Funnel Control With Saturation: Nonlinear SISO Systems

Norman Hopfe, Achim Ilchmann, and Eugene P. Ryan

***Abstract*—Tracking—by the system output—of a reference signal (assumed bounded with essentially bounded derivative) is considered in the context of a class of nonlinear, single-input, single-output systems modelled by functional differential equations and subject to input saturation. Prespecified is a parameterized performance funnel within which the tracking error is required to evolve; transient and asymptotic behaviour of the tracking error is influenced through choice of parameter values which define the funnel. The control structure is a saturating error feedback with time-varying nonmonotone gain designed to evolve in such a way as to preclude contact with the funnel boundary. A feasibility condition—formulated in bounds of the plant data, the saturation bound, the funnel data, the reference signal, and the initial data—is presented under which the tracking objective is achieved, whilst maintaining boundedness of the state and gain function.**

***Index Terms*—Input saturation, nonlinear systems, output feedback, tracking, transient behavior.**

NOMENCLATURE

$\mathbb{R}_+ := [0,\infty)$; $C(I,\mathbb{R}^\ell)$, $I \subset \mathbb{R}$, the space of continuous functions $I \to \mathbb{R}^\ell$; $L^\infty(I,\mathbb{R}^\ell)$ the space of measurable, essentially bounded functions $f\colon I \to \mathbb{R}^\ell$, with norm $\|f\|_\infty := \text{ess-sup}_{t\in I}\|y(t)\|$; the space of measurable, locally essentially bounded functions $f\colon I \to \mathbb{R}^\ell$ is denoted by $L^\infty_{\rm loc}(I,\mathbb{R}^\ell)$; if $\ell = 1$, we simply write $C(I)$, $L^\infty(I)$, and $L^\infty_{\rm loc}(I)$; $W^{1,\infty}(\mathbb{R}_+)$ is the space of absolutely continuous functions $r\colon \mathbb{R}_+ \to \mathbb{R}$ with $r, \dot r \in L^\infty(\mathbb{R}_+)$. A function $\beta\colon \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{K}$ function if it is continuous, strictly increasing, $\beta(0) = 0$; the class of *unbounded* $\mathcal{K}$ functions is denoted by $\mathcal{K}_\infty$. A continuous function $\alpha\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{KL}$ function if $\alpha(\cdot,t) \in \mathcal{K}$ for all $t \in \mathbb{R}_+$ and, for all $s \in \mathbb{R}_+$, $\alpha(s,\cdot)$ is nonincreasing with $\alpha(s,t) \to 0$ as $t \to \infty$.

I. INTRODUCTION

IN common with its precursor [1], we investigate funnel control in the presence of input constraints. In contrast with [1], the systems to be controlled are nonlinear and are described by functional differential equations. We restrict attention to SISO systems. By way of motivation, consider a system of two interconnected nonlinear subsystems

$$\left.\begin{aligned} \dot y(t) &= f_1(d(t),y(t),z(t)) + \mathrm{sat}_{\hat u}(u(t)), & y(0) &= \eta \\ \dot z(t) &= f_2(y(t),z(t)), & z(0) &= \zeta \end{aligned}\right\} \tag{1.1}$$

where $f_1\colon \mathbb{R}^p \times \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}$ and $f_2\colon \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ are locally Lipschitz, $d \in L^\infty(\mathbb{R}_+,\mathbb{R}^p)$ is a disturbance, and $\mathrm{sat}_{\hat u}$ is the saturation function, parameterized by $\hat u > 0$, given by

$$\mathrm{sat}_{\hat u}\colon \mathbb{R} \to [-\hat u, \hat u], \quad u \mapsto \begin{cases} \hat u\,\mathrm{sgn}(u), & |u| > \hat u \\ u, & |u| \le \hat u. \end{cases} \tag{1.2}$$

Regarding the second subsystem in (1.1) as an independent system with (continuous) input y, let $\varphi(\cdot;\zeta,y)$ denote the unique maximal solution of the initial-value problem

$$\dot z(t) = f_2(y(t),z(t)), \quad z(0) = \zeta. \tag{1.3}$$

Now assume that (1.3) is input-to-state stable (ISS) and so there exist $\alpha \in \mathcal{KL}$ and $\beta \in \mathcal{K}_\infty$ such that, for all $(\zeta,y) \in \mathbb{R}^{n-1} \times C(\mathbb{R}_+)$, the unique maximal solution $z(\cdot) = \varphi(\cdot;\zeta,y)$ is global (i.e., is defined on $\mathbb{R}_+$) and satisfies the ISS estimate

$$\|z(t)\| \le \alpha(\|\zeta\|,t) + \text{ess-sup}_{s\in[0,t]}\beta(|y(s)|)\ \forall t \ge 0. \tag{1.4}$$

Therefore, with (1.3), we may associate an operator $T_\zeta\colon C(\mathbb{R}_+) \to C(\mathbb{R}_+,\mathbb{R}^{n-1})$ (parameterized by the initial data ζ) defined by the property that $T_\zeta y = \varphi(\cdot,\zeta,y)$ is the unique global solution of (1.3). Equipped with this operator, system (1.1) may be expressed in the form of a functional differential equation

$$\dot y(t) = f_1(d(t),y(t),(T_\zeta y)(t)),\ y(0) = \eta \tag{1.5}$$

which provides the prototype for the general system class underpinning the paper. Note that in a sense the zero dynamics of (1.1) are captured by the operator T_ζ.

