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Simple adaptive stabilization of high-gain stabilizable systems

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Abstract: It is shown that the simple adaptive feedback strategy $u(t) = \ln k(t) \cos\sqrt{\ln k(t)}$, $\dot{k}(t) = y(t)^2$ is a universal adaptive stabilizer for the class of single-input, single-output, finite-dimensional, linear systems which are stabilizable by either negative or positive high-gain feedback.

Keywords: Adaptive control systems; adaptive and robust stabilization; high-gain feedback, feedback control.

1. Introduction

In this paper we present a solution to an open problem in the field of non-identifier-based, smooth, universal adaptive stabilization. Since the early 1980's it has been well known that each system belonging to the class of scalar input-output, relative degree one, minimum phase systems with positive high frequency gain

$$\begin{aligned} &\dot{x}(t) = Ax(t) + bu(t), \qquad y(t) = cx(t), \qquad x(0) \in \mathbb{R}^n \\ &(A, b, c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^{1\times n}, \qquad cb > 0, \qquad (A, b, c) \text{ is minimum phase,} \end{aligned} \tag{1.1}$$

is output stabilizable by the feedback strategy

$$u(t) = -k(t)y(t), \qquad \dot{k}(t) = y(t)^2, \qquad k(0) \in \mathbb{R}. \tag{1.2}$$

(1.2) is a *universal adaptive stabilizer* for the class (1.1) in the sense that, if (1.2) is applied to an arbitrary system (A, b, c) belonging to (1.1), then the trajectory of the closed-loop system tends to zero, $\lim_{t\to\infty} x(t) = 0$, and the gain converges to a finite limit, $\lim_{t\to\infty} k(t) = k_\infty < \infty$.

The open question, see Morse [4], as to whether the knowledge of the sign of the high frequency gain is a necessary condition for adaptive stabilization was answered by Nussbaum [6], who, by introducing the concept of switching functions, presented an adaptive stabilizer for first order, single-input, single-output systems where the sign of cb is unknown. This idea was then used by Willems and Byrnes [9], who showed that the class of systems

$$\begin{aligned} &\dot{x}(t) = Ax(t) + bu(t), \qquad y(t) = cx(t), \qquad x(0) \in \mathbb{R}^n \\ &(A, b, c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^{1\times n}, \qquad cb \neq 0, \qquad (A, b, c) \text{ is minimum phase} \end{aligned}$$

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can be stabilized by the universal adaptive stabilizer

$$u(t) = N(k(t))y(t), \qquad \dot{k}(t) = y(t)^2, \qquad k(0) \in \mathbb{R} \tag{1.3}$$

where $N(\cdot):[k', \infty) \to \mathbb{R}$ is a *Nussbaum function*, i.e. a piecewise right continuous function satisfying the following so called *Nussbaum condition*

$$\sup_{k>k_0} \frac{1}{k-k_0}\int_{k_0}^{k} N(\tau)\, d\tau = +\infty \quad \text{and} \quad \inf_{k>k_0} \frac{1}{k-k_0}\int_{k_0}^{k} N(\tau)\, d\tau = -\infty \tag{1.4}$$

for some $k_0 \geq k'$.

Examples are given by $N(k) = k\cos\sqrt{|k|}$, $N(k) = k^2\cos(k)$, $N(k) = \cos(\frac{1}{2}\pi k)e^{k^2}$.

It then remained an open question whether (1.3) would also be a universal adaptive stabilizer for the class of *positive or negative high-gain stabilizable systems*, i.e. the class

$$\begin{aligned} &\dot{x}(t) = Ax(t) + bu(t), \qquad y(t) = cx(t), \\ &(A, b, c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^{1\times n}, \qquad x(0) \in \mathbb{R}^n, \\ &\text{there exist } k^* > 0,\ s \in \{-1, +1\} \text{ such that } \sigma(A - skbc) \subset \mathbb{C}_- \text{ for all } k \geq k^*. \end{aligned} \tag{1.5}$$

The difficulty is that (1.5) contains a subclass of relative degree two systems. Under the additional assumption, that for every system belonging to (1.5) the sign of the high frequency gain is positive, Morse [5] and Corless [1,2] have proved that the feedback strategy (1.2) is a universal adaptive stabilizer for (1.5). Morse [5] also conjectured that (1.3) is a universal adaptive stabilizer for the class (1.5). Recently, Corless and Ryan [3] have shown that if the adaptation law $\dot{k}(t) = y(t)^2$ is modified, then

$$u(t) = N(k(t))y(t), \qquad k(t) = \varepsilon y(t)^2 + \int_0^t y(s)^2\, ds + k(0), \quad k(0) \in \mathbb{R}, \tag{1.6}$$

is a universal adaptive stabilizer for all relative degree 2 systems belonging to (1.5) provided that, additionally, some

$$\varepsilon > -\frac{1}{2}\frac{cAb}{cA^2b}$$

is *known* to the designer. It is not known whether (1.6) is actually a universal adaptive stabilizer for relative degree 1 systems belonging to (1.5).

