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ROBUSTNESS IN THE GRAPH TOPOLOGY OF A COMMON ADAPTIVE CONTROLLER*

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Abstract. For any m-input, m-output, finite-dimensional, linear, minimum-phase plant P with first Markov parameter having spectrum in the open right half complex plane, it is well known that the adaptive output feedback control C, given by $u = -ky$, $\dot{k} = \|y\|^2$, yields a closed-loop system $[P, C]$ for which the state converges to zero, the signal k converges to a finite limit, and all other signals are of class L^2. It is first shown that these properties continue to hold in the presence of L^2-input and L^2-output disturbances. Working within the conceptual framework of the nonlinear gap metric approach to robust stability, and by establishing gain function stability of an appropriate closed-loop operator, it is proved that these properties also persist when the plant P is replaced with a stabilizable and detectable linear plant P_1 within a sufficiently small neighborhood of P in the graph topology, provided that the plant initial data and the L^2 magnitude of the disturbances are sufficiently small. Example 9 of Georgiou and Smith [*IEEE Trans. Automat. Control*, 42 (1997), pp. 1200–1221] is revisited. Unstable behavior for large initial conditions and/or large L^2 disturbances is shown, demonstrating that the bounds obtained from the L^2 theory are qualitatively tight: this contrasts with the L^∞-robustness analysis of Georgiou and Smith, which is insufficiently tight, to predict the stable behavior for small initial conditions and zero disturbances.

Key words. adaptive control, gap metric, robust stability

AMS subject classifications. 93D21, 93D09, 93C40, 93D25

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1. Introduction.

In an important paper in 1997, the well-established concept of the gap metric for linear systems (see, e.g., [19, 5, 6, 18, 2]) was extended to a nonlinear setting by Georgiou and Smith [7]. Further developments can be found in [8, 16, 1, 4]. The central property analyzed in the nonlinear gap framework is that of robust stability, i.e., the property that, if $\mathcal{W}$ is some requisite class (for example, L^∞ or L^2) to which the signals of a nominal closed-loop plant/controller configuration belong, then the closed-loop signals remain in $\mathcal{W}$ if the nominal plant is replaced with another plant which is sufficiently close in the gap sense. Gain function stability (gf-stability) a concept made precise in subsection 2.3, of the closed-loop operator mapping external disturbances to the input and output of the nominal plant provides a sufficient condition for robust stability (however, in contrast with the results in the linear setting, gf-stability is not a necessary condition for robust stability in the nonlinear setting). The nonlinear gap framework has been used to investigate the robustness (or lack of robustness) of certain classical adaptive controllers and variants thereof.

In an L^∞ setting, Example 9 of [7] (see also [9]) considers the controller (ubiqui-

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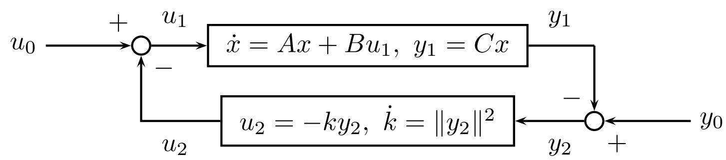

FIG. 1.1. *The adaptive closed-loop system.*

tous in the adaptive control literature)

$$u = -ky, \quad \dot{k} = y^2 \tag{1.1}$$

applied to the scalar linear plant $\dot{y} = ay + u$, for some $a \in \mathbb{R}$, and shows that the closed-loop operator mapping external disturbances onto the input and output of the nominal plant is not gain function stable (gf-stable). While the lack of gf-stability does not preclude robust stability, numerical and other informal evidence was presented which suggested that, with nonzero initial conditions, the closed-loop system is not robustly stable, even in the absence of disturbances. One consequence of the results of the present paper is to clarify the latter suggestion: we prove that, in the absence of disturbances, the closed-loop system is—with sufficiently small initial data—robustly stable but fails to be robustly stable for large initial data.

In [4], the nonlinear gap framework is used in an L^2 setting to establish robust stability properties of the controller $u = -k^{\frac{1}{4}}y$, $\dot{k} = y^2$ when applied to (a) single-input, single-output, linear, minimum-phase, relative-degree-one, nominal plants with positive high-frequency gain, and (b) a class of perturbed plants, where the gap metric distance between the nominal and perturbed plants is constrained by a function of the norms of the external disturbances. The present paper shows that the analysis developed in [4] can also be applied to the more familiar adaptive controller (1.1) (and its multivariable counterpart). This is considered to be important, as such controllers form the basis for many adaptive designs; see, e.g., [13, 14]. For example, adaptive backstepping designs [14, p. 100] reduce to (1.1) when applied to first-order linear plants. From an applied point of view, many application studies of such controllers exist in the literature (see, for example, [14] and the bibliography therein).

The analysis of the controller (1.1) differs significantly from that of the controllers in [4], requiring a different technique to establish the required gf-stability (Propositions 3.3, 3.5). In particular, in an L^2 setting, we establish a robust stability result for nominal m-input, m-output, finite-dimensional, stabilizable and detectable linear plants (A, B, C) which are minimum phase and are such that the first Markov parameter CB has its spectrum in the open right half complex plane (we denote the class of such plants by $\mathcal{M}$). With reference to Figure 1.1, in the absence of external disturbances (that is, with $u_0 = 0 = y_0$), it is well known (see, for example, [11, Theorem 4.2.1]) that, for every plant in $\mathcal{M}$ and all initial plant/controller data $(x^0, k^0) \in \mathbb{R}^n \times \mathbb{R}_+$ ($\mathbb{R}_+ := [0, \infty)$), the closed loop is such that (i) $u_1, y_1 \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, (ii) $x(t) \to 0$ as $t \to \infty$, and (iii) $k(t) \to k^\infty \in \mathbb{R}_+$ as $t \to \infty$. First, we show that properties (i)–(iii) persist under external disturbances $u_0, y_0 \in L^2(\mathbb{R}_+, \mathbb{R}^m)$. Second, we consider the question of robust stability of the closed loop with respect to both external L^2 disturbances and perturbations of the plant (A, B, C), i.e., to what extent do the above properties (i)–(iii) persist if $(A, B, C) \in \mathcal{M}$ is perturbed to another m-input, m-output, linear, finite-dimensional, stabilizable and detectable plant $(A_p, B_p, C_p) \notin \mathcal{M}$? An appropriate conceptual framework in which to pose and answer such questions is provided by the gap metric. We show that properties (i)–(iii) persist if (A, B, C)

and (A_p, B_p, C_p) are sufficiently close in the gap metric. The associated bounds on the robust stability margin have a semiglobal nature insofar as they depend on the "size" of the external disturbances and initial data. In the case of zero initial conditions, the linear gap metric δ_0 measures the size of the smallest stable coprime factor perturbation between plants. Thus, the stability results of the present paper can be interpreted within the framework of linear robust control, where coprime factor perturbations form the widely accepted model for unstructured uncertainties.

We emphasize that the principal contribution of [4] and the present paper is to obtain robust stability results for adaptive controllers for this general description of unmodeled dynamics. This contrasts with other results in adaptive control (see, for example, [13, Chapters 8 and 9], [10]), which are far more restricted in the class of unmodeled dynamics considered and which typically consider robust "modifications" to the underlying adaptive law, which can introduce conservativeness (see [4] for a full discussion).

For purposes of illustration, one expression for the linear gap δ_0 is given in the frequency domain as follows. Let (A_1, B_1, C_1) and (A_2, B_2, C_2) be single-input, single-output stabilizable linear systems with respective transfer functions $\hat{P}_1$ and $\hat{P}_2$. Then $\hat{P}_1$ and $\hat{P}_2$ admit normalized right coprime factorizations over RH^∞, the class of rational functions that are analytic and bounded on the open half plane $\mathbb{C}_+ := \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) > 0\}$. In particular, there exist $N_i, D_i \in RH^\infty$ such that

$$\hat{P}_i = N_i D_i^{-1}, \qquad N_i^* N_i + D_i^* D_i = 1, \qquad i = 1, 2. \tag{1.2}$$

For $(i, j) = (1, 2), (2, 1)$, define the directed gap

$$\vec{\delta}_0(\hat{P}_i, \hat{P}_j) := \inf\left\{\|(\Delta_N, \Delta_D)\|_{H^\infty} \,\middle|\, \Delta_N, \Delta_D \in RH^\infty,\ \hat{P}_j = (N_i + \Delta_N)(D_i + \Delta_D)^{-1}\right\}$$

(with the convention $\inf \emptyset := +\infty$). The linear gap between $\hat{P}_1$ and $\hat{P}_2$ is given by

$$\delta_0(\hat{P}_1, \hat{P}_2) \equiv \delta_0(\hat{P}_2, \hat{P}_1) = \max\left\{\vec{\delta}_0(\hat{P}_1, \hat{P}_2), \vec{\delta}_0(\hat{P}_2, \hat{P}_1)\right\}. \tag{1.3}$$

We remark that the gap between the following plants $\hat{P}_1$ and $\hat{P}_2$ tends to zero as $\varepsilon \to 0$:

	$\hat{P}_1(s)$	$\hat{P}_2(s)$		Reference
(i)	$\frac{1}{s-\theta}$	$\frac{\lvert\lambda\rvert^2}{(s-\lambda)(s-\bar{\lambda})(s-\theta)}$,	$\mathrm{Re}(\lambda) \le -\varepsilon^{-1}$	[4]
(ii)	$\frac{1}{s-\theta}$	$\frac{N(M-s)}{(N+s)(M+s)(s-\theta)}$,	$N, M \ge \varepsilon^{-1}$	section 3.4, Example 3.10
(iii)	$\frac{1}{s-\theta}$	$\frac{(M-s)}{(M+s)(s-\theta)}$,	$M \ge \varepsilon^{-1}$	section 4.5; see also [7]

Example (i) is the classical Rohrs' example [15] which first drew the attention of the adaptive control community to the robustness issue. As observed in [4], example (ii) is of particular interest since $\hat{P}_2$ exhibits none of the classical assumptions of adaptive control: in particular, the sign of the high-frequency gain and the relative degree of $\hat{P}_2$ differ from those of the nominal plant $\hat{P}_1$ and, moreover, $\hat{P}_2$ is not minimum phase. Example (ii) is considered in more detail in section 3.4. Example (iii) is comprised of an all-pass factor in series with the nominal plant and is considered extensively in section 4. Example (iii) also coincides with Example 9 in [7], to which our general L^2 theory applies to conclude robust stability provided the initial data and L^2 disturbance norms are sufficiently small. In section 4, we additionally prove the lack of robustness when the initial data or the L^2 disturbances are large. Moreover, we clarify some of the informal arguments in the L^∞ setting of [7].

The paper is structured as follows. In section 2 we introduce the notation and background theory: notions of well posedness and gf-stability of plant/controller interconnections are made precise, and the nonlinear gap is defined; this material is based on [7, section II] and [4, section 2]. Section 3 establishes the main robust stability result for the adaptive controller (1.1) (Theorem (3.8)) via the general framework of [4]. By considering example (iii) above in detail in section 4, we show that the bounds obtained in Theorem 3.8 are qualitatively tight in the sense that examples (i)–(iii) show that the robustness margin necessarily depends on the size of the disturbances and initial conditions.

2. Background concepts and terminology.

2.1. Preliminaries. While our goal is to establish stability of various configurations of plant and controller, the nonlinear nature of the controller is such that finite-time blow up of solutions of the closed-loop system cannot be ruled out a priori. To accommodate the potential for such behavior in the analysis, we introduce the following notions. Let $\mathcal{X}$ be a nonempty set and, for $0 < \omega \leq \infty$, let $\mathcal{S}_\omega$ denote the set of locally integrable maps $[0, \omega) \to \mathcal{X}$. For simplicity, we write $\mathcal{S} := \mathcal{S}_\infty$. For $0 < \tau < \omega \leq \infty$, $T_\tau : \mathcal{S}_\omega \to \mathcal{S}$ denotes the operator given by

$$T_\tau v \ := \begin{cases} v(t), & t \in [0, \tau), \\ 0, & \text{otherwise}. \end{cases}$$

With $\mathcal{V} \subset \mathcal{S}$ we associate spaces as follows: $\mathcal{V}_e = \{v \in \mathcal{S} \big|\ T_\tau v \in \mathcal{V} \text{ for all } \tau > 0\}$, *the extended space*; $\mathcal{V}_\omega = \{v \in \mathcal{S}_\omega \big|\ T_\tau v \in \mathcal{V} \text{ for all } \tau \in (0, \omega)\}$, $0 < \omega \leq \infty$; $\mathcal{V}_a = \cup_{\omega \in (0,\infty]} \mathcal{V}_\omega$, *the ambient space.* If $v, w \in \mathcal{V}_a$ with $v|_I = w|_I$ on $I = \mathrm{dom}(v) \cap \mathrm{dom}(w)$, then we write $v = w$. Note that $\mathcal{V} \subset \mathcal{V}_e \subset \mathcal{V}_a$ are strict inclusions and $\mathcal{V}_\infty = \mathcal{V}_e$. For $(f, g) \in \mathcal{V}_a \times \mathcal{V}_a$, the domains of f and g may be different; we adopt the convention $\mathrm{dom}(f, g) := \mathrm{dom}(f) \cap \mathrm{dom}(g)$. We define $\mathcal{V} \subset \mathcal{S}$ to be a *signal space* if and only if it is a vector space. In our applications, frequently $\mathcal{V}$ will be a normed signal space, such as $L^p(\mathbb{R}_+, \mathbb{R}^m)$ for $1 \leq p \leq \infty$, in which case $\mathcal{V}_e = L^p_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$, $\mathcal{V}_\omega = L^p_{\mathrm{loc}}([0, \omega), \mathbb{R}^m)$ for $\omega \in (0, \infty]$, and $\mathcal{V}_a = \cup_{0<\omega\leq\infty} L^p_{\mathrm{loc}}([0, \omega), \mathbb{R}^m)$. It is important to note that $\mathcal{V}_\omega \neq L^p([0, \omega), \mathbb{R}^m)$. Throughout the paper we consider only those normed signal spaces $\mathcal{V}$ which have the property that $\sup_{\tau\geq 0} \|T_\tau x\| < \infty$ implies $x \in \mathcal{V}$. We observe that $L^p(\mathbb{R}_+, \mathbb{R}^m)$ for $1 \leq p \leq \infty$ is such a normed signal space. We will often write $\|x\|_\tau = \|T_\tau x\|$. For a normed signal space $\mathcal{U}$ and a Euclidean space $\mathbb{R}^n$, we will also consider subsets of spaces of the form $\mathcal{V} = \mathbb{R}^n \times \mathcal{U}$ which, on identifying each $\theta \in \mathbb{R}^n$ with the constant signal $t \mapsto \theta$, can be thought of as a normed signal space with norm given by $\|(\theta, x)\| = \sqrt{|\theta|^2 + \|x\|^2_{\mathcal{U}}}$.

