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Brief Paper

Universal λ-Tracking for Nonlinearly-Perturbed Systems in the Presence of Noise*

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Key Words—Adaptive control; feedback; nonlinear systems; robustness; servomechanisms; tracking systems; stabilization.

Abstract—For a class $\mathcal{S}$ of multivariable, nonlinearly-perturbed, linear, minimum-phase systems of relative degree one, with output measurement noise of Sobolev class $W^{1,\infty}$ (absolutely continuous on compact intervals, and bounded with essentially bounded derivative), the following servomechanism problem is solved. Determine a $\mathcal{S}$-universal adaptive strategy to control the output to track any reference signal in $W^{1,\infty}$, with tracking error asymptotic to a ball of arbitrary prescribed radius $\lambda > 0$. The control strategy is simple and does not invoke an internal model principle.

Nomenclature
$\|x\|_P := \sqrt{\langle x, Px\rangle}$ for $x \in \mathbb{R}^N$, $P = P^T \in \mathbb{R}^{N\times N}$ positive-definite, $\|x\|_I \equiv \|x\|$;
For $\Lambda, \lambda \ge 0$,

$$D_\Lambda(x) := \begin{cases} \|x\|_P - \Lambda, & \text{if } \|x\|_P \ge \Lambda \\ 0, & \text{if } \|x\|_P < \Lambda \end{cases}$$

and

$$d_\lambda(x) := \begin{cases} \|x\| - \lambda, & \text{if } \|x\| \ge \lambda \\ 0, & \text{if } \|x\| < \lambda \end{cases}$$

$\mathbb{C}_+(\mathbb{C}_-)$, open right-half (left-half) complex plane;
$\sigma(A)$, the spectrum of the matrix $A \in \mathbb{R}^{N\times N}$;
$L^\infty(J, \mathbb{R}^N)$, the vector space of measurable functions $f: J \to \mathbb{R}^N$, $J \subset \mathbb{R}$ an interval, with $\|f(\cdot)\|_\infty := \operatorname{ess\,sup}_{t\in J} \|f(t)\| < \infty$;
$W^{1,\infty}(\mathbb{R}, \mathbb{R}^M)$, the Sobolev space of functions $\xi: \mathbb{R} \to \mathbb{R}^M$ which are absolutely continuous on compact intervals and $\xi, \dot\xi \in L^\infty(\mathbb{R}, \mathbb{R}^M)$, $W^{1,\infty}(\mathbb{R}, \mathbb{R}) \equiv W^{1,\infty}(\mathbb{R})$;
$\|\xi\|_{1,\infty} := \|\xi\|_\infty + \|\dot\xi\|_\infty$, $\xi \in W^{1,\infty}(\mathbb{R}, \mathbb{R}^M)$.

1. *Introduction*

THE SEMINAL WORKS of Byrnes, Mareels, Mårtensson, Morse, Nussbaum, Willems, and others in the mid 1980's, initiated numerous theoretical advances in the area of *universal stabilization* [see Ilchmann, 1991 for a survey and bibliography]. A (valid) practical criticism of many universal stabilizers is their lack of robustness with respect to noise: specifically, the crucial property of bounded control gain can be lost in the presence of output measurement noise because, loosely speaking, the differential equation generating the monotone control gain is then driven by the variance of the noise signal. Here, we develop bounded-gain adaptive controllers which are robust with respect to a certain class of measurement noise signals. The price we pay is the following: we cannot guarantee that the output tends asymptotically to zero; instead, we guarantee output behaviour asymptotic to a ball of *arbitrarily small prespecified radius* $\lambda > 0$. However, this enables us to subsume the stabilization problem in one of tracking with a much larger class of reference signals than those allied to approaches based on an internal model principle, and allow for system nonlinearities of a fairly general nature.