In this paper, we will show that Morse's conjecture is true if the Nussbaum function is chosen appropriately. One possibility for the Nussbaum function is $N(k) = \ln k \cos\sqrt{\ln k}$. The paper is organized as follows. In Section 2, we present a convenient state space form for systems belonging to (1.5), and prove a crucial lemma. In Section 3, we present our main result, i.e. (1.3) is a universal adaptive stabilizer for the class (1.5).

2. Preliminaries

In this section we derive some basic properties of the class of systems (1.5).

The system

$$\dot{x}(t) = Ax(t) + bu(t), \quad x(0) \in \mathbb{R}^n, \qquad y(t) = cx(t) \tag{2.1}$$

where $A \in \mathbb{R}^{n\times n}$, b, $c^T \in \mathbb{R}^n$, is called *minimum phase* if

$$\det\begin{bmatrix} sI - A & b \\ c & 0 \end{bmatrix} \neq 0 \quad \text{for all } s \in \overline{\mathbb{C}}_+. \tag{2.2}$$

It is well known that (2.1) is minimum phase if and only if it is stabilizable, detectable and the zero polynomial of the transfer function $c(sI_n - A)^{-1}b$ has no roots in the closed right half plane $\overline{\mathbb{C}}_+$.

The following two lemmata present convenient state space forms for systems of relative degree 1 and 2.

2.1. Lemma. *If the system* (2.1) *is of relative degree* 1, *i.e.* $cb \neq 0$, *then the transformation* $S^{-1}x = (y, z^{\mathrm{T}})^{\mathrm{T}}$, *where* $S := [b(cb)^{-1}, V]$ *and* $V \in \mathbb{R}^{n\times(n-1)}$ *is of full rank with* $\ker c = V\cdot\mathbb{R}^{n-1}$, *applied to* (2.1) *yields*

$$\begin{aligned} \dot y(t) &= A_{11}y(t) + A_{12}z(t) + cbu(t), \\ \dot z(t) &= A_{21}y(t) + A_{22}z(t), \end{aligned} \qquad \begin{pmatrix} y(0) \\ z(0) \end{pmatrix} = S^{-1}x_0. \tag{2.3}$$

Here $A_{11} \in \mathbb{R}$, A_{12}^{T}, $A_{21} \in \mathbb{R}^{n-1}$, $A_{22} \in \mathbb{R}^{(n-1)\times(n-1)}$. *If* (A, b, c) *is minimum phase, then* $\sigma(A_{22}) \subset \mathbb{C}_-$.

The proof is straightforward and therefore omitted.

2.2. Lemma. *If the system* (2.1) *is of relative degree* 2, *i.e.* $cb = 0$, $cAb \neq 0$, *then there exists a coordinate transformation* $S \in \mathrm{GL}_n(\mathbb{R})$, *such that* $S^{-1}x = (y, \dot y, z^{\mathrm{T}})^{\mathrm{T}}$ *applied to* (2.1) *yields*

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y(t) \\ \dot y(t) \\ z(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ a_2 & cA^2b/cAb & a_4^{\mathrm{T}} \\ a_5 & 0 & A_6 \end{bmatrix}\begin{pmatrix} y(t) \\ \dot y(t) \\ z(t) \end{pmatrix} + \begin{bmatrix} 0 \\ cAb \\ 0 \end{bmatrix} u(t). \tag{2.4}$$

Here $a_2 \in \mathbb{R}$, a_4, $a_5 \in \mathbb{R}^{n-2}$, $A_6 \in \mathbb{R}^{(n-2)\times(n-2)}$. *If* (2.1) *is minimum phase, then* $\sigma(A_6) \subset \mathbb{C}_-$.