2.2. Well-posedness. A mapping $Q \colon \mathcal{X}_1 \to \mathcal{X}_2$ between signal spaces is said to be *causal* if and only if for all $\tau > 0$, $x, y \in \mathcal{X}_1$, $T_\tau x = T_\tau y$ implies $T_\tau Q x = T_\tau Q y$. Let $\mathcal{U}$ and $\mathcal{Y}$ be normed signal spaces and let $P \colon \mathcal{U}_a \to \mathcal{Y}_a$ and $C \colon \mathcal{Y}_a \to \mathcal{U}_a$ be causal mappings representing a plant and controller, respectively. Our central concern is the system of equations

$$[P, C] \ : \quad y_1 = P u_1, \quad u_2 = C y_2, \quad u_0 = u_1 + u_2, \quad y_0 = y_1 + y_2 \tag{2.1}$$

corresponding to the closed-loop feedback configuration as depicted in Figure 2.1. By a solution of (2.1) we mean the following. For $w_0 = (u_0, y_0) \in \mathcal{W} := \mathcal{U} \times \mathcal{Y}$, a pair

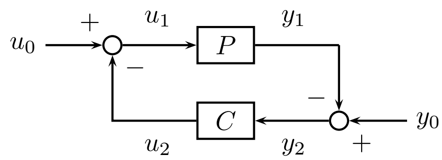

FIG. 2.1. *The closed-loop system* $[P, C]$.

$(w_1, w_2) = \big((u_1, y_1), (u_2, y_2)\big) \in \mathcal{W}_a \times \mathcal{W}_a$, $\mathcal{W}_a := \mathcal{U}_a \times \mathcal{Y}_a$, is a *solution* of (2.1) if and only if (2.1) holds on $\mathrm{dom}(w_1, w_2)$. The (possibly empty) set of all solutions is denoted by $\mathcal{X}_{w_0} := \big\{(w_1, w_2) \in \mathcal{W}_a \times \mathcal{W}_a \,|\, (w_1, w_2) \text{ solves (2.1)}\big\}$. The closed-loop system $[P, C]$, given by (2.1), is said to have (a) the *existence property* if and only if $\mathcal{X}_{w_0} \neq \emptyset$; (b) the *uniqueness property* if and only if for each $w_0 \in \mathcal{W}$,

$$(\hat{w}_1, \hat{w}_2), (\tilde{w}_1, \tilde{w}_2) \in \mathcal{X}_{w_0} \Rightarrow (\hat{w}_1, \hat{w}_2) = (\tilde{w}_1, \tilde{w}_2) \ \text{ on } \ \mathrm{dom}(\hat{w}_1, \hat{w}_2) \cap \mathrm{dom}\,(\tilde{w}_1, \tilde{w}_2).$$

Assume that $[P, C]$ has the existence and uniqueness properties. For each $w_0 \in \mathcal{W}$, define ω_{w_0}, $0 < \omega_{w_0} \leq \infty$, by the property $[0, \omega_{w_0}) := \cup_{(\hat{w}_1, \hat{w}_2) \in \mathcal{X}_{w_0}} \mathrm{dom}(\hat{w}_1, \hat{w}_2)$, and define $(w_1, w_2) \in \mathcal{W}_a \times \mathcal{W}_a$, with $\mathrm{dom}(w_1, w_2) = [0, \omega_{w_0})$, by the property $(w_1, w_2)|_{[0,t)} \in \mathcal{X}_{w_0}$ for all $t \in [0, \omega_{w_0})$. This construction induces an operator $H_{P,C} : \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a$, $w_0 \mapsto (w_1, w_2)$. The closed-loop system $[P, C]$, given by (2.1), is said to be

- *locally well posed* if and only if it has the existence and uniqueness properties and the operator $H_{P,C} : \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a$, $w_0 \mapsto (w_1, w_2)$, is causal;
- *globally well posed* if and only if it is locally well posed and $\mathrm{im}\, H_{P,C} \subset \mathcal{W}_e \times \mathcal{W}_e$;
- $\mathcal{W}$*-stable* if and only if it is locally well posed and $\mathrm{im}\, H_{P,C} \subset \mathcal{W} \times \mathcal{W}$;
- *regularly well posed* if and only if it is locally well posed and

$$\forall\, w_0 \in \mathcal{W} \quad \big[\, \omega_{w_0} < \infty \quad \Longrightarrow \quad T_{\omega_{w_0}} H_{P,C}(w_0) \notin \mathcal{W} \times \mathcal{W} \big]\,. \tag{2.2}$$

If $[P, C]$ is globally well posed, then for each $w_0 \in \mathcal{W}$ the solution $H_{P,C}(w_0)$ exists on the half line $\mathbb{R}_+$. Regular well posedness means that if the closed-loop system has a finite escape time $\omega > 0$ for some disturbance $(u_0, y_0) \in \mathcal{W}$, then at least one of the signals (u_1, y_1) or (u_2, y_2) is not a restriction to $[0, \omega)$ of a function in $\mathcal{W}$. If $[P, C]$ is regularly well posed and satisfies

$$\forall\, w_0 \in \mathcal{W} \quad \big[\, \omega_{w_0} < \infty \quad \Longrightarrow \quad T_{\omega_{w_0}} H_{P,C}(w_0) \in \mathcal{W} \times \mathcal{W} \big]\,,$$

then there does not exist a solution of $[P, C]$ with a finite escape time, and therefore $[P, C]$ is globally well posed. However, global well posedness does not guarantee that each solution belongs to $\mathcal{W} \times \mathcal{W}$; the latter is ensured by $\mathcal{W}$-stability of $[P, C]$. Note also that neither regular nor global well posedness implies the other. Our main concern will be the situation wherein the closed-loop system $[P, C]$ is generated by a system of (nonlinear) differential equations. In this context, a globally well-posed system is a system with the property of existence and uniqueness of solutions and for which finite-time blow up does not occur: all (forward) solutions have a maximal interval of existence $[0, \infty)$. Regular well posedness usually follows from standard existence theory for differential equations when $\mathcal{W} = L^\infty \times L^\infty$. However, when $\mathcal{W} \neq L^\infty \times L^\infty$ (in this paper we are primarily interested in $\mathcal{W} = L^2 \times L^2$), stronger properties of the underlying differential equations are required. As will be shown, all closed-loop systems considered in this paper are regularly well posed.

2.3. Graphs and gf-stability. In our investigation of robustness of stability properties of a closed-loop system, the concept of graphs and gf-stability will play a central role. Corresponding to a plant operator P (respectively, the controller operator C) is a subset of $\mathcal{W}$, called the *graph* of the plant $\mathcal{G}_P$ (respectively, the controller $\mathcal{G}_C$), defined as

$$\mathcal{G}_P = \left\{ \begin{pmatrix} u \\ Pu \end{pmatrix} \;\middle|\; u \in \mathcal{U},\ Pu \in \mathcal{Y} \right\} \subset \mathcal{W}, \quad \mathcal{G}_C = \left\{ \begin{pmatrix} Cy \\ y \end{pmatrix} \;\middle|\; Cy \in \mathcal{U},\ y \in \mathcal{Y} \right\} \subset \mathcal{W}.$$

Note that in general, $\mathcal{G}_P, \mathcal{G}_C \neq \mathcal{W}$. A causal operator $F\colon \mathcal{X} \to \mathcal{V}_a$, where $\mathcal{X}, \mathcal{V}$ are subsets of normed signal spaces, is said to be gf-stable if and only if $\operatorname{im} F \subset \mathcal{V}$ and if the following nonlinear so-called *gain function* is well defined:

$$\text{(2.3)}\quad g[F]\colon (r_0, \infty) \to \mathbb{R}_+, \qquad r \mapsto g[F](r) = \sup\left\{ \|T_\tau F x\| \,\middle|\, x \in \mathcal{X}\ ,\ \|T_\tau x\| \in (r_0, r],\ \tau > 0 \right\},$$

where $r_0 := \inf_{x\in\mathcal{X}} \|x\| < \infty$. Observe that $\|Fx\|_\tau \leq g[F](\|x\|_\tau)$. A closed-loop system $[P, C]$ is said to be *gf-stable* if and only if it is globally well posed and $H_{P,C}\colon \mathcal{W} \to \mathcal{W}_e \times \mathcal{W}_e$ is gf-stable. Note the following facts: (i) global well posedness of $[P, C]$ implies that $\operatorname{im} H_{P,C} \subset \mathcal{W}_e \times \mathcal{W}_e$; (ii) gf-stability of $[P, C]$ implies $\mathcal{W}$-stability of $[P, C]$; (iii) if $[P, C]$ is $\mathcal{W}$-stable, then $H_{P,C} : \mathcal{W} \to \mathcal{G}_P \times \mathcal{G}_C$ is a bijective operator with inverse $H_{P,C}^{-1} : (w_1, w_2) \mapsto w_1 + w_2$. To see (iii), note that $\operatorname{im} H_{P,C} \subset \mathcal{W} \times \mathcal{W}$ implies that $\operatorname{im} H_{P,C} \subset \mathcal{G}_P \times \mathcal{G}_C$, and since for any $w_1 \in \mathcal{G}_P \subset \mathcal{W}$, $w_2 \in \mathcal{G}_C \subset \mathcal{W}$ we have $w_1 + w_2 \in \mathcal{W}$, it follows that $\operatorname{im} H_{P,C} \supset \mathcal{G}_P \times \mathcal{G}_C$. Therefore, we can think of a gf-stable $H_{P,C}$ as a surjective operator $H_{P,C}\colon \mathcal{W} \to \mathcal{G}_P \times \mathcal{G}_C$. The inverse of $H_{P,C} : \mathcal{W} \to \mathcal{G}_P \times \mathcal{G}_C$ is obviously $H_{P,C}^{-1} : (w_1, w_2) \mapsto w_1 + w_2$. Finally, with a closed-loop system $[P, C]$ we associate the following two parallel projection operators: $\Pi_{P//C} : \mathcal{W} \to \mathcal{W}_a$, $w_0 \mapsto w_1$ and $\Pi_{C//P} : \mathcal{W} \to \mathcal{W}_a$, $w_0 \mapsto w_2$. Clearly, $H_{P,C} = \left(\Pi_{P//C}\,, \Pi_{C//P}\right)$ and $\Pi_{P//C} + \Pi_{C//P} = I$. Therefore, gf-stability of one of the operators $\Pi_{P//C}$ and $\Pi_{C//P}$ implies the gf-stability of the other, and so gf-stability of either operator implies gf-stability of the closed-loop system $[P, C]$.

