

(i) Precisely one maximal solution $x : [0, \omega) \to \mathbb{R}^n$ exists and this solution is *global* (i.e., $\omega = \infty$).

(ii) The global solution x is bounded and the tracking error $e = Cx - r$ evolves within the performance funnel $\mathcal{F}(\psi)$; more precisely, for all $t \geq 0$

$$\psi(t) - \|e(t)\| \geq \varepsilon := \min\left\{\frac{\lambda}{2}, \frac{\lambda}{2\widehat{u}}, \psi(0) - \|e(0)\|\right\}. \qquad (4.7)$$

(iii) The gain function k is bounded, with $\|k\|_\infty \leq 1/\varepsilon$.

(iv) The input is unsaturated at some time; i.e., there exists $\tau \geq 0$ such that $\|u(\tau)\| < \widehat{u}$.
If the input is unsaturated at time τ, then it remains unsaturated thereafter, i.e.,

$$\tau \geq 0, \quad \|u(\tau)\| < \widehat{u} \Longrightarrow \|u(t)\| < \widehat{u} \quad \forall\, t \geq \tau.$$

The input is globally unsaturated (i.e., $\|u(t)\| < \widehat{u}$ for all $t \geq 0$) if, and only if

$$\|e(0)\| < \psi(0)\frac{\widehat{u}}{(1+\widehat{u})}. \qquad (4.8)$$

(In which case, the first of equations (4.6) takes the simple form $u(t) = -k(t)e(t)$).

Remark 4.2:

(a) Hypothesis (4.2) is simply the assumption that CB is positive definite: symmetry of CB is not required.

(b) In view of the potential singularity in (4.6), some care must be exercised in formulating the closed-loop initial-value problem (2.1), (4.6). This is done in Step 1 of the proof, wherein the closed-loop initial-value problem is posed as

$$\dot{x}(t) = F(t, x(t)), \quad x(0) = x^0, \quad (0, x^0) \in \mathcal{D} \qquad (4.9)$$

for suitable $F : \mathcal{D} \to \mathbb{R}^n$ with appropriately defined domain $\mathcal{D}$. By a solution of (4.9) we mean a continuously differentiable function $x : [0, \omega) \to \mathbb{R}^n$ which satisfies (4.9) and has graph in $\mathcal{D}$; x is maximal if it has no right extension that is also a solution; x is global if $\omega = \infty$. Assertion (i) of the theorem confirms the existence of precisely one maximal solution x of (4.9) and, moreover, this solution is global. The requirement that $\mathrm{graph}(x)$ is in $\mathcal{D}$ implies that the graph of the tracking error $e = Cx - r$ is in $\mathcal{F}(\psi)$: this—together with boundedness of x— is the content of Assertion (ii). Assertion (iii) establishes boundedness of the control gain function $k(\cdot)$. Assertion (iv) implies that the control input cannot remain saturated for all $t \geq 0$ and, when it becomes unsaturated, then it remains so thereafter; furthermore, if the control is initially unsaturated, then the saturation bound is never attained.

(c) Condition (4.3) is necessary for attainment of the control objective and is equivalent to the requirement that $(0, x^0) \in \mathcal{D}$.

(d) In conjunction with the other hypotheses, the feasibility condition (4.4) is a sufficient condition for attainment of the control objective. It quantifies and exhibits the interplay between the saturation bound (sufficiently large to ensure performance) and bounds on the plant data, funnel data, initial data and reference signal data. In practice, the choice of funnel involves a compromise between the competing goals of high performance versus satisfaction of the feasibility condition. ◇

B. Componentwise Saturation Constraints

Next, we turn our attention to the case in which the saturation constraint is imposed *componentwise* in the sense that, for some $\widehat{u} = (\widehat{u}_1, \ldots, \widehat{u}_m)$, $\widehat{u}_i > 0$, the input $u = (u_1, \ldots, u_m)$ is required to satisfy (1.4). To conform with this componentwise structure, we impose a componentwise performance funnel, as in (3.3). In particular, for prescribed parameters λ_i, Λ_i and functions $\psi_i \in \mathcal{G}(\lambda_i, \Lambda_i)$, $i = 1, \ldots, m$, we seek a control structure which ensures that for any given reference signal $r \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$, the output y is such that the tracking error $e = Cx - r$ evolves componentwise (components e_i, $i = 1, \ldots, m$) in the funnel, that is

$$\mathrm{graph}(e_i) \subset \{(t, \eta) \in \mathbb{R}_+ \times \mathbb{R} |\, |\eta| < \psi_i(t)\} \quad \forall i = 1, \ldots, m.$$