Proof. Choose $V \in \mathbb{R}^{n\times(n-2)}$ of full rank so that $\ker\left[\begin{smallmatrix} c \\ cA \end{smallmatrix}\right] = V\cdot\mathbb{R}^{n-2}$. Let $S_1 := [Ab, b, V](cAb)^{-1}$. It is easily verified that the inverse of S_1 is given by

$$S_1^{-1} = \begin{bmatrix} c \\ cA - (cA^2b/cAb)c \\ cAbV^*\left[I_n - \left(Ab - (cA^2b/cAb)b\right)(cAb)^{-1}c - bcA(cAb)^{-1}\right] \end{bmatrix}, \qquad V^* := (V^{\mathrm{T}}V)^{-1}V^{\mathrm{T}},$$

and it follows that

$$\bar c := cS_1 = [1, 0, \ldots, 0], \qquad \bar b := S_1^{-1}b = \begin{bmatrix} 0 \\ cAb \\ 0 \end{bmatrix},$$

$$\bar A := S_1^{-1}AS_1 = \begin{bmatrix} * & 1 & * \\ * & * & * \\ * & * & * \end{bmatrix}, \qquad \bar c\bar A\bar b = cAb.$$

Applying to $\bar A$ elementrary row and column operations of the form

$$S_2^{-1} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & 0 & I_{n-2} \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 & 0 \\ * & 1 & * \\ 0 & 0 & I_{n-2} \end{bmatrix}$$

yields

$$\hat A := (S_1S_2)^{-1}AS_1S_2 = \begin{bmatrix} 0 & 1 & 0 \\ a_2 & a_3 & a_4^{\mathrm{T}} \\ a_5 & 0 & A_6 \end{bmatrix}.$$

Moreover, $\bar{c}S_2 = \bar{c}$, $S_2^{-1}\bar{b} = \bar{b}$, and $cA^2b = \bar{c}\hat{A}^2\bar{b} = a_3 cAb$, and hence (2.4) holds with $S = S_1S_2$. Using (2.4) again, we have

$$\begin{vmatrix} \lambda I_n - A & b \\ c & 0 \end{vmatrix} = \begin{vmatrix} \lambda & -1 & 0 & 0 \\ -a_2 & \lambda - a_3 & a_4^{\mathrm{T}} & cAb \\ -a_5 & 0 & \lambda I_{n-2} - A_6 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -cAb \cdot | \lambda I_{n-2} - A_6 |.$$

If (2.1) is minimum phase, then it follows that $| \lambda I_{n-2} - A_6 | \neq 0$ for all $\lambda \in \bar{\mathbb{C}}_+$, and hence A_6 is stable. This completes the proof. □

For the sake of completeness, we show that the state space forms (2.3) and (2.4) immediately imply the following characterization of the systems class (1.5).

2.3. Proposition. *If $c(sI_n - A)^{-1}b \not\equiv 0$, then (A, b, c) belongs to* (1.5) *if and only if the following conditions are satisfied*:

(i) *(A, b, c) is minimum phase,*
(ii) *either $cb \neq 0$ or $[cAb \neq 0$ and $cA^2b/cAb < 0]$.*

Proof. Under high-gain feedback, $n-r$ poles are tending to the $n-r$ zeros of the system, and the remaining r poles tend to infinity. Moreover, the angles of the asymptotes are $\pm 360/r$ deg. See [7], p.369. Therefore (A, b, c) is necessarily of relative degree 1 or 2, and minimum phase.

If $cb \neq 0$, then it follows from (2.3) that

$$| \lambda I_n - A + skbc | = (\lambda - A_{11} + skcb) \cdot \left| \lambda I_{n-1} - A_{22} - A_{21}^{\mathrm{T}}(\lambda - A_{11} + skcb)^{-1} A_{12} \right|$$

whence

$$\sigma(A - skbc) \to \{A_{11} - skcb\} \cup \sigma(A_{22}) \quad \text{as } k \to \infty.$$

Since A_{22} is stable, the proposition is proved for relative degree 1 systems.

If (A, b, c) is of relative degree 2, then (2.4) yields

$$| \lambda I_n - A + skbc | = | \lambda I_{n-2} - A_6 | \cdot \left| \lambda \left(\lambda - \frac{cA^2b}{cAb} \right) + skcAb - a_2 - a_4^{\mathrm{T}}(\lambda I_{n-2} - A_6)^{-1} a_5 \right|.$$

It follows that, if $cA^2b = 0$, then the system is not positive or negative high-gain stabilizable, whereas, if $cA^2b \neq 0$, then

$$\sigma(A - skbc) \to \sigma(A_6) \cup \left\{ \frac{cA^2b}{2cAb} \pm \sqrt{\left(\frac{cA^2b}{2cAb} \right)^2 - skcAb} \right\} \quad \text{as } k \to \infty.$$

This proves the proposition. □

The following lemma is crucial for the proof of our main result. We shown that if the feedback $u(t) = N(t)y(t)$ is applied to a relative degree 2 system belonging to (1.5), then the state decays exponentially in each interval where $N(t)$ and $\dot{N}(t)$ have 'correct' sign and $| N(t) |$ is sufficiently large.