2.4. The nonlinear gap. The essence of the paper is a study of robust stability in a specific adaptive control context. Robust stability is the property that the stability properties of a globally well-posed closed-loop system $[P, C]$ persist under "sufficiently small" perturbations of the plant. In other words robust stability is the property that $[P_1, C]$ inherits the stability properties of $[P, C]$ when the plant P is replaced with any plant P_1 sufficiently "close" to P. In the context of this paper, plants P_1 and P_2 are deemed to be close if and only if the nonlinear gap [7] between P_1 and P_2 is small, where the nonlinear gap is defined as follows. Let $\Gamma := \left\{ P : \mathcal{U}_a \to \mathcal{Y}_a \,\middle|\, P \text{ is causal} \right\}$ and, for $P_1, P_2 \in \Gamma$, define the (possibly empty) set $\mathcal{O}_{P_1,P_2} := \left\{ \Phi\colon \mathcal{G}_{P_1} \to \mathcal{G}_{P_2} \,\middle|\, \Phi \text{ is causal, bijective, and } \Phi(0) = 0 \right\}$. Write

$$\vec{\delta}(P_1, P_2) := \inf_{\Phi\in\mathcal{O}_{P_1,P_2}} \sup_{x\in\mathcal{G}_{P_1}\setminus\{0\},\ \tau>0} \left(\frac{\|(\Phi - I)|_{\mathcal{G}_{P_1}} x\|_\tau}{\|x\|_\tau} \right),$$

with the convention that $\vec{\delta}(P_1, P_2) := \infty$ if $\mathcal{O}_{P_1,P_2} = \emptyset$. The *nonlinear gap* δ is

$$\text{(2.4)}\qquad \delta\colon \Gamma \times \Gamma \to [0, \infty]\,, \qquad (P_1, P_2) \mapsto \delta(P_1, P_2) := \max\{\vec{\delta}(P_1, P_2), \vec{\delta}(P_2, P_1)\}.$$

The nonlinear gap provides a generalization of the standard definition of the linear gap δ_0 (previously discussed briefly in the introduction). To explain this, some notation is needed. For $q, m \in \mathbb{N}$, let $R_{q,m}$ denote the set of proper, rational, $(q \times m)$-matrix-valued functions and let $H^\infty_{q,m}$ denote the set of analytic and bounded $\mathbb{C}^{q\times m}$-valued functions on the open right half plane $\mathbb{C}_+ := \{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) > 0\}$. By $RH^\infty_{q,m}$, we denote the class of functions in $R_{q,m}$ that are analytic in $\mathbb{C}_+$. It is known (see, for example, [17, pp. 74–75; 261–262]) that any $P \in R_{q,m}$ has a normalized right coprime factorization, that is, $P = ND^{-1}$, where $N \in RH^\infty_{q,m}$, $D \in RH^\infty_{m,m}$, D has an inverse in $R_{m,m}$, and $N^*N + D^*D = I_m$, where $N^*(s) := \overline{N(-\bar{s})}^T$. Let $\mathcal{U} = L^2(\mathbb{R}_+, \mathbb{R}^m)$, $\mathcal{Y} = L^2(\mathbb{R}_+, \mathbb{R}^q)$ for some $m, q \in \mathbb{N}$ and associate with $\hat{P} \in R_{q,m}$ the linear operator $P\colon \mathcal{U}_e \to \mathcal{Y}_e,\ u \mapsto y := \mathbf{L}^{-1}(\hat{P}) \star u$, where $\mathbf{L}$ denotes the Laplace transform and $\star$ denotes convolution. We refer to P as a linear plant with associated transfer function $\hat{P} \in R_{q,m}$. For $i = 1, 2$, let $P_i\colon \mathcal{U}_e \to \mathcal{Y}_e$ be linear plants with associated strictly proper rational transfer functions $\hat{P}_i = N_i D_i^{-1}$, where $(N_i, D_i) \in RH^\infty_{q,m} \times RH^\infty_{m,m}$ are right coprime factors (analogous to (1.2)), and let $\Pi_i : \mathcal{U}_e \times \mathcal{Y}_e \to \mathcal{G}_{P_i}$ denote the orthogonal projection onto the closed subspace $\mathcal{G}_{P_i}$. The linear gap between these plants is defined as in (1.3), with

$$\vec{\delta}_0(\hat{P}_i, \hat{P}_j) := \inf\big\{ \|(\Delta_N, \Delta_D)\|_{H^\infty} \big|\ \Delta_N \in RH^\infty_{q,m},\ \Delta_D \in RH^\infty_{m,m},\ \hat{P}_j = (N_i + \Delta_N)(D_i + \Delta_D)^{-1} \big\}.$$

In [7, Proposition 5] it is shown that if $\max\{\|(\Pi_2 - \Pi_1)\,\Pi_1\|, \|(\Pi_1 - \Pi_2)\,\Pi_2\|\} < 1$, then $\vec{\delta}(P_1, P_2) = \|(\Pi_2 - \Pi_1)\,\Pi_1\|$, and in [5, Lemma 2] it is shown that $\|(\Pi_2 - \Pi_1)\,\Pi_1\| = \vec{\delta}_0(\hat{P}_1, \hat{P}_2)$ if $\vec{\delta}_0(\hat{P}_1, \hat{P}_2) < 1$, the conjunction of which yields

$$\vec{\delta}(P_1, P_2) = \vec{\delta}_0(\hat{P}_1, \hat{P}_2) \qquad \text{if } \delta_0(\hat{P}_1, \hat{P}_2) < 1. \tag{2.5}$$

The topology induced on $R_{q,m}$ by the gap δ_0 is called the graph topology [17, p. 235]; note that the graph topology on Γ induces the graph topology on $R_{q,m}$ via the subset topology and the Laplace transform $\mathbf{L}$. Robust stability with respect to gap perturbations has been extensively studied in the linear setting; see, e.g., [18] and the references therein. In a nonlinear setting, sufficient conditions for robust gain stability were given in [7]. The adaptive controllers considered in this paper and in [4] do not meet any of the sufficient conditions of [7] and hence require a further development of the robust stability theory in section 5 of [4], as given in Theorem 3.11 of the present paper.

3. System classes and the adaptive controller. We are interested in the control of linear m-input, m-output stabilizable n-dimensional state space realizations of transfer functions in $R_{m,m}$, i.e., systems of the form

$$\dot{x}(t) = A\,x(t) + B\,u_1(t), \qquad x(0) \in \mathbb{R}^n, \qquad y_1(t) = Cx(t). \tag{3.1}$$

Henceforth, we fix (arbitrarily) the number $m \geq 1$ of inputs/outputs but allow for a variation in the state space dimension n. We denote this system class by

$$\mathcal{P}_n = \{(A, B, C) \in E_n \mid n \geq m,\ (A, B, C) \text{ is stabilizable and detectable}\}, \tag{3.2}$$

where $E_n := \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$, and write $\mathcal{P} := \cup_{n\geq m} \mathcal{P}_n$. We define the subclass of minimum-phase systems with "high-frequency gain" CB having spectrum in $\mathbb{C}_+$,

$$\widetilde{\mathcal{M}}_n = \left\{ (A, B, C) \in \mathcal{P}_n \,\middle|\, \sigma(CB) \subset \mathbb{C}_+,\quad \det\begin{bmatrix} sI_n - A & B \\ C & 0 \end{bmatrix} \neq 0\ \forall s \in \overline{\mathbb{C}}_+ \right\}. \tag{3.3}$$

Observe that for each $(A, B, C) \in \widetilde{\mathcal{M}}_n$, there exists an element of its similarity orbit $\{(TAT^{-1}, TB, CT^{-1}) | \ T \in \mathbb{R}^{n\times n} \text{ invertible}\}$ such that

$$TAT^{-1} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad TB = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ CB \end{bmatrix}, \quad CT^{-1} = \begin{bmatrix} 0 & I \end{bmatrix},$$

where $\sigma(B_2) \subset \mathbb{C}_+$ and, by the minimum-phase property, A_1 has a spectrum in the open left half complex plane $\mathbb{C}_-$. Therefore, we introduce

$$\mathcal{M}_n := \left\{ (A, B, C) \in \mathcal{P}_n \ \middle| \ \begin{array}{l} A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \ C = [0 \ \ I], \\ B_2, A_4 \in \mathbb{R}^{m\times m}, \ \sigma(A_1) \subset \mathbb{C}_-, \ \sigma(B_2) \subset \mathbb{C}_+ \end{array} \right\} \tag{3.4}$$

and write $\mathcal{M} := \cup_{n\geq m} \mathcal{M}_n$. For a system of class $\mathcal{M}_n$, (3.1) may be expressed in the equivalent form

$$\left. \begin{array}{ll} \dot{z}(t) = A_1 z(t) + A_2 y_1(t), & z(0) = z^0 \in \mathbb{R}^{n-m} \\ \dot{y_1}(t) = A_3 z(t) + A_4 y_1(t) + B_2 u_1(t), & y_1(0) = y_1^0 \in \mathbb{R}^m \end{array} \right\}, \quad (z^0, y_1^0) = x^0. \tag{3.5}$$

We will have occasion to identify $\mathcal{P}_n$ with a subspace of a Euclidean space $\mathbb{R}^{n^2+2mn}$ by identifying a plant $\theta = (A, B, C)$ with a vector θ consisting of the elements of the plant matrices, ordered lexicographically. With normed signal spaces $\mathcal{U}$ and $\mathcal{Y}$ and $(\theta, x^0) \in \mathcal{P}_n \times \mathbb{R}^n$, we associate the causal plant operator

$$\widetilde{P}(\theta, x^0) : \ \mathcal{U}_a \to \mathcal{Y}_a, \quad u_1 \mapsto \widetilde{P}(\theta, x^0)(u_1) := y_1\,, \tag{3.6}$$

where, for $u_1 \in \mathcal{U}_a$ with $\mathrm{dom}(u_1) = [0, \omega)$, we have $y_1 = Cx$, x being the unique solution of (3.1) on $[0, \omega)$. Note that $\widetilde{P}$ is a map from $\bigcup_{n\geq m}(\mathcal{P}_n \times \mathbb{R}^n)$ to the space of maps $\mathcal{U}_a \to \mathcal{Y}_a$. Furthermore, for $(\theta, x^0) = (A, B, C, x^0) \in \mathcal{P}_n \times \mathbb{R}^n$, $\widetilde{P}(\theta, x^0)$ corresponds to a stabilizable and detectable realization of $C(sI_n - A)^{-1}B \in R_{m,m}$. Our objective is to study, in the context of systems of form (3.1), the adaptive strategy

$$u_2(t) = -k(t) y_2(t), \quad \dot{k}(t) = \|y_2(t)\|^2, \quad k(0) = k^0 \in \mathbb{R}_+\,, \tag{3.7}$$

with the associated control operator, parameterized by k^0, denoted by

$$\widetilde{C}(k^0) : \mathcal{Y}_a \to \mathcal{U}_a, \qquad y_2 \mapsto \widetilde{C}(k^0)(y_2) := u_2. \tag{3.8}$$

Note that $\widetilde{C}$ is a map from $\mathbb{R}_+$ to the space of causal maps $\mathcal{Y}_a \to \mathcal{U}_a$. In particular and with reference to Figure 2.1, we will study properties of the closed-loop system $[\widetilde{P}(\theta, x^0), \widetilde{C}(k^0)]$, generated by the application of the controller (3.8) to system (3.1), in the presence of disturbances $(u_0, y_0) \in \mathcal{W}$ satisfying the interconnection equations

$$u_0 \ = \ u_1 + u_2, \qquad y_0 \ = \ y_1 + y_2\,. \tag{3.9}$$

Results will be given for systems (3.5) of class $\mathcal{M}_n$ (such systems will later play the role of the nominal plant) and for the more general class of systems (3.1), (3.2) (such systems will later play the role of the perturbed plant).

3.1. Properties of the interconnection of the adaptive controller with the general linear plant. In this section we investigate the interconnection of the adaptive controller (3.7) (with associated operator $\widetilde{C}(k^0)$) and any plant in the form (3.1) (with associated operator $\widetilde{P}(\theta, x^0)$), where $(\theta, x^0, k^0) \in \mathcal{P}_n \times \mathbb{R}^n \times \mathbb{R}_+$.

PROPOSITION 3.1. *Let $n \geq m$, $(\theta, x^0, k^0) \in \mathcal{P}_n \times \mathbb{R}^n \times \mathbb{R}_+$, and $u_0, y_0 \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^m)$. Then the closed-loop initial-value problem given by* (3.1)*,* (3.7)*,* (3.9) *has the following properties:*

(i) *There exists a unique maximal solution $(x, k) : [0, \omega) \to \mathbb{R}^n \times \mathbb{R}_+$, $0 < \omega \leq \infty$;*

(ii) *if $k \in L^\infty([0, \omega), \mathbb{R}_+)$, then $\omega = \infty$;*

(iii) *if $y_2 \in L^2([0, \omega), \mathbb{R}^m)$, then $\omega = \infty$.*

Proof. (i) follows from the theory of ordinary differential equations.

(ii) If $k \in L^\infty([0, \omega), \mathbb{R}_+)$, then consider the following subsystem of the initial-value problem (3.1), (3.7): $\dot{x}(t) = Ax(t) + B\big[u_0(t) + k(t)y_2(t)\big]$. Integration, together with elementary estimates, yields the existence of constants $c_0, c_1 > 0$ such that

$$\|x(t)\| \leq c_0 \left(e^{c_1 t} + \int_0^t e^{c_1(t-s)} \big(\|u_0(s)\| + \|y_2(s)\| \big) \mathrm{d}s \right) \quad \forall\ t \in [0, \omega). \tag{3.10}$$

Suppose $\omega < \infty$. Since $y_2 \in L^2([0, \omega), \mathbb{R}^m)$ (which is equivalent to $k \in L^\infty([0, \omega), \mathbb{R}_+)$) and since $u_0 \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^m)$, it follows from the convolution in (3.10) that the right-hand side of (3.10) is bounded on $[0, \omega)$, which contradicts maximality of the solution x. Therefore, $\omega = \infty$.