Example 1.1: As a highly specialized example of a system (1.1), consider the following:

$$\left.\begin{aligned} &\dot y(t) = d(t) + a|y(t)|^b y(t) + z(t) + \mathrm{sat}_{\hat u}(u(t)), \\ &\dot z(t) = -z(t) - z(t)^3 + [1 + z(t)^2]\,y(t), \\ &(y(0),z(0)) = (\eta,\zeta) \end{aligned}\right\} \tag{1.6}$$

with real parameters a and $b \ge 0$, disturbance $d \in L^\infty(\mathbb{R}_+)$ and initial values $\eta, \zeta \in \mathbb{R}$. To see that the second subsystem in (1.6) is ISS, consider the subsystem in isolation with input $y \in C(\mathbb{R}_+)$ and let $z(\cdot) = \varphi(\cdot;\zeta,y)$ denote its unique solution on its maximal interval of existence $[0,\omega)$, $0 < \omega \le \infty$. Writing $V(t) := z^2(t)/2$ for all $t \in [0,\omega)$, we have

$$\begin{aligned} \dot V(t) &= -2V(t) - z(t)^2 + z(t)\,y(t) + z(t)^3\,y(t) \\ &\le -2V(t) + \frac{y(t)^2}{2} + \frac{y(t)^4}{4}\ \forall\, t \in [0,\omega). \end{aligned}$$

By Gronwall's Lemma (e.g., [8, Lem. VII.29.VI]), it follows that, for all $t \in [0,\omega)$

$$V(t) \le e^{-2t}V(0) + \frac{1}{4}\int_0^t \Big(2y(s)^2 + y(s)^4\Big)e^{-2(t-s)}\,\mathrm{d}s$$

from which we may infer that $\omega = \infty$ and, for all $t \ge 0$,

$$|z(t)| \le e^{-t}|\zeta| + \frac{1}{2\sqrt{2}}\ \underset{s\in[0,t]}{\text{ess-sup}}\Big(|y(s)|\sqrt{2 + |y(s)|^2}\Big).$$

Therefore, the ISS estimate (1.4) holds with

$$\alpha\colon (\zeta,t) \mapsto e^{-t}\zeta \ \text{ and } \ \beta\colon \rho \mapsto \frac{1}{2\sqrt{2}}\,\rho\sqrt{2 + \rho^2}. \tag{1.7}$$

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Manuscript received September 11, 2009; revised December 22, 2009 and March 25, 2010; accepted May 25, 2010. Date of publication June 01, 2010; date of current version September 09, 2010. This work was supported by the Deutsche Forschungsgemeinschaft under Grant IL 25/4-1. Recommended by Associate Editor A. Loria.

N. Hopfe and A. Ilchmann are with the Institute of Mathematics, Technical University Ilmenau, Ilmenau 98693, Germany (e-mail: norman.hopfe@tu-ilmenau.de; achim.ilchmann@tu-ilmenau.de).

E. P. Ryan is with the Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, U.K. (e-mail: epr@maths.bath.ac.uk).

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Digital Object Identifier 10.1109/TAC.2010.2051735

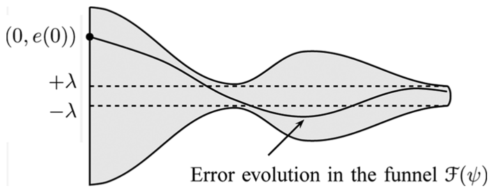

Fig. 1. Prescribed performance funnel $\mathcal{F}(\psi)$.

Returning to the prototype system (1.1) (or its equivalent (1.5)), the control objective is formulated in terms of a *performance funnel* (see Fig. 1)

$$\mathcal{F}(\psi) := \{(t,e) \in \mathbb{R}_+ \times \mathbb{R} \mid |e| < \psi(t)\} \tag{1.8}$$

determined by a bounded function $\psi: \mathbb{R}_+ \to [\lambda, \infty)$ which is globally Lipschitz with Lipschitz constant $\Lambda > 0$ and is bounded away from 0 by $\lambda > 0$, that is, ψ is a function of the class shown in

$$\mathcal{G}(\Lambda,\lambda) := \left\{ \psi: \mathbb{R}_+ \to [\lambda,\infty) \,\middle|\, \psi \text{ bounded and Lipschitz with Lipschitz constant } \Lambda \right\} \tag{1.9}$$

The *control objective* is: determine a feedback which ensures that, for a given reference signal $r \in W^{1,\infty}(\mathbb{R}_+)$, the output tracking error $e = y - r$ evolves within the funnel (i.e., $\operatorname{graph}(e) \subset \mathcal{F}(\psi)$): transient and asymptotic behavior of the tracking error is influenced through choice of the function ψ. The proposed control structure is an error feedback of the form $u(t) = -k(t)e(t)$, wherein the gain function $k: t \mapsto 1/(\psi(t) - |e(t)|)$ evolves so as to preclude contact with the funnel boundary. A feasibility condition (formulated in terms of the plant data, the funnel data, the reference signal r, the disturbance signal d, and the initial state η) is presented under which the tracking objective is achieved, whilst maintaining boundedness of all signals.

Given $\lambda > 0$ (arbitrarily small) and $\Lambda > 0$, a wide variety of funnels are possible. For example, if $\alpha, \beta > 0$ are chosen such that $\alpha > \lambda$ and $\alpha\beta \le \Lambda$, then the function $t \mapsto \psi(t) := \max\{\alpha e^{-\beta t}, \lambda\}$ is in $\mathcal{G}(\Lambda,\lambda)$ and evolution within the associated funnel ensures a prescribed exponential decay in the transient phase $[0, \ln(\alpha/\lambda)/\beta]$ and tracking accuracy $\lambda > 0$ thereafter (we stress that λ may be taken arbitrarily small). Monotonicity of the funnel boundary is not required: nonmonotone funnels may be advantageous in applications for which it is known *a priori* when perturbations or set-point changes may occur—in this sense, nonmonotone funnels have the connotation of re-initialization of the control structure.