2.4. Lemma. *Suppose the system* (2.1) *is minimum phase with*

$$cb = 0, \quad cAb \neq 0, \quad \frac{cA^2b}{cAb} < 0 \tag{2.5}$$

and $N : I \to \mathbb{R}$ *is a differentiable function on an interval* $I \subset \mathbb{R}$ *with*

$$cAbN(t) > M \quad \text{for all } t \in I, \tag{2.6}$$

$$cAb\dot{N}(t) \geq 0 \quad \text{for all } t \in I, \tag{2.7}$$

for some $M > 0$.

If feedback of the form $u(t) = -N(t)y(t)$ *is applied to* (2.1) *and* M *is sufficiently large (depending on the entries of* A, b, c*), then there exist* $\hat{M}$, $\omega > 0$ *(independent of* $N(\cdot)$*) such that*

$$y(t)^2 \leq \hat{M}\, \mathrm{e}^{-\omega(t-t_0)} \| x(t_0) \|^2 \quad \text{for all } t \geq t_0,\ t_0,\ t \in I. \tag{2.8}$$

Proof. By using Lemma 2.2, we may assume that the closed-loop system (2.4), $u(t) = -N(t)y(t)$, is of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y(t) \\ \dot{y}(t) \\ z(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ a_2 - cAbN(t) & -a_3 & a_4^{\mathrm{T}} \\ a_5 & 0 & A_6 \end{bmatrix} \begin{pmatrix} y(t) \\ \dot{y}(t) \\ z(t) \end{pmatrix} \tag{2.9}$$

where

$$a_2 \in \mathbb{R}, \quad a_4, a_5 \in \mathbb{R}^{n-2}, \quad a_3 = -\frac{cA^2b}{cAb} > 0, \quad \sigma(A_6) \subset \mathbb{C}_-. \tag{2.10}$$

In order to obtain a Lyapunov-like function for (2.9), we apply the time-varying transformation

$$U(t)^{-1}\begin{pmatrix} y(t) \\ \dot{y}(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ \eta(t) \\ z(t) \end{pmatrix}, \quad U(t)^{-1} := \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2}r(t)a_3 & r(t) & 0 \\ 0 & 0 & I_{n-2} \end{bmatrix} \quad \text{for all } t \in I,$$

where, if $M > a_2 + \frac{1}{4}a_3^2$,

$$0 < r(t) := \left(cAbN(t) - a_2 - \frac{a_3^2}{4} \right)^{-1/2} < \left(cAbM - a_2 - \frac{a_3^2}{4} \right)^{-1/2} \quad \text{for all } t \in I. \tag{2.11}$$

This transformation, due to Corless [2], has the benefit that the first two off-diagonal terms in the transformed system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y(t) \\ \eta(t) \\ z(t) \end{pmatrix} = \begin{bmatrix} -\frac{1}{2}a_3 & r(t)^{-1} & 0 \\ -r(t)^{-1} & -\frac{1}{2}a_3 + \dot{r}(t)/r(t) & r(t)a_4^{\mathrm{T}} \\ a_5 & 0 & A_6 \end{bmatrix} \begin{pmatrix} y(t) \\ \eta(t) \\ z(t) \end{pmatrix} \tag{2.12}$$

are $-r(t)^{-1}$ and $r(t)^{-1}$. Choosing the Lyapunov-like function

$$V(y, \eta, z) := \tfrac{1}{2}y^2 + \tfrac{1}{2}\eta^2 + \tfrac{1}{2}\alpha\langle z, Pz \rangle$$

where $P = P^{\mathrm{T}} \in \mathbb{R}^{(n-2)\times(n-2)}$ is the positive definite solution of

$$PA_6 + A_6^{\mathrm{T}}P = -2I_{n-2}$$

and $\alpha > 0$ is to be specified later, the derivative of V along the solution of (2.12) is

$$\frac{\mathrm{d}}{\mathrm{d}t}V(y(t),\eta(t),z(t)) = -\tfrac{1}{2}a_3\left[y(t)^2+\eta(t)^2\right] - \alpha\|z(t)\|^2 + \frac{\dot{r}(t)}{r(t)}\eta(t)^2 + \eta(t)r(t)a_4^{\mathrm{T}}z(t) + \alpha z(t)^{\mathrm{T}}Pa_5y(t).$$