(iii) By assumption, $y_2 \in L^2([0, \omega), \mathbb{R}^m)$, and so $t \mapsto k(t) = k^0 + \|y_2\|^2_{L^2([0,t),\mathbb{R}^m)}$ is bounded on $[0, \omega)$, and thus assertion (iii) follows from (ii). ∎

COROLLARY 3.2. *Let $\mathcal{U} = \mathcal{Y} = L^2(\mathbb{R}_+, \mathbb{R}^m)$, $n \geq m$, $(\theta, x^0, k^0) \in \mathcal{P}_n \times \mathbb{R}^n \times \mathbb{R}_+$. Then the closed-loop initial-value problem $[\widetilde{P}(\theta, x^0), \widetilde{C}(k^0)]$ given by* (2.1)*,* (3.6)*, and* (3.8) *is regularly well posed.*

Proof. Let $\mathcal{W} = L^2(\mathbb{R}_+, \mathbb{R}^m) \times L^2(\mathbb{R}_+, \mathbb{R}^m)$. The closed-loop $[\widetilde{P}(\theta, x^0), \widetilde{C}(k^0)]$ is locally well posed by Proposition 3.1(i). To prove that $[\widetilde{P}(\theta, x^0), \widetilde{C}(k^0)]$ is regularly well posed, it suffices to show that (2.2) holds. Let $w_0 = (u_0, y_0) \in \mathcal{W}$. Consider $(w_1, w_2) = H_{\widetilde{P}(\theta,x^0),\widetilde{C}(k^0)}(w_0)$, where $\operatorname{dom}(w_1, w_2) = [0, \omega)$ and is maximal. Suppose $T_\omega(w_1, w_2) \in \mathcal{W} \times \mathcal{W}$. Then we have $y_1 \in L^2([0, \omega), \mathbb{R}^m)$, which, in view of Proposition 3.1(iii) yields $\omega = \infty$. Hence as $w_0 \in \mathcal{W}$ is arbitrary, it follows that the closed-loop system is regularly well posed. ∎

3.2. Properties of the interconnection of the adaptive controller with the nominal plant. In this section we consider the closed-loop behavior of the nominal plant and controller interconnection given by (3.5), (3.7), and perturbations u_0, y_0 satisfying (3.9). The L^2 bounds obtained in [4] do not directly generalize to the controller (1.1), even in the single-input, single-output (SISO) case. Hence the results in this subsection are obtained via an alternative (and tighter) series of bounds than those considered in the results of [4].

PROPOSITION 3.3. *Let $n \geq m$, $(A, B, C) \in \mathcal{M}_n$, $(x^0, k^0) \in \mathbb{R}^n \times \mathbb{R}_+$, and $u_0, y_0 \in L^2([0, \infty), \mathbb{R}^m)$. Then the closed-loop initial-value problem* (3.5)*,* (3.7)*,* (3.9) *has the following properties:*

(i) *There exists a unique solution $(z, y_1, k) : [0, \infty) \to \mathbb{R}^n \times \mathbb{R}_+$;*

(ii) *the limit $\lim_{t\to\infty} k(t)$ exists and is finite;*

(iii) *$u_1, y_1 \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, $z \in L^2(\mathbb{R}_+, \mathbb{R}^{n-m})$;*

(iv) $\lim_{t\to\infty}(z(t), y_1(t)) = 0$.

Proof. The closed-loop equations (3.5), (3.7), (3.9) may be expressed as

$$\tag{3.11} \begin{cases} \dot z(t) = A_1 z(t) + A_2 y_1(t)\,, \\ \dot y_1(t) = A_3 z(t) + A_4 y_1(t) + B_2\left(u_0(t) + k(t)\left[y_0(t) - y_1(t)\right]\right), \\ \dot k(t) = \|y_0(t) - y_1(t)\|^2, \\ u_1(t) = u_0(t) + k(t)\left[y_0(t) - y_1(t)\right], \\ (z(0), y_1(0), k(0)) = (z^0, y_1^0, k^0) \in \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}_+\,, \end{cases}$$

where $x^0 = (z^0, y_1^0)$. By Proposition 3.1 there exists a unique maximal solution $(z, y_1, k) : [0,\omega) \to \mathbb{R}^n \times \mathbb{R}_+$ of the initial-value problem (3.11) for some $\omega \in (0,\infty]$. To prove the proposition, we claim that it suffices to show that $y_1 \in L^2([0,\omega), \mathbb{R}^m)$. To argue this claim, assume that $y_1 \in L^2([0,\omega), \mathbb{R}^m)$ and first note that, by Proposition 3.1, $\omega = \infty$, and so assertion (i) holds. Assertion (ii) follows from the third of equations (3.11). Since $u_0, y_0, y_1 \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ and k is bounded, we have $u_0 + k[y_0 - y_1] = u_1 \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ and, since $\sigma(A_1) \subset \mathbb{C}_-$, it follows from the first of equations (3.11) that $z \in L^2(\mathbb{R}_+, \mathbb{R}^{n-m})$ and $z(t) \to 0$ as $t \to \infty$, whence we get assertion (iii). Finally, writing the second of equations (3.11) in the form

$$\dot y_1(t) = -y_1(t) + f(t), \quad f : t \mapsto (I + A_4) y_1(t) + A_3 z(t) + B_2\big(u_0(t) + k(t)[y_0(t) - y_1(t)]\big),$$

and noting that $f \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, we conclude $y_1(t) \to 0$ as $t \to \infty$, and so (iv) holds. We proceed to show that $y_1 \in L^2([0,\omega), \mathbb{R}^m)$. First, we assemble some useful inequalities. Recalling that $\sigma(A_1) \subset \mathbb{C}_-$, we have

$$M_p := \left(\int_{\mathbb{R}_+} \|e^{A_1 t}\|^p dt\right)^{1/p} < \infty \;\; \forall\, p \in [1,\infty), \quad M_\infty := \sup_{t \in \mathbb{R}_+} \|e^{A_1 t}\| < \infty,$$

and, by the first of equations (3.11) (together with elementary estimates),

$$\tag{3.12} \|z(t)\|^2 \;\le\; 2\left[M_\infty^2 \|z^0\|^2 + M_2^2 \|A_2\|^2 \|y_1\|^2_{L^2([0,t),\mathbb{R}^m)}\right] \qquad \forall\, t \in [0,\omega).$$

Introduce the convolution operator $L : L^2(\mathbb{R}_+, \mathbb{R}^{n-m}) \to L^2(\mathbb{R}_+, \mathbb{R}^{n-m})$, given by

$$(Lv)(t) := \int_0^t e^{A_1(t-\tau)} v(\tau) d\tau\,.$$

Then $\|Lv\|_{L^2(I,\mathbb{R}^{n-m})} \le M_1 \|v\|_{L^2(I,\mathbb{R}^{n-m})}$ for every bounded interval $I \subset \mathbb{R}_+$ and all $v \in L^2_e(\mathbb{R}_+, \mathbb{R}^{n-m})$, which, with the first of equations (3.11), yields

$$\|z\|^2_{L^2([s,t],\mathbb{R}^{n-m})} \le 2M_1^2\left[\|z(s)\|^2 + \|A_2\|^2 \|y_1\|^2_{L^2([s,t],\mathbb{R}^m)}\right]$$

for all s, t with $0 \le s \le t < \omega$. Writing

$$\tag{3.13} c_1 := \frac{1}{2} + M_1^2\left[1 + \|A_2\|^2\right],$$

we may infer

$$\tag{3.14} \begin{aligned} \int_s^t \|z(\tau)\| \|y_1(\tau)\| d\tau &\le \frac{1}{2}\left[\|z\|^2_{L^2([s,t],\mathbb{R}^{n-m})} + \|y_1\|^2_{L^2([s,t],\mathbb{R}^m)}\right] \\ &\le c_1\left[\|z(s)\|^2 + \|y_1\|^2_{L^2([s,t],\mathbb{R}^m)}\right] \;\text{ for all } s, t \text{ with } 0 \le s \le t < \omega. \end{aligned}$$

Also, observe that, for all s, t with $0 \le s \le t < \infty$,

$$k(t) = k(s) + \|y_0 - y_1\|^2_{L^2([s,t],\mathbb{R}^m)} \le k(s) + \|y_0\|^2_{L^2([s,t],\mathbb{R}^m)} + \|y_1\|^2_{L^2([s,t],\mathbb{R}^m)}. \tag{3.15}$$

Since $\sigma(B_2) \subset \mathbb{C}_+$, the Lyapunov equation $QB_2 + B_2^T Q - 2I = 0$ has a unique positive-definite symmetric solution Q. From the second of equations (3.11), noting that $\|QB_2\| = 1$ and invoking elementary estimates, we have

$$\begin{aligned}\langle y_1(t), Q\dot{y}_1(t)\rangle \le \|QA_3\|\|z(t)\|\|y_1(t)\| - \frac{1}{2}\big[k(t) - 2\|QA_4\| - 1\big]\|y_1(t)\|^2 \\ + \frac{1}{2}k(t)\|y_0(t)\|^2 + \frac{1}{2}\|u_0(t)\|^2 \ \forall\, t \in [0,\omega)\end{aligned}$$

which, on integration, using (3.14), (3.15) and invoking monotonicity of k, yields

$$\begin{aligned}0 \le \langle y_1(t), Qy_1(t)\rangle \le \langle y_1(s), Qy_1(s)\rangle + 2c_1\|QA_3\|\big(\|z(s)\|^2 + \|y_1\|^2_{L^2([s,t],\mathbb{R}^m)}\big) \\ + \big(k(s) + \|y_0\|^2_{L^2([s,t],\mathbb{R}^m)} + \|y_1\|^2_{L^2([s,t],\mathbb{R}^m)}\big)\|y_0\|^2_{L^2(\mathbb{R}_+,\mathbb{R}^m)} + \|u_0\|^2_{L^2(\mathbb{R}_+,\mathbb{R}^m)} \\ - \big(k(s) - 2\|QA_4\| - 1\big)\|y_1\|^2_{L^2([s,t],\mathbb{R}^m)} \ \ \forall\, s,t \text{ with } 0 \le s \le t < \omega.\end{aligned}$$

Defining

$$c_2 := 2c_1\|QA_3\| + 2\|QA_4\| + 2 + \|y_0\|^2_{L^2(\mathbb{R}_+,\mathbb{R}^m)}, \tag{3.16}$$

we have

$$\begin{aligned}0 \le \langle y_1(t), Qy_1(t)\rangle \le \langle y_1(s), Qy_1(s)\rangle + 2c_1\|QA_3\|\|z(s)\|^2 + \|u_0\|^2_{L^2(\mathbb{R}_+,\mathbb{R}^m)} \\ - \big(k(s) - c_2 + 1\big)\|y_1\|^2_{L^2([s,t],\mathbb{R}^m)} \ \forall\, s,t \text{ with } 0 \le s \le t < \omega.\end{aligned} \tag{3.17}$$

Next, observe that

$$\|y_1\|^2_{L^2([0,t),\mathbb{R}^m)} \le \|y_0\|^2_{L^2([0,t),\mathbb{R}^m)} + \|y_2\|^2_{L^2([0,t),\mathbb{R}^m)} = \|y_0\|^2_{L^2([0,t),\mathbb{R}^m)} + k^0 + k(t)$$

for all $t \in [0,\omega)$. We consider two possible cases.

Case 1. Assume $k(t) \le c_2$ for all $t \in [0,\omega)$. Then $\|y_1\|^2_{L^2([0,\omega),\mathbb{R}^m)} \le \|y_0\|^2_{L^2(\mathbb{R}_+,\mathbb{R}^m)} + c_2$.

Case 2. Assume $k(\tau) = c_2$ for some $\tau \in [0,\omega)$. Then, by (3.17) with $s = \tau$, we have

$$\|y_1\|^2_{L^2([\tau,\omega),\mathbb{R}^m)} \le \langle y_1(\tau), Qy_1(\tau)\rangle + 2c_1\|QA_3\|\|z(\tau)\|^2 + \|u_0\|^2_{L^2(\mathbb{R}_+,\mathbb{R}^m)}.$$

By monotonicity, $k(t) \le c_2$ for all $t \in [0,\tau]$, and so $\|y_1\|^2_{L^2([0,\tau],\mathbb{R}^m)} \le \|y_0\|^2_{L^2(\mathbb{R}_+,\mathbb{R}^m)} + c_2$. By (3.17) with $s = 0$ and $t = \tau$, we now have

$$\begin{aligned}\langle y_1(\tau), Qy_1(\tau)\rangle \le c_3 := \langle y_1^0, Qy_1^0\rangle + 2c_1\|QA_3\|\|z^0\|^2 + \|u_0\|^2_{L^2(\mathbb{R}_+,\mathbb{R}^m)} \\ + (c_2 + 1)\big(\|y_0\|^2_{L^2(\mathbb{R}_+,\mathbb{R}^m)} + c_2\big).\end{aligned} \tag{3.18}$$

By (3.12), we have

$$\|z(\tau)\|^2 \le c_4 := 2\big[M_\infty^2\|z^0\|^2 + M_2^2\|A_2\|^2\|y_0\|^2_{L^2(\mathbb{R}_+,\mathbb{R}^m)} + c_2 M_2^2\|A_2\|^2\big]. \tag{3.19}$$

We may now conclude that

$$\begin{aligned}(3.20)\quad &\|y_1\|^2_{L^2([0,\tau],\mathbb{R}^m)} + \|y_1\|^2_{L^2([\tau,\omega),\mathbb{R}^m)} \\ &\qquad \le c_5 := c_2 + c_3 + 2c_1c_4\|QA_3\| + \|(u_0,y_0)\|^2_{L^2(\mathbb{R}_+,\mathbb{R}^{2m})}\,.\end{aligned}$$

Therefore, in both Cases 1 and 2, we have $\|y_1\|^2_{L^2([0,\omega),\mathbb{R}^m)} \le c_5$. ◻

Proposition 3.3 immediately implies the following.

