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Performance funnels and tracking control

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Tracking of an absolutely continuous reference signal (assumed bounded with essentially bounded derivative) is considered in the context of a class of non-linear, single-input, single-output, dynamical systems modelled by functional differential equations satisfying certain structural hypotheses (which, interpreted in the highly specialised case of linear systems, translate into assumptions of (i) relative degree one, (ii) positive high-frequency gain and (iii) stable zero dynamics). The control objective is evolution of the tracking error within a prespecified funnel, thereby guaranteeing prescribed transient performance and prescribed asymptotic tracking accuracy. This objective is achieved by a control which takes the form of linear error feedback with time-varying gain. The gain is generated by a non-linear feedback law in which the reciprocal of the distance of the tracking error to the funnel boundary plays a central role. In common with many established adaptive control methodologies, the overall feedback structure exploits an intrinsic high-gain property of the system, but differs from these methodologies in two fundamental respects: the funnel control gain is not dynamically generated and is not necessarily monotone. The main distinguishing feature of the present article vis à vis its various precursors is twofold: (a) non-linearities of a general nature can be tolerated in the input channel; (b) a more general formulation of prescribed transient behaviour is encompassed (including, for example, practical (M,μ)-stability wherein, for prescribed parameter values $M>1$, $\mu>0$ and $\lambda>0$, the tracking error $e(\cdot)$ is required to satisfy $|e(t)| < \max\{Me^{-\mu t}|e(0)|, \lambda\}$ for all $t \geq 0$).

Keywords: output feedback; non-linear systems; functional differential equations; transient behaviour; tracking

1. Introduction

Feedback stabilisation or tracking for non-linear systems is investigated in many textbooks; see, for example, Narendra and Annaswamy (1989), Isidori (1995), Marino and Tomei (1995), Isidori (1999), Sastry (1999). Restricting attention to single-input, single-output systems of relative degree one (the latter means, loosely speaking, that the input u appears explicitly in the expression for the first derivative of the output y), many authors study systems in the following Byrnes–Isidori normal form (or variants thereof):

$$\begin{aligned}\dot{y}(t) &= a(y(t), z(t)) + b(y(t), z(t))\, u(t),\\ \dot{z}(t) &= c(y(t), z(t)), \quad (y(0), z(0)) = (\xi, \zeta).\end{aligned} \tag{1.1}$$

Under suitable assumptions on the continuous functions $a, b : \mathbb{R}\times\mathbb{R}^{n-1}\to\mathbb{R}$ and $c:\mathbb{R}\times\mathbb{R}^{n-1}\to\mathbb{R}^{n-1}$, the objective is a dynamic control law of the form

$$u(t) = k\big(y(t), \eta(t)\big)\, y(t), \quad \dot{\eta}(t) = p(y(t), z(t)), \tag{1.2}$$

where $k:\mathbb{R}\times\mathbb{R}^{\ell}\to\mathbb{R}$ and $p:\mathbb{R}\times\mathbb{R}^{\ell}\to\mathbb{R}^{\ell}$ are continuous, which ensures (semi) global (practical) stabilisation of the closed-loop system; see, to name but two, Isidori (1995, pp. 143, 174, 189) and Isidori (1999, p. 79). In approaches to achieving the above objective, the following assumptions are ubiquitous: (i) the continuous function b is bounded away from zero (the *relative-degree-one* assumption); (ii) *stable zero dynamics*, that is in particular, 0 is a (globally asymptotically) stable equilibrium of the system $\dot{z} = c(0, z)$. Assuming that the subsystem $\dot{z} = c(y, z)$ generates a controlled semi-flow ϕ in the sense that, if we temporarily regard y as an independent (continuous) input, then, for each $(\zeta, y(\cdot)) \in \mathbb{R}^{n-1}\times C(\mathbb{R}_+)$, where $\mathbb{R}_+ := [0,\infty)$, the initial-value problem $\dot{z} = c(y, z), z(0) = \zeta$, has unique solution $z:\mathbb{R}_+\to\mathbb{R}^{n-1}$ given by $z(t) := \phi(t;\ \zeta, y)$. Thus, with the equation $\dot{z} = c(y, z)$, we may associate a family of operators $T_\zeta : C(\mathbb{R}_+)\to C(\mathbb{R}_+)$, parameterised by the initial data ζ, given by $(T_\zeta y)(t) := \phi(t;\ \zeta, y(\cdot))$. Introducing the operator T defined by $(Ty)(t) := (y(t), (T_\zeta y)(t))$, the initial-value problem (1.1) may be reformulated (in terms of the input and output variables) as

$$\dot{y}(t) = a\big((Ty)(t)\big) + b\big((Ty)(t)\big)\, u(t), \quad y(0) = \xi.$$

The above reformulation of (1.1) as an initial-value problem for a functional differential equation may be

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regarded as a prototype for the *system class* considered in the present article (and made precise in § 2) which consists of systems of the form

$$\dot{y}(t) = a(d_1(t), (Ty)(t)) + b(d_2(t), (Ty)(t))\, g(u(t) + d_3(t)) \tag{1.3}$$

wherein a, b and g are continuous functions, d_1, d_2 and d_3 are disturbances, and T is a causal operator (and is locally Lipschitz in a sense to be made precise in § 2). This class affords considerably more generality than that of system (1.1). Firstly, in (1.1) the variable u occurs affine linearly on the right-hand side and thus (with the aforementioned assumption (i) that the function b is bounded away from zero) this system has relative degree one: by contrast, the allowable input non-linearities g in (1.3) (to be made precise in § 2) are such the system does not necessarily have a well-defined relative degree (for definition of the latter see Isidori (1995, p. 137) or, more generally, Liberzon, Morse, and Sontag (2002)): for example, g may be supported only on a set of finite measure. Secondly, (1.1) is finite-dimensional whilst the system class of the present article encompasses – via the generality of the operators T allowable in (1.3) – infinite-dimensionality (e.g. delays, both point and distributed) and hysteretic effects (e.g. backlash, Prandtl and Preisach hysteresis): moreover, the aforementioned assumption (ii) of asymptotically stable zero dynamics has a weaker counterpart in the form of a bounded-input, bounded-output hypothesis on T. We elaborate on such examples in Appendix A.

Many approaches in the literature, both adaptive or non-adaptive, are concerned with asymptotic behaviour of solutions of the feedback system. In contrast, the present article is concerned with both asymptotic and transient performance of the feedback system. In particular, the *control objective* is to ensure that the tracking error, i.e. the difference between output and reference signal, evolves within a prespecified funnel, which is 'shaped' to ensure the requisite transients and asymptotics. With reference to Figure 1, we remark that the funnel radius is not permitted to shrink to zero at infinity: it may, however, approach an arbitrarily small prescribed value $\lambda > 0$, thereby ensuring tracking with prescribed asymptotic accuracy λ. The main contribution of the article is to establish that the error evolution within the funnel is achieved by the application of a variant of a so-called 'funnel controller', introduced in Ilchmann, Ryan, and Sangwin (2002). Novel features of funnel control include the following:

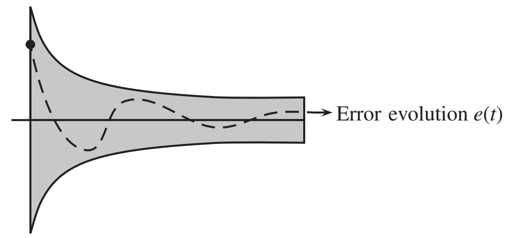

Figure 1. Performance funnel $\mathcal{F}_{\varphi}$.

(a) The control law is a simple time-varying error feedback of the form $u(t) = -k(t)\, e(t)$, where $e = y - r$ denotes the tracking error between the output y and a given reference signal r and the gain function k is generated by a feedback of the form $k(t) = f(t, e(t), |e(0)|)$. The intuition underpinning this control structure is the exploitation of an inherent high-gain property of the system class in order to maintain the error evolution within the funnel by ensuring that, if the error approaches the funnel boundary, then the gain takes values sufficiently large to preclude contact with the boundary. Whilst the control exploits an inherent high-gain property, it is not a high-gain controller in the usual sense: in particular, and in contrast to high-gain adaptive control methodologies, k is not monotone and decreases as the error recedes from the funnel boundary.

(b) The gain k is adapted, but the control $u(t) = -k(t)\, e(t)$ is not an adaptive strategy in the conventional sense: in particular, and in contrast to (1.2), it is non-dynamic.

(c) The approach does not invoke any identification mechanism or internal model principle.

Funnel control has been applied to temperature control in chemical reactor models (Ilchmann and Trenn 2004), even in the presence of input constraints, and to speed control of electric drives (Ilchmann and Schuster 2009), the latter has been tested successfully in the laboratory.

To provide a simple introductory illustration of the principles of funnel control and the attendant stability analysis, we focus briefly on a scalar linear system of the form

$$\dot{y}(t) = ay(t) + bu(t), \quad y(0) = y^0, a, b \in \mathbb{R}, b > 0. \tag{1.4}$$

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a bounded, non-decreasing, continuously differentiable (C^1) function with bounded derivative, and with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. Consider the control

$$u(t) = -k(t)y(t), \quad k(t) = \frac{\varphi(t)}{1 - \varphi(t)|y(t)|}. \tag{1.5}$$

Introducing the set $\Omega = \{(t,z) \in \mathbb{R}_+ \times \mathbb{R} \mid \varphi(t)|z| < 1\}$, the closed-loop initial-value problem (1.4) and (1.5) takes the form

$$\begin{aligned} &\dot{y}(t) = F(t, y(t)), y(0) = y^0, \\ &F : \Omega \to \mathbb{R}, (t,z) \mapsto \big(a - b\varphi(t)/(1-\varphi(t)|z|)\big)z. \end{aligned} \tag{1.6}$$

By a solution of (1.6) we mean a C^1 function $y : [0,\omega) \to \mathbb{R}$, satisfying (1.6), with $0 < \omega \le \infty$ and $(t, y(t)) \in \Omega$ for all $t \in [0,\omega)$: a solution is maximal if it has no extension that is also a solution; a solution is global if it exists on $\mathbb{R}_+$. Observe that F is locally Lipschitz in the sense that, for every $(t,z) \in \Omega$, there exists an open neighbourhood $B(t,z)$ of (t,z) and $L > 0$ such that

$$|F(\tau,\zeta) - F(\tau,\xi)| \le L|\zeta - \xi| \quad \forall (\tau,\zeta),(\tau,\xi) \in \Omega \cap B(t,z).$$