Since, by assumption (2.7), $\dot{r}(t)/r(t) = -\tfrac{1}{2}r(t)^2cAb\dot{N}(t) < 0$, and

$$\eta(t)r(t)a_4^{\mathrm{T}}z(t) \le \tfrac{1}{2}\sqrt{\alpha}\,\|z(t)\|2\frac{1}{\sqrt{\alpha}}r(t)\left\|a_4^{\mathrm{T}}\right\|\|\eta(t)\| \le \tfrac{1}{4}\alpha\|z(t)\|^2 + \frac{4}{\alpha}r(t)^2\left\|a_4^{\mathrm{T}}\right\|^2\eta(t)^2,$$

$$z(t)^{\mathrm{T}}Pa_5y(t) \le \tfrac{1}{2}\|z(t)\|2\|Pa_5\|\,|y(t)| \le \tfrac{1}{4}\|z(t)\|^2 + 4\|Pa_5\|^2y(t)^2,$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}V(y(t),\eta(t),z(t)) \le \left(-\tfrac{1}{2}a_3 + 4\alpha\|Pa_5\|^2\right)y(t)^2 + \left(-\tfrac{1}{2}a_3 + \frac{4}{\alpha}r(t)^2\|a_4\|^2\right)\eta(t)^2 - \tfrac{1}{2}\alpha\|z(t)\|^2.$$

First choose $\alpha > 0$ sufficiently small so that

$$-\tfrac{1}{2}a_3 + 4\alpha\|Pa_5\|^2 < -\tfrac{1}{4}a_3$$

and then let M in (2.6) be sufficiently large, see also (2.11), such that

$$-\tfrac{1}{2}a_3 + \frac{4}{\alpha}r(t)^2\|a_4\|^2 < -\tfrac{1}{4}a_3 \quad \text{for all } t\in I.$$

These assumptions yield for all $t \in I$,

$$\frac{\mathrm{d}}{\mathrm{d}t}V(y(t),\eta(t),z(t)) \le -\frac{a_3}{4}\left[y(t)^2+\eta(t)^2\right] - \tfrac{1}{2}\alpha\|z(t)\|^2 \le -\omega V(y(t),\eta(t),z(t)) \tag{2.13}$$

where $\omega = \min\{\tfrac{1}{2}a_3, 1/\|P\|\}$. Therefore,

$$\tfrac{1}{2}y(t)^2 \le V(y(t),\eta(t),z(t)) \le \mathrm{e}^{-\omega(t-t_0)}V(y(t_0),\eta(t_0),z(t_0)) \quad \text{for all } t\ge t_0,\ t_0,\ t\in I.$$

Now it is straightforward to see that

$$V(y(t_0),\eta(t_0),z(t_0)) \le M_1\left\|\left(y(t_0),\dot{y}(t_0),z(t_0)^{\mathrm{T}}\right)^{\mathrm{T}}\right\|^2$$

where $M_1 := \max_{t\in I}\{\tfrac{1}{2} + r(t)^2(\tfrac{1}{4}a_3^2+1) + \tfrac{1}{2}\alpha\|P\|\}$ is, by (2.11), independent of $N(\cdot)$. Finally, taking into account the coordinate transformation used in Lemma 2.2, we obtain (2.8) and the proof is complete. □

3. Main result

The main result of this paper is that the simple feedback strategy

$$u(t) = -\ln k(t)\cos\sqrt{\ln k(t)}\,y(t), \qquad \dot{k}(t) = y(t)^2, \quad k(0) = k_0, \tag{3.1}$$

is a universal adaptive stabilizer for the class of systems (1.5).

First we show that $N(k) := \ln k\,\cos\sqrt{\ln k}$ does, in fact, satisfy (1.4).

3.1. Lemma.

$$N(k):[1,\infty)\to\mathbb{R},\quad k\mapsto \ln k\,\cos\sqrt{\ln k} \tag{3.2}$$

is a Nussbaum function.

Proof. We may assume that $k_0=1$. The substitution $\ln\sigma=\omega^2$, $\ln k:=\tau^2$ yields

$$\begin{aligned}\int_1^k N(\sigma)\,\mathrm{d}\sigma &= \int_0^\tau 2\omega^3\,\mathrm{e}^{\omega^2}\cos\omega\,\mathrm{d}\omega\\ &= \mathrm{e}^{1/4}\,\mathrm{Re}\int_0^\tau 2\omega^3\,\mathrm{e}^{(\omega+\mathrm{i}/2)^2}\mathrm{d}\omega\\ &= \mathrm{e}^{1/4}\,\mathrm{Re}\left[\int_0^\tau \omega^2 2(\omega+\tfrac{1}{2}\mathrm{i})\,\mathrm{e}^{(\omega+\mathrm{i}/2)^2}\mathrm{d}\omega-\mathrm{i}\int_0^\tau\omega^2\,\mathrm{e}^{(\omega+\mathrm{i}/2)^2}\,\mathrm{d}\omega\right],\end{aligned}$$