COROLLARY 3.4. *Let $n \ge m$, $\mathcal{U} = \mathcal{Y} = L^2(\mathbb{R}_+,\mathbb{R}^m)$, $\theta = (A,B,C) \in \mathcal{M}_n$, and $(x^0,k^0) \in \mathbb{R}^n \times \mathbb{R}_+$. Then the closed loop $[\widetilde{P}(\theta,x^0), \widetilde{C}(k^0)]$ given by* (3.1) (*equivalently,* (3.5)), (3.6), (3.8), (3.9) *is globally well posed and $(\mathcal{U}\times\mathcal{Y})$-stable.*

PROPOSITION 3.5. *Let $n \ge m$ and define*

$$(3.21)\qquad \mathcal{D} := \mathcal{M}_n \times \mathbb{R}^n \times \mathbb{R}_+ \times L^2(\mathbb{R}_+,\mathbb{R}^m) \times L^2(\mathbb{R}_+,\mathbb{R}^m).$$

There exists a continuous map $\nu : \mathcal{D} \to \mathbb{R}_+$ such that, for all

$$d = (A,B,C,x^0,k^0,u_0,y_0) \in \mathcal{D},$$

the closed-loop system (3.11) *is such that $\|(u_1,y_1)\|_{L^2(\mathbb{R}_+,\mathbb{R}^{2m})} \le \nu(d)$.*

Proof. Observe that the parameters c_i, $i = 1,\dots,5$, defined in (3.13), (3.16), (3.18), (3.19), and (3.20), depend continuously on the data $d = (A,B,C,x^0,k^0,u_0,y_0)$. In particular, the map $\hat\nu : d \mapsto \sqrt{c_5}$ is continuous. Let $d \in \mathcal{D}$ be arbitrary. Then, by Proposition 3.3 (and recalling (3.20)), we have $\|y_1\|_{L^2(\mathbb{R}_+,\mathbb{R}^m)} \le \hat\nu(d)$. Now,

$$k(t) = k^0 + \|y_0 - y_1\|^2_{L^2([0,t),\mathbb{R}^m)} \le \nu^*(d) := k^0 + 2\|y_0\|^2_{L^2(\mathbb{R}_+,\mathbb{R}^m)} + 2(\hat\nu(d))^2 \quad \forall\, t \in \mathbb{R}_+.$$

Therefore,

$$\begin{aligned}\|u_1\|_{L^2([0,t),\mathbb{R}^m)} &= \|u_0 + k(t)[y_0(t) - y_1(t)]\|_{L^2([0,t),\mathbb{R}^m)} \\ &\le \tilde\nu(d) := \|u_0\|_{L^2(\mathbb{R}_+,\mathbb{R}^m)} + \nu^*(d)\big(\|y_0\|_{L^2(\mathbb{R}_+,\mathbb{R}^m)} + \hat\nu(d)\big) \quad \forall\, t \in [0,\omega)\,.\end{aligned}$$

We may now infer that $\|(u_1,y_1)\|_{L^2(\mathbb{R}_+,\mathbb{R}^{2m})} \le \nu(d) := \sqrt{(\hat\nu(d))^2 + (\tilde\nu(d))^2}$. ◻

Remark 3.6. It is worthwhile to note that $\nu(d) \to \infty$ as the data approach the boundary of $\mathcal{M}_n$: for example, if some eigenvalues of A_1 approach the imaginary axis, then $\|L\| \to \infty$ and so c_1, given by (3.13), grows unboundedly; if $\|B_2\| \to 0$, then $\|Q\| \to \infty$ and so c_2, given by (3.16), grows unboundedly. Specifically, there exists a *bounded* sequence (d_j) in $\mathcal{D}$ such that $\nu(d_j) \to \infty$ as $j \to \infty$. However, if $\Omega \subset \mathcal{M}_n$ is closed and (d_j) is a bounded sequence in $\Omega \times \mathbb{R}^n \times \mathbb{R}_+ \times L^2(\mathbb{R}_+,\mathbb{R}^m) \times L^2(\mathbb{R}_+,\mathbb{R}^m) \subset \mathcal{D}$, then $(\nu(d_j))$ is bounded.

3.3. Construction of a gain function. To establish gap margin results, we will need to construct augmented plant and controller operators, as in [4]. Reiterating earlier remarks, we may consider $\mathcal{M}_n$ to be a subset of the Euclidean space $E_n = \mathbb{R}^{n^2+2mn}$, with the standard Euclidean norm, by identifying a plant $\theta = (A,B,C) \in \mathcal{M}_n$ with a vector $\theta \in E_n$ consisting of the $n^2 + 2mn$ elements of the plant matrices ordered lexicographically. Note that $0 \notin \mathcal{M}_n$. Let $\mathcal{U} = \mathcal{Y} = L^2(\mathbb{R}_+,\mathbb{R}^m)$ and define $\widetilde{\mathcal{U}} := \mathbb{R}^{n^2+2mn} \times \mathcal{U}$ and $\widetilde{\mathcal{W}} := \widetilde{\mathcal{U}} \times \mathcal{Y}$, which can be considered as signal spaces by identifying $\theta \in \mathbb{R}^{n^2+2mn}$ with the constant function $t \mapsto \theta$ and endowing $\widetilde{\mathcal{U}}$ with the norm $\|(\theta,u)\|_{\widetilde{\mathcal{U}}} := \sqrt{\|\theta\|^2 + \|u\|^2_{L^2(\mathbb{R}_+,\mathbb{R}^m)}}$. For given $\widetilde{P}(\theta,0)$ as in (3.6), we define the (augmented) plant operator as

$$(3.22)\qquad P :\ \widetilde{\mathcal{U}}_a \to \mathcal{Y}_a, \quad (\theta,u_1) = \tilde u_1 \mapsto y_1 = P(\tilde u_1) := \widetilde{P}(\theta,0)(\tilde u_1)\,.$$

Fix $k^0 \geq 0$ and define, for $\widetilde{C}(k^0)$ as in (3.8), the (augmented) controller operator as

$$C: \mathcal{Y}_a \to \widetilde{\mathcal{U}}_a, \quad y_2 \mapsto \widetilde{u}_2 = C(y_2) := (0, \widetilde{C}(k^0)(y_2)). \tag{3.23}$$

For each nonempty $\Omega \subset \mathcal{M}_n$, define $\mathcal{W}^\Omega := (\Omega \times \mathcal{U}) \times \mathcal{Y}$ and $H^\Omega_{P,C} := H_{P,C}|_{\mathcal{W}^\Omega}$. It easily follows from Corollary 3.4 that $H^\Omega_{P,C}\colon \mathcal{W}^\Omega \to \widetilde{\mathcal{W}} \times \widetilde{\mathcal{W}}$ is a causal operator for any $\Omega \subset \mathcal{M}_n$. We now establish gf-stability.

PROPOSITION 3.7. *Let $n \geq m$, $k^0 \geq 0$, and assume $\Omega \subset \mathcal{M}_n$ is closed. Then, for the closed-loop system $[P, C]$ given by* (2.1), (3.22), *and* (3.23), *the operator $H^\Omega_{P,C}$ is gf-stable.*

Proof. For $\nu : \mathcal{D} \to \mathbb{R}_+$ as in Proposition 3.5 we have, for all $(\theta, u_0, y_0) \in \mathcal{W}^\Omega$,

$$\begin{aligned} \|H^\Omega_{P,C}(\theta, u_0, y_0)\|_{\widetilde{\mathcal{W}}\times\widetilde{\mathcal{W}}} &\leq \|(\theta, u_0, y_0)\|_{\widetilde{\mathcal{W}}} + 2\|(\theta, u_1, y_1)\|_{\widetilde{\mathcal{W}}} \\ &\leq \|(u_0, y_0)\|_{\mathcal{W}} + 3\,\|\theta\| + 2\,\nu(\theta, 0, k^0, u_0, y_0)\,, \end{aligned}$$

and so, for $r_0 := \inf_{w\in\mathcal{W}^\Omega} \|w\|_{\widetilde{\mathcal{W}}}$ and $\alpha \in (r_0, \infty)$, closedness of Ω yields

$$\gamma(\alpha) := \sup\left\{ \|(u_0, y_0)\|_{\mathcal{W}} + 3\|\theta\| + 2\,\nu(\theta, 0, k^0, u_0, y_0) \,\middle|\, \begin{array}{l} (\theta, u_0, y_0) \in \mathcal{W}^\Omega, \\ \|(\theta, u_0, y_0)\|_{\widetilde{\mathcal{W}}} \leq \alpha \end{array} \right\} < \infty.$$

Therefore, a gain function for $H^\Omega_{P,C}$ exists, and the proof is complete. ∎

3.4. Robust stability. In Proposition 3.3 and Corollary 3.4 we have established that, for $k^0 \geq 0$, $(\theta, x^0) \in \mathcal{M}_n \times \mathbb{R}^n$ for some $n \geq m$, and $u_0, y_0 \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, the closed-loop system $[\widetilde{P}(\theta, x^0), \widetilde{C}(k^0)]$ is globally well posed and has desirable stability properties. The purpose of this section is to determine conditions under which these properties are maintained when the plant $\widetilde{P}(\theta, x^0)$ is perturbed to a plant $\widetilde{P}(\theta_1, x^0_1)$, where $(\theta_1, x^0_1) \in \mathcal{P}_q \times \mathbb{R}^q$ for some $q \geq m$, in particular when $\theta_1 \notin \mathcal{M}_q$. The essence of the main result, Theorem 3.8, is that the stability properties persist if (a) the plants $\widetilde{P}(\theta_1, 0)$ and $\widetilde{P}(\theta, 0)$ are sufficiently close (in the gap sense) and (b) the initial data x^0_1 and disturbance $w_0 = (u_0, y_0)$ are sufficiently small.

THEOREM 3.8. *Let $m, n, q \in \mathbb{N}$ with $n \geq m$, $q \geq m$, $\mathcal{U} = \mathcal{Y} = L^2(\mathbb{R}_+, \mathbb{R}^m)$, $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$, and $\theta \in \mathcal{M}_n$. For (ϑ, x^0) in $\mathcal{P}_q \times \mathbb{R}^q$ or $\mathcal{P}_n \times \mathbb{R}^n$ and $k^0 \geq 0$, consider $\widetilde{P}(\vartheta, x^0)\colon \mathcal{U}_a \to \mathcal{Y}_a$ and $\widetilde{C}(k^0)\colon \mathcal{Y}_a \to \mathcal{U}_a$ as defined by* (3.6) *and* (3.8), *respectively. There exist a continuous function $\eta\colon \mathbb{R}_+ \to (0, \infty)$ and a function $\lambda : \mathcal{P}_q \to (0, \infty)$ such that the following holds. For all $(\theta_1, x^0_1, w_0, r) \in \mathcal{P}_q \times \mathbb{R}^q \times \mathcal{W} \times (0, \infty)$,*

$$\left. \begin{array}{l} \vec{\delta}(\widetilde{P}(\theta, 0), \widetilde{P}(\theta_1, 0)) \leq \ \eta(r) \\ \lambda(\theta_1)\|x^0_1\| + \|w_0\|_{\mathcal{W}} \leq r \end{array} \right\} \quad \Longrightarrow \quad H_{\widetilde{P}(\theta_1, x^0_1), \widetilde{C}(k^0)}(w_0) \ \in \ \mathcal{W} \times \mathcal{W}. \tag{3.24}$$

Remark 3.9. In the setup of Theorem 3.8, if $H_{\widetilde{P}(\theta_1, x^0_1), \widetilde{C}(k^0)}(w_0) \in \mathcal{W} \times \mathcal{W}$ with $\theta_1 = (A, B, C) \in \mathcal{P}_q$, then the following hold:

(i) If $u_1, y_1 \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, then detectability of (A, C) yields that the solution x of (3.1) belongs to $x \in L^2(\mathbb{R}_+, \mathbb{R}^m)$. Since $x, \dot{x} \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, it follows that $x(t) \to 0$ as $t \to \infty$. Moreover, the monotone controller gain k, given by (3.7), converges to a finite limit.

(ii) If $u_0, y_0 \in (L^2 \cap L^\infty)(\mathbb{R}_+, \mathbb{R}^m)$, then $u_1, y_1 \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$. This follows from the fact that $x \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ by (i), and so $y_1 \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$. Furthermore, $y_2 \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, and so $k \in L^\infty(\mathbb{R}_+, \mathbb{R})$, which, by $u_2 = k\, y_2$, yields $u_2 \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$, whence $u_1 \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$.

(iii) If $u_0, y_0 \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ and $\lim_{t\to\infty} u_0(t) = \lim_{t\to\infty} y_0(t) = 0$, then $\lim_{t\to\infty} u_1(t) = 0$. This is a consequence of (i), which gives $\lim_{t\to\infty} y_1(t) = 0$, and therefore $\lim_{t\to\infty} y_2(t) = 0$, which, together with boundedness of k, yields $\lim_{t\to\infty} u_2(t) = 0$ and $\lim_{t\to\infty} u_1(t) = 0$.