Example 1.2 (Example 1.1 Revisited): In the highly specialized context of example (1.6), the main result of the paper translates into the following: for arbitrary $\Lambda, \lambda \ge 0$, $\psi \in \mathcal{G}(\Lambda,\lambda)$, any absolutely continuous reference signal $r: \mathbb{R}_+ \to \mathbb{R}$ with essentially bounded derivative, and writing $e(t) = y(t) - r(t)$, the simple control strategy

$$u(t) = -k(t)e(t), \quad k(t) = [\psi(t) - |e(t)|]^{-1}$$

applied to (1.6) ensures attainment of the tracking objective (and, moreover, the gain function k is bounded) provided that the initial data satisfy $|\eta - r(0)| < \psi(0)$ and the following feasibility assumption holds: $\hat{u} > L + \Lambda + \|\dot{r}\|_\infty$, where

$$L := \|d\|_\infty + a(\|\psi\|_\infty + \|r\|_\infty)^{b+1} + \|\zeta\| + \frac{1}{2\sqrt{2}}(\|\psi\|_\infty + \|r\|_\infty)\sqrt{2 + (\|\psi\|_\infty + \|r\|_\infty)^2}.$$

■

The concept of funnel control was introduced in [4]. For several generalizations and substantial bibliography pertaining to that literature, see the survey article [3]. For experimental results on controlling the speed of electric devices using the funnel control methodology, see [5]. The control problem to be considered in the present note is the analogue—in a context of nonlinear single-input, single-output systems—of the problem considered (in a context of linear multi-input, multi-output systems) in its precursor [1].

II. The System Class

Generalizing the prototype (1.1), or its equivalent (1.5), we consider single-input, single-output systems described by a functional differential equation

$$\left.\begin{aligned} \dot{y}(t) &= f\Big(d(t), (Ty)(t)\Big) + g\Big(u(t)\Big), \\ y|_{[-h,0]} &= \eta \in C[-h,0] \end{aligned}\right\} \tag{2.1}$$

wherein $h \ge 0$ quantifies the memory in the system (delay elements are encompassed by the formulation), the functions $f: \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are continuous, $d \in L^\infty(\mathbb{R}_+, \mathbb{R}^p)$ is a disturbance, and the operator T satisfies the following assumptions of causality **(C)**, bounded-input/bounded-output **(B)** and Lipschitz-like continuity **(L)**.

(C) $T: C[-h,\infty) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^q)$ is a causal operator with the properties:

(B) there exists $\mu \in C(\mathbb{R}_+)$ such that, for all $c_1, \omega > 0$ and all $y \in C[-h,\omega)$,

$$\sup_{t \in [-h,\omega)} |y(t)| \le c_1 \Rightarrow \sup_{t \in [0,\omega)} |(Ty)(t)| \le \mu(c_1)$$

(L) for all $t \ge 0$ and all $w \in C[-h,t]$, there exist $\tau > t$ and $\delta, c_0 > 0$ such that, for all $y,\ z \in \mathcal{C}(w; h, t, \tau, \delta)$,

$$\operatorname*{ess\text{-}sup}_{s \in [t,\tau]} \|(Ty)(s) - (Tz)(s)\| \le c_0 \max_{s \in [t,\tau]} |y(s) - z(s)|,$$

where

$$\mathcal{C}(w; h, t, \tau, \delta) := \left\{ v \in C[-h,\tau] \,\middle|\, \begin{smallmatrix} v|_{[-h,t]} = w \\ |v(s) - w(t)| \le \delta \ \forall\ s \in [t,\tau] \end{smallmatrix} \right\}$$

i.e., the space of all continuous extensions v of $w \in C[-h,t]$ to the interval $[-h,\tau]$ such that that $|v(s) - w(t)| \le \delta$ for all $s \in [t,\tau]$.

The continuous input nonlinearity $g: \mathbb{R} \to \mathbb{R}$ satisfies:

(G) g is nondecreasing with $g(0) = 0$ and

$$\gamma^+ := \sup_{v \ge 0} g(v) \in (0,\infty],$$
$$\gamma^- := -\inf_{v \le 0} g(v) \in (0,\infty].$$

Remark 2.1 (on the Input Nonlinearity g): The prototype input nonlinearity is $g = \operatorname{sat}_{\hat{u}}$ as in (1.2), with $\hat{u} > 0$, in which case $\gamma^+ = \hat{u} = \gamma^-$. Note that g allows for saturation nonlinearities with nonsymmetric bounds. Note also that g might yield a system (1.1) which does not have a relative degree; see [6, Ex. 4.1.1]. However, systems with relative degree greater than one are not covered.

Example 2.2 (Particular Cases Subsumed by (2.1)):

(a) The prototype system (1.1) can be written in the form (2.1) and Assumptions (C), (B), (L), and (G) are valid, provided that the ISS estimate (1.4) holds. This is shown in Appendix A.

(b) Linear systems

$$\dot{x} = Ax + Bu, \quad y = Cx$$

of relative degree one, as investigated in the precursor [1], are encompassed by (2.1) in the single-input, single-output case.

(c) In [2, Ex. 2.3], it is shown that classes of linear retarded minimum-phase systems of the form

$$\dot{x} = \mathrm{d}A * x + bu, \quad y = cx$$

with relative degree one can be written in the form (2.1) and satisfy (B) and (L).

(d) In [4, Sec. 4.5], it is shown that a class of systems with hysteresis (for example relay and backlash) can be written in the form (2.1) and satisfy (B) and (L).

(e) In [4, Sec. 4.2], it is shown that a class of infinite-dimensional regular linear systems can be written in the form (2.1) and satisfy (B) and (L).

III. The Main Result

We now arrive at the main result, the proof of which may be found in the Appendix B.