and integration by parts gives

$$\int_1^k N(\sigma)\,\mathrm{d}\sigma=\mathrm{e}^{1/4}\,\mathrm{Re}\Big[\big[\omega^2\mathrm{e}^{(\omega+\mathrm{i}/2)^2}\big]_0^\tau-\mathrm{i}I(\tau)\Big]$$

where

$$\begin{aligned}I(\tau)&:=\int_0^\tau(\omega^2-2\mathrm{i}\omega)\mathrm{e}^{(\omega+\mathrm{i}/2)^2}\,\mathrm{d}\omega\\ &=\tfrac{1}{2}\int_0^\tau\omega 2(\omega+\tfrac{1}{2}\mathrm{i})\,\mathrm{e}^{(\omega+\mathrm{i}/2)^2}\,\mathrm{d}\omega-\tfrac{5}{2}\int_0^\tau\mathrm{i}(\omega+\tfrac{1}{2}\mathrm{i})\,\mathrm{e}^{(\omega+\mathrm{i}/2)^2}\,\mathrm{d}\omega-\tfrac{5}{4}\int_0^\tau\mathrm{e}^{(\omega+\mathrm{i}/2)^2}\,\mathrm{d}\omega.\end{aligned}$$

Again, integration by parts yields

$$I(\tau)=\frac{1}{2}\big[\omega\,\mathrm{e}^{(\omega+\mathrm{i}/2)^2}\big]_0^\tau-\tfrac{7}{4}\int_0^\tau\mathrm{e}^{(\omega+\mathrm{i}/2)^2}\,\mathrm{d}\omega-\tfrac{5}{4}\mathrm{i}\big[\mathrm{e}^{(\omega+\mathrm{i}/2)^2}\big]_0^\tau.$$

Therefore, we obtain for $k>1$,

$$\begin{aligned}\frac{1}{k-1}\int_1^k N(\sigma)\,\mathrm{d}\sigma=\frac{k}{k-1}\Bigg[&(\ln k-\tfrac{5}{4})\cos\sqrt{\ln k}+\tfrac{1}{2}\sqrt{\ln k}\,\sin\sqrt{\ln k}\\ &+\frac{5}{4k}-\tfrac{7}{4}\int_0^{\sqrt{\ln k}}\mathrm{e}^{-(\ln k-\ln\sigma)}\sin\sqrt{\ln\sigma}\,\mathrm{d}\sigma\Bigg],\end{aligned}$$

which proves the lemma. □

3.2. Theorem (Universal Adaptive Stabilization). *The feedback strategy* (3.1) *applied to any system belonging to the class* (1.5) *yields for arbitrary* $x_0\in\mathbb{R}^n$, $k_0>1$, *a closed-loop system*

$$\begin{aligned}\dot{x}(t)&=\big[A-\ln k(t)\cos\sqrt{\ln k(t)}\,bc\big]x(t), & x(0)&\in\mathbb{R}^n,\\ \dot{k}(t)&=(cx(t))^2, & k(0)&=k_0,\end{aligned}\tag{3.3}$$

which satisfies the properties

(i) *the solution* x(t), k(t) *exists for all* $t\geq 0$,
(ii) $x(\cdot)\in L_2(0,\infty)\cap L_\infty(0,\infty)$,
(iii) $\lim_{t\to\infty}k(t)=k_\infty$ *exists and is finite*,
(iv) $\lim_{t\to\infty}x(t)=0$.

If (A, b, c) is of relative degree one, a proof is known and follows essentially from the Nussbaum property (1.4). However in the relative degree two case similar arguments are not applicable. Instead, we prove directly that an unbounded gain $k(\cdot)$ yields a contradiction as follows. The Nussbaum function $N(k(t))$ satisfies the assumptions (2.6) (2.7) for a sequence of intervals $I_1, I_2, \ldots$ and on each interval the state is, by virtue of Lemma 2.4, exponentially decaying. Due to, loosely speaking, the logarithm in the argument of the cosine, the intervals become larger and larger. Eventually it is possible to estimate crudely (ignoring that the state was decaying on certain intervals) the whole past of the state, and show that there exists an interval where $x(t)$ decays to 0 so that the integral of $y(t)^2$ converges.

Proof. Without loss of generality we may assume that (3.3) is of one of the special forms (2.3), (2.4).

(a) Since the right hand side of the closed-loop system (3.3) is continuous and locally Lipschitz in (x, k), existence and uniqueness of a solution $(x(t), k(t))$ on a maximal interval $[0, \omega)$, $\omega > 0$, follows from the classical theory of differential equations.