Therefore, the standard theory of ordinary differential equations (see, e.g., Walter (1998), Theorem II.6.VII) can be invoked to conclude that, for each $y^0 \in \mathbb{R}$, (1.6) has unique maximal solution y: $[0,\omega) \to \mathbb{R}$ and, moreover, if $t \mapsto 1/(1-\varphi(t)|y(t)|)$ is bounded, then the solution is global. Let $y^0 \in \mathbb{R}$ be arbitrary and let $y : [0,\omega) \to \mathbb{R}$ be the unique maximal solution of (1.6). Then $(t, y(t)) \in \Omega$ for all $t \in [0,\omega)$ and so, in particular, the function y is bounded. Define $\kappa : [0,\omega) \to \mathbb{R}_+$ by $\kappa(t) := \varphi(t)|y(t)|/(1-\varphi(t)|y(t)|) = k(t)|y(t)| = |u(t)|$ and note that $\kappa(0) = 0$. Seeking a contradiction, suppose that the κ is unbounded. Then there exists a strictly increasing sequence (t_n) in $[0,\omega)$ such that $\kappa(t_n) = n$ for all $n \in \mathbb{N}$ and $\varphi(t_n)|y(t_n)| \to 1$ as $n \to \infty$. For each $n \in \mathbb{N}$, define

$$s_n := \sup\{t \in [t_n, t_{n+1}] \mid \quad \kappa(t) = n\}.$$

Since $\kappa(t_n) = n$ and $\kappa(t_{n+1}) = n+1$, we have $t_n \le s_n < t_{n+1}$ and $\kappa(t) = k(t)|y(t)| \ge n$ for all $t \in [s_n, t_{n+1}]$. Therefore, $|y(t)| > 0$ for all $t \in [s_n, t_{n+1}]$ and all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $\sigma_n = \operatorname{sgn}(y(s_n))$, and so $\sigma_n y(t) > 0$ for all $t \in [s_n, t_{n+1}]$. Observe that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\big(\varphi(t)|y(t)|\big) &= \frac{\mathrm{d}}{\mathrm{dt}}\big(\sigma_n \varphi(t) y(t)\big) \\ &= \sigma_n\big(\dot{\varphi}(t)y(t) + \varphi(t)\dot{y}(t)\big) \\ &= c_n(t) - \sigma_n b k(t)\varphi(t)y(t) \\ &= c_n(t) - b\varphi(t)\kappa(t) \quad \forall t \in [s_n, t_{n+1}] \quad \forall n \in \mathbb{N}, \end{aligned}$$

where c_n: $[0,1) \to \mathbb{R}$ is the continuous function given by $c_n(t) := \sigma_n(\dot{\varphi}(t)y(t) + a\varphi(t)y(t))$. By boundedness of $\varphi, \dot{\varphi}$ and y, there exists $d_0 > 0$ such that

$$|c_n(t)| \le d_0 \quad \forall t \in [0,\omega) \quad \forall n \in \mathbb{N}.$$

Since $t_1 > 0$ and invoking properties of φ, we have

$$\varphi(t) \ge \varphi(t_1) =: d_1 > 0 \quad \forall t \ge t_1.$$

Now choose $n \in \mathbb{N}$ such that $d_1 b n > d_0$, in which case, we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\big(\varphi(t)|y(t)|\big) \le d_0 - b\varphi(t)\kappa(t) \le d_0 - d_1 b n < 0 \quad \forall t \in [s_n, t_{n+1}].$$

Therefore,

$$\begin{aligned} \dot{\kappa}(t) &= \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\varphi(t)|y(t)|}{1-\varphi(t)|y(t)|}\right) \\ &= \frac{1}{(1-\varphi(t)|y(t)|)^2}\frac{\mathrm{d}}{\mathrm{dt}}\big(\varphi(t)|y(t)|\big) < 0 \quad \forall t \in [s_n, t_{n+1}], \end{aligned}$$

whence the contradiction: $0 > \kappa(t_{n+1}) - \kappa(s_n) = n+1-n = 1$. Therefore, $\kappa(\cdot)$ is bounded and so $\sup_{t\in[0,\omega)}\varphi(t)|y(t)| < 1$. Equivalently, there exists $\varepsilon > 0$ such that $\varphi(t)|y(t)| \le 1-\varepsilon$ and so the function $t \mapsto 1/(1-\varphi(t)|y(t)|)$ is bounded, whence $\omega = \infty$. Thus, via elementary arguments, we have shown that, for every y^0, the simple control structure (1.5) ensures that the closed-loop initial-value problem (1.6) has unique global solution y. We emphasise that the control u and gain k are bounded functions; moreover, $|y(t)| \le \varphi(t)$ for all $t \ge 0$ and so transient and asymptotic behaviour of the solution may be prespecified through choice of the function φ.

The main concern of the present article is to extend the above ideas to tracking control problems in the setting of functional differential equations of the type (1.3). The distinguishing feature of the paper vis à vis its precursor (Ilchmann et al. 2002) is twofold: (a) nonlinearities g of a general nature can be tolerated in the input channel; (b) a more general formulation of prescribed transient behaviour is encompassed, including, for example, a variant of (M,μ)-stability (see Hinrichsen and Pritchard (2005), Section 5.5).

The article is organised as follows. In § 2, we make precise the underlying system class and the control objective is formulated in § 3: illustrative examples are provided in Appendix A. The main result is presented in § 4, wherein the closed-loop system gives rise to an initial-value problem for a functional differential equation: the nature of this problem is such that it falls outside the scope of existence theories in the literature known to the authors. For this reason, an existence theory – of sufficient generality to include the closed-loop initial-value problem – is developed in Appendix B. The article concludes with a numerical simulation in § 5.

2. System class

We consider single-input, single-output systems described by a functional differential equation of the form (1.3) and having the general structure depicted in Figure 2, wherein T is a causal operator and d_1, d_2 and

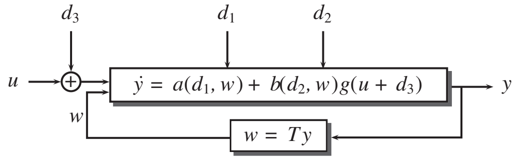

Figure 2. The open loop system.

d_3 are extraneous disturbances. For example, as already shown in § 1, the initial-value problem (1.1) may be reformulated (in terms of the input and output variables) as an initial-value problem for a disturbance-free system of the form depicted in Figure 2 with $g = \mathrm{id}$ (the identity map on $\mathbb{R}$). Other examples can be found in Appendix A.

We proceed to a description of the general system class, first making precise the associated class of operators T. Throughout, $L^\infty(\mathbb{R}_+, \mathbb{R}^\ell)$ is the space of measurable, essentially bounded functions $\mathbb{R}_+ \to \mathbb{R}^\ell$, with norm given by $\|y\|_\infty := \operatorname{ess\,sup}_{t\in\mathbb{R}_+}\|y(t)\|$; the space of measurable, locally essentially bounded functions $\mathbb{R}_+ \to \mathbb{R}^\ell$ is denoted by $L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^\ell)$; $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^\ell)$ is the space of absolutely continuous functions $y : \mathbb{R}_+ \to \mathbb{R}^\ell$ with $y, \dot{y} \in L^\infty(\mathbb{R}_+, \mathbb{R}^\ell)$.

Definition 2.1 (Operator class $\mathcal{T}^q_h$): For $h,\ t \in \mathbb{R}_+$, $w \in C[-h, t]$, $\tau > \mathrm{t}$ and $\delta > 0$, define

$$\mathcal{C}(w; h, t, \tau, \delta) := \big\{ v \in C[-h, \tau] \,\big|\, v|_{[-h,t]} = w, \\ |v(s) - w(t)| \le \delta \quad \forall s \in [t, \tau] \big\},$$

that is, the space of all continuous extensions v of $w \in C[-h, t]$ to the interval $[-h, \tau]$ with the property that $|v(s) - w(t)| \le \delta$ for all $s \in [t, \tau]$.

An operator T is said to be of class $\mathcal{T}^q_h$, for some $q \in \mathbb{N}$, if, and only if the following hold:

(i) $T : C[-h, \infty) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^q)$.
(ii) T is a causal operator.
(iii) For each $t \ge 0$ and each $w \in C[-h, t]$, there exist $\tau > t$, $\delta > 0$ and $c_0 > 0$ such that

$$\operatorname{ess\text{-}sup}_{s\in[t,\tau]}\|(Ty)(s) - (Tz)(s)\| \\ \le c_0 \max_{s\in[t,\tau]}|y(s) - z(s)| \\ \forall y, z \in \mathcal{C}(w; h, t, \tau, \delta).$$

(iv) For every $c_1 > 0$ there exists $c_2 > 0$ such that, for all $y \in C[-h, \infty)$,

$$\sup_{s\in[-h,\infty)}|y(s)| \le c_1 \Longrightarrow \|(Ty)(t)\| \le c_2 \quad \text{for a.a. } t \ge 0.$$

Remark 2.2: Property (iii) is a technical assumption of local Lipschitz type which is required for well-posedness of the closed-loop system. To interpret (iii) correctly, we need to give meaning to Ty, for a function $y \in C(I)$ on a bounded interval I of the form $[-h, \rho)$ or $[-h, \rho]$, where $0 < \rho < \infty$. This we do by showing that T 'localises', in a natural way, to an operator $\tilde{T} : C(I) \to L^\infty_{\mathrm{loc}}(J, \mathbb{R}^q)$, where $J := I \setminus [-h, 0)$. Let $y \in C(I)$. For each $\sigma \in J$, define $y_\sigma \in C[-h, \infty)$ by

$$y_\sigma(t) := \begin{cases} y(t), & t \in [-h, \sigma], \\ y(\sigma), & t > \sigma. \end{cases}$$

By causality, we may define $\tilde{T}y \in L^\infty_{\mathrm{loc}}(J, \mathbb{R}^q)$ by the property $\tilde{T}y|_{[0,\sigma]} = Ty_\sigma|_{[0,\sigma]}$ for all $\sigma \in J$. Henceforth, we will not distinguish notationally an operator T and its 'localisation' $\tilde{T}$: the correct interpretation being clear from context.

Property (iv) is a bounded-input, bounded-output assumption on the operator T. This assumption is a weak counterpart of the 'stable zero dynamics' assumption ubiquitous in the context of high-gain control of linear systems.

Definition 2.3 (System class $\Sigma^{p,q}_h$): Let $p, q \in \mathbb{N}$ and $h \ge 0$. The functional differential equation

$$\dot{y}(t) = a\big(d_1(t), (Ty)(t)\big) + b\big(d_2(t), (Ty)(t)\big)\, g\big(u(t) + d_3(t)\big), \tag{2.1}$$

defines a system of class $\Sigma^{p,q}_h$, written $(a, b, g, T, d_1, d_2, d_3) \in \Sigma^{p,q}_h$, if, and only if, the following hold:

(i) $a : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$ is continuous.
(ii) $b : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$ is continuous and sign definite, that is,

$$b(d, s) \ne 0 \qquad \forall (d, s) \in \mathbb{R}^p \times \mathbb{R}^q. \tag{2.2}$$

(iii) $g : \mathbb{R} \to \mathbb{R}$ is continuous with

$$\left.\begin{array}{ll} \text{(a)} & \limsup_{t\to\infty} b_1 g(v) = +\infty \\ \text{(b)} & \limsup_{v\to\infty} b_1 g(-v) = -\infty \end{array}\right\} \tag{2.3}$$

where $b_1 := \mathrm{sgn}(b)$, the polarity of the sign-definite function b.
(iv) $T \in \mathcal{T}^q_h$.
(v) $d_1,\ d_2 \in L^\infty(\mathbb{R}_+, \mathbb{R}^p)$, $d_3 \in W^{1,\infty}(\mathbb{R}_+)$.