Example 3.10. As an illustrative example, we consider example (ii) from the table in the introduction, where P and P_1 are specified in the frequency domain by the associated transfer functions

$$\hat{P}_1(s) = \frac{1}{s-\theta}, \qquad \hat{P}_2(s) = \frac{N(M-s)}{(N+s)(M+s)(s-\theta)}, \quad N, M \geq \varepsilon^{-1}\,,\ \varepsilon > 0.$$

Note that $\hat{P}_1$ has a realization $\widetilde{P}(\theta, 0) \in \mathcal{M}_1$ and $\hat{P}_2$ has a realization $\widetilde{P}(\theta_1, 0) \in \mathcal{P}_3 \setminus \mathcal{M}_3$. We claim that $\vec{\delta}_0(\hat{P}_1, \hat{P}_2) \to 0$ as $\varepsilon \to 0+$. To prove this assertion, note that

$$A(s) = \frac{s-\theta}{s+\sqrt{\theta^2+1}}, \qquad B(s) = \frac{1}{s+\sqrt{\theta^2+1}}$$

satisfy $\hat{P}_1(s) = B(s)A(s)^{-1}$, $A, B \in H^\infty$, and $A^*(s)A(s) + B^*(s)B(s) = I$. Therefore, A, B form a normalized right coprime factorization of $\hat{P}_1$. Since

$$\hat{P}_2(s) = \frac{B(s) + \Delta_B(s)}{A(s)}, \quad \text{where} \quad \Delta_B(s) = \left(\frac{N(M-s)}{(s+N)(s+M)} - 1\right) B(s),$$

and $\Delta_B \in H^\infty$, by (2.5) it suffices to show that

$$\left\| \begin{pmatrix} 0 \\ \Delta_B \end{pmatrix} \right\|_{H^\infty} = \left\| \left(\frac{N(M-s)}{(s+N)(s+M)} - 1 \right) B(s) \right\|_{H^\infty} \to 0 \quad \text{as } \varepsilon \to 0+, \tag{3.25}$$

and this follows from a routine calculation. Thus the claim is proved. To apply Theorem 3.8 to conclude robust stability, it would suffice to show that $\vec{\delta}(\hat{P}_1, \hat{P}_2) \to 0$ as $\varepsilon \to 0^+$. In view of the equivalence (2.5), the latter could be shown by establishing that the directed gap $\vec{\delta}_0(\hat{P}_2, \hat{P}_1)$ is less than 1 (recall that $\vec{\delta}_0(\hat{P}_1, \hat{P}_2) \to 0$ as $\varepsilon \to 0^+$). Alternatively, anticipating Lemma 4.5, we can adapt the proof of that lemma (to the case wherein A, B are defined as above and $A' = A$, $B' = B + \Delta_B$, $(V, U) := (A^*, B^*)$), and invoke (3.25), to conclude that

$$\vec{\delta}(\widetilde{P}(\theta, 0), \widetilde{P}(\theta_1, 0)) \leq \left\| \begin{pmatrix} 0 \\ \Delta_B \end{pmatrix} \right\| \cdot \|(A^*, B^*)\|_{H^\infty} = \left\| \begin{pmatrix} 0 \\ \Delta_B \end{pmatrix} \right\| \to 0 \quad \text{as } \varepsilon \to 0+\,.$$

Thus, under the conditions of Theorem 3.8, the controller $\widetilde{C}(k^0)\colon \mathcal{Y}_a \to \mathcal{U}_a$ defined by (3.6), (3.8) stabilizes any stabilizable and detectable realization of $\hat{P}_2$. As observed in the introduction, $\hat{P}_2$ is an example of a plant which violates all the classical assumptions of adaptive control.

To prove Theorem 3.8 we need to show how the gf-stability of the augmented closed loop $[P, C]$ as given in (3.22) and (3.23) yields the robustness property (3.24) for the unaugmented closed loop $[\widetilde{P}(\theta_1, x_1^0), \widetilde{C}(k^0)]$. This follows from the next result, which is a direct consequence of Theorems 5.2 and 5.3 in [4].

THEOREM 3.11. *Let $m, n, q \in \mathbb{N}$ with $n \geq m$, $q \geq m$, $\mathcal{U} = \mathcal{Y} = L^p(\mathbb{R}_+, \mathbb{R}^m)$, $1 \leq p \leq \infty$, and $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$. Let $\widetilde{K}\colon \mathcal{Y}_a \to \mathcal{U}_a$ be causal, and consider $\widetilde{P}(\vartheta, 0)\colon \mathcal{U}_a \to \mathcal{Y}_a$ defined in* (3.6) *for (ϑ, x^0) in $\mathcal{P}_q \times \mathbb{R}_q$ or $\mathcal{P}_n \times \mathbb{R}_n$. Assume that $[\widetilde{P}(\vartheta, 0), \widetilde{K}]$ is regularly well posed for all $\vartheta \in \mathcal{P}_q$ and let $\Omega \subset \mathcal{M}_n$ be closed. Define*

$$\begin{array}{lrcl} P\colon \mathcal{P}_n \times \mathcal{U}_a \to \mathcal{Y}_a, & (\vartheta, u_1) \mapsto P(\vartheta, u_1), & = & \widetilde{P}(\vartheta, 0)(u_1), \\ C\colon \mathcal{Y}_a \to \mathcal{P}_n \times \mathcal{U}_a, & y_2 \mapsto C(y_2) & = & (0, \widetilde{K}(y_2)). \end{array}$$

If $H_{P,C}|_{\Omega\times\mathcal{W}}$ is gf-stable and $T_\tau\Pi_{P//C}$ is continuous for all $\tau > 0$, then there exist a continuous function $\mu\colon \mathbb{R}_+ \times \Omega \to (0,\infty)$ and a function $\lambda : \mathcal{P}_q \to (0,\infty)$ such that, for all $(\theta_1, \theta, x_1^0, w_0, r) \in \mathcal{P}_q \times \Omega \times \mathbb{R}^q \times \mathcal{W} \times (0,\infty)$,

$$\left.\begin{array}{l}\vec{\delta}(\widetilde{P}(\theta,0), \widetilde{P}(\theta_1,0)) \le \mu(r\,,\,\theta)\\ \lambda(\theta_1)\|x_1^0\| + \|w_0\|_{\mathcal{W}} \le r\end{array}\right\} \quad\Longrightarrow\quad H_{\widetilde{P}(\theta_1,x_1^0),\widetilde{K}}(w_0) \in \mathcal{W}\times\mathcal{W}. \tag{3.26}$$

We are now in a position to prove Theorem 3.8.

Proof of Theorem 3.8. Let $\theta \in \mathcal{M}_n$ and define $\Omega = \{\theta\}$. Consider Theorem 3.11 with $p = 2$ and $\widetilde{K} = \widetilde{C}(k^0)$, where $\widetilde{C}(k^0)$ is given by (3.8). Note that by Corollary 3.2, the closed loop $[\widetilde{P}(\vartheta,0), \widetilde{C}(k^0)]$ is regularly well posed for all $\vartheta \in \mathcal{P}_q$. For P and C as defined in Theorem 3.11, the operator $H^{\Omega}_{P,C}$ is gf-stable by Proposition 3.7. By, for example, the proof of Theorem 4.D in [20], $T_\tau\Pi_{\widetilde{P}(\theta,0)//\widetilde{C}(k^0)}$ is continuous for all $\tau > 0$, and therefore $T_\tau\Pi_{P//C}|_{\Omega\times\mathcal{W}}$ is continuous for all $\tau > 0$. Now all hypotheses of Theorem 3.11 are in place and so there exist a continuous function $\mu\colon \mathbb{R}_+\times\Omega \to (0,\infty)$ and a function $\lambda : \mathcal{P}_q \to (0,\infty)$ such that (3.26) holds. Statement (3.24) follows on setting $\eta(\cdot) = \mu(\cdot,\theta)$. ◻

4. Georgiou and Smith's example revisited. In this section we reconsider Example 9 in the paper by Georgiou and Smith [7] (see also [9]). This serves two purposes, which are to clarify some of the informal arguments therein in relation to robustness with respect to initial data, and to demonstrate that, in the L^2 setting of the present paper, the robustness bound in Theorem 3.8 is qualitatively tight in the sense that it is necessarily dependent on the data u_0, y_0, x_1^0. In those cases where we demonstrate a lack of stability, namely for large initial conditions, large L^2 disturbances, or L^∞ disturbances, the instability is caused by the adaptive gain k undergoing the well-known phenomena of "parameter drift" [13, p. 545], leading to k passing some critical value, thus inducing instability. The conditions given for stability, namely small initial conditions and small L^2 disturbances, prevent this instability mechanism by ensuring that the adaptive gain k never reaches criticality. In general, the issue of "drift-inducing" disturbances has led to various "robustifying" modifications to adaptive designs: one such modification is the introduction of a dead-zone in the second of equations (1.1), to become $\dot{k} = |y|\max\{0, |y| - \lambda\}$, with parameter $\lambda > 0$. Such a modification has led to practical applications to, for example, exothermic chemical reactors [12]. The robustness in the graph topology of such modifications remains to be investigated.

4.1. The nominal and perturbed closed-loop systems. After appropriate rescaling and relabeling of variables, the first-order linear plant $\widetilde{P}(a, y_1^0)$, considered in [7, Example 9], can be parameterized by $a \in \mathbb{R}$ and $y_1^0 \in \mathbb{R}$ and expressed as

$$\widetilde{P}(a, y_1^0)\colon \mathcal{U}_a \to \mathcal{Y}_a, \quad u_1 \mapsto y_1, \quad \text{where } \dot{y}_1 = ay_1 + u_1, \quad y_1(0) = y_1^0, \tag{4.1}$$

and so, for $u_1 \in \mathcal{U}_a$, $y_1 = \widetilde{P}(a, y_1^0)(u_1) : \mathrm{dom}(u_1) \to \mathbb{R}$ is the unique maximal solution of the initial-value problem in (4.1). The controller, parameterized by $k^0 \in \mathbb{R}_+$, is

$$\widetilde{C}(k^0)\colon\ \mathcal{Y}_a \to \mathcal{U}_a,\ y_2 \mapsto u_2 := -k\,y_2, \quad \text{where } \dot{k} = y_2^2, \quad k(0) = k^0, \tag{4.2}$$

and so, for $y_2 \in \mathcal{Y}_a$, $-ky_2 = u_2 = \widetilde{C}(k^0)(y_2) : \mathrm{dom}(y_2) \to \mathbb{R}$, where k is the unique maximal solution of the initial-value problem in (4.2). The closed-loop system $[\widetilde{P}(a, y_1^0), \widetilde{C}(k^0)]$ will be analyzed in the two settings of $\mathcal{U} = \mathcal{Y} = L^2(\mathbb{R}_+, \mathbb{R})$ and

$\mathcal{U} = \mathcal{Y} = L^\infty(\mathbb{R}_+, \mathbb{R})$. In view of Proposition 3.1(i), for all $a, y_1^0 \in \mathbb{R}$ and $k^0 \in \mathbb{R}_+$, the closed-loop system $[\widetilde{P}(a, y_1^0), \widetilde{C}(k^0)]$ is locally well posed in both settings; moreover, by Proposition 3.3, in the former L^2 setting, the closed-loop system is globally well posed, and the signals y_1 and k are bounded, with $y_1(t) \to 0$ and $k(t) \to k^\infty \in \mathbb{R}$ as $t \to \infty$. As in [7, Example 9], consider a perturbation of the plant $\widetilde{P}(a, y_1^0)$ consisting of the series connection of $1/(s-a)$ (i.e., the transfer function associated with $\widetilde{P}(a, y_1^0)$) and an all-pass factor $(M-s)/(M+s)$ with $M > 0$. As a realization of this series connection, consider

$$\dot{y}_1 = a\, y_1 + z - u_1\,, \quad \dot{z} = -M\,[z - 2\,u_1]\,, \quad (y_1(0), z(0)) = (y_1^0, z^0)\,.$$

This series connection is denoted by

$$\widetilde{P}_1\left(a, M, y_1^0, z^0\right) : \mathcal{U}_a \to \mathcal{Y}_a, \quad u_1 \mapsto y_1\,. \tag{4.3}$$

The closed-loop equations for $[\widetilde{P}_1\left(a, M, y_1^0, z^0\right), \widetilde{C}(k^0)]$ are given by

$$\left\{\begin{array}{ll} \dot{y}_1(t) = [a + k(t)]\, y_1(t) + z(t) - k(t) y_0(t) - u_0(t)\,, & y_1(0) = y_1^0, \\ \dot{z}(t) = -2Mk(t) y_1(t) - Mz(t) + 2Mk(t) y_0(t) + 2Mu_0(t)\,, & z(0) = z^0, \\ \dot{k}(t) = (y_0(t) - y_1(t))^2\,, & k(0) = k^0, \\ u_1(t) = u_0(t) + k(t)\, y_0(t) - k(t)\, y_1(t). & \end{array}\right. \tag{4.4}$$

For fixed (but arbitrary) $a \in \mathbb{R}$ and $k^0 \in \mathbb{R}_+$, and applying Theorem 3.8, we may conclude the existence of a continuous function $\eta : \mathbb{R}_+ \to (0, \infty)$ and a function $\lambda_1 : \mathbb{R} \to (0, \infty)$ such that, if $\vec{\delta}(\widetilde{P}(a, 0), \widetilde{P}_1(a, M, 0, 0)) \le \eta(r)$ for some $r > 0$, then, for all initial data $x_1^0 = (y_1^0, z^0)$ and all disturbances $u_0, y_0 \in L^2(\mathbb{R}_+, \mathbb{R})$ satisfying $\lambda_1(M)\|x_1^0\| + \|(u_0, y_0)\|_{L^2(\mathbb{R}_+, \mathbb{R}^2)} \le r$, the closed-loop system is globally well posed and is such that $(y_1(t), z(t)) \to (0, 0)$ as $t \to \infty$, and the monotone gain k converges to a finite limit. At this point, we briefly digress to prove a technicality which will prove convenient in later analyses.

LEMMA 4.1. *Let $M > 0$, $a = 0$, $u_0, y_0 \in L^2(\mathbb{R}_+, \mathbb{R}) \cup L^\infty(\mathbb{R}_+, \mathbb{R})$, and $(y_1^0, z^0, k^0) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$. Let $(y_1, z, k) : [0, \omega) \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$ be the unique maximal solution to the closed-loop initial-value problem* (4.4). *If there exists $T \in [0, \omega)$ such that $k(T) \ge 4M$, $(y_0(T) - y_1(T), z(T)) \ne (0, 0)$, and $(u_0(t), y_0(t)) = (0, 0)$ for all $t \in [T, \infty)$, then*

(i) $u_1, y_1 \notin L^\infty([0, \omega), \mathbb{R})$ *and*

(ii) $k \notin L^\infty([0, \omega), \mathbb{R}_+)$.