In particular, we consider a class $\mathcal{S}$ of nonlinearly-perturbed multivariable linear systems, denoted by $\Sigma = (A, B, C, f, g) \in \mathcal{S}$, with output corrupted by noise $n(\cdot) \in \mathcal{N}$ and reference signal to be tracked $y_{\text{ref}}(\cdot) \in \mathcal{R}$:

$$\begin{aligned} \dot x(t) &= Ax(t) + B[u(t) + f(t, x(t))] + g(t, x(t)), \\ &\qquad\qquad x(t) \in \mathbb{R}^N, \quad u(t) \in \mathbb{R}^M, \\ x(t_0) &= x^0, \\ y(t) &= Cx(t) + n(t), \quad y(t) \in \mathbb{R}^M, \quad n(\cdot) \in \mathcal{N}, \\ e(t) &= y(t) - y_{\text{ref}}(t), \quad y_{\text{ref}}(\cdot) \in \mathcal{R}. \end{aligned} \tag{1}$$

Assumptions 1–4 below implicitly determine the class $\mathcal{S}$. The signals $e(\cdot)$ and $y_{\text{ref}}(\cdot)$, only, are available for feedback. We assume

$$\mathcal{N} = W^{1,\infty}(\mathbb{R}, \mathbb{R}^M) = \mathcal{R},$$

that is, both noise and reference signals are assumed to belong to the Sobelev space $W^{1,\infty}$ of *bounded* functions $\mathbb{R} \to \mathbb{R}^M$ that are absolutely continuous on compact intervals and have *essentially bounded derivative*. We equip this space with the norm

$$\|\xi\|_{1,\infty} := \|\xi\|_\infty + \|\dot\xi\|_\infty,$$

where $\|\ \|_\infty$ denotes the L^∞ norm.

Although the noise and reference signals are of the same class, our tacit physical assumption is that the former signals have a high-frequency content while the latter are of a low-frequency nature: note that, whilst these signals are assumed bounded with essentially bounded derivatives, no *a priori* bounds are imposed and so the formulation encompasses noise of finite bandwidth but with arbitrarily large cutoff frequency (see Fig. 5 for a realization of noise of class $W^{1,\infty}$).

We construct a *universal adaptive λ-regulator*

$$u(t) = U_\lambda(k(t), e(t), y_{\text{ref}}(t)), \quad \dot k(t) = K_\lambda(e(t), y_{\text{ref}}(t)), \quad k(t_0) = k^0 \in \mathbb{R} \tag{2}$$

which solves the following servomechanism problem. For any fixed $\lambda > 0$, the output feedback strategy (2) applied to any system $\Sigma \in \mathcal{S}$ has the properties: for each $(t_0, x^0, k^0) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$, every solution $(x(\cdot), k(\cdot))$ of the initial-value