Remark 2.4: Some remarks on the nature of the input non-linearity are warranted. The function g can be interpreted in two distinct ways:

(i) The function g may form part of the overall control structure in the sense that it is a synthesisable element which may be designed to compensate for lack of knowledge of the polarity $b_1 = \mathrm{sgn}(b)$ of the input connection function b. In this context, the role of g is akin to that of a so-called 'Nussbaum function' in adaptive control, see Nussbaum (1983). For example, the choice $g : u \mapsto u \cos u$ ensures that properties (2.3) hold.

(ii) Alternatively, g may be regarded as an uncertain intrinsic component of the plant, in which case, assumption (2.3) places some restrictions on the manner in which the functions g and b interact. In this context, note that g may influence/reverse the polarity of a control input u in a somewhat arbitrary manner. Note also that (assuming $d_3 = 0$ for simplicity) a control input u is nullified on the zero set $g^{-1}(0) \subset \mathbb{R}$ of g and the measure of this set may be infinite in the sense that the function g may be supported only on a set of finite Lebesgue measure: a simple example of such a function is a continuous unbounded odd function $g : \mathbb{R} \to \mathbb{R}$ with $g(u) := \sum_{n=1}^{\infty} b_1 g_n(u)$ for all $u \in \mathbb{R}_+$ (and so $g(u) = -g(-u)$ for all $u < 0$), where, for each $n \in \mathbb{N}$, $g_n : \mathbb{R}_+ \to \mathbb{R}_+$ is a locally Lipschitz function supported on $I_n := [n, n + 2^{-n}]$ in which case, the measure of the set $\mathbb{R}\backslash g^{-1}(0)$ is bounded from above by the quantity $2\sum_{n=1}^{\infty} |I_n| = 2\sum_{n=1}^{\infty} 2^{-n} = 2$. In more extreme cases, the function g may reverse the intended polarity of the control input: moreover, the 'bad' set on which the polarity is reversed may be large in comparison with the 'good' set on which polarity is maintained. In particular, assume (2.3) holds and let g^+ and g^- denote the positive and negative parts of $b_1 g$, in which case $b_1 g = g^+ - g^-$; the Lebesgue measure of the support of g^- (the 'bad' polarity-reversing set) may be infinite, whilst the support of g^+ (the 'good' set) may have only finite measure. Since, in this second context, knowledge of g is not available to the controller, it is perhaps counter-intuitive that the approximate tracking objective, as described in the next section, is achievable in the presence of input non-linearities of such generality.

3. The control objective and performance funnel

Let $(a, b, g, T, d_1, d_2, d_3) \in \Sigma_h^{p,q}$ and consider the initial-value problem

$$\begin{aligned}\dot{y}(t) &= a\big(d_1(t), (Ty)(t)\big) \\ &\quad + b\big(d_2(t), (Ty)(t)\big)\, g\big(u(t) + d_3(t)\big), \\ y|_{[-h,0]} &= y^0 \in C[-h, 0].\end{aligned}$$

The control objective is to design a simple tracking error feedback controller of the form $u(t) = -k(t)e(t)$, with gain $k(t)$ also generated by feedback of the error $e(t) = y(t) - r(t)$, so that, for all initial functions $y^0 \in C[-h, 0]$ and all reference signals $r \in W^{1,\infty}(\mathbb{R}_+)$, every solution of the closed-loop initial-value problem is bounded and approximate tracking with prescribed asymptotic accuracy and transient behaviour is achieved in the sense that the tracking error satisfies an a priori bound and asymptotically approaches a prescribed (arbitrarily small) neighbourhood of zero. The prescription of asymptotic and transient behaviour will be formulated via the following class Ψ of functions $\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$.

Definition 3.1 (Function class Ψ): A continuous function $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class Ψ if for all $\zeta \ge 0$, the following hold:

(i) $\psi(t, \zeta) > 0 \quad \forall t > 0$
(ii) $\liminf_{t\to\infty} \psi(t, \zeta) > 0$;
(iii) $\psi(\cdot, \zeta) \in W^{1,\infty}(\mathbb{R}_+)$;
(iv) $\zeta\ \psi(0, \zeta) < 1$.

Let $\psi \in \Psi$, $y^0 \in C[-h, 0]$ and $r \in W^{1,\infty}(\mathbb{R}_+)$ be arbitrary, and write $e^0 := y^0(0) - r(0)$. Then the performance objective of prescribed asymptotic and transient behaviour of the tracking error e is now specified in a predetermined manner through choice of the function Ψ and captured by the requirement that

$$\psi(t, |e^0|)|e(t)| < 1 \quad \forall t \in \mathbb{R}_+. \tag{3.1}$$

In addition, if for some prescribed $\lambda > 0$ arbitrarily small, ψ satisfies

$$\limsup_{t\to\infty} [\psi(t, \zeta)]^{-1} \le \frac{1}{\lambda} \quad \forall\, \zeta \ge 0\,, \tag{3.2}$$

then asymptotic tracking accuracy $\limsup_{t\to\infty} |e(t)| \le \lambda$ is achieved. With reference to Figure 1 and writing $\varphi(\cdot) = \psi(\cdot, |e^0|) \in W^{1,\infty}(\mathbb{R}_+)$, we see that (3.1) may, in turn, be identified as the requirement that the tracking error should evolve within a performance funnel

$$\mathcal{F}_\varphi := \text{graph}\ \Big(t \mapsto \big\{z \in \mathbb{R} \big| \varphi(t)|z| < 1\big\}\Big),$$

i.e. the graph of a set-valued map defined on $\mathbb{R}_+$, the value of which, at $t \in \mathbb{R}_+$, is the interval $(-1/\varphi(t), 1/\varphi(t))$. Note that the boundary of $\mathcal{F}_\varphi$ is determined by the reciprocal of φ.

Example 3.2:

(A) Fix $\lambda > 0$ and choose $\varphi \in W^{1,\infty}(\mathbb{R}_+)$ such that $\varphi(0) = 0$, $\varphi(t) \in (0, 1/\lambda)$ for all $t > 0$ and $\lim_{t\to\infty} \varphi(t) = 1/\lambda$. Define $\psi \in \Psi$ by the property that $\psi(\cdot, \zeta) = \varphi(\cdot)$ for all $\zeta \in \mathbb{R}_+$. Then satisfaction of (3.1) (equivalently, error evolution within the funnel $\mathcal{F}_\varphi$) implies that

$$|e(t)| < 1/\varphi(t) \quad \forall\, t > 0.$$

For example, the choice

$$t \mapsto \varphi(t) = \frac{\min\{t/\tau, 1\}}{\lambda}, \quad \text{with } \tau, \lambda > 0,$$

ensures that the modulus of the error decays at rate $\tau\lambda/t$ in the 'initial (transient) phase' $(0,\tau]$, and, since (3.2) holds, is bounded by λ in the 'terminal phase' $[\tau,\infty)$.

(B) In this second example, and in contrast with Example (A) above, the function ψ has non-trivial dependence on its second argument: for $M>1, \mu>0$ and $\lambda>0$, define $\psi\in\Psi$ by

$$\psi(t,\zeta) := 1/\max\{Me^{-\mu t}\zeta,\, \lambda\}, \quad \forall t,\zeta\in\mathbb{R}_+.$$

This choice of ψ corresponds to the control objective of 'practical (M,μ)-stability' of the tracking error in the sense that, for every $y^0\in C[-h,0]$ and $r\in W^{1,\infty}(\mathbb{R}_+)$, the tracking error $e=y-r$ (with $e^0=e(0)$) is required to satisfy

$$|e(t)| < \max\left\{Me^{-\mu t}|e^0|,\,\lambda\right\} \qquad \forall t\geq 0.$$

For example, if $M|e^0|>\lambda$ and (3.1) holds, then, defining $\tau := \ln(M|e^0|/\lambda)/\mu$, the tracking error decays at prescribed exponential rate in the 'initial (transient) phase' $[0,\tau]$ (that is, $|e(t)|<Me^{-\mu t}|e^0|$ for all $t\in[0,\tau]$), and is bounded by λ in the 'terminal phase' $[\tau,\infty)$ (that is $|e(t)|<\lambda$ for all $t\geq\tau$).

4. Main result: funnel output feedback

Loosely speaking, funnel control exploits an inherent benign high-gain property of the system by designing – with appropriate choice of $\psi\in\Psi$ – a proportional error feedback $u(t)=-k(t)\,e(t)$ in such a way that $k(t)$ becomes large if $|e(t)|$ approaches the performance funnel boundary (equivalently, if $\psi(t,|e(0)|)|e(t)|$ approaches the value 1), thereby precluding contact with the funnel boundary. We emphasise that the gain is non-monotone and decreases as the error recedes from the funnel boundary. The essence of the proof of the main result lies in showing that the closed-loop system is well posed in the sense that u and k are bounded functions and the error evolves strictly within the performance funnel.