Assume given a reference signal $r \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$ and an m-input, m-output system (A, B, C) with initial data $x^0 \in \mathbb{R}^n$. If the high-frequency gain CB is positive definite and *diagonal*, then the problem essentially decomposes into m single-input, single-output (SISO) subproblems, to each of which Theorem 4.1 (specialized to the SISO case) may be applied. Of more interest is the case in which CB has non-zero off-diagonal entries. We impose that

$$\left.\begin{aligned} \det(CB) \neq 0, \quad [CB]_{ii} > 0, \quad i \in \{1, \ldots, m\}, \\ [CB]_{ij} \leq 0, \quad i, j \in \{1, \ldots, m\},\, i \neq j. \end{aligned}\right\}. \qquad (4.10)$$

We also require a type of "diagonal dominance" condition

$$\sum_{j=1}^{m} [CB]_{ij}\widehat{u}_j - (L + \Lambda_i) =: \Delta_i > 0 \quad \forall i \in \{1, \ldots, m\} \qquad (4.11)$$

with L given by (4.5) (wherein $\psi = (\psi_1, \ldots \psi_m)$). The arguments used in establishing Theorem 4.1 are now readily modified to conclude the following.

Theorem 4.3: Assume that system (2.1) is minimum phase, i.e., (2.2) holds, and is such that (4.10) holds. Let $\Lambda_i \geq 0$, $\lambda_i > 0$ and $\psi_i \in \mathcal{G}(\Lambda_i, \lambda_i)$, $i = 1, \ldots, m$ define the performance funnel given by (3.3) with $\psi = (\psi_1, \ldots, \psi_m)$. Assume that the initial data $x^0 \in \mathbb{R}^n$ and reference signal $r \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$ are such that the initial error $e(0) = Cx^0 - r(0)$ satisfies the componentwise inequalities $|e_i(0)| < \psi_i(0)$, $i = 1, \ldots, m$. Let $\widehat{u}_i > 0$, $i = 1, \ldots, m$, denote the componentwise saturation constraints and, adopting the notation of (2.4) –(2.7), assume that (4.11) holds. Then application of the componentwise feedback strategy

$$\left.\begin{aligned} u_i(t) &= -\mathrm{sat}_{\widehat{u}_i}(k_i(t)e_i(t)), \\ k_i(t) &= \Big[\psi_i(t) - |e_i(t)|\Big]^{-1}, \; e_i(t) = y_i(t) - r_i(t), \end{aligned}\right\} \qquad (4.12)$$

to (2.1) yields a closed-loop initial-value problem with the following properties.

(i) Precisely one maximal solution $x : [0, \omega) \to \mathbb{R}^n$ exists and this solution is *global* (i.e., $\omega = \infty$).

(ii) The global solution x is bounded and, for each $i \in \{1, \ldots, m\}$, the tracking error component e_i evolves within its associated performance funnel; more precisely, for all $t \geq 0$ and all $i = 1, \ldots, m$

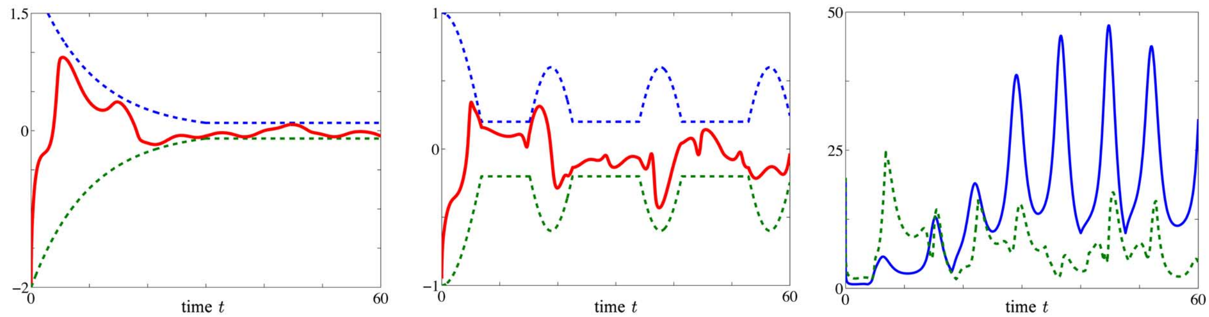
$$\psi_i(t) - |e_i(t)| \geq \varepsilon_i := \min\left\{\frac{\lambda_i}{2}, \frac{\lambda_i}{2\widehat{u}_i}, \psi_i(0) - |e_i(0)|\right\}. \qquad (4.13)$$

(iii) The gain functions k_i are bounded, with $\|k_i\|_\infty \leq 1/\varepsilon_i$, $i = 1, \ldots, m$.