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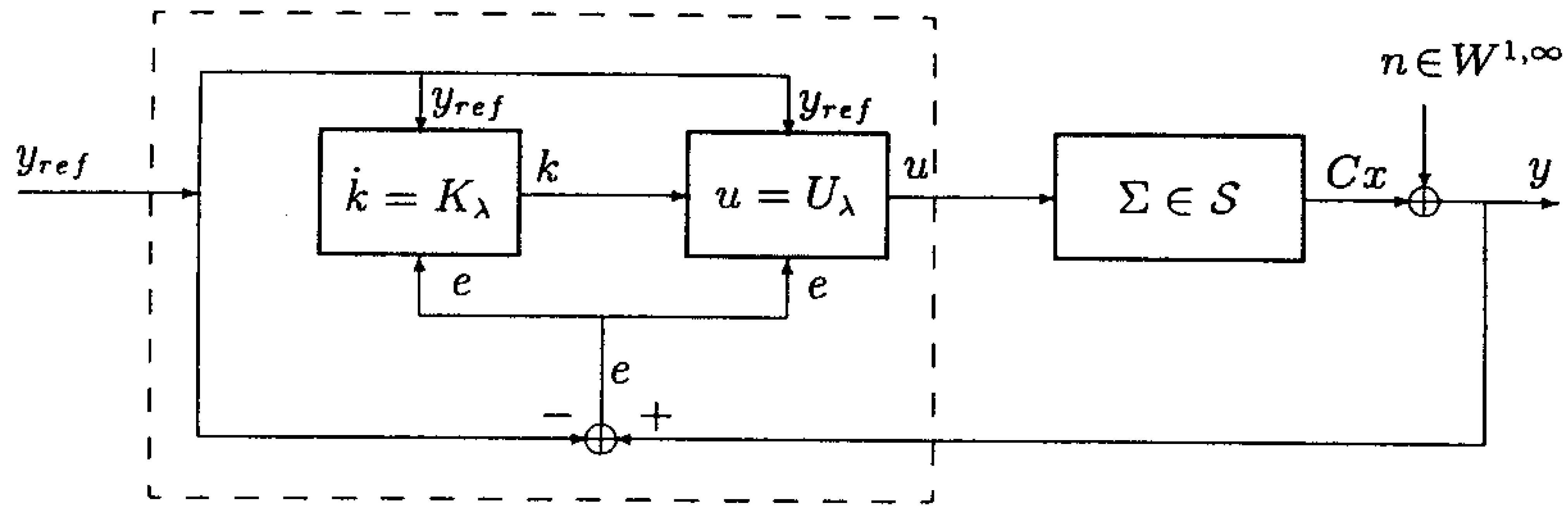

FIG. 1. Closed-loop system.

problem (1–2) is bounded on its maximal interval of existence $[t_0, \infty)$ and the output $y(\cdot)$ λ-tracks the reference signal $y_{ref}(\cdot)$ in the sense that the error $e(t) = y(t) - y_{ref}(t)$ tends, as $t \to \infty$, to the closed ball $\bar{B}_\lambda(0)$ of radius λ centred at $0 \in \mathbb{R}^M$ (Fig. 1).

In view of the interpretation of $n(\cdot)$ as measurement noise, the *true* underlying tracking objective is that of causing the *uncorrupted* output $Cx(\cdot)$ to track the reference $y_{ref}(\cdot)$. In effect, the *actual* objective attained by our strategy is to cause $Cx(\cdot)$ to λ-track the signal $y_{ref}(\cdot) - n(\cdot)$; in this context, if an *a priori* bound on the noise magnitude, $\hat{n} \geq \|n\|_\infty$, is available, then a controller parameter value $\lambda > \hat{n}$ should be employed.

The class $\mathcal{S}$ of systems $\Sigma = (A, B, C, f, g)$ is implicitly defined through Assumptions 1–4 below.

Assumption 1. The unperturbed linear system $(A, B, C, 0, 0)$ satisfies the minimum phase condition:

$$\det\begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} \neq 0 \quad \forall s \in \bar{\mathbb{C}}_+$$

that is, the linear system is stabilizable, detectable and has no zeros in the closed right-half complex plane $\bar{\mathbb{C}}_+$.

Assumption 2. *CB is such that one of the following holds*:
(i) $\sigma(CB)$ is known *a priori* to lie in $\mathbb{C}_+$.
(ii) $\sigma(CB)$ is known *a priori* to lie in $\mathbb{C}_-$.
(iii) A positive definite $P = P^T \in \mathbb{R}^{M \times M}$ is known *a priori* such that $PCB + (CB)^T P$ is sign definite (but of unknown sign).

Remarks. The number M of inputs and outputs is fixed but the state dimension $N \geq M$ is not prescribed. Assumption 2(i–ii) is a multivariable counterpart of knowledge of the sign of the non-zero high-frequency gain in the single-input, single-output case $M = 1$; Assumption 2(iii) is a partial generalization, to the multivariable case, of the weaker single-variable condition that the high-frequency gain is non-zero (but of unknown sign). A more satisfactory multivariable extension of the latter scalar condition would be an assumption that $\sigma(CB)$ lies in one or the other of the half planes $\mathbb{C}_+$ and $\mathbb{C}_-$ (but which half plane actually contains the spectrum is unknown): however, we cannot handle this case.