For $\psi\in\Psi$, the 'funnel controller' can be expressed in its simplest form as

$$u(t) = -k(t)\,e(t), \quad k(t) = \frac{\psi(t,|e(0)|)}{1-\psi(t,|e(0)|)|e(t)|}. \tag{4.1}$$

This form is a special case of a more general structure

$$u(t) = -k(t)\,e(t), \quad k(t) = \alpha\big(\psi(t,|e(0)|)|e(t)|\big)\,\psi(t,|e(0)|), \tag{4.2}$$

wherein α is any continuously differentiable unbounded injection $[0,1)\to\mathbb{R}_+$ with $\alpha(0)>0$. Note that $\alpha(s)\to\infty$ as $s\uparrow 1$ and the choice $\alpha: s\mapsto 1/(1-s)$ yields (4.1). Let $p,q\in\mathbb{N}$, $h\geq 0$ and consider control (4.2) applied to the system $(a,b,g,T,d_1,d_2,d_3)\in\Sigma_h^{p,q}$. In view of the potential for 'blow up' in the gain generation in (4.2), some care must be exercised in formulating the closed-loop system. Introducing

$$\Omega := \{(t,z,\zeta)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+ \mid \psi(t,\zeta)|z|<1\}, \tag{4.3}$$

we define

$$f:\Omega\to\mathbb{R}, \quad (t,z,\zeta)\mapsto f(t,z,\zeta) := \alpha\big(\psi(t,\zeta)|z|\big)\psi(t,\zeta), \tag{4.4}$$

in which case, the control (4.1) can be interpreted, in explicit feedback form, as

$$u(t) = -f(t,e(t),|e(0)|)\; e(t). \tag{4.5}$$

Let $y^0\in C[-h,0]$ and $r\in W^{1,\infty}(\mathbb{R}_+)$ be arbitrary and write $e^0 := y^0(0)-r(0)$. The closed-loop initial-value problem now takes the form

$$\left.\begin{aligned} \dot y(t) &= a\big(d_1(t),(Ty)(t)\big) + b\big(d_2(t),(Ty)(t)\big) \\ &\quad \times g\big(d_3(t) - f(t, y(t)-r(t), |e^0|)e(t)\big) \\ y|_{[-h,0]} &= y^0 \in C[-h,0]. \end{aligned}\right\} \tag{4.6}$$

Setting $\varphi(\cdot) := \psi(\cdot,|e^0|)$ (with associated performance funnel $\mathcal{F}_\varphi$), (4.6) may, in turn, be rewritten as

$$\dot y(t) = F(t,y(t),(Ty)(t)), \quad y|_{[-h,0]} = y^0 \in C[-h,0], \tag{4.7}$$

where

$$F:\mathcal{D}\times\mathbb{R}^q\to\mathbb{R}, \quad \mathcal{D} := \{(t,v)\in\mathbb{R}_+\times\mathbb{R} \mid (t, v-r(t))\in\mathcal{F}_\varphi\} \tag{4.8}$$

is a Carathéodory function (as defined in Appendix B) given by

$$\begin{aligned} F(t,v,w) &:= a(d_1(t),w) + b(d_2(t),w) \\ &\quad \times g\big(d_3(t) - f(t, v-r(t), |e^0|)(v-r(t))\big). \end{aligned} \tag{4.9}$$

By a *solution* of (4.7) we mean a function $y\in C[-h,\omega)$, $0<\omega\leq\infty$, such that $y|_{[-h,0]}=y^0$, $y|_{[0,\omega)}$ is locally absolutely continuous, with $(t,y(t))\in\mathcal{D}$ for all $t\in[0,\omega)$ and $\dot y(t) = F(t,y(t),(Ty)(t))$ for almost all $t\in[0,\omega)$. A solution is said to be *maximal* if it has no proper right extension that is also a solution. A solution defined on $[-h,\infty)$ is said to be *global*.

In Appendix B, we develop an existence theory of sufficient generality to encompass the closed-loop initial-value problem (4.7): this theory is a variant of that in Ilchmann et al. (2002) – the distinguishing feature of the present article resides in the nature of the domain of the function F in (4.8) which places the initial-value problem (4.7) outside the scope of the existence theory in Ilchmann et al. (2002).

Now we are in a position to state the main result.

Theorem 4.1: *Let* $p, q \in \mathbb{N}$, $h \geq 0$, $y^0 \in C[-h, 0]$ *and* $r \in W^{1,\infty}(\mathbb{R}_+)$ *be arbitrary. Let* $\psi \in \Psi$ *and let* $\alpha : [0, 1) \to \mathbb{R}_+$ *be a continuously differentiable unbounded injection with* $\alpha(0) > 0$. *Then the control* (4.2) *applied to any system* $(a, b, g, T, d_1, d_2, d_3) \in \Sigma_h^{p,q}$ *is such that the resulting closed-loop initial-value problem* (4.3)–(4.6) *has a solution and every solution can be extended to a global solution. Every global solution y is such that, with* $e := y - r$, *the following hold*:

(a) *the control and gain functions u and k* (*given by* (4.2)) *are bounded;*
(b) *there exists* $\varepsilon \in (0, 1)$ *such that*

$$|e(t)| \leq \frac{1-\varepsilon}{\psi(t, |e(0)|)} \quad \forall t > 0.$$

Remark 4.2: Assertion (b) of Theorem 4.1 is its essence. Writing $\varphi(\cdot) := \psi(\cdot, |e(0)|)$, it asserts that the tracking error evolves within the performance funnel $\mathcal{F}_\varphi$ as depicted in Figure 1; moreover, the error evolution is strictly bounded away from the funnel boundary, thereby ensuring that the gain function k and the control function u in (4.1) are bounded. Therefore, the control objective is attained, whilst maintaining boundedness of all signals.

Proof of Theorem 4.1: Let $p, q \in \mathbb{N}$, $h \geq 0$, $(a, b, g, T, d_1, d_2, d_3) \in \Sigma_h^{p,q}$, $r \in W^{1,\infty}(\mathbb{R}_+)$ and $y^0 \in C[-h, 0]$ be arbitrary. Write $e^0 := e(0) = y^0(0) - r(0)$ and $\varphi(\cdot) := \psi(\cdot, |e^0|) \in W^{1,\infty}(\mathbb{R}_+)$ (by Definition 3.1(ii)). We have seen that the closed-loop initial-value problem (4.3)–(4.6) may be expressed in the form (4.7). Invoking Theorem B.1 of Appendix B, we may conclude that (4.7) has a solution and every solution can be extended to a maximal solution; moreover (noting that F is locally essentially bounded), if $y : [-h, \omega) \to \mathbb{R}$ is a maximal solution, then the closure of graph$(y|_{[0,\omega)})$ is not a compact subset of $\mathcal{D}$.

Let $y : [-h, \omega) \to \mathbb{R}$, $0 < \omega \leq \infty$, be a maximal solution. Then $e := y - r$ is bounded with $\varphi(t)|e(t)| < 1$, for all $t \in [0, \omega)$. Since r is bounded, it follows that $y = e + r$ is bounded and so, by property (iv) of $T \in \mathcal{T}_h^q$, the function $x := Ty$ is also bounded. Define $c : [0, \omega) \to \mathbb{R}$ by $c(t) := a(d_1(t), x(t)) - \dot{r}(t)$. By continuity of a, boundedness of x, and essential boundedness of $\dot{r}$ and d_1, it follows that $c \in L^\infty[0, \omega)$. By (4.7) and (4.9), we have

$$\dot{e}(t) = c(t) + b\big(d_2(t), x(t)\big)\, g\big(u(t) + d_3(t)\big) \quad \text{for a.a. } t \in [0, \omega). \tag{4.10}$$

Let $\beta : [0, 1) \to \mathbb{R}_+$ be the continuously differentiable bijection given by $\beta(s) := s\alpha(s)$; we record that $\beta'(s) = \alpha(s) + s\alpha'(s) \geq \alpha(0) > 0$ for all $s \in \mathbb{R}_+$. Write

$$\kappa(t) := \beta(\varphi(t)|e(t)|) = \alpha(\varphi(t)|e(t)|)\varphi(t)|e(t)| = k(t)|e(t)| = |u(t)| \quad \forall t \in [0, \omega).$$

Seeking a contradiction, suppose that the function κ is unbounded on $[0, \omega)$. Then there exists a strictly increasing sequence (t_n) in $[0, \omega)$, with $t_n \uparrow \omega$ as $n \to \infty$, such that the sequence $(\kappa(t_n))$ is a strictly increasing unbounded sequence in $\mathbb{R}_+$ and $(\varphi(t_n)|e(t_n)|)$ is a sequence in $(0, 1)$ with $\varphi(t_n)|e(t_n)| \to 1$ as $n \to \infty$. Since φ is bounded with $\varphi(t) > 0$ for all $t \in (0, \omega)$ and, if necessary, passing to a subsequence of (t_n) (which we do not relabel), we may infer the existence of $c_0 \in \{-1, 1\}$ such that $c_0 e(t_n) = -|e(t_n)| < 0$ for all $n \in \mathbb{N}$. By property (2.2) of the continuous function b, together with boundedness of x and essential boundedness of d_2, there exists $b_0 > 0$ such that

$$|b(d_2(t), x(t))| \geq b_0 \quad \text{for a.a. } t \in [0, \omega). \tag{4.11}$$

By properties (2.3) of g, there exist strictly increasing unbounded sequences (u_n) and (v_n) in $\mathbb{R}_+$ such that

$$b_1 g(u_n) \to \infty \quad \text{and} \quad -b_1 g(v_n) \to \infty \quad \text{as} \quad n \to \infty. \tag{4.12}$$

(Recall that $b_1 = \mathrm{sgn}(b)$, the polarity of the sign-definite function b.) Define the sequence (s_n) and the continuous function $\gamma : \mathbb{R} \to \mathbb{R}$ by

$$s_n := \begin{cases} u_n, & \text{if } c_0 = +1 \\ -v_n, & \text{if } c_0 = -1 \end{cases}, \quad \gamma(s) := b_0 b_1 c_0\, g(s) \quad \forall s \in \mathbb{R}.$$

Clearly,

$$c_0 = +1 \Rightarrow \gamma(s_n) = b_0 b_1 g(u_n) \quad \text{and}$$
$$c_0 = -1 \Rightarrow \gamma(s_n) = -b_0 b_1 g(-v_n),$$

and so, invoking (4.12) and recalling that $b_0 > 0$, we have $\gamma(s_n) \to \infty$ as $n \to \infty$. Passing to a subsequence (no relabelling) if necessary, we may assume that $(\gamma(s_n))$ is a strictly increasing sequence in $\mathbb{R}_+$. Now define the sequence $(\kappa_n) := (c_0 s_n)$. Observe that, if $c_0 = +1$, then $(\kappa_n) = (u_n)$ or, if $c_0 = -1$, then $(\kappa_n) = (v_n)$. Therefore (κ_n) is a strictly increasing sequence in $\mathbb{R}_+$ and so, extracting a subsequence (which we do not relabel), we may assume that $\kappa_n \geq 1 + \kappa(0) + \|d_3\|_\infty$ for all $n \in \mathbb{N}$. Again passing to a subsequence (without relabelling) of (t_n) if necessary, we may also assume that $\kappa(t_n) \geq \kappa_{n+1} + \|d_3\|_\infty$ for all $n \in \mathbb{N}$. We now have

$$\kappa(0) + c_0 d_3(0) < \kappa_n \quad \text{and}$$
$$\kappa(t_n) + c_0 d_3(t_n) \geq \kappa(t_n) - \|d_3\|_\infty \geq \kappa_{n+1} \forall\, n \in \mathbb{N}.$$

Therefore, the following is well defined for each $n \in \mathbb{N}$:

$$\tau_n := \inf\{t \in [0, t_n] |\ \kappa(t) + c_0 d_3(t) = \kappa_{n+1}\} > 0.$$

Noting that, for each $n \in \mathbb{N}$, there exists $t \in (0, \tau_n]$ such that $\kappa(t) + c_0 d_3(t) = \kappa_n$ (and so $c_0\kappa(t) + d_3(t) = s_n$), the following is also well defined for each $n \in \mathbb{N}$:

$$\sigma_n := \sup\{t \in (0, \tau_n] |\ \gamma(c_0\kappa(t) + d_3(t)) = \gamma(s_n)\} < \tau_n,$$

wherein the strict inequality $\sigma_n < \tau_n$ holds because

$$\gamma(c_0\kappa(\tau_n) + d_3(\tau_n)) = \gamma(c_0\kappa_{n+1}) = \gamma(s_{n+1}) > \gamma(s_n).$$