(iv) Each input u_i is unsaturated at some time $\tau_i \geq 0$ and remains unsaturated thereafter. An input u_i is globally unsaturated if, and only if

$$|e_i(0)| < \psi_i(0)\, \frac{\widehat{u}_i}{(1+\widehat{u}_i)}. \qquad (4.14)$$

(In which case, $u_i(t) = -k_i(t)e_i(t)$).



(a) (b) (c)

Fig. 2. Behavior of the closed-loop system (5.1), (4.12). (a) Funded and tracking error e_1. (b) Funnel and tracking error e_2. (c) Gains: k_1 solid line and k_2 dashed line.

V. EXAMPLE

For purposes of illustration, we choose a multi-input, multi-output system of the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y(t) \\ z(t)\end{pmatrix} &= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & 0 & 0 & -3\end{bmatrix}\begin{pmatrix} y(t) \\ z(t)\end{pmatrix} \\ &\quad + \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \\ 0 & 0 \\ 0 & 0\end{bmatrix}\begin{pmatrix} u_1(t) \\ u_2(t)\end{pmatrix}, \\ \begin{pmatrix} y(0) \\ z(0)\end{pmatrix} &= \begin{pmatrix} y^0 \\ z^0\end{pmatrix}\end{aligned} \tag{5.1}$$

where y^0, $z^0 \in \mathbb{R}^2$ and the control inputs are subject to the saturation constraints

$$\begin{aligned}|u_1(t)| &\le \widehat{u}_1 := 20 \quad \forall t \ge 0 \quad \text{and} \\ |u_2(t)| &\le \widehat{u}_2 := 30 \quad \forall t \ge 0.\end{aligned}$$

In this case, $\|A_1\| = 1$, $\|A_2\| = 2$, $\|A_3\| = \sqrt{2}$, and (2.7) holds with $\alpha = (5-\sqrt{2})/2$ and $\beta = 1$. As reference signal we choose $r(\cdot) = [\xi_1(\cdot),\ -\xi_2(\cdot)]^\top$, the first and second components of the solution of the Lorentz system

$$\begin{aligned}\dot{\xi}_1 &= \xi_2 - \xi_1, \quad \dot{\xi}_2 = \left(\frac{28\xi_1}{10}\right) - \left(\frac{\xi_2}{10}\right) - \xi_1\xi_3, \\ \dot{\xi}_3 &= \xi_1\xi_2 - \left(\frac{8\xi_3}{30}\right)\end{aligned}$$

with the initial data $(\xi_1(0),\ \xi_2(0),\ \xi_3(0)) = (1,\ 0,\ 3)$. This solution is bounded with bounded derivative (see, [9, App.C]: numerical computation over a long period yields $\|r_1\|_\infty \le 9/5$, $\|r_2\|_\infty \le 5/2$, $\|\dot{r}_1\|_\infty \le 6/5$, $\|\dot{r}_2\|_\infty \le 12/5$, and so $\|r\|_\infty \le \sqrt{949}/10$ and $\|\dot{r}\|_\infty \le 6/\sqrt{5}$. Setting $(\lambda_1,\ \lambda_2) = (0.1,\ 0.2)$ and $(\Lambda_1,\ \Lambda_2) = (0.2,\ 0.2)$, the funnel $\mathcal{F}(\psi)$ is determined by $\psi = (\psi_1, \psi_2)$, with the functions $\psi_i \in \mathcal{G}(\lambda_i, \Lambda_i)$ given by

$$\begin{aligned}\psi_1(t) &:= \max\{2e^{-0.1\,t},\ 0.1\}, \\ \psi_2(t) &:= \begin{cases}\max\left\{\cos\left(\frac{t}{5}\right),\ 0.2\right\}, & t \in [0, 10] \\ \max\left\{\frac{3}{5}\,\cos\left(\frac{t}{3}\right),\ 0.2\right\}, & \text{otherwise}\end{cases}\end{aligned}$$