Some structure is imposed on the nonlinearities f and g.

Assumption 3. $g: \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ has the properties:
(i) for each $x \in \mathbb{R}^N$, the function $g(\cdot, x)$ is Lebesgue measurable;
(ii) for almost all $t \in \mathbb{R}$, the function $g(t, \cdot)$ is continuous;
(iii) for some unknown scalar γ, $\|g(t, x)\| \leq \gamma[1 + \|Cx\|]$ for all x and almost all t.

Assumption 4. $f: \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^M$ has the properties:
(i) for each $x \in \mathbb{R}^N$, the function $f(\cdot, x)$ is Lebesgue measurable;
(ii) for almost all $t \in \mathbb{R}$, the function $f(t, \cdot)$ is continuous;
(iii) for some unknown scalar α and function $\hat{f}: \mathbb{R}^M \to [0, \infty)$

$$\|f(t, x)\| \leq \alpha[\|x\| + \hat{f}(Cx)] \quad \text{for all } x \text{ and almost all } t;$$

(iv) there exists a known continuous function $\bar{f}: \mathbb{R}^M \to [0, \infty)$ with the property that, for every $R > 0$, there is an unknown scalar α_R such that, for all $\xi \in \mathbb{R}^M$,

$$\hat{f}(\xi + v) \leq \alpha_R \bar{f}(\xi) \quad \text{for all } v \in \mathbb{R}^M \text{ with } \|v\| \leq R.$$

Remarks. Assumptions 3 and 4 ensure that f and g are Carathéodory functions, and so the classical existence theory can be brought to bear on the initial-value problem (1–2); we do not require uniqueness of solutions. The function g is uniformly bounded: the bound γ can be arbitrarily large. The function f is assumed bounded, modulo an unknown scalar multiplier α, by the map $x \mapsto \|x\| + \hat{f}(Cx)$. The function $\hat{f}$ (a function of the noise-free output variables) is, in turn, bounded in the particular sense that, for each $R > 0$, there exists α_R such that $\max_{\|v\| \leq R} \hat{f}(\xi + v) \leq \alpha_R \bar{f}(\xi)$ for all $\xi \in \mathbb{R}^M$, where $\bar{f}$ is known and continuous. For example, suppose that f has polynomial dependence (of degree not exceeding d, say) on the components of Cx with t-dependent coefficients of L^∞ class, then Assumption 4 (iii) and (iv) hold with $\hat{f} = \bar{f}: \xi \mapsto 1 + \|\xi\|^d$. We stress that the only *a priori* information on f and g available to the controller is the continuous function $\bar{f}$. In particular, the scalars γ, α and α_R need not be known.

The adaptive λ-regulator we are about to construct has a simple form. For example, if we specialize to the class of *linear*, single-input, single-output systems $(A, B, C, 0, 0) \in \mathcal{S}$ with positive high-frequency gain $CB > 0$, then the feedback regulator (2), for fixed $\lambda > 0$, has the form

$$\begin{aligned} u(t) &= -k(t)e(t), \\ \dot{k}(t) &= \begin{cases} (|e(t)| - \lambda)\,|e(t)|, & \text{if } |e(t)| \geq \lambda \\ 0, & \text{if } |e(t)| < \lambda \end{cases}, \\ k(t_0) &= k^0 \in \mathbb{R}. \end{aligned} \tag{3}$$

The intuition underlying the strategy is clear: the adaptation mechanism is switched off whenever the error is acceptably small (quantified by the controller parameter $\lambda > 0$). If it is known only that $CB \neq 0$, then the feedback law (3) should be modified to

$$u(t) = \nu(k(t))e(t), \tag{4}$$

where $\nu(\cdot): \mathbb{R} \to \mathbb{R}$ is any continuous function of *Nussbaum type* (Nussbaum, 1983), that is, ν has the properties

$$\limsup_{k \to \infty} \frac{1}{k} \int_0^k \nu(s)\,ds = +\infty \quad \text{and} \quad \liminf_{k \to \infty} \frac{1}{k} \int_0^k \nu(s)\,ds = -\infty. \tag{5}$$