Theorem 3.1: Let $\Lambda, \lambda > 0$ and $\psi \in \mathcal{G}(\Lambda, \lambda)$ define the performance funnel

$$\mathcal{F}(\psi) = \{(t, e) \in \mathbb{R}_+ \times \mathbb{R} | \, |e| < \psi(t)\}.$$

Let $h \geq 0$, $\eta \in C([-h, 0])$, $d \in L^\infty(\mathbb{R}_+, \mathbb{R}^p)$ and $r \in W^{1,\infty}(\mathbb{R}_+)$. Consider a system (2.1) satisfying (C), (B) (with associated function $\mu \in C(\mathbb{R}_+)$), (L) and (G). Define

$$L := \sup \left\{ |f(\rho, \sigma)| \,\middle|\, \begin{array}{l} (\rho,\sigma)\in\mathbb{R}^p\times\mathbb{R}^q,\ \|\rho\|\leq\|d\|_\infty, \\ |\sigma|\leq\mu(\|\eta\|_\infty+\|\psi\|_\infty+\|r\|_\infty) \end{array} \right\} \tag{3.1}$$

If the initial data η and the reference signal r are such that

$$|\eta(0) - r(0)| < \psi(0), \tag{3.2}$$

and the feasibility condition

$$\gamma := \max\{\gamma^-, \gamma^+\} > L + \Lambda + \|\dot{r}\|_\infty =: \Gamma \tag{3.3}$$

holds, then application of the feedback strategy

$$\left.\begin{array}{l} u(t) = -k(t)\, e(t), \\ k(t) = [\psi(t) - |e(t)|]^{-1}, \; e(t) = y(t) - r(t) \end{array}\right\} \tag{3.4}$$

to (2.1) yields a closed-loop initial-value problem with the following properties.

(a) The closed-loop initial-value problem (2.1), (3.4) has a solution $y : [-h, \omega) \to \mathbb{R}$ and every solution can be extended to a global solution, i.e., $\omega = \infty$.

(b) There exists $\varepsilon > 0$ such that every global solution y satisfies

$$|y(t) - r(t)| \leq \psi(t) - \varepsilon \; \forall\, t \geq 0.$$

(c) The function $u(\cdot)$ is bounded and the following hold:

(i) $|g(u(\tau))| < \gamma$ for some $\tau \in \mathbb{R}_+$.

(ii) $\exists\, \tau \geq 0 : \; |g(u(\tau))| < \gamma \Rightarrow |g(u(t))| < \gamma \; \forall\, t \geq \tau$.

Remark 3.2:

(i) Assertion (b) is the essence of the result: it asserts that, if (3.2) and the feasibility condition (3.3) hold, then the funnel control (3.4) ensures achievement of the control objectives; in particular, the tracking error $e = y - r$ remains uniformly bounded away from the funnel boundary and the gain function k is bounded, with $\|k\|_\infty \leq 1/\varepsilon$.

(ii) Assertion (c) has nontrivial content only in the case wherein γ is finite and either the supremum γ^+ or the infimum $-\gamma^-$ of g is attained, that is, the case wherein the input may saturate (the prototype being the saturation function $g = \mathrm{sat}_{\hat{u}}$). Assertion (c)(i) implies that the control input cannot remain saturated for all $t \geq 0$ and, when it becomes unsaturated, then Assertion (c)(ii) implies that the signal remains unsaturated thereafter. If the initial data is such that the signal $g(u(\cdot))$ is initially unsaturated, i.e., $|g(u(0))| < \gamma$, then the saturation bound is never attained [see Assertion (c)(ii)]. If, on the other hand, the signal $g(u(\cdot))$ is initially saturated, i.e., $|g(u(0))| = \gamma$, then the conjunction of Assertions (c)(i) and (c)(ii) ensures that it remains so only on a finite interval $[0, \tau]$ and is unsaturated on (τ, ∞).

(iii) The condition (3.2) is necessary for attainment of the control objective and is equivalent to the requirement that $(0, \eta(0)) \in \mathcal{D}$. The feasibility condition (3.3) is a sufficient condition for attainment of the control objective (of course, in the case $\gamma = \infty$, i.e., in the absence of saturation, (3.3) holds trivially). It quantifies and exhibits the interplay between the saturation bound γ (sufficiently large to ensure performance) and bounds of the plant data, funnel data, initial data, reference signal data, and disturbance signal data.

The nature of the dependence of the saturation bound on these data is not surprising:

1) it is to be expected that tracking of "large and rapidly varying" reference signals r would require control inputs capable of taking sufficiently large values;
2) transient and asymptotic behavior of the tracking error is influenced by the choice of funnel $\mathcal{F}(\psi)$ determined by the globally Lipschitz function ψ;
3) it is to be expected that the saturation bound depends on the disturbance signal d.

IV. Simulations

For purposes of illustration, consider system (1.6) subject to the saturation constant $\hat{u} := 25$. Example 2.2(i) shows that the systems (1.6) can be written in the form (2.1) and that the conditions (C) and (G) are satisfied.

As reference signal we choose $r(\cdot) = \xi_1(\cdot)$, the first component of the solution of the Lorenz system

$$\dot{\xi}_1 = \xi_2 - \xi_1, \quad \dot{\xi}_2 = \left(\frac{28\xi_1}{10}\right) - \left(\frac{\xi_2}{10}\right) - \xi_1\xi_3,$$
$$\dot{\xi}_3 = \xi_1\xi_2 - \left(\frac{8\xi_3}{30}\right), \quad (\xi_1(0), \xi_2(0), \xi_3(0)) = (1, 0, 3).$$

It is shown in [7, App. C] that the solutions are chaotic and bounded with bounded derivatives. Note that $r(0) = 1$ and numerical computation over a long period yields $\|r\|_\infty \leq 9/5$, and $\|\dot{r}\|_\infty \leq 6/5$.

As disturbance signal we choose $d(\cdot) = -\xi_2(\cdot)$; again, numerical simulation over a long period gives $\|d\|_\infty \leq 2.4$.

Setting $\lambda = 0.1$, the funnel $\mathcal{F}(\psi)$ is determined by $\psi \in \mathcal{G}(\Lambda, \lambda)$ given by

$$\psi(t) := \begin{cases} 2e^{-0.1\,t}, & t \in [0,\, 10 \ln 20] \\ \max\left\{\frac{3}{5}\cos\left(\frac{t}{3}\right), \lambda\right\}, & \text{otherwise.} \end{cases}$$

Then $\|\psi\|_\infty = 2$ and it prescribes an exponential (exponent 0.1) decay of the tracking error in the transient phase $[0, T]$, where $T = 10 \ln 20 \approx 30$, and a tracking accuracy quantified by $\lambda = 0.1$ and $3/5 \cos(t/3)$ thereafter; ψ is nonmonotone with global Lipschitz constant $\Lambda = 0.2$.