Suppose, for contradiction, that $\kappa(\sigma_n) + c_0 d_3(\sigma_n) \geq \kappa(\tau_n) + c_0 d_3(\tau_n)$ for some $n \in \mathbb{N}$. Then

$$\begin{aligned}\kappa(0) + c_0 d_3(0) < \kappa_{n+1} &= \kappa(\tau_n) + c_0 d_3(\tau_n)\\ &\leq \kappa(\sigma_n) + b_1 c_0 d_3(\sigma_n)\end{aligned}$$

and so, by continuity, there exists $s \in (0, \sigma_n]$ such that $\kappa(s) + c_0 d_3(s) = \kappa_{n+1}$, whence the contradiction:

$$\tau_n = \inf\{t \in [0, t_n] |\ \kappa(t) + c_0 d_3(t) = \kappa_{n+1}\} \leq s \leq \sigma_n < \tau_n.$$

Therefore, $\kappa(\sigma_n) + c_0 d_3(\sigma_n) < \kappa(\tau_n) + c_0 d_3(\tau_n)$ for all $n \in \mathbb{N}$. Since d_3 is bounded with essentially bounded derivative, there exists $c_1 > 0$ such that

$$\begin{aligned}\kappa(\sigma_n) &< \kappa(\tau_n) + c_0(d_3(\tau_n) - d_3(\sigma_n))\\ &\leq \kappa(\tau_n) + (\tau_n - \sigma_n)c_1 \quad \forall n \in \mathbb{N}.\end{aligned} \tag{4.13}$$

By definition of σ_n and noting that $\gamma(c_0\kappa(\tau_n) + d_3(\tau_n)) = \gamma(s_{n+1}) > \gamma(s_n)$, we have

$$\gamma(c_0\kappa(t) + d_3(t)) > \gamma(s_n) \quad \forall t \in (\sigma_n, \tau_n] \quad \forall n \in \mathbb{N}. \tag{4.14}$$

We also record that

$$-|e(t)| = c_0 e(t) < 0 \quad \forall t \in [\sigma_n, \tau_n] \quad \forall n \in \mathbb{N}. \tag{4.15}$$

We may now conclude that

$$\begin{aligned}&-c_0 b(d_2(t), w(t)) g(u(t) + d_3(t))\\ &\quad = -|b(d_2(t), w(t))| \gamma(c_0\kappa(t) + d_3(t))/b_0\\ &\quad \leq -\gamma(c_0\kappa(t) + d_3(t))\\ &\quad \leq -\gamma(s_n)\ \forall t \in [\sigma_n, \tau_n]\ \forall\ n \in \mathbb{N}.\end{aligned} \tag{4.16}$$

Observe that, for all $n \in \mathbb{N}$ and almost all $t \in [\sigma_n, \tau_n]$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi(t)|e(t)|\big) &= -c_0\big(\dot\varphi(t)e(t) + \varphi(t)\dot e(t)\big)\\ &= -c_0\big(\dot\varphi(t)e(t) + \varphi(t)c(t)\\ &\qquad + \varphi(t)b(d_2(t), w(t))g(u(t) + d_3(t))\big).\end{aligned}$$

By boundedness of φ and e, together with essential boundedness of $\dot\varphi$ and c, and invoking (4.16), we may conclude that, for some constant $c_2 > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi(t)|e(t)|\big) \leq c_2 - \varphi(t)\gamma(s_n) \quad \text{for a.a. } t \in [\sigma_n, \tau_n] \quad \forall n \in \mathbb{N}. \tag{4.17}$$

Define $c_3 := \inf_{t\in[\sigma_1,\infty)}\varphi(t) = \inf_{[\sigma_1,\infty)}\psi(t, |e^0|)$. Since $\sigma_1 > 0$ and invoking property (ii) of $\psi \in \Psi$, it follows that $c_3 > 0$. Fix $n \in \mathbb{N}$ sufficiently large so that $\alpha(0)(c_2 - c_3\gamma(s_n)) < -c_1$, in which case we have

$$\begin{aligned}\dot\kappa(t) &= \beta'(\varphi(t)|e(t)|)\frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi(t)|e(t)|\big) \leq \alpha(0)\big(c_2 - c_3\gamma(s_n)\big)\\ &< -c_1 \quad \text{for a.a. } t \in [\sigma_n, \tau_n]\end{aligned}$$

whence $\kappa(\tau_n) - \kappa(\sigma_n) < -(\tau_n - \sigma_n)c_1$, which contradicts (4.13). This proves boundedness of κ. By boundedness of $t \mapsto \kappa(t) = \beta(\varphi(t)|e(t)|)$, we may conclude that $\sup_{t\in[0,\omega)}\varphi(t)|e(t)| < 1$, equivalently, there exists $\varepsilon \in (0, 1)$ such that $\varphi(t)|e(t)| \leq 1 - \varepsilon$ for all $t \in [0, \omega)$. Therefore, the gain k and input u functions are bounded.

It remains only to show that the solution $y : [-h, \omega) \to \mathbb{R}$ is global. Seeking a contradiction, suppose $\omega < \infty$. Then $\mathcal{K} := \{(t, y) \in \mathcal{D} |\ t \in [0, \omega], \varphi(t)|y - r(t)| \leq 1 - \varepsilon\}$ is a compact subset of $\mathcal{D}$ with the property $(t, y(t)) \in \mathcal{K}$ for all $t \in [0, \omega)$, which contradicts the fact that the closure of graph$(y|_{[0,\omega)})$ is not a compact subset of $\mathcal{D}$. Therefore, $\omega = \infty$. □

5. Illustrative simulation

Consider the system shown in Figure 3 consisting of a linear, single-input, single-output system (c, A, b) with state space $\mathbb{R}^n$, disturbance $d \in L^\infty(\mathbb{R}_+)$, a non-linearity g in the input channel, and a feedback loop containing a hysteretic non-linearity H:

$$\begin{aligned}\dot x(t) &= Ax(t) + b\big(d(t) + (H(cx))(t) + g(u(t))\big),\\ x(0) &= x^0 \in \mathbb{R}^n, \quad y(t) = cx(t).\end{aligned} \tag{5.1}$$

As discussed in §A.2 of Appendix A, under the assumptions that the linear system (c, A, b) has positive high-frequency gain $cb > 0$ and is minimum-phase, there exists a similarity transformation S that takes the triple into the form $(\hat c, \hat A, \hat b)$, with

$$\hat c = cS^{-1} = \begin{pmatrix}1 & 0\end{pmatrix}, \quad \hat A = SAS^{-1} = \begin{pmatrix}A_1 & A_2\\ A_3 & A_4\end{pmatrix},$$
$$\hat b = Sb = \begin{pmatrix}cb\\ 0\end{pmatrix},$$

and, in view of the minimum-phase assumption, $A_4 \in \mathbb{R}^{(n-1)\times(n-1)}$ is a Hurwitz matrix. Writing $\binom{y^0}{z^0} := Sx^0$, defining the operator T_1 by

$$(T_1 y)(t) := A_1 y(t) + A_2\int_0^t (\exp A_4(t - s))A_3 y(s)\,\mathrm{d}s$$

and writing $T_2 := cbH$, we see that (5.1) can be reformulated as

$$\begin{aligned}\dot y(t) &= ((T_1 + T_2)(y)) + d_1(t) + cb\,g(u(t)),\\ y(0) &= y^0, \quad d_1(t) := A_2\big(\exp A_4 t\big)z^0 + cb\,d(t).\end{aligned} \tag{5.2}$$

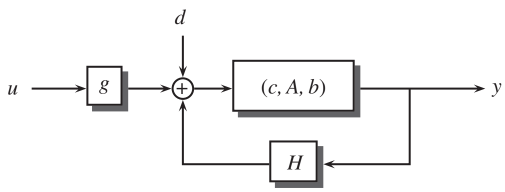

Figure 3. Linear system with hysteretic feedback loop and input non-linearity.

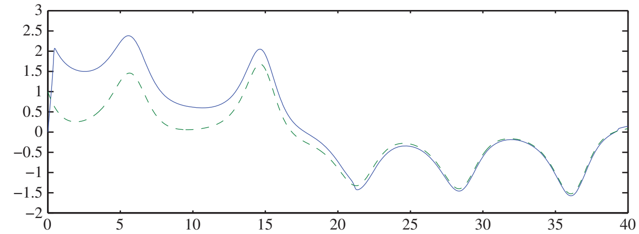

Figure 4. The output y (solid line) and reference r (dashed line).

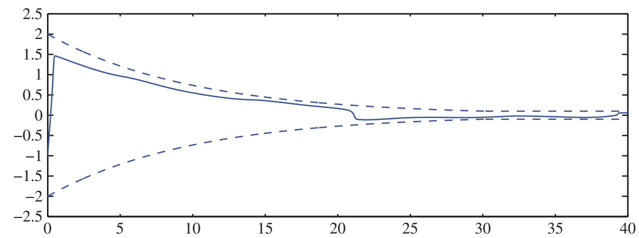

Figure 5. Error evolution within funnel.

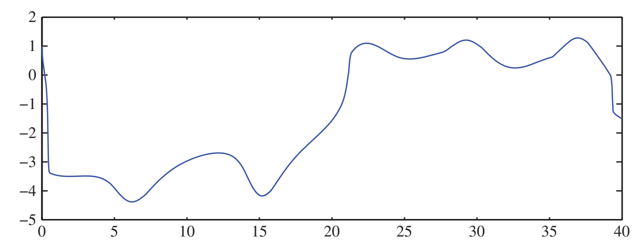

Figure 6. The control u.

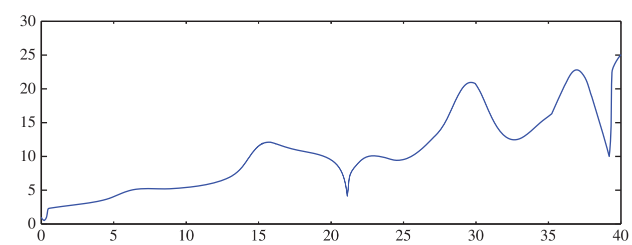

Figure 7. The gain function k.