(b) Suppose $k(\cdot) \in L_\infty(0, \omega)$ and (2.1) is of relative degree 2. It follows from the classical theory of differential equations that $\omega = \infty$. Thus $y(\cdot) \in L_2(0, \infty)$, and from the second equation in (2.4) we derive, since $a_3 > 0$, $\dot{y}(\cdot) \in L_2(0, \infty)$. Using again (2.4) yields $(y(\cdot), z(\cdot)^T)^T, (\dot{y}(\cdot), \dot{z}(\cdot)^T)^T \in L_2(0, \infty)$, whence $\lim_{t\to\infty}(y(t), z(t)^T) = 0$. This proves the statements (i)–(iv) in case of boundedness of $k(\cdot)$ and for relative degree 2 systems.

The proof for relative degree 1 systems, which is omitted, follows by the same arguments by using the state space form (2.3).

(c) Suppose (A, b, c) is of relative degree 2, $k(\cdot) \notin L_\infty(0, \omega)$, and $N(k)$ is given as in (3.2). Let $M > 0$ be sufficiently large, depending on (A, b, c), so that the assumptions of Lemma 2.4 hold true. Since $N(k(\cdot)) \notin L_\infty(0, \omega)$, there exists a $p \in \mathbb{N}$ and sequence

$$t_{2p} < \hat{t}_{2p} < t_{2p+1} < t_{2(p+1)} < \hat{t}_{2(p+1)} < \hat{t}_{2(p+1)+1} < \cdots < \omega,$$

such that for all $j \geq p$ we have

$$\begin{aligned} &\ln k(t_{2j}) \cos\sqrt{\ln k(t_{2j})} = M, \qquad \sqrt{\ln k(t_{2j})} \geq (2j - \tfrac{1}{2})\pi \\ &\sqrt{\ln k(\hat{t}_{2j})} = (2j - \tfrac{1}{4})\pi, \qquad \sqrt{\ln k(t_{2j+1})} = 2j\pi. \end{aligned} \tag{3.4}$$

By definition of $k(\cdot)$ and Lemma 2.4, we have for all $j \geq p$,

$$k(t_{2j+1}) - k(t_{2j}) = \int_{t_{2j}}^{t_{2j+1}} y(s)^2 \, ds \leq \frac{\hat{M}}{\omega} \| x(t_{2j}) \|^2. \tag{3.5}$$

By definition of the systems class (1.5), there exists an $N^* \in \mathbb{R}$ such that $\sigma(A^*) \subset \mathbb{C}_-$ where $A^* := A - N^*bc$. The first equation in (2.1) can be rewritten as

$$\dot{x}(t) = A^*x(t) - [N(k(t)) - N^*]by(t), \quad x(0) \in \mathbb{R}^n$$

and since A^* is exponentially stable there exist $L, \lambda > 0$ such that

$$\| x(t) \| \leq L \, e^{-\lambda t} \| x(0) \| + L \| b \| [\, | N(k(\cdot)) |_{L_\infty(0,t)} + | N^* | \,] \int_0^t e^{-\lambda(t-s)} | y(s) | \, ds. \tag{3.6}$$

By using Hölder's inequality, we obtain

$$\int_0^t e^{-\lambda(t-s)} | y(s) | \, ds \leq \left(\int_0^t e^{-\lambda(t-s)} \, ds \right)^{1/2} \left(\int_0^t e^{-\lambda(t-s)} y(s)^2 \, ds \right)^{1/2} \leq \frac{1}{\sqrt{\lambda}} \left(\int_0^t y(s)^2 \, ds \right)^{1/2}. \tag{3.7}$$

Taking square roots in (3.5), and inserting (3.6) and (3.7) with $M_1 := \sqrt{\hat{M}/\omega\, L}\, \| x(0) \|$, $M_2 := \sqrt{\hat{M}/\omega\, L}\, \| b \| / \sqrt{\lambda}$ yields

$$\sqrt{k(t_{2j+1}) - k(t_{2j})} \le M_1 + M_2 \left[\, | N(k(\cdot)) |_{L_\infty(0,t_{2j})} + | N^* | \right] \sqrt{k(t_{2j}) - k(0)}$$

or equivalently

$$1 \le \frac{M_1}{\sqrt{k(t_{2j+1}) - k(t_{2j})}} + M_2 \left[\, | N(k(\cdot)) |_{L_\infty(0,t_{2j})} + | N^* | \right] \sqrt{\frac{k(t_{2j})}{k(t_{2j+1}) - k(t_{2j})}} . \tag{3.8}$$