For $a = b = 1$ and $\zeta = 1$ and L as in Example 1.2, we have $L + \|\dot{r}\|_\infty + \Lambda = 24.68 < 25 = \hat{u}$, and so the feasibility condition (3.3) is satisfied. The condition (3.2), i.e., $|e(0)| = |\eta - r(0)| < 2$, implies $\eta \in (-1, 3)$. To illustrate the occurrence of saturation of the

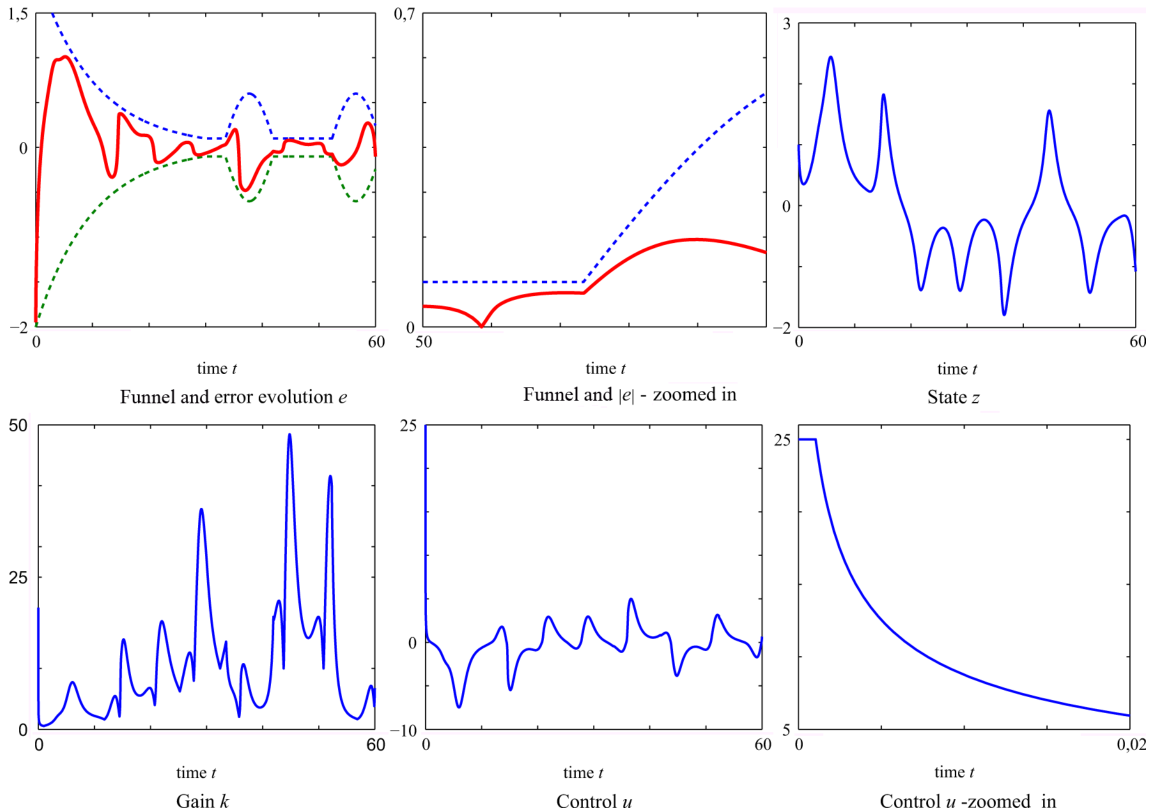

Fig. 2. Behavior of the closed-loop system (1.6), (3.4).

control input, we choose η to be such that Assertion (c)(ii) fails to hold for $\tau = 0$ (in which case, there exists $\tau > 0$ such that the control u is saturated on $[0, \tau)$). Note that

$$|\mathrm{sat}_{\hat{u}}(u(0))| < \hat{u} \quad \iff \quad |\eta - r(0)| < \frac{\psi(0)\hat{u}}{1+\hat{u}}$$

and so the input is saturated at the beginning if, and only if, $|e(0)| = |\eta - r(0)| \geq 25/13$. Hence, we choose $\eta = -0.95$, and so $\varepsilon = \lambda/(2\hat{u}) = 0.002$.

Fig. 2 depicts the behavior of the closed-loop system (1.6), (3.4). The result of Theorem 3.1 are confirmed: the tracking error remains uniformly bounded away from the funnel boundary; moreover, the second picture suggests that the calculated bound $\varepsilon = 0.002$ is conservative. Nonmonotonicity of gain function k is also evident: it increases when the error approaches the funnel boundary and decreases when the error recedes from the boundary. The last two simulations show that the input is initially saturated: it remains so on an interval of short duration and remains unsaturated thereafter.