Since A_4 is Hurwitz, it is readily verified that T_1 is in the operator class $\mathcal{T}_0^1$ and $d_1 \in L^\infty(\mathbb{R}_+)$. If we assume that the hysteresis operator H is also of class $\mathcal{T}_0^1$ (as discussed in § A.3 of Appendix A, many commonly encountered hysteretic components – including backlash and, more generally, Preisach operators – are of class $\mathcal{T}_0^1$), then $T := T_1 + T_2$ is of class $\mathcal{T}_0^1$. Defining $a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\tilde{b} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $a(d, w) := \mathrm{d} + w$ and $\tilde{b}(d, w) = cb$, (5.2) may be expressed as

$$\dot{y}(t) = a(d_1(t), (Ty)(t)) + \tilde{b}(0, (Ty)(t))g(u(t)), \quad y(0) = y^0,$$

which is an initial-value problem for the system $(a, \tilde{b}, g, T, d_1, 0, 0)$ of class $\Sigma_0^{1,1}$. For purposes of illustration, as reference signal $r \in W^{1,\infty}(\mathbb{R}_+)$ and disturbance $d \in W^{1,\infty}(\mathbb{R}_+)$, we take $r = \zeta_1$ and $d = \zeta_3$, where ζ_1 and ζ_3 are the first and third components of the (chaotic) solution of the following initial-value problem for the Lorenz system:

$$\left.\begin{aligned} \dot{\zeta}_1(t) &= \zeta_2(t) - \zeta_1(t), & \zeta_1(0) &= 1, \\ \dot{\zeta}_2(t) &= c_0\zeta_1(t) - c_1\zeta_2(t) - \zeta_1(t)\zeta_3(t), & \zeta_2(0) &= 0, \\ \dot{\zeta}_3(t) &= \zeta_1(t)\zeta_2(t) - c_2\zeta_3(t), & \zeta_3(0) &= 3. \end{aligned}\right\} \quad (5.3)$$

with parameter values $c_0 = 28/10$, $c_1 = 1/10$ and $c_2 = 8/30$. It is well known that the unique global solution of (5.3) is bounded with bounded derivative; see, for example, Sparrow (1982).

Let (c, A, b) be given by

$$c = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

let g be given by $g(u) := (1 + u)|u|$, and let $H = \mathcal{B}_{\sigma,\xi}$ be the backlash hysteresis operator of § A.3 in Appendix A, with parameter values $\sigma = 1/2$ and $\xi = 0$. We adopt the objective of 'practical (M, μ)-stability', as described in Example 3.2 (B), with parameter values $\lambda = 0.02$, $\mu = 0.2$ and $M = 2$, and the simple control structure given by (4.1). For initial data $x^0 = 0$, Figure 4 depicts the evolution of the output y and reference signal r; Figure 5 depicts the error evolution within the funnel; Figures 6 and 7 show the control signal u and the gain k.

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Appendix A. Examples of the system class $\Sigma_h^{p,q}$ and the operator class $\mathcal{T}_h^q$

A.1 *Finite-dimensional non-linear prototype*

Consider again the initial-value problem for the non-linear prototype system (1.1). In §1, we have seen that, if the equation $\dot z = c(y, z)$ is assumed to generate a controlled semiflow ϕ, then with this equation we may associate a family of operators $T_\zeta : C(\mathbb{R}_+) \to C(\mathbb{R}_+, \mathbb{R}^{n-1})$, parameterised by the initial data $\zeta \in \mathbb{R}^{n-1}$, given by $(T_\zeta y)(t) := \phi(t; \zeta, y(\cdot))$. Introducing $T : C(\mathbb{R}_+) \to C(\mathbb{R}_+, \mathbb{R} \times \mathbb{R}^{n-1})$ defined by $(Ty)(t) := (y(t), (T_\zeta)(t))$, the initial-value problem (1.1) may be reformulated (in terms of the input and output variables) as

$$\dot y(t) = a\big((Ty)(t)\big) + b\big((Ty)(t)\big)u(t), \quad y(0) = \xi. \tag{A.1}$$

If, in addition, we assume that the system $\dot z = c(y, z)$ is input-to-state stable (ISS) (see Sontag (2007)), then it is readily verified that the operator T is of class $\mathcal{T}_0^n$. Assuming that the function b is positive-valued and bounded away from zero and introducing the functions $\tilde a : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $\tilde b : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ (these are simply convenient artifacts) given by

$$\tilde a(d, w) := d + a(w), \quad \tilde b(d, w) := d + b(w),$$

we see that (A.1) is equivalent to

$$\dot y(t) = \tilde a\big(0, (Ty)(t)\big) + \tilde b\big(0, (Ty)(t)\big)u(t), \quad y(0) = \xi,$$

which is an initial-value problem for the system $(\tilde a, \tilde b, \mathrm{id}, T, 0, 0, 0)$ of class $\Sigma_0^{1,n}$. Therefore, under the assumptions that the system $\dot z = c(y, z)$ is ISS and b is positive-valued and bounded away from zero, Theorem 4.1 implies that, for all $(\zeta, \xi) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and all $r \in W^{1,\infty}(\mathbb{R}_+)$, the control (4.2) applied to (1.1) ensures attainment of the tracking objectives.

A.2 *Linear (retarded) systems with input non-linearities*

Let $h > 0$, let A be an $n \times n$-matrix with entries in $BV[0, h]$ (the space of real-valued functions of bounded variation on $[a, b] \subset \mathbb{R}$) and let $b, c^T \in \mathbb{R}^n$. Consider the linear retarded system with non-linearity g in the input channel

$$\dot x = \mathrm{d}A * x + bg(u), \quad x|_{[-h,0]} = x^0 \in C([-h, 0], \mathbb{R}^n), \quad y = cx, \tag{A.2}$$

where $(\mathrm{d}A * x)(t) := \int_0^h \mathrm{dA}(\tau)\mathrm{x}(\mathrm{t} - \tau)$ for all $t \in \mathbb{R}_+$, satisfying

- *minimum-phase* condition, i.e.
$$\det\begin{pmatrix} sI - \hat A(s) & -b \\ c & 0 \end{pmatrix} \neq 0 \quad \forall s \in \mathbb{C}, \mathrm{Re}(s) > 0,$$
$$\text{where } \hat A(s) := \int_0^h \exp(-s\tau)\mathrm{d}A(\tau)$$
- *positive high-frequency gain* condition, i.e. $cb > 0$
- $\lim\sup_{v\to\infty} g(v) = +\infty$, $\lim\inf_{v\to\infty} g(-v) = -\infty$.

It is well known that, under these assumptions, there exists a similarity transformation which takes system (A.2) into the form

$$\dot y = \mathrm{d}A_1 * y + \mathrm{d}A_2 * z + cb\, g(u), \quad y|_{[-h,0]} = y^0, \tag{A.3a}$$

$$\dot z = \mathrm{d}A_3 * y + \mathrm{d}A_4 * z, \quad z|_{[-h,0]} = z^0, \tag{A.3b}$$

where, by the minimum-phase condition, A_4 has the property that

$$\det(sI - \hat A_{22}(s)) \neq 0 \quad \forall s \in \mathbb{C}, \mathrm{Re}(s) > 0, \tag{A.4}$$

see Ilchmann and Logemann (1998) for details. For given $z^0 \in C([-h, 0], \mathbb{R}^{n-1})$ and given $\xi \in C[-h, \infty)$, let $z(\cdot\,; z_0, \xi)$ denote the unique solution of the initial-value problem

$$\dot z = \mathrm{d}A_4 * z + \mathrm{d}A_3 * \xi, \quad z|_{[-h,0]} = z^0.$$

Defining the operator T and function d_1 by

$$T(\xi) := \mathrm{d}A_1 * \xi + \mathrm{d}A_2 * z(\cdot\,; 0, \xi)\,, \quad d_1 := \mathrm{d}A_2 * z(\cdot\,; z_0, 0)\,,$$

Equation (A.3a) can be expressed as

$$\dot{y} = d_1 + T(y) + cb\, g(u)\,, \quad y^0 = cx^0. \tag{A.5}$$

By the standard theory of retarded functional differential equations (Hale and Verduyn Lunel 1993, Corollary 6.1, p. 215), (A.4) implies that the zero solution of the retarded equation $\dot{z} = \mathrm{d}A_4 * z$ is exponentially stable, so that there exists $K > 0$ such that, for all $z^0 \in C([-h, 0], \mathbb{R}^{n-1})$ and all $\xi \in C[-h, \infty)$,

$$\sup_{t \in [0,\infty)} |z(t; z_0, \xi)| \le K\left(\sup_{t \in [-h,0]} |z^0(t)| + \sup_{t \in [-h,\infty)} |\xi(t)|\right).$$

We conclude that d is bounded and that $T \in \mathcal{T}_h^1$. Finally, defining $a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\tilde{b} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (as in the previous example, these are simply artifacts) by $a(d, w) := d + w$ and $\tilde{b}(d, w) = cb$, we see that (A.5) is equivalent to

$$\dot{y}(t) = a\big(d_1(t), (Ty)(t)\big) + \tilde{b}\big(0, (Ty)(t)\big)u(t), \quad y^0 = cx^0\,,$$

which is an initial-value problem for the system $(a, \tilde{b}, g, T, d_1, 0, 0)$ of class $\Sigma_h^{1,(n-1)}$.

The above example is readily modified to include the non-retarded case $h = 0$. In this case, we simply replace $\mathrm{d}A * x$ by Ax ($A \in \mathbb{R}^{n \times n}$), and $\mathrm{d}A_1 * y, \mathrm{d}A_2 * z, \mathrm{d}A_3 * y, \mathrm{d}A_4 * z$ by $A_1, \dots, A_4$ at the appropriate places. In this way, we see that the class of linear, single-input, single-output, minimum-phase, relative-degree-one systems (c, A, b), with $cb > 0$ and with non-linearity g in the input channel, is subsumed by our system class $\Sigma_0^{1,1}$.

A.3 *Systems with hysteresis*

An operator $T : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ is a *hysteresis operator* if it is causal and rate independent. Here *rate independence* means that $T(y \circ \zeta) = (Ty) \circ \zeta$ for every $y \in C(\mathbb{R}_+)$ and every time transformation ζ, where $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *time transformation* if it is continuous, non-decreasing and surjective. The so-called Preisach operators are among the most general and most important hysteresis operators: in particular, they can model complex hysteresis effects such as nested loops in input–output characteristics.