whence, $\|\psi\|_\infty = \sqrt{5}$. Taking $z^0 = 0$, a direct calculation, using (4.5), gives $L < 17$. Noting that $[CB]_{11} = 1 = [CB]_{22}$, $[CB]_{21} = -1/2$ and $[CB]_{12} = 0$, (4.10) holds; moreover

$$\begin{gathered}\sum_{j=1}^{2}[CB]_{ij}\widehat{u}_j - (L+\Lambda_i) = 20 - (L+\Lambda_i) > 2.5 > 0, \\ i = 1, 2\end{gathered}$$

and so (4.11) also holds. The requirement that $|e_i(0)| < \psi_i(0)$ for $i = 1, 2$ holds if $y^0 \in (-1, 3) \times (-1, 1)$; in this case all hypotheses of Theorem 4.3 are in place. Setting $y^0 = [-0.95,\ -0.95]^\top$ (in which case control component u_1 is initially saturated), Fig. 2 depicts the behavior of the closed-loop system (5.1), (4.12).

APPENDIX
PROOF OF THEOREMS 4.1 AND 4.3

A. *Proof of Theorem 4.1*

Step 1: Some care must be exercised in formulating the initial-value problem (2.1), (4.6) or, equivalently, (2.5), (4.6). Define

$$\begin{aligned}&\kappa : \mathcal{F}(\psi) \to \mathbb{R},\ (t, \eta) \mapsto \kappa(t, \eta) := (\psi(t) - \|\eta\|)^{-1}, \\ &\mathcal{D} := \left\{(t, \eta, \zeta) \in \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^{n-m})|(t, \eta - r(t)) \in \mathcal{F}(\psi)\right\}, \\ &f : \mathcal{D} \to \mathbb{R}^m,\ (t, \eta, \zeta) \mapsto f(t, \eta, \zeta) := A_1\eta + A_2\zeta \\ &\qquad\qquad - CB\,\mathrm{sat}_{\hat{u}}(\kappa(t, \eta - r(t))(\eta - r(t))).\end{aligned}$$

Then the initial-value problem (2.5), (4.6) may be expressed in the form

$$\left.\begin{aligned}\dot{y}(t) &= f(t, y(t), z(t)), & y(0) &= Cx^0 \\ \dot{z}(t) &= A_3y(t) + A_4z(t), & z(0) &= Nx^0\end{aligned}\right\}. \tag{6.1}$$

Clearly, $(y, z) : [0, \omega) \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ is a (maximal) solution of (6.1) if, and only if, $x = B(CB)^{-1}y + Vz : [0, \omega) \to \mathbb{R}^n$ is a (maximal) solution of (4.9).

Now, it is readily verified that $F : (t, \mu, \zeta) \mapsto \Big(f(t, \mu, \zeta),\ A_3\mu + A_4\zeta\Big)$ satisfies a local Lipschitz condition on the (relatively open) domain $\mathcal{D} \subset \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^{n-1}$, in the sense that, for each $(t, \mu, \zeta) \in \mathcal{D}$, there exists an open neighbourhood U of (t, μ, ζ) and a constant K such that

$$\|F(t, y, z) - F(t, \mu, \zeta)\| \le K\Big(\|y - \mu\| + \|z - \zeta\|\Big) \quad \forall (t, y, z) \in U.$$

By the standard theory of ordinary differential equations (see, e.g., [10, Theorem III.10.VI], the initial-value problem (6.1) has a unique maximal solution $(y,z):[0,\omega)\to\mathbb{R}^m\times\mathbb{R}^{n-m}$, $0<\omega\le\infty$; moreover, $\mathrm{graph}((y,z))=\{(t,y(t),z(t))\,|\,t\in[0,\omega)\}\subset\mathcal{D}$ does not have compact closure in $\mathcal{D}$.