Appendix
Proofs

Proof of the Claim in Example 2.2 (i): That condition (G) holds is an immediate consequence of Remark 2.1. As before, regarding the first subsystem in (1.1) in isolation with input $y \in C(\mathbb{R}_+)$, let $\varphi(\cdot;\zeta,y)$ denote the unique maximal solution of the initial-value problem (1.3): in view of (1.4), we know that this solution is global (i.e., exists on $\mathbb{R}_+$). For each $\zeta \in \mathbb{R}^{n-1}$, we define a causal operator $T_\zeta : C(\mathbb{R}_+) \to C(\mathbb{R}_+, \mathbb{R}^{n-1})$ by $\Big(T_\zeta(y)\Big)(t) := \varphi(t;\zeta,y)$ for all $t \in \mathbb{R}_+$. Setting

$$T: C(\mathbb{R}_+) \to C(\mathbb{R}_+, \mathbb{R}^n),\ y \mapsto \Big(y, \Big(T_\zeta(y)\Big)\Big)$$
$$f: \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R},\ (\delta,\xi) = \Big(\delta, (\xi_1,\xi_2)\Big) \mapsto f_1(\delta,\xi_1,\xi_2)$$

the original initial-value problem (1.1) may be expressed in the form (2.1). Causality of T_ζ ensures that T satisfies (C). Assumption (B) holds since, by virtue of (1.4), we may infer that, for all $c_1 > 0$ and $y \in C(\mathbb{R}_+)$,

$$\|y\|_\infty \leq c_1 \implies \|Ty\|_\infty \leq \mu(c_1) := c_1 + \alpha(|\zeta|, 0) + \beta(c_1). \qquad (5.1)$$

It remains to show that T satisfies (L). Let $w \in C(\mathbb{R}_+)$, fix $t \geq 0$, $\delta > 0$ and $\tau > 0$ arbitrarily, and, for notational convenience, set $\Delta := \delta + \sup_{s\in[0,t]} |w(s)|$. Define the compact set

$$K := \Big\{(\rho,\theta) \in \mathbb{R} \times \mathbb{R}^{n-1} \,\Big|\, |\rho| \leq \Delta,\ \|\theta\| \leq \alpha(|\zeta|, 0) + \beta(\Delta)\Big\}.$$

By the local Lipschitz property of f_2, there exists $c > 0$ such that

$$\|f_2(\rho,\theta) - f_2(\varrho,\vartheta)\| \leq c\Big[|\rho - \theta| + \|\varrho - \vartheta\|\Big]\ \forall (\rho,\theta), (\varrho,\vartheta) \in K.$$

Let $y_1, y_2 \in C(\mathbb{R}_+)$ be such that $y_1(s) = w(s) = y_2(s)$ for all $s \in [0,t]$, and $|y_1(s)|, |y_2(s)| \leq \delta$ for all $s \in [t, t+\tau]$. Then, for all $s \in [t, t+\tau]$

$$\begin{aligned}&\|(T_\zeta y_1)(s) - (T_\zeta y_2)(s)\|\\ &\leq \int_0^s \|f_2(y_1(\sigma), z(\sigma;\zeta,y_1)) - f_2(y_2(\sigma), z(\sigma;\zeta,y_2))\| \mathrm{d}\sigma\\ &\leq c\int_t^s \Big[|y_1(\sigma) - y_2(\sigma)| + \|(T_\zeta y_1)(\sigma) - (T_\zeta y_2)(\sigma)\|\Big] \mathrm{d}\sigma.\end{aligned}$$

By Gronwall's Lemma (e.g., [8, Lem. VII.29.VI]), it follows that, for all $s \in [t, t+\tau]$

$$\|(T_\zeta y_1)(s) - (T_\zeta y_2)(s)\| \leq c\int_t^s e^{L(s-\sigma)} |y_1(\sigma) - y_2(\sigma)| \mathrm{d}\sigma$$

whence

$$\sup_{s\in[t,t+\tau]} \|(T_\zeta y_1)(s)-(T_\zeta y_2)(s)\| \le c\tau e^{L\tau} \sup_{s\in[t,t+\tau]} |y_1(s)-y_2(s)|.$$

We may now conclude that Assumption (L) holds with $c_0 = 1+c\tau e^{L\tau}$.

Proof of Theorem 3.1: We preface the proof with some remarks. To interpret (B) and (L) correctly, we need to give meaning to Ty, for a function $y \in C(I)$ on a bounded interval I of the form $[-h,\rho)$ or $[-h,\rho]$, where $0<\rho<\infty$. This we do by showing that T "localizes" to an operator $\tilde{T}: C(I) \to L^\infty_{\mathrm{loc}}(J,\mathbb{R}^q)$, where $J := I \setminus [-h,0)$. Let $y \in C(I)$. For each $\sigma \in J$, define $y_\sigma \in C[-h,\infty)$ by

$$y_\sigma(t) := \begin{cases} y(t), & t\in[-h,\sigma] \\ y(\sigma), & t>\sigma. \end{cases}$$

By causality, we may define $\tilde{T}y \in L^\infty_{\mathrm{loc}}(J,\mathbb{R}^q)$ by the property $\tilde{T}y|_{[0,\sigma]} = Ty_\sigma|_{[0,\sigma]}$ for all $\sigma \in J$. Henceforth, we will not distinguish notationally an operator T and its "localization" $\tilde{T}$: the correct interpretation being clear from context.

In view of the potential singularity in (3.4), some care is required in formulation the closed-loop initial-value problem (2.1), (3.4). Define

$$\mathcal{D} := \{(t,v)\in\mathbb{R}_+\times\mathbb{R} \mid (t,v-r(t))\in\mathcal{F}(\psi)\} \tag{5.2}$$

and define $F: \mathcal{D}\times\mathbb{R}^q \to \mathbb{R}$ by

$$F(t,v,w) := f(d(t),w) + g\left(\frac{-(v-r(t))}{\psi(t)-|v-r(t)|}\right) \tag{5.3}$$

in which case, the closed-loop, initial-value problem (2.1), (3.4) is formulated as

$$\dot{y}(t) = F(t,y(t),(Ty)(t)),\ y|_{[-h,0]} = \eta. \tag{5.4}$$

By a *solution* of (5.4) we mean a function $y\in C[-h,\omega), 0<\omega\le\infty$, such that $y|_{[-h,0]}=\eta$, $y|_{[0,\omega)}$ is locally absolutely continuous, with $(t,y(t))\in\mathcal{D}$ for all $t\in[0,\omega)$ and $\dot{y}(t) = F(t,y(t),(Ty)(t))$ for almost all $t\in[0,\omega)$. A solution is said to be *maximal* if it has no proper right extension that is also a solution. A solution defined on $[-h,\infty)$ is said to be *global*.