A basic building block for these operators is the *backlash* operator. A discussion of the *backlash* operator (also called *play* operator) can be found in a number of references; see, for example, Krasnosel'skii and Pokrovskii (1989), Brokate and Sprekels (1996) and Logemann and Mawby (2001). Let $\sigma \in \mathbb{R}_+$ and introduce the function $b_\sigma : \mathbb{R}^2 \to \mathbb{R}$ given by

$$b_\sigma(v_1, v_2) := \max\big\{v_1 - \sigma,\ \min\{v_1 + \sigma, v_2\}\big\}.$$

Let $C_{\mathrm{pm}}(\mathbb{R}_+)$ denote the space of continuous piecewise monotone functions defined on $\mathbb{R}_+$. For all $\sigma \in \mathbb{R}_+$ and $\xi \in \mathbb{R}$, define the operator $\mathcal{B}_{\sigma,\xi} : C_{\mathrm{pm}}(\mathbb{R}_+) \to C(\mathbb{R}_+)$ by

$$\mathcal{B}_{\sigma,\xi}(y)(t) = \begin{cases} b_\sigma(y(0), \xi) & \text{for } t = 0\,, \\ b_\sigma(y(t), (\mathcal{B}_{\sigma,\xi}(u))(t_i)) & \text{for } t_i < t \le t_{i+1}, i = 0, 1, 2, \dots\,, \end{cases}$$

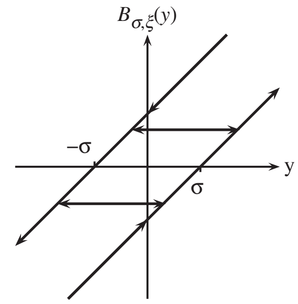

Figure A1. Backlash hysteresis.

where $0 = t_0 < t_1 < t_2 < \cdots$, $\lim_{n\to\infty} t_n = \infty$ and u is monotone on each interval $[t_i, t_{i+1}]$. We remark that ξ plays the role of an 'initial state'. It is not difficult to show that the definition is independent of the choice of the partition (t_i). Figure A1 illustrates how $\mathcal{B}_{\sigma,\xi}$ acts. It is well known that $\mathcal{B}_{\sigma,\xi}$ extends to a Lipschitz continuous operator on $C(\mathbb{R}_+)$ (with Lipschitz constant $L = 1$), the so-called backlash operator, which we shall denote by the same symbol $\mathcal{B}_{\sigma,\xi}$. It is well known that $\mathcal{B}_{\sigma,\xi}$ is a hysteresis operator.

Let $\xi : \mathbb{R}_+ \to \mathbb{R}$ be a compactly supported and globally Lipschitz function with Lipschitz constant 1. Let μ be a signed Borel measure on $\mathbb{R}_+$ such that $|\mu|(K) < \infty$ for all compact sets $K \subset \mathbb{R}_+$, where $|\mu|$ denotes the total variation of μ. Denoting Lebesgue measure on $\mathbb{R}$ by μ_L, let $w : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be a locally $(\mu_L \otimes \mu)$-integrable function and let $w_0 \in \mathbb{R}$. The operator $\mathcal{P}_\xi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by

$$(\mathcal{P}_\xi(y))(t) = \int_0^\infty \int_0^{(\mathcal{B}_{\sigma,\xi(\sigma)}(y))(t)} w(s, \sigma)\mu_L(\mathrm{d}s)\mu(\mathrm{d}\sigma) + w_0 \qquad \forall\, y \in C(\mathbb{R}_+)\,, \quad \forall\, t \in \mathbb{R}_+, \tag{A.6}$$

is called a *Preisach* operator (cf. Brokate and Sprekels 1996, p. 55). It is well known that $\mathcal{P}_\xi$ is a hysteresis operator (this follows from the fact that $\mathcal{B}_{\sigma,\xi(\sigma)}$ is a hysteresis operator for every $\sigma \ge 0$). Under the assumption that the measure μ is finite and w is essentially bounded, the operator $\mathcal{P}_\xi$ is Lipschitz continuous with Lipschitz constant $L = |\mu|(\mathbb{R}_+)\|w\|_\infty$ (Logemann and Mawby 2001) in the sense that

$$\sup_{t \in \mathbb{R}_+} |\mathcal{P}_\xi(y_1)(t) - \mathcal{P}_\xi(y_2)(t)| \le L \sup_{t \in \mathbb{R}_+} |y_1(t) - y_2(t)| \quad \forall\, y_1, y_2 \in C(\mathbb{R}_+).$$

This property ensures that the Preisach operator belongs to our operator class $\mathcal{T}_0^1$.

Setting $w(\cdot, \cdot) = 1$ and $w_0 = 0$ in (A.6), yields the *Prandtl* operator $\mathcal{P}_\xi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ given by

$$\mathcal{P}_\xi(y)(t) = \int_0^\infty (\mathcal{B}_{\sigma,\xi(\sigma)}(y))(t)\mu(\mathrm{d}\sigma) \quad \forall\, y \in C(\mathbb{R}_+)\,, \quad \forall\, t \in \mathbb{R}_+. \tag{A.7}$$

For $\xi \equiv 0$ and μ given by $\mu(E) = \int_E \chi_{[0,5]}(\sigma)\mathrm{d}\sigma$ (where $\chi_{[0,5]}$ denotes the indicator function of the interval $[0, 5]$), the Prandtl operator is of class $\mathcal{T}_0^1$ and is illustrated in Figure A2. These examples serve to illustrate that systems (1.3) incorporating rather general hysteresis operators T fall within the scope of our theory.

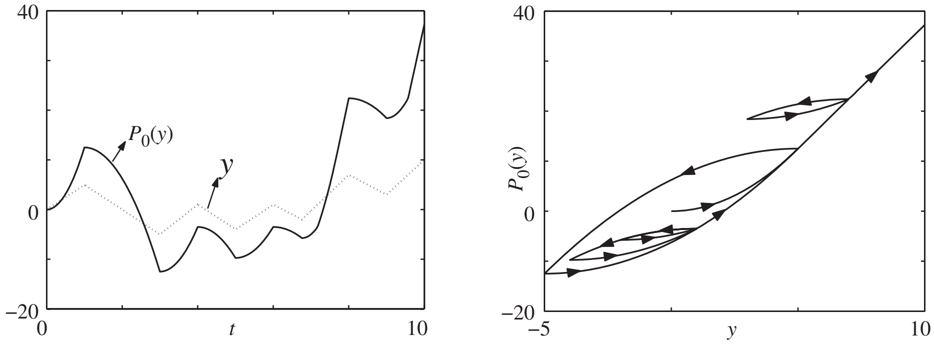

Figure A2. Example of Prandtl hysteresis.

Appendix B. Existence theory

Let $\mathcal{D}$ be a domain in $\mathbb{R}_+ \times \mathbb{R}$ (that is, a non-empty, connected, relatively open subset of $\mathbb{R}_+ \times \mathbb{R}$). Let $q \in \mathbb{N}$ and assume that $F : \mathcal{D} \times \mathbb{R}^q \to \mathbb{R}$ is a Carathéodory function.[1] Let $T \in \mathcal{T}_h^q$ and $t_0 \in \mathbb{R}_+$. Consider the initial-value problem

$$\dot{y}(t) = F(t, y(t), (Ty)(t)), \quad y|_{[-h,t_0]} = y^0 \in C[-h, t_0], \quad (t_0, y^0(t_0)) \in \mathcal{D}. \tag{B.1}$$

A *solution* of (B.1) is a function $y \in C(I)$ on an interval of the form $I = [-h, \rho]$, $t_0 < \rho < \infty$, or $[-h, \omega)$, $t_0 < \omega \le \infty$, such that $y|_{[-h,t_0]} = y^0$, $y|_J$ is locally absolutely continuous, with $(t, y(t)) \in \mathcal{D}$ for all $t \in J$ and $\dot{y}(t) = F(t, y(t), (Ty)(t))$ for almost all $t \in J$, where $J := I \backslash [-h, t_0]$. A solution is *maximal* if it has no proper right extension that is also a solution.

Theorem B.1: *For all initial data* $(t_0, y^0) \in \mathbb{R}_+ \times C[-h, t_0]$ *with* $(t_0, y^0(t_0)) \in \mathcal{D}$

(i) *the initial-value problem* (B.1) *has a solution*,
(ii) *every solution can be extended to a maximal solution* $y \in C[-h, \omega)$,
(iii) *if F is locally essentially bounded and* $y \in C[-h, \omega)$ *is a maximal solution, then the closure of* graph$(y|_{[t_0,\omega)})$ *is not a compact subset of* $\mathcal{D}$.

Proof: By property (iii) of the class $\mathcal{T}_h^q$, there exist $\tau > t_0, \delta > 0$ and $c_0 > 0$ such that

$$\text{ess}-\sup_{s\in[t,\tau]} \|(Ty)(s) - (Tz)(s)\| \le c_0 \max_{s\in[t,\tau]} |y(s) - z(s)| \quad \forall y, z \in \mathcal{C}(y^0; h, t, \tau, \delta).$$

We may assume that $\delta \in (0, 1)$ and $\tau - t_0 > 0$ are sufficiently small so that

$$\mathcal{D}_0 := [t_0, \tau] \times [y^0(t_0) - \delta\, ,\; y^0(t_0) + \delta] \subset \mathcal{D}.$$

By property (iv) of $\mathcal{T}_h^q$, there exists $c_2 > 0$ such that

$$\forall y \in C[-h, \infty) \text{ and } \forall t \in [t_0, \tau] : \quad \sup_{t\in[-h,\infty)} |y(t)| < c_1 := \delta + \|y^0\|_\infty \|(Ty)(t)\| < c_2.$$

Since F is a Carathéodory function, there exists integrable $\gamma : [t_0, \tau] \to \mathbb{R}_+$ such that

$$|F(t, \xi, \zeta)| \le \gamma(t) \quad \forall (t, \xi, \zeta) \in \mathcal{D}_0 \times \{\zeta \in \mathbb{R}^q \mid \|\zeta\| < c_2\}. \tag{B.2}$$

Define $\Gamma \in C[-h, \tau]$ by

$$\Gamma(t) := \begin{cases} 0, & t \in [-h, t_0) \\ \int_{t_0}^t \gamma(s)\mathrm{d}s, & t \in [t_0, \tau]. \end{cases}$$

Since Γ is continuous and non-decreasing with $\Gamma(t_0) = 0$, there exists $\rho \in (t_0, \tau)$ such that $\Gamma(\rho) \in [0, \delta)$. We will establish the existence of a solution of the initial-value problem (B.1) on the interval $[-h, \rho]$. This we do by constructing a sequence (y_n) in $C[-h, \rho]$ with a subsequence converging to a solution $y \in C[-h, \rho]$ of (B.1). Let $n \in \mathbb{N}$ be arbitrary and define

$$\rho_m := t_0 + m\Delta_n \quad \text{for } m = 0, \ldots, n, \quad \Delta_n := (\rho - t_0)/n.$$

For each $m \in \{1, \ldots, n\}$ let $P(m)$ be the statement

$$P(m): \begin{cases} \text{there exists } y_m \in C[-h, \rho_m] \text{ such that} \\ |y_m(t)| < c_1 \;\; \forall t \in [-h, \rho_m], \;\; |y_m(t) - y^0(t_0)| < \delta \;\; \forall t \in [t_0, \rho_m] \\ y_m(t) = y^0(t) \;\; \forall t \in [-h, t_0], \;\; y_m(t) = y^0(t_0) \;\; \forall t \in (t_0, \rho_1) \\ y_m(t) = y^0(t_0) + \int_{t_0}^{t-\Delta_n} F(s, y_m(s), (Ty_m(s)))\mathrm{d}s \;\; \forall t \in [\rho_1, \rho_m]. \end{cases}$$