Step 2: We show that the absolutely continuous tracking error e, defined by $e(t):=y(t)-r(t)$ for all $t\in[0,\omega)$, satisfies

$$\langle e(t),\dot e(t)\rangle\le -\|e(t)\|\Big[\Lambda+\Delta+\gamma\Big(\|u(t)\|-\widehat u\Big)\Big]\quad\text{for almost all } t\in[0,\omega) \tag{6.2}$$

where γ is as in (4.2) and

$$\begin{aligned} u&:[0,\omega)\to\mathbb{R}^m, & u(t)&:=-\mathrm{sat}_{\hat u}(k(t)e(t)),\\ k&:[0,\omega)\to\mathbb{R}_+, & k(t)&:=(\psi(t)-\|e(t)\|)^{-1}.\end{aligned}$$

Since $\mathrm{graph}((y,z))$ is in $\mathcal{D}$, it follows that $\mathrm{graph}(e)$ is in $\mathcal{F}(\psi)$ and so

$$\|e(t)\|<\psi(t)\le\|\psi\|_\infty\quad\forall\, t\in[0,\omega). \tag{6.3}$$

In view of (2.8), it follows that, for all $t\in[0,\omega)$

$$\|z(t)\|\le\beta\|Nx^0\|+\frac{\beta}{\alpha}\|A_3\|\Big[\|\psi\|_\infty+\|r\|_\infty\Big]=:M. \tag{6.4}$$

The conjunction of (4.5), (6.3) and (6.4). give

$$\|A_1\|\|e(t)\|+\|A_2\|\|z(t)\|\le L-\|A_1\|\|r\|_\infty-\|\dot r\|_\infty\quad\text{for almost all } t\in[0,\omega). \tag{6.5}$$

By absolute continuity of e and the first subsystem in (6.1), we have, for almost all $t\in[0,\omega)$

$$\dot e(t)=A_1\Big(e(t)+r(t)\Big)+A_2z(t)+CBu(t)-\dot r(t)$$

from which, on invoking (6.5), we may infer that, for almost all $t\in[0,\omega)$

$$\langle e(t),\dot e(t)\rangle\le L\|e(t)\|-\langle e(t),CB\mathrm{sat}_{\hat u}(k(t)e(t))\rangle$$

which, in turn and via a straightforward calculation using (4.2) and (4.4), yields (6.2).

Step 3: We show that, for ε as in (4.7)

$$\psi(t)-\|e(t)\|\ge\varepsilon\quad\forall\, t\in[0,\omega). \tag{6.6}$$

Seeking a contradiction, suppose there exists $t_1\in[0,\omega)$ such that $\psi(t_1)-\|e(t_1)\|<\varepsilon$. Since $\psi(0)-\|e(0)\|\ge\varepsilon$, the following is well defined $t_0:=\max\{t\in[0,t_1)\,|\,\psi(t)-\|e(t)\|=\varepsilon\}\in(0,t_1)$. Moreover

$$\|e(t)\|\ge\psi(t)-\varepsilon\ge\lambda-\varepsilon\ge\frac{\lambda}{2}\quad\forall\, t\in[t_0,t_1]$$

and so

$$k(t)\|e(t)\|=\|e(t)\|[\psi(t)-\|e(t)\|]^{-1}\ge\frac{\lambda}{2\varepsilon}\ge\widehat u\quad\forall\, t\in[t_0,t_1].$$

Therefore, in view of (6.2), we may infer that $\langle e(t),\dot e(t)\rangle\le-\Lambda\|e(t)\|$ for almost all $t\in[t_0,t_1]$. Integration, together with the Lipschitz property of ψ, now yields

$$\begin{aligned}\|e(t_1)\|-\|e(t_0)\|&\le-\Lambda[t_1-t_0]\le-|\psi(t_1)-\psi(t_0)|\\&\le\psi(t_1)-\psi(t_0)\end{aligned}$$

whence the contradiction: $\varepsilon=\psi(t_0)-\|e(t_0)\|\le\psi(t_1)-\|e(t_1)\|<\varepsilon$. Therefore, (6.6) holds.

Step 4: It immediately follows that the function k is bounded, with $k(t)\le 1/\varepsilon$ for all $t\in[0,\omega)$. Moreover, in view of (6.3) and (6.4) and boundedness of r, we may infer boundedness of the solution $x:[0,\omega)\to\mathbb{R}^n$, $t\mapsto x(t)=B(CB)^{-1}y(t)+Vz(t)$. To establish Assertions (i)-(iii), it remains only to show that $\omega=\infty$. Suppose that $\omega<\infty$ and define

$$\mathcal{C}:=\Big\{(t,\eta,\zeta)\in[0,\omega]\times\mathbb{R}\times\mathbb{R}^{n-1}\,|\;\psi(t)-|\eta|\ge\varepsilon,\ |\eta|\le\|\psi\|_\infty,\ \|\zeta\|\le M\Big\}.$$

Then, in view of (6.3), (6.4) and (6.6), $\mathcal{C}$ is a compact set which contains $\mathrm{graph}((e,z))=\{(t,e(t),z(t))\,|\,t\in[0,\omega)\}$, thereby contradicting the fact that the closure of the latter is not a compact subset of $\mathcal{D}$. Therefore, $\omega=\infty$.