That (5.4) has a solution, and that every solution can be extended to a maximal solution, is an immediate consequence of [3, Th. 7.1] which also implies that, if $y\in C[-h,\omega)$ is a maximal solution, then the closure of $\mathrm{graph}\left(y|_{[0,\omega)}\right) = \{(t,y(t)) \mid t\in[0,\omega)\}\subset\mathcal{D}$ is not a compact subset of $\mathcal{D}$.

Let $y:[-h,\omega)\to\mathbb{R}$, $0<\omega\le\infty$, be any maximal solution (5.4). The proof of Theorem 3.1 now proceeds in four steps.

Step 1: We show that the tracking error e satisfies

$$e(t)\dot{e}(t) \le -|e(t)|\Big(\Lambda - \Gamma + |g(u(t))|\Big) \quad \text{for almost all } t\in[0,\omega). \tag{5.5}$$

Observe that

$$|y(t)|\le|e(t)|+|r(t)|\le\|\psi\|_\infty+\|r\|_\infty \quad \forall t\in[0,\omega)$$

and so

$$|y(t)|\le c_1 := \|\eta\|_\infty+\|\psi\|_\infty+\|r\|_\infty \quad \forall t\in[-h,\omega).$$

By Property (B) , we may infer that $\|(Ty)(t)\|\le\mu(c_1)$ for almost all $t\in[0,\omega)$, and so (3.1) yields

$$\left|f\Big(d(t),(Ty)(t)\Big)\right|\le L \quad \text{for almost all } t\in[0,\omega).$$

Therefore, for almost all $t\in[0,\omega)$

$$\begin{aligned} e(t)\dot{e}(t) &= e(t)\left(f\Big(d(t),(Ty)(t)\Big) + g(u(t)) - \dot{r}(t)\right) \\ &\le |e(t)|\Big(L + \|\dot{r}\|_\infty - |g(u(t))|\Big) \end{aligned}$$

and, since $L+\|\dot{r}\|_\infty = \Gamma-\Lambda$, (5.5) follows.

Step 2: Choose $\varepsilon>0$ sufficiently small so that

$$\left.\begin{aligned} \varepsilon \le \min\left\{\tfrac{\lambda}{2},\ \psi(0)-|e(0)|\right\} \\ g\left(\tfrac{\lambda}{2\varepsilon}\right)\ge\Gamma,\ -g\left(\tfrac{-\lambda}{2\varepsilon}\right)\ge\Gamma. \end{aligned}\right\} \tag{5.6}$$

We will show that

$$\psi(t)-|e(t)|\ge\varepsilon \quad \forall t\in[0,\omega). \tag{5.7}$$

Seeking a contradiction, suppose that there exists $t_1\in(0,\omega)$ such that $\psi(t_1)-|e(t_1)|<\varepsilon$. Since $\psi(0)-|e(0)|\ge\varepsilon$, the number $t_0 := \max\{t\in[0,t_1) \mid \psi(t)-|e(t)|=\varepsilon\}\in[0,t_1)$ is well defined. It follows that $\psi(t)-|e(t)|\le\varepsilon$ for all $t\in[t_0,t_1]$ and so $|e(t)|\ge\psi(t)-\varepsilon\ge\lambda-\varepsilon\ge\lambda/2$ for all $t\in[t_0,t_1]$, whence

$$|u(t)| = k(t)\,|e(t)| \ge \frac{\lambda}{2\varepsilon} \quad \forall t\in[t_0,t_1].$$

Therefore, by monotonicity of g and (5.6), we get

$$s_0 g(s_0|u(t)|)\ge s_0 g\left(\frac{s_0\lambda}{(2\varepsilon)}\right)\ge\Gamma\ \forall s_0\in\{-1,1\}\ \forall t\in[t_0,t_1].$$

Noting that

$$|g(u(t))| = \begin{cases} g(|u(t)|), & \text{if } u(t)\ge 0 \\ -g(-|u(t)|), & \text{if } u(t)<0 \end{cases} \tag{5.8}$$

and invoking (5.5), we may infer that

$$\begin{aligned} e(t)\dot{e}(t) &\le -|e(t)|(\Lambda-\Gamma+\Gamma) \\ &= -\Lambda|e(t)|\ \forall t\in[t_0,t_1]. \end{aligned}$$

which, on integration, gives $|e(t_1)|-|e(t_0)|<-\Lambda\,|t_1-t_0|$. By the Lipschitz property of ψ, it follows that

$$|e(t_1)|-|e(t_0)|\le-|\psi(t_1)-\psi(t_0)|\le\psi(t_1)-\psi(t_0).$$

We now arrive at the contradiction:

$$\varepsilon = \psi(t_0)-|e(t_0)|\le\psi(t_1)-|e(t_1)|<\varepsilon.$$

Therefore, (5.7) holds.

Step 3: We establish Assertions (a) and (b). In view of (5.6), boundedness of r implies boundedness of y. To establish Assertions (a), (b), it remains only to show that $\omega=\infty$. Seeking a contradiction, suppose that $\omega<\infty$. Then $\{(t,y)\in\mathcal{D} \mid t\in[0,\omega],\ \psi(t)-|y-r(t)|\ge\varepsilon\}$ is a compact subset of $\mathcal{D}$ and contains the graph of $y|_{[0,\omega)}$: this contradicts the fact that the closure of the graph is not a compact subset of $\mathcal{D}$. Therefore, $\omega=\infty$.

Step 4: We establish Assertion (c). Boundedness of u is an immediate consequence of Assertion (b). If $\gamma=\infty$, then Assertion (c) trivially holds. Assume $\gamma<\infty$.