In order to derive a contradiction, we show that the right hand side of (3.8) tends to 0 as $j \to \infty$. The first term on the right hand side of (3.8) tends to 0. We shall prove that the second term tends to 0 as well. Using the equations in (3.4) we have

$$\begin{aligned} \left[\, | N(k(\cdot)) |_{L_\infty(0,t_{2j})} + | N^* | \right]^2 \frac{k(t_{2j})}{k(t_{2j+1}) - k(t_{2j})} &< \left[4j^2\pi^2 + | N^* | \right]^2 \frac{k(\hat{t}_{2j})}{e^{4j^2\pi^2} - k(\hat{t}_{2j})} \\ &= \left[4j^2\pi^2 + | N^* | \right]^2 \frac{e^{(-j+1/16)\pi^2}}{1 - e^{(-j+1/16)\pi^2}} . \end{aligned}$$

The right hand side tends to 0 as $j \to \infty$, thus contradicting (3.8). Therefore, $k(\cdot) \in L_\infty(0, \omega)$.

(d) Suppose $cb \ne 0$ and $k(\cdot) \notin L_\infty(0, \omega)$. Let $P = P^{\mathrm{T}} \in \mathbb{R}^{(n-1)\times(n-1)}$ be the positive definite solution of

$$PA_{22} + A_{22}^{\mathrm{T}} P = -2 I_{n-1},$$

and define the Lyapunov-like function

$$V(y, z) := \tfrac{1}{2} y^2 + \tfrac{1}{2} \langle z, Pz \rangle .$$

Differentiation of $V(y(t), z(t))$, for $t \in [0, \omega)$, along the solution of (3.3) yields, with $N(k)$ defined as in (3.2),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V(y(t), z(t)) &\le (A_{11} - cbN(k(t))) y^2(t) \\ &\quad + \sqrt{2} \left(\| A_{12} \| + \| PA_{21} \| \right) | y(t) | \frac{1}{\sqrt{2}} \| z(t) \| - \| z(t) \|^2 \\ &\le (M_1 - cbN(k(t))) \dot{k}(t) - \tfrac{1}{2} \| z(t) \|^2 \end{aligned}$$

where $M_1 := A_{11} + (\| A_{12} \| + \| PA_{21} \|)^2$. Integration over $[0, t)$, $0 < t \le \omega$, so that $k(t) > k(0)$, and changing variables yields

$$\begin{aligned} V(y(t), z(t)) &\le V(y(0), z(0)) + \int_{k(0)}^{k(t)} (M_1 - cbN(\sigma)) \, \mathrm{d}\sigma \\ &= V(y(0), z(0)) + [k(t) - k(0)] \left[M_1 - \frac{cb}{k(t) - k(0)} \int_{k(0)}^{k(t)} N(\sigma) \, \mathrm{d}\sigma \right] . \end{aligned}$$

By the Nussbaum property (1.4) of $N(\cdot)$, the right hand side takes arbitrary large negative values, contradicting the positivity of the left hand side.

This completes the proof of the theorem. □

We complete this section with some remarks on the different Nussbaum functions: $k \cos\sqrt{k}$ known in the literature for relative degree one systems, and $\ln k \cos\sqrt{\ln k}$ used in Theorem 3.2.

3.3. Remarks. (i) If for the class of positive or negative high-gain stabilizable systems the sign of the high frequency gain is known to be positive, then $u(t) = -k(t)y(t)$, $\dot{k}(t) = y(t)^2$ is a universal adaptive controller. This was proved in [1,2,5] but also follows from the proof of Theorem 3.2. Part (a), (b), (d) go through without changes, (c) becomes simpler: There exists a t_{2j} so that the assumptions of Lemma 2.4 are satisfied. Then (3.8) holds true for arbitrary $t_{2j+1} > t_{2j}$. Fixing t_{2j} and choosing $k(t_{2j+1})$ large enough yields a contradiction.

(ii) In Theorem 3.2, the derivative of the Nussbaum function tends to 0 as $k \to \infty$, as opposed to diverging derivative for the standard Nussbaum function $N(k) = k \cos\sqrt{k}$. The different switching mechanism takes into account the delayed response of a relative degree 2 system, whose natural frequency of occillation is faster than the exponential decay. This feature is emphasized if one changes variables in the gain parameter. If $h := \ln(k)$, then

$$u(t) = -h(t) \cos\sqrt{h(t)}\, y(t), \quad \dot{h}(t) = e^{-h(t)} y(t)^2.$$

We believe, that the feedback strategy $u = k \cos\sqrt{k}\, y$, $\dot{k} = y^2$ is *not* a universal adaptive stabilizer for the class (1.5).

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