Step 5: Finally, we proceed to establish Assertion (iv).

Step 5a: We establish the existence of $\tau\ge0$ such that $\|u(\tau)\|<\widehat u$. Seeking a contradiction, suppose that $\|u(t)\|=\widehat u$ for all $t\ge0$. Then, by (6.2), we have $\langle e(t),\dot e(t)\rangle\le-\Delta\|e(t)\|$ for all $t\ge0$. Integration now yields $0\le\|e(t)\|\le\|e(0)\|-\Delta t$ for all $t\ge0$, which contradicts the fact that, by (4.4), $\Delta>0$.

Step 5b: Assume that $\tau\ge0$ is such that $\|u(\tau)\|<\widehat u$; we show that the input remains unsaturated for all $t\ge\tau$. Suppose that there exists $t_1>\tau$ such that $\|u(t_1)\|=\widehat u$. Choose $\delta>0$ sufficiently small so that $\|u(\tau)\|\le(1-\delta)\widehat u$ and $\delta\gamma\widehat u\le\Delta/2$. Define $t_0:=\sup\{t\in[\tau,t_1]\,|\,\|u(t)\|=(1-\delta)\widehat u\}$. Then, invoking (4.4), we have

$$\gamma\widehat u\ge\gamma\|u(t)\|\ge(1-\delta)\gamma\widehat u\ge L+\Lambda+\frac{\Delta}{2}\quad\forall\, t\in[t_0,t_1].$$

By (6.2), we may now infer that $\langle e(t),\dot e(t)\rangle\le-\Lambda\|e(t)\|$ for almost all $t\in[t_0,t_1]$, which, on integration and invoking the Lipschitz property of ψ, yields

$$\begin{aligned}\|e(t_1)\|-\|e(t_0\|&\le-\Lambda\Big[t_1-t_0\Big]\le-|\psi(t_1)-\psi(t_0)|\\&\le\psi(t_1)-\psi(t_0)\end{aligned}$$

whence the contradiction

$$\begin{aligned}\widehat u&=\|u(t_1)\|=\|e(t_1)\|\,[\psi(t_1)-\|e(t_1)\|]^{-1}\\&<\|e(t_0)\|\,[\psi(t_0)-\|e(t_0)\|]^{-1}=\|u(t_0)\|<\widehat u.\end{aligned}$$

Step 5c: Finally, we turn to the last claim in Assertion (iv). Note that $\|u(0)\|=\|e(0)\|/(\psi(0)-\|e(0)\|)<\widehat u$ is equivalent to $\|e(0)\|<\psi(0)\,\widehat u/(1+\widehat u)$ and so the claim follows from Step 5b and setting $\tau=0$. This completes the proof. □

B. Sketch of Proof of Theorem 4.3

The structure of the proof of Theorem 4.3 closely resembles that of Theorem 4.1. For brevity, we do not include a full proof. Instead, we remark that the essential difference in the two cases is that, in the proof of Theorem 4.3, one argues *componentwise*: a key feature is the following counterpart of (6.2), the derivation of which invokes (4.10) and (4.11) and wherein $\gamma_i:=[CB]_{ii}$, for all $i=1,\ldots,m$

$$e_i(t)\dot e_i(t)\le-|e_i(t)|\Big[\Lambda_i+\Delta_i+\gamma_i(|u_i(t)|-\widehat u_i\Big)\Big]\quad\text{for almost all } t\in[0,\omega).$$

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Observer Design for $(\max, +)$ Linear Systems

Laurent Hardouin, Carlos Andrey Maia, Bertrand Cottenceau, and Mehdi Lhommeau

***Abstract*—This technical note deals with the state estimation for max-plus linear systems. This estimation is carried out following the ideas of the observer method for classical linear systems. The system matrices are assumed to be known, and the observation of the input and of the output is used to compute the estimated state. The observer design is based on the residuation theory which is suitable to deal with linear mapping inversion in idempotent semiring.**