Step 4a: First, we establish Assertion (c)(i) . Seeking a contradiction, suppose $|g(u(t))|\ge\gamma$ for all $t\ge 0$. Recalling that $\gamma>\Gamma$ and invoking (5.5), we have $e(t)\dot{e}(t)\le-\Lambda|e(t)|$ for all $t\ge0$, which, on integration, yields the contradiction:

$$0\le|e(t)|\le|e(0)|-\Lambda t\ \forall t\ge 0.$$

Therefore, there exists $\tau \geq 0$ such that $|g(u(\tau))| < \gamma$. This establishes Assertion (c)(i).

Step 4b: Next, we show that Assertion (c)(ii) holds. Let $\tau \in \mathbb{R}_+$ be such that $|g(u(\tau))| < \gamma$. Again seeking a contradiction, suppose that $|g(u(t))| \geq \gamma$ for some $t \in [\tau, \infty)$. Define $t_1 := \min\{t \in [\tau,\infty) | \, |g(u(t))| = \gamma\}$. Choose $\rho \in [\Gamma, \gamma)$ such that $|g(u(\tau))| \leq \rho$ and define $t_0 := \max\{t \in [\tau, t_1) | \, |g(u(t))| = \rho\}$. Observe that

$$\tau \leq t_0 < t_1 \text{ and } |g(u(t))| \geq \rho \geq \Gamma \; \forall \, t \in [t_0, t_1].$$

Therefore, $|u(t)| > 0$ for all $t \in [t_0, t_1]$ and so, by continuity of u, we have $u(t) = s_0|u(t)|$ for all $t \in [t_0, t_1]$, where $s_0 := \mathrm{sgn}(u(t_0))$. By (5.8), it now follows that

$$|g(u(t))| = s_0 g(s_0|u(t)|) \; \forall \, t \in [t_0, t_1]. \tag{5.9}$$

Invoking (5.5), we have $e(t)\dot{e}(t) \leq -\Lambda|e(t)|$ for all $t \geq [t_0, t_1]$, which, on integration, yields $|e(t_1)| - |e(t_0)| \leq -\Lambda|t_1 - t_0|$. The latter inequality in conjunction with the global Lipschitz property of ψ gives

$$\begin{aligned}\psi(t_0) - |e(t_0)| - (\psi(t_1) - |e(t_1)|) \\ \leq |\psi(t_1) - \psi(t_0)| - \Lambda|t_1 - t_0| \leq 0.\end{aligned}$$

Therefore

$$\begin{aligned}|u(t_1)| &= \frac{|e(t_1)|}{\psi(t_1) - |e(t_1)|} \\ &< \frac{|e(t_0)|}{\psi(t_0) - |e(t_0)|} = |u(t_0)|\end{aligned}$$

which, in conjunction with (5.9) and monotonicity of g, yields the contradiction

$$\begin{aligned}\gamma = |g(u(t_1))| &= s_0 g(s_0|u(t_1)|) \\ &\leq s_0 g(s_0|u(t_0)|) \\ &= |g(u(t_0))| = \rho < \gamma.\end{aligned}$$

Therefore, $|g(u(t))| < \gamma$ for all $t \in [\tau, \infty)$. This completes the proof. □

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Performance Improvement in Adaptive Control of Linearly Parameterized Nonlinear Systems

Veronica Adetola and Martin Guay

***Abstract*—This technical note demonstrates how the finite-time identification procedure [1] can be used to improve the overall performance of adaptive control systems. First, we develop an adaptive compensator which guarantees exponential convergence of the estimation error provided the integral of a filtered regressor matrix is positive definite. The approach does not involve online checking of matrix invertibility and computation of matrix inverse nor switching between parameter estimation methods. The convergence rate of the parameter estimator is directly proportional to the adaptation gain and a measure of the system's excitation. The adaptive compensator is then combined with existing adaptive controllers to guarantee exponential stability of the closed-loop system. The effectiveness of the proposed method is illustrated with a simulation example.**

***Index Terms*—Adaptive control, finite-time identifier, nonlinear systems.**

I. INTRODUCTION

There are two major approaches to online parameter identification of nonlinear systems. The first is the identification of parameters as a part of a state observer while the second deals with parameter identification as a part of a controller. In the first approach, the observer is designed to provide state derivatives information and the parameters are estimated via estimation methods such as least squares method [15] and dynamic inversion [3]. The second trend of parameter identification is more widespread, as it allows identification of systems with unstable dynamics. Algorithms in this area include parameter identification methods based on variable structure theory [18], [19] and those based on the notion of passivity [9].

In conventional adaptive control algorithms, the focus is on the tracking of a given reference trajectory and in most cases parameter estimation errors are not guaranteed to converge to zero due to a lack of excitation [6]. Parameter convergence is an important issue as it enhances the overall stability and robustness properties of the closed loop adaptive systems [10]. Moreover, there are control problems whereby the reference trajectory is not known *a priori* but depends on the unknown parameters of the system dynamics. For example, in adaptive extremum seeking control problems, the desired target is the operating setpoint that optimizes an uncertain cost function [4], [17].

Assuming the satisfaction of appropriate excitation conditions, asymptotic and exponential parameter convergence results are available for both linear and nonlinear systems. Some lower bounds which depend (nonlinearly) on the adaptation gain and the level of excitation in the system have been provided for some specific control and estimation algorithms [7], [13], [16]. However, it is not always easy to characterize the convergence rate.

A parameter estimation scheme that allows exact reconstruction of the unknown parameters in finite-time was developed in [1]. The finite-time (FT) identification method has two distinguishing features. First, the true

Manuscript received October 03, 2008; revised October 06, 2008; accepted May 21, 2010. Date of publication June 07, 2010; date of current version September 09, 2010. This work was supported by the Natural sciences and Engineering Research Council of Canada. Recommended by Associate Editor X. Chen.

M. Guay is with the Department of Chemical Engineering, Queen's University, Kingston, ON K7L 3N6, Canada (e-mail: guaym@chee.queensu.ca)

V. Adetola is with the United Technologies Research Center, East Hartford, CT 06118-1127 USA (e-mail: adetolav@gmail.com).

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Digital Object Identifier 10.1109/TAC.2010.2052149