Proof. Writing $y_2 := y_0 - y_1$, then, by the hypotheses, we get

$$\left.\begin{array}{l} \dot{y}_2(t) = k(t) y_2(t) - z(t), \quad \dot{z}(t) = 2Mk(t) y_2(t) - Mz(t) \\ \dot{k}(t) = \big(y_2(t)\big)^2 \end{array}\right\} \quad \forall\ t \in [T, \omega). \tag{4.5}$$

Defining $\eta := y_2 - z/(2M)$, we have

$$\left.\begin{array}{l} \dot{y}_2(t) = \big[k(t) - 2M\big] y_2(t) + 2M\eta(t) \\ \dot{\eta}(t) = -My_2(t) + M\eta(t), \quad \dot{k}(t) = \big(y_2(t)\big)^2 \end{array}\right\} \quad \forall\ t \in [T, \omega). \tag{4.6}$$

Introduce $W : [T, \omega) \to \mathbb{R}_+$, $t \mapsto \frac{1}{2}\big[y_2^2 + 2\eta^2\big](t)$. By hypothesis, $(y_2(T), z(T)) \ne (0, 0)$ and so $W(T) > 0$. Moreover, since $k(T) \ge 4M$, we have

$$\dot{W}(t) = \big(k(t) - 2M\big)(y_2(t))^2 + 2M(\eta(t))^2 \ge 2MW(t) \quad \forall\ t \in [T, \omega)\,.$$

Therefore,

$$W(t) \geq e^{2M(t-T)} W(T) \quad \forall\, t \in [T, \omega), \qquad W(T) > 0. \tag{4.7}$$

Seeking a contradiction, suppose that y_2 is bounded on $[T, \omega)$. Then $\omega = \infty$ (to see this, simply note that, if ω is finite, then y_2 is square integrable and so k is bounded, which, together with Proposition 3.1(ii), yields a contradiction). Let $c_0 > 0$ be such that $\big(y_2(t)\big)^2 \leq c_0$ for all $t \in [T, \infty)$, and so $0 \leq k(t) \leq k(T) + c_0[t - T]$ for all $t \geq 0$. By (4.5), there exists $c_1 > 0$ such that $|z(t)| \leq c_1\big[1 + t\big]$ for all $t \in [T, \infty)$. Hence, there exists $c_2 > 0$ such that $|\eta(t)| \leq c_2[1 + t]$ for all $t \in [T, \infty)$. Therefore, it follows that $W(t) \leq \frac{1}{2}\big[c_0 + 2c_2^2(1 + t)^2\big]$ for all $t \geq T$, which contradicts (4.7). Therefore, y_2 is unbounded on $[T, \omega)$ and so, since $y_1(t) = -y_2(t)$ for all $t \in [T, \omega)$, we have $y_1 \notin L^\infty([0, \omega), \mathbb{R})$. Finally, and again seeking a contradiction, suppose that k is bounded. Then, by Proposition 3.1(ii), $\omega = \infty$. By the third of equations (4.5), $y_2 \in L^2([T, \infty), \mathbb{R})$. By the second of equations (4.5), we may conclude that $z \in L^2([T, \infty), \mathbb{R})$ and $z(t) \to 0$ as $t \to \infty$. Rewriting the first of equations (4.5) in the form $\dot{y}_2(t) = -y_2(t) + \zeta(t)$, with $\zeta(t) := [1 + k(t)]y_2(t) - z(t)$, and noting that $\zeta \in L^2(\mathbb{R}_+, \mathbb{R})$, it follows that $y_2(t) \to 0$ as $t \to \infty$. Therefore, $W(t) \to 0$ as $t \to \infty$, which contradicts (4.7). Therefore, k is unbounded, and so property (ii) holds. Since both k, y_1 are unbounded and k is monotone, and $u_1(t) = k(t)y_1(t)$ for all $t \geq T$, it follows that $u_1 \notin L^\infty([0, \omega), \mathbb{R})$. Therefore, property (i) holds. ∎

4.2. Nonrobustness with respect to large initial conditions.

PROPOSITION 4.2. *For $M, a, y_1^0, z^0, k^0 \in \mathbb{R}$ and $u_0, y_0 \in L^2(\mathbb{R}_+, \mathbb{R})$, consider the closed-loop system $[\widetilde{P}_1\left(M, a, y_1^0, z^0\right), \widetilde{C}(k^0)]$ defined by* (2.1), (4.4), (4.2), (4.3) *in the specific case wherein $M > 0$, $a = z^0 = k^0 = 0$, and $u_0 = y_0 = 0$.*

There exists $\chi > 0$ such that, if $(y_1^0)^2 > \chi$, then the unique maximal solution $(y_1, z, k) : [0, \omega) \to \mathbb{R}^2 \times \mathbb{R}_+$, $0 < \omega \leq \infty$, of the closed-loop system $[\widetilde{P}_1\left(M, 0, y_1^0, 0\right),$ $\widetilde{C}(0)]$ has the following properties:

(i) $u_1, y_1 \notin L^\infty([0, \omega), \mathbb{R})$;

(ii) $k \notin L^\infty([0, \omega), \mathbb{R}_+)$.

Proof. Under the above hypothesis, the initial-value problem (4.4) is given by

$$\left\{ \begin{array}{l} \dot{y}_1(t) = k(t)y_1(t) + z(t), \quad \dot{z}(t) = -2Mk(t)y_1(t) - Mz(t), \;\; \dot{k}(t) = (y_1(t))^2, \\ (y_1(0), z(0), k(0)) = (y_1^0, 0, 0). \end{array} \right. \tag{4.8}$$

Let $(y_1, z, k) : [0, \omega) \to \mathbb{R}^2 \times \mathbb{R}_+$ be the unique maximal solution of (4.8) with

$$\big(y_1^0\big)^2 > \chi := \big(32M^2\big(1 + 64M^2\big) + 4\big(1 + 8M^2\big)^2\big) / \big(1 - e^{-4M}\big). \tag{4.9}$$

We will consider separately the two possible cases (a) $\omega < \infty$, and (b) $\omega = \infty$.

Case (a). Assume $\omega < \infty$. Then, by Proposition 3.1, the monotone function k is unbounded. This, in turn, implies that $y_1 \notin L^2([0, \omega), \mathbb{R})$, and so $y_1 \notin L^\infty([0, \omega), \mathbb{R})$. Therefore, properties (i) and (ii) hold.

Case (b). Now assume $\omega = \infty$. For later convenience, we observe that, by (4.8),

$$\|z\|_{L^2([0,t);\mathbb{R})} \leq 2\, \|ky_1\|_{L^2([0,t),\mathbb{R})} \quad \forall\, t \geq 0. \tag{4.10}$$

First, we will show that $k(1) > 4M$. For contradiction, suppose otherwise. Then, $\|y_1\|^2_{L^2([0,1],\mathbb{R})} = k(1) \leq 4M$ and, by monotonicity of k, $k(t) \in [0, 4M]$ for all $t \in [0, 1]$. From (4.10), it now follows that

$$\|y_1\|^2_{L^2([0,1],\mathbb{R})} + \|z\|^2_{L^2([0,1];\mathbb{R})} \leq 4M\big[1 + 64M^2\big]. \tag{4.11}$$

Define $V : [0,1] \to \mathbb{R}_+$, $t \mapsto \frac{1}{2}\big(y_1^2 + z^2\big)(t)$. Then,

$$\begin{aligned}\dot V(t) &= k(t)(y_1(t))^2 + \big(1 - 2Mk(t)\big)y_1(t)z(t) - M(z(t))^2\\ &\ge -(1+8M^2)|y_1(t)z(t)| - M(z(t))^2\\ &\ge -2M(z(t))^2 - (4M)^{-1}\big[(1+8M^2)y_1(t)\big]^2\\ &\ge -4MV(t) - (4M)^{-1}\big[(1+8M^2)y_1(t)\big]^2 \quad \forall\, t \in [0,1].\end{aligned}$$

Therefore,

$$\begin{aligned}\|V\|_{L^1([0,1],\mathbb{R}_+)} &\ge (4M)^{-1}\big(1 - e^{-4M}\big)\, V(0) - (4M)^{-2}(1+8M^2)^2\, \|y_1\|^2_{L^2([0,1],\mathbb{R})}\\ &\ge (8M)^{-1}\big(1 - e^{-4M}\big)\,\chi - (4M)^{-1}(1+8M^2)^2 > 2M\big(1+64M^2\big)\,,\end{aligned}$$

which, in conjunction with (4.11), yields the contradiction

$$4M\big(1+64M^2\big) \ge \|y_1\|^2_{L^2([0,1],\mathbb{R})} + \|z\|^2_{L^2([0,1],\mathbb{R})} = 2\|V\|_{L^1([0,\tau],\mathbb{R}_+)} > 4M\big(1+64M^2\big).$$

Therefore, $k(1) > 4M$. Moreover, since $y_1^0 \ne 0$, we may infer from (4.8) that $(y_1(1), z(1)) \ne (0,0)$. The result follows by application of Lemma 4.1 (with $T = 1$). □

4.3. Nonrobustness with respect to large L^2 disturbances.

PROPOSITION 4.3. *For $M, a, y_1^0, z^0, k^0 \in \mathbb{R}$ and $u_0, y_0 \in L^2(\mathbb{R}_+, \mathbb{R})$, consider the closed-loop system $[\widetilde{P}_1\left(M, a, y_1^0, z^0\right), \widetilde{C}(k^0)]$ defined by* (2.1), (4.4), (4.2), (4.3) *in the specific case wherein $M > 0$, $a = y_1^0 = z^0 = k^0 = 0$, and $y_0 = 0$.*

There exists $u_0 \in L^2(\mathbb{R}_+, \mathbb{R})$ such that the unique maximal solution $(y_1, z, k) : [0,\omega) \to \mathbb{R}^2 \times \mathbb{R}_+$ of the closed-loop system $[\widetilde{P}_1\left(M,0,0,0\right), \widetilde{C}(0)]$ has the following properties:

(i) $y_1 \notin L^\infty([0,\omega),\mathbb{R})$;

(ii) $k \notin L^\infty([0,\omega),\mathbb{R}_+)$.

Proof. Let $a = y_1^0 = z^0 = k^0 = 0$, $M > 0$, and $y_0 = 0$. Fix $r \ne 0$, and denote, by $(\tilde y_1, \tilde z, \tilde k) : [0,\tilde\omega) \to \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+$, the unique maximal solution of (4.4) with u_0 given by $u_0(t) = r$ for all $t \ge 0$, in which case we have

$$\left\{\begin{aligned}&\tfrac{d}{dt}\,[\tilde y_1(t) - r] = -[\tilde y_1(t) - r] + [1 + \tilde k(t)]\,\tilde y_1(t) + [\tilde z(t) - 2r],\\ &\tfrac{d}{dt}\,[\tilde z(t) - 2r] = -M\,[\tilde z(t) - 2r] - 2M\tilde k(t)\,\tilde y_1(t),\\ &\tfrac{d}{dt}\,\tilde k(t) = (\tilde y_1(t))^2, \qquad (\tilde y_1(0), \tilde z(0), \tilde k(0)) = (0,0,0).\end{aligned}\right. \tag{4.12}$$

For contradiction, suppose that the component $\tilde k$ is bounded. Then, by Proposition 3.1, $\tilde\omega = \infty$. Since $\tilde k \in L^\infty(\mathbb{R}_+)$, it follows that $\tilde y_1 \in L^2(\mathbb{R}_+)$ and so, by the second differential equation in (4.12), we may infer that $\tilde z(\cdot) - 2r \in L^2(\mathbb{R}_+)$. Noting that $[1 + \tilde k(\cdot)]\tilde y_1(\cdot) + [\tilde z(\cdot) - 2r] \in L^2(\mathbb{R}_+)$, it follows from the first equation in (4.12) that $\tilde y_1(t) \to r \ne 0$ as $t \to \infty$, which contradicts the fact that $\tilde y_1 \in L^2(\mathbb{R}_+)$. Therefore, the solution component $\tilde k$ is unbounded. Unboundedness of $\tilde k$, together with the third differential equation in (4.12), implies the existence of $T \in [0,\tilde\omega)$ such that $k(T) > 4M$ and $\tilde y_1(T) \ne 0$. Let $u_0 \in L^2(\mathbb{R}_+,\mathbb{R})$ be the piecewise constant function $u_0 := T_T r$ (viz. $u_0(t) = r$ on $[0,T)$ and $u_o(t) = 0$ on $[T,\infty)$), and denote by $(y_1, z, k) : [0,\omega) \to \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+$ the unique maximal solution of

$$\begin{aligned}&\dot y_1(t) = k(t)\,y_1(t) + z(t) - u_0(t), && y_1(0) = 0,\\ &\dot z(t) = -2Mk(t)\,y_1(t) - Mz(t) + 2Mu_0(t), && z(0) = 0,\\ &\dot k(t) = y_1(t)^2, && k(0) = 0.\end{aligned}$$

Clearly, $\tilde{\omega} \leq \omega$ and $(\tilde{y}_1(t), \tilde{z}(t), \tilde{k}(t)) = (y_1(t), z(t), k(t))$ for all $t \in [0, \tilde{\omega})$. Therefore, $y_1(T) = \tilde{y}_1(T) \neq 0$ and $k(T) = \tilde{K}(T) > 4M$. An application of Lemma 4.1 completes the proof. ∎

4.4. Nonrobustness with respect to L^∞ disturbances. The initial calculation in [7, Example 9] (see also [9]) shows that arbitrarily small L^∞ disturbances u_0, $y_0 \in L^\infty(\mathbb{R}_+, \mathbb{R})$ can cause $\|u_1\|_{L^\infty(\mathbb{R}_+,\mathbb{R})}$ to be arbitrarily large, and so $H_{P,C}$ is not gf-stable in an L^∞ sense, whence we get the claim that the (L^∞) "robustness margin… should be assigned the value zero." Note that the gf-stability of $H_{P,C}$ is only a sufficient condition for robust stability. In this context, we next show that any constant nonzero input disturbance (and zero output disturbance) leads to unbounded signals in the perturbed closed loop $[\widetilde{P}_1(a, M, y_1^0, z^0), \widetilde{C}(k^0)]$. This is not surprising in view of [7, Example 9], where it is shown that the unperturbed closed loop $[\widetilde{P}(a, y_1^0), \widetilde{C}(k^0)]$ is nonrobust with respect to L^∞ disturbances.