***Index Terms*—Dioid, discrete event dynamics systems, idempotent semirings, max-plus algebra, observer, residuation theory, state estimation, timed event graphs (TEGs).**

I. Introduction

Many discrete event dynamic systems, such as transportation networks [12], [21], communication networks, manufacturing assembly lines [3], are subject to synchronization phenomena. Timed event graphs (TEGs) are a subclass of timed Petri nets and are suitable tools to model these systems. A timed event graph is a timed Petri net of which all places have exactly one upstream transition and one downstream transition. Its description can be transformed into a $(\max, +)$ or a $(\min, +)$ linear model and *vice versa* [1], [5]. This property has advantaged the emergence of a specific control theory for these systems, and several control strategies have been proposed, e.g., optimal open loop control [4], [16], [19], [20], and optimal feedback control in order to solve the model matching problem [6], [14], [18], [19] and also [22]. This technical note focuses on observer design for $(\max, +)$ linear systems. The observer aims at estimating the state for a given plant by using input and output measurements. The state trajectories correspond to the transition firings of the corresponding timed event graph, their estimation is worthy of interest because it provides insight into internal properties of the system. For example these state estimations are sufficient to reconstruct the marking of the graph, as it is done in [10] for Petri nets without temporization. The state estimation has many potential applications, such as fault detection, diagnosis, and state feedback control.

The $(\max, +)$ algebra is a particular idempotent semiring, therefore Section II reviews some algebraic tools concerning these algebraic structures. Some results about the residuation theory and its applications over semiring are also given. Section III recalls the description of timed event graphs in a semiring of formal series. Section IV presents and develops the proposed observer. It is designed by analogy with the classical Luenberger [17] observer for linear systems. It is done under the assumption that the system behavior is $(\max, +)$-linear. This assumption means the model represents the fastest system behavior, in other words it implies that the system is unable to be accelerated, and consequently the disturbances can only reduce the system performances i.e., they can only delay the events occurrence. They can be seen as machine breakdown in a manufacturing system, or delay due to an unexpected crowd of people in a transport network. In the opposite, the disturbances which increase system performances, i.e., which anticipate the events occurrence, could give an upper estimation of the state, in this sense the results obtained are not equivalent to the observer for the classical linear systems. Consequently, it is assumed that the model and the initial state correspond to the fastest behavior (e.g., ideal behavior of the manufacturing system without extra delays or ideal behavior of the transport network without traffic holdup and with the maximal speed) and that disturbances only delay the occurrence of events. Under these assumptions a sufficient condition allowing to ensure equality between the state and the estimated state is given in proposition 4 in spite of possible disturbances, and proposition 3 yields some weaker sufficient conditions allowing to ensure equality between the asymptotic slopes of the state and the one of the estimated state, that means the error between both is always bounded. We invite the reader to consult the following link http://www.istia.univ-angers.fr/~hardouin/Observer.html to discover a dynamic illustration of the observer behavior.

II. Algebraic Setting

An idempotent semiring $\mathcal{S}$ is an algebraic structure with two internal operations denoted by $\oplus$ and $\otimes$. The operation $\oplus$ is associative, commutative and idempotent, that is, $a \oplus a = a$. The operation $\otimes$ is associative (but not necessarily commutative) and distributive on the left and on the right with respect to $\oplus$. The neutral elements of $\oplus$ and $\otimes$ are represented by ε and e respectively, and ε is an absorbing element for the law $\otimes (\forall a \in \mathcal{S},\ \varepsilon \otimes a = a \otimes \varepsilon = \varepsilon)$. As in classical algebra, the operator $\otimes$ will be often omitted in the equations, moreover, $a^i = a \otimes a^{i-1}$ and $a^0 = e$. In this algebraic structure, a partial order

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L. Hardouin, B. Cottenceau, and M. Lhommeau are with the Laboratoire d’Ingénierie des Systèmes Automatisés, Université d’Angers, Angers 49100, France (e-mail: laurent.hardouin@univ-angers.fr; bertrand.cottenceau@univ-angers.fr; mehdi.lhommeau@univ-angers.fr)

C.A. Maia is with the Departamento de Engenharia Elétrica, Universidade Federal de Minas Gerais, , Belo Horizonte 31270-010, MG, Brazil (e-mail: maia@cpdee.ufmg.br).

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