Let $m \in \{1, \ldots, (n-1)\}$ and assume that $P(m)$ is a true statement. Then,

$$\big(s, y_m(s), (Ty_m)(s)\big) \in \mathcal{D}_0 \times \{\zeta \in \mathbb{R}^q \mid \|\zeta\| < c_2\} \quad \forall s \in [t_0, \rho_m]$$

and so, by (B.2),

$$\left| \int_{t_0}^{t-\Delta_n} F(s, y_m(s), (Ty_m)(s))\mathrm{d}s \right| \le \int_{t_0}^{t-\Delta_n} \gamma(s)\mathrm{d}s = \Gamma(t - \Delta_n) \le \Gamma(\rho_m) < \delta \quad \forall t \in [\rho_m, \rho_{m+1}].$$

[1]Let $\mathcal{D}$ be a *domain* in $\mathbb{R}_+ \times \mathbb{R}$ (that is, a non-empty, connected, relatively open subset of $\mathbb{R}_+ \times \mathbb{R}$). A function $F : \mathcal{D} \times \mathbb{R}^q \to \mathbb{R}$, is deemed to be a *Carathéodory function* if, for every 'rectangle' $[a, b] \times [c, d] \subset \mathcal{D}$ and every compact set $K \subset \mathbb{R}^q$, the following hold: (i) $F(t, \cdot, \cdot) : [c, d] \times K \to \mathbb{R}$ is continuous for all $t \in [a, b]$; (ii) $F(\cdot, x, w) : [a, b] \to \mathbb{R}$ is measurable for each fixed $(x, w) \in [c, d] \times K$; (iii) there exists an integrable function $\gamma : [a, b] \to \mathbb{R}_+$ such that $|F(t, x, w)| \le \gamma(t)$ for almost all $t \in \mathbb{R}_+$ and all $(x, w) \in [c, d] \times K$.

Now, define $y_{m+1} : [-h, \rho_{m+1}] \to \mathbb{R}$ by

$$y_{m+1}(t) := \begin{cases} y^0(t), & t \in [-h, t_0] \\ y^0(t_0), & t \in (t_0, \rho_1) \\ y^0(t_0) + \int_{t_0}^{t-\Delta_n} F(s, y_m(s), (Ty_m)(s))\,\mathrm{d}s, & t \in [\rho_1, \rho_{m+1}]. \end{cases}$$

It immediately follows that $|y_{m+1}(t)| < c_1$ for all $t \in [-h, \rho_{m+1}]$ and $|y_{m+1}(t) - y^0(t_0)| < \delta$ for all $t \in [t_0, \rho_{m+1}]$. Clearly, y_{m+1} is continuous at all points $t \in [-h, \rho_{m+1}]$ with $t \neq \rho_1$. Moreover, since $\rho_1 - \Delta_n = \rho_0 = t_0$, we see that $y_{m+1}(\rho_1) = y^0(t_0)$, whence continuity at $t = \rho_1$. Therefore, $y_{m+1} \in C[-h, \rho_{m+1}]$. Finally, observing that $y_{m+1}(t) = y_m(t)$ for all $t \in [-h, \rho_m]$ and invoking causality of T, we may infer that

$$y_{m+1}(t) = y^0(t_0) + \int_{t_0}^{t-\Delta_n} F(s, y_{m+1}(s), (Ty_{m+1})(s))\mathrm{d}s \quad \forall t \in [\rho_1, \rho_{m+1}].$$

We have now established the following:

$P(m)$ true for some $m \in \{1, \ldots, (n-1)\} \Rightarrow P(m+1)$ true.

Defining $y_1 \in C[-h, \rho_1]$ by

$$y_1(t) := \begin{cases} y^0(t), & t \in [-h, \rho_0) \\ y^0(t_0), & t \in [\rho_0, \rho_1], \end{cases}$$

we see that $P(1)$ is a true statement. Therefore, $P(m)$ is true for $m = 1, \ldots, n$. We may now conclude that, for each $n \in \mathbb{N}$, there exists $y_n \in C[-h, \rho]$ such that

$$y_n(t) = \begin{cases} y^0(t), & t \in [-h, t_0] \\ y^0(t_0), & t \in (t_0, \rho_1) \\ y^0(t_0) + \int_{t_0}^{t-\Delta_n} F(s, y_n(s), (Ty_n)(s))\,\mathrm{d}s, & t \in [\rho_1, \rho]. \end{cases}$$

Moreover, $\max_{t \in [-h,\rho]} |y_n(t)| < c_1$ for all $n \in \mathbb{N}$ and so (y_n) is a bounded sequence in the Banach space $C[-h, \rho]$ with norm $\|y\|_\infty = \max_{t \in [-h,\rho]} |y(t)|$. We proceed to prove that the bounded sequence (y_n) is also equicontinuous. Let $\varepsilon > 0$ be arbitrary. By uniform continuity of $\Gamma \in C[-h, \rho]$, there exists $\bar{\delta} > 0$ such that

$$t, s \in [-h, \rho] \text{ with } |t - s| < \bar{\delta} \Rightarrow |\Gamma(t) - \Gamma(s)| < \varepsilon.$$

Let $t,\ s \in [t_0, \rho]$ be such that $|t - s| < \bar{\delta}$. Without loss of generality, we may assume that $s \leq t$. Observe that

(a) $t, s \in [t_0, \rho_1) \Rightarrow |y_n(t) - y_n(s)| = 0$,

(b) $s \leq \rho_1 \leq t \Rightarrow t - \rho_1 < \bar{\delta}$ and

$$|y_n(t) - y_n(s)| = |y_n(t) - y^0(t_0)| \leq \Gamma(t - \Delta_n) = |\Gamma(t - \rho_1 + t_0) - \Gamma(t_0)| < \varepsilon,$$

(c) $t, s \in [\rho_1, \rho] \Rightarrow |y_n(t) - y_n(s)|$

$$\leq |\Gamma(t - \Delta_n) - \Gamma(s - \Delta_n)| < \varepsilon.$$

Therefore, the sequence $(y_n|_{[t_0,\rho]})$ is equicontinuous. Since $y_n|_{[-h,t_0]} = y^0$ for all n, it follows that (y_n) is an equicontinuous sequence in $C[-h, \rho]$. By the Arzelà–Ascoli Theorem, it follows that (y_n) has a subsequence (which we do not relabel) converging to $y \in C[-h, \rho]$. Clearly, $y|_{[-h,t_0]} = y^0$. By property (iii) of $\mathcal{T}_h^q$, $\lim_{n\to\infty}(Ty_n)(t) = (Ty)(t)$ for almost all $t \in [t_0, \rho]$ and so, by continuity of $(\xi, \zeta) \mapsto F(t, \xi, \zeta)$,

$$\lim_{n\to\infty} F(t, y_n(t), (Ty_n)(t)) = F(t, y(t), (Ty)(t)) \quad \text{for a.a. } t \in [t_0, \rho].$$

By the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \int_{t_0}^{t} F(s, y_n(s), (Ty_n)(s))\,\mathrm{d}s = \int_{t_0}^{t} F(s, y(s), (Ty)(s))\,\mathrm{d}s \quad \forall t \in [t_0, \rho].$$

Noting that

$$\begin{aligned} y_n(t) &= y^0(t_0) + \int_{t_0}^{t-\Delta_n} F(s, y_n(s), (Ty_n)(s))\,\mathrm{d}s \\ &= y^0(t_0) + \left(\int_{t_0}^{t} - \int_{t-\Delta_n}^{t}\right) F(s, y_n(s), (Ty_n)(s))\,\mathrm{d}s \\ &\quad \forall t \in [t_0 + \Delta_n, \rho], \end{aligned}$$

and since $\Delta_n \downarrow 0$ as $n \to \infty$, we may conclude that

$$y(t) = \begin{cases} y^0(t), & t \in [-h, t_0] \\ y^0(t_0) + \int_{t_0}^{t} F(s, y(s), (Ty)(s))\,\mathrm{d}s, & t \in (t_0, \rho]. \end{cases}$$

Therefore, $y \in C[-h, \rho]$ is a solution of the initial-value problem (B.1). This establishes assertion (i) of the theorem.

Let $y \in C(I)$ be a solution of (B.1). Define

$$\mathcal{E} := \{(\omega, z) | \omega = \sup J, J \supset I, z \in C(J) \text{ is a solution of (B.1)}, z|_I = y\},$$

and so, for $(\omega, z) \in \mathcal{E}$, either $z = y$ or z is a solution which extends y. On this non-empty set, define a partial order $\preceq$ by

$$(\omega_1, z_1) \preceq (\omega_2, z_2) \Longleftrightarrow \omega_1 \leq \omega_2 \text{ and } z_1(t) = z_2(t) \quad \forall t \in [-h, \omega_1).$$

Assertion (ii) follows if we can establish that $\mathcal{E}$ has a maximal element. This we do by an application of Zorn's Lemma, as follows. Let $\mathcal{O}$ be a totally ordered subset of $\mathcal{E}$. Let $\omega^* := \sup\{\omega | \ (\omega, z) \in \mathcal{E}\}$ and define $z^* \in C[0, \omega^*)$ by the property that, for every $(\omega, z) \in \mathcal{O}, z^*|_{[-h,\omega)} = z$. Then (ω^*, z^*) is in $\mathcal{E}$ and is an upper bound for $\mathcal{O}$ (that is, $(\omega, z) \preceq (\omega^*, z^*)$ for all $(\omega, z) \in \mathcal{O}$). By Zorn's Lemma, it follows that $\mathcal{E}$ contains at least one maximal element.

Finally, we prove assertion (iii). Assume that F is locally essentially bounded and let $y \in C[-h, \omega)$ be a maximal solution of (B.1). Seeking a contradiction, suppose that $G := \text{graph}(y|_{[t_0,\omega)})$ has compact closure $\overline{G}$ in $\mathcal{D}$. Then, by boundedness of y, property (iv) of $\mathcal{T}_h^q$ and local essential boundedness of F, there exists $c_3 > 0$ such that $|\dot{y}(t)| \leq c_3$ for almost all $t \in [t_0, \omega)$. We may now conclude that y is uniformly continuous on the bounded interval $[-h, \omega)$ and so extends to a function $\hat{y} \in C[-h, \omega]$ with $\text{graph}(\hat{y}|_{[t_0,\omega]}) \subset \overline{G} \subset \mathcal{D}$. In particular, we have $(\omega, \hat{y}(\omega)) \in \mathcal{D}$. An application of assertion (i) (the roles of t_0 and y^0 now being taken by ω and $\hat{y}$) yields the existence $z \in C[-h, \rho]$ with $\omega < \rho$ and $z|_{[-h,\omega]} = \hat{y}$ such that $\dot{z}(t) = F(t, z(t), (\tilde{T}z)(t))$ for almost all $t \in [\omega, \rho]$. Therefore, z is a solution of the initial-value problem (B.1) and is an extension of y. This contradicts maximality of the solution y. $\square$