PROPOSITION 4.4. *For $M, a, y_1^0, z^0, k^0 \in \mathbb{R}$ and $u_0, y_0 \in L^2(\mathbb{R}_+, \mathbb{R})$, consider the closed-loop system $[\widetilde{P}_1\left(M, a, y_1^0, z^0\right), \widetilde{C}(k^0)]$ defined by* (2.1), (4.4), (4.2), (4.3) *in the specific case wherein $M > 0$, $a = y_1^0 = z^0 = k^0 = 0$, and $y_0 = 0$.*

For any $r > 0$, there exists $u_0 \in L^\infty(\mathbb{R}_+, \mathbb{R})$ with $\|u_0\|_{L^\infty(\mathbb{R}_+,\mathbb{R})} \leq r$, such that the unique maximal solution $(y_1, z, k) : [0, \omega) \to \mathbb{R}^2 \times \mathbb{R}_+$ of the closed-loop system $[\widetilde{P}_1\left(M, 0, 0, 0\right), \widetilde{C}(0)]$ has the following properties:

(i) $u_1, y_1 \notin L^\infty([0, \omega), \mathbb{R})$;

(ii) $k \notin L^\infty([0, \omega), \mathbb{R}_+)$.

Proof. This follows directly from the proof of Proposition 4.3, wherein it was shown that for each, $r > 0$, there exists $T \in (0, \omega)$ such that the disturbance $u_0 \in L^\infty(\mathbb{R}_+, \mathbb{R})$, given by $u_0 := T_T r$ and with norm $\|u_0\|_{L^\infty(\mathbb{R}_+,\mathbb{R})} = r$, is such that properties (i) and (ii) hold. ∎

Interestingly, our analysis in subsections 4.2, 4.3, and 4.4 has not established whether a finite escape time can occur within these closed loops. Simulation evidence presented in [7] makes this plausible, but the question remains open.

4.5. Robustness with respect to small L^2 disturbances and small initial conditions. Having noted in subsection 4.4 that the L^∞ robustness margin should be assigned the value zero, our next task is to show that, in the L^2 setting of the current paper, the situation is less pessimistic. In the L^∞ framework, one last remark is warranted. In [7, Example 9], and based on informal numerical evidence, there is a suggestion that—even with zero disturbances—the closed-loop system fails to be robustly stable if the initial conditions are nonzero. Proposition 4.2 confirms this in the case of *large* initial conditions. However, Proposition 4.6 below subsumes the following observation: With zero disturbances, the closed-loop system is robustly stable for *sufficiently small* initial conditions. As noted in [7, Example 8], in the L^∞ framework, $\vec{\delta}\big(\widetilde{P}(a, 0), \widetilde{P}_1(a, M, 0, 0)\big) \to 0$ as $M \to \infty$. We now show that this result also holds true in the L^2 framework.

LEMMA 4.5. *For $M, a, y_1^0, z^0 \in \mathbb{R}$ and $u_0, y_0 \in L^2(\mathbb{R}_+, \mathbb{R})$, consider $\widetilde{P}(a, y_1^0)$ and $\widetilde{P}_1\left(M, a, y_1^0, z^0\right)$ given by* (4.1) *and* (4.3)*, respectively, in the specific case wherein $M > 0$ and $y_1^0 = z^0 = 0$. Then $\vec{\delta}\big(\widetilde{P}(a, 0), \widetilde{P}_1(a, M, 0, 0)\big) \to 0$ as $M \to \infty$.*

Proof. It will be convenient to utilize a frequency domain representation of linear operators. Firstly, let $c > 0$, and define the rational functions A, A', B, B' by $A(s) = A'(s) := (s-a)/(s+c)$, $B(s) := 1/(s+c)$, and $B'(s) := (M-s)/(s+c)(s+M)$. For $n, m \in \mathbb{N}$, let H^2 denote the set of all analytic functions $f : \mathbb{C}_+ \to \mathbb{C}^{n\times m}$ so that $\int_\mathbb{R} \|f(\alpha+i\beta\|^2 \mathrm{d}\beta$ is finite for all $\alpha > 0$. Since, by Paley and Wiener (see, for example, [3, Thm. 11.9]), the extended Laplace transform $\mathbf{L}$ on $L^2(\mathbb{R}_+, \mathbb{R})$ yields an isometric isomorphism between $L^2(\mathbb{R}_+, \mathbb{R})$ and H^2, we may observe (see [17, p. 234]) that the graphs corresponding to $\widetilde{P}(a, 0)$, $\widetilde{P}_1(M, a, 0, 0)$ can be written as:

$$\mathbf{L}\left(\mathcal{G}_{\widetilde{P}}\right) = \begin{pmatrix} A \\ B \end{pmatrix} H^2 \quad \text{and} \quad \mathbf{L}\left(\mathcal{G}_{\widetilde{P}_1}\right) = \begin{pmatrix} A' \\ B' \end{pmatrix} H^2.$$

Let $(V, U) := (1, c + a)$, and define the mappings $\Phi, \tilde{\Phi}$:

$$\tilde{\Phi}\colon \mathbf{L}\left(\mathcal{G}_{\widetilde{P}(a,0)}\right) \to \mathbf{L}\left(\mathcal{G}_{\widetilde{P}_1(M,a,0,0)}\right), \quad \begin{pmatrix} u \\ y \end{pmatrix} \mapsto \tilde{\Phi}\begin{pmatrix} u \\ y \end{pmatrix} := \begin{pmatrix} A' \\ B' \end{pmatrix} (V, U) \begin{pmatrix} u \\ y \end{pmatrix},$$

$$\Phi\colon \mathcal{G}_{\widetilde{P}(a,0)} \to \mathcal{G}_{\widetilde{P}_1(M,a,0,0)}, \qquad \Phi := \mathbf{L}^{-1}\tilde{\Phi}\mathbf{L}.$$

Since $(V, U)\binom{A}{B} = 1$, A', B' are coprime, and $\mathbf{L}$ is an isometric isomorphism, it follows that $\Phi \in \mathcal{O}_{\widetilde{P}, \widetilde{P}_1}$. Additionally, since every element of $\mathbf{L}\left(\mathcal{G}_{\widetilde{P}}\right)$ is of the form $y = \binom{A}{B}x$, where $x \in H^2$, it follows that

$$(\tilde{\Phi} - I)|_{\mathbf{L}(\mathcal{G}_{\widetilde{P}})} y = \left(I - \begin{pmatrix} A' \\ B' \end{pmatrix} (V, U)\right) \begin{pmatrix} A \\ B \end{pmatrix} x = \begin{pmatrix} A - A' \\ B - B' \end{pmatrix} x = \begin{pmatrix} A - A' \\ B - B' \end{pmatrix} (V, U) y,$$

and since $(\Phi - I)|_{\mathcal{G}_{\widetilde{P}}} = \mathbf{L}^{-1}(\tilde{\Phi} - I)|_{\mathbf{L}(\mathcal{G}_{\widetilde{P}})}\mathbf{L}$, we have

$$\vec{\delta}(\widetilde{P}, \widetilde{P}_1) \leq \left\| \mathbf{L}^{-1} \left(\begin{pmatrix} A - A' \\ B - B' \end{pmatrix} (V, U) \right) \mathbf{L} \right\|_{L^2} = \left\| \begin{pmatrix} A - A' \\ B - B' \end{pmatrix} (V, U) \right\|_{H^\infty}.$$

Hence,

$$\vec{\delta}(\widetilde{P}, \widetilde{P}_1) \leq \left\| \begin{pmatrix} 0 \\ \frac{2s}{(s+c)(s+M)} \end{pmatrix} (1, c + a) \right\|_{H^\infty} \leq 2\sqrt{1 + (c + a)^2} \left\| \frac{s}{(s + c)(s + M)} \right\|_{H^\infty}.$$

A straightforward computation confirms that the right-hand side in the above goes to 0 as M tends to ∞. This completes the proof. ∎

The final result states that for any disturbance level, M can be chosen to ensure stability of the perturbed closed-loop system; furthermore, this stability is local with respect to initial conditions.

PROPOSITION 4.6. *Let $a \in \mathbb{R}$. For any $M > 0$, $y_1^0, z^0 \in \mathbb{R}$, $k^0 > 0$, consider the closed-loop system $[\widetilde{P}_1\left(M, a, y_1^0, z^0\right), \widetilde{C}(k^0)]$ as defined by* (2.1), (4.4), (4.3), *and* (4.2)*. For any $r > 0$ there exists $\hat{M} > 0$, and for any $M \geq \hat{M}$ there exists $\varepsilon > 0$ such that, if $\|(y_1^0, z^0)\| \leq \varepsilon$ and $\|(u_0, y_0)\|_{L^2(\mathbb{R}_+, \mathbb{R}^2)} \leq r$, then the closed-loop system $[\widetilde{P}_1\left(M, a, y_1^0, z^0\right), \widetilde{C}(k^0)]$ has the following properties:*

(i) *There exists a unique solution $(y_1, z, k) : \mathbb{R}_+ \to \mathbb{R}^2 \times \mathbb{R}_+$;*
(ii) *$(u_1, y_1) \in L^2(\mathbb{R}_+, \mathbb{R}^2)$;*
(iii) *$\lim_{t\to\infty}(y_1(t), z(t)) = 0$;*
(iv) *$k \in L^\infty(\mathbb{R}_+, \mathbb{R}_+)$.*

Proof. Properties (i), (ii), and (iv) follow from Lemma 4.5 and Theorem 3.8. Therefore, $y_1, y_0, u_0, ky_0, ky_1 \in L^2(\mathbb{R}_+)$. Invoking the second differential equation of (4.4), we have $z \in L^2(\mathbb{R}_+)$ and $\lim_{t\to\infty} z(t) = 0$. By the first equation in (4.4) we have $\dot{y}_1 \in L^2(\mathbb{R}_+)$. It now follows that $\lim_{t\to\infty} y_1(t) = 0$, whence we get property (iii). ∎

5. Summary. In subsection 3.4, we developed a general result establishing a robust stability margin (whose size is dependent on the L^2 disturbance level and size of the initial condition) for the class of m-input, m-output, relative degree one, nonminimum phase plants, whose first Markov parameter lies in the open right half plane, when controlled by the "standard" adaptive output feedback controller (1.1). In section 4 we have given a qualitative analysis of a first-order system $\widetilde{P}(0, y_1^0)$ perturbed by an all-pass factor $\frac{M-s}{M+s}$ and controlled by a standard adaptive controller as considered by Georgiou and Smith [7, Ex. 9]. The results of section 4 are summarized in the following table:

Disturbances and initial data	Stability	Internal signals	Controller gain
For any small L^2 disturbance (u_0, y_0) and any small initial condition (y_1^0, z^0)	stable	$(u_1, y_1) \in L^2$	$k \in L^\infty$
There exists large L^2 disturbance $(u_0, 0)$	unstable	$(u_1, y_1) \notin L^\infty$	$k \notin L^\infty$
For any large initial condition $(y_1^0, 0)$	unstable	$(u_1, y_1) \notin L^\infty$	$k \notin L^\infty$
There exists an L^∞ disturbance $(u_0, 0)$ (of any nonzero size)	unstable	$(u_1, y_1) \notin L^\infty$	$k \notin L^\infty$

It is worth noting that the L^2 analysis in this paper provides a mechanism to prove the stability of the disturbance-free system in the presence of small initial conditions. The informal plausibility arguments presented in [7] for the lack of robustness of a closed-loop system in the presence of nonzero initial conditions do not predict the stable behavior of the closed-loop system when the initial conditions are small. This case study highlights the critical role played by the choice of signal space—alternative signal spaces may give different robust stability guarantees (in particular, in the disturbance-free case, we have seen that the L^∞ analysis does not give any indication of the robustness of the closed-loop system under gap perturbations with nonzero initial conditions; however, the L^2 analysis does establish robustness). This highlights the importance of an L^2 analysis for considering response to initial conditions. The second and third entries in the above table illustrate that the sufficient conditions for robust stability given by the gap analysis in section 3.4 cannot be improved qualitatively and emphasize the complementary role of initial conditions and disturbances. It is also important to note that zero L^∞ robustness margins are not inevitable. An example of an adaptive controller exhibiting a nonzero L^∞ margin is given in [4].

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