For example, $\nu: \tau \mapsto \tau^2 \cos \tau$ suffices. For linear systems $(A, B, C, 0, 0) \in \mathcal{S}$ without noise $(n(\cdot) \equiv 0)$ and with reference signal $y_{ref}(\cdot) \equiv 0$, the problem of universal adaptive asymptotic stabilization has been solved by Willems and Byrnes (1984) and Byrnes and Willems (1984). Interpreted in our context, the latter problem is equivalent to finding a universal adaptive $(\lambda = 0)$-regulator: while we present our analysis for $\lambda > 0$, we remark that the proofs in the present paper can be modified to recover the adaptive stabilization results for linear systems achieved by Byrnes and Willems in the case $\lambda = 0$. To our knowledge, no previous results are available which extend the Byrnes–Willems adaptive stabilizer in the presence of noise.

The universal adaptive asymptotic *tracking* problem $(\lambda = 0)$ for linear systems, in the noise-free case, has been solved for a class of reference signals which correspond to solutions of known *linear* differential equations, see Mareels (1984), Helmke *et al.* (1990), Miller and Davison (1991a), Townley and Owens (1991). All of these results use continuous feedbacks and invoke the *Internal Model Principle*: 'a regulator is structurally stable only if the controller utilizes feedback of the regulated variable, and incorporates in the feedback loop a suitably reduplicated model of the dynamic structure of the exogenous signals which the regulator is required to process', see Wonham (1979).

Adaptive tracking, without using an internal model and thereby allowing a much larger class of reference signals, is feasible (a) if the class of feedbacks is widened to allow discontinuous controls (as in Ryan, 1992), or (b) if asymptotic tracking is replaced by the weaker requirement of λ-tracking, as in the present paper. Here, we do not use any identification mechanisms or probing signals: the approach is in the spirit of Willems and Byrnes, the feedback depends continuously on the error and the reference signal, the gain is a monotonic nondecreasing differentiable function.

Furthermore, the approach of this paper differs from the many existing adaptive strategies, which use switching gain controllers with stepwise constant gain (see, for example, Mårtensson (1986) and Miller and Davison (1988, 1989)). Based on a stepwise constant gain-tuning algorithm, Miller and Davison (1991) introduced an adaptive λ-tracking regulator which is applicable to the class of systems $\mathcal{S}$ considered in the present paper *when restricted to single-input, single-output systems.* For this sub-class of $\mathcal{S}$, their results are sharper in the sense that the regulator achieves prescribed tracking error within a prespecified time, and with prescribed overshoot bound. In comparison, the regulator presented in the present paper is simpler, allows for large class of nonlinear perturbations, and is applicable to multivariable systems.

The paper is organized as follows. In Section 2, some basic properties of the system class are established. In Section 3, we introduce a simple λ-tracking regulator for single-input, single-output, linear, disturbance-free systems with positive high-frequency gain. Although this result is subsumed by the more general results in Sections 4 and 5, we provide a separate proof in the simple linear case in order to illustrate the analytical structure of the general proof (which otherwise is somewhat obscured by technicalities). In Section 4, the general problem is solved for those systems of class $\mathcal{S}$ which satisfy Assumption 2 (i) or (ii); systems satisfying Assumption 2 (iii) are treated in Section 5. A two-dimensional example is presented in Section 6, wherein an uncertain system with output noise is required to λ-track a 'chaotic' reference signal generated by a Lorenz system of nonlinear differential equations: numerical simulations confirm the efficacy of the strategy.

2. *Properties of the system class*

In this section we derive some basic properties of the class $\mathcal{S}$ of systems (1) satisfying Assumptions 1–4.

By Assumption 2, CB is invertible. Now let the linear map $L:\mathbb{R}^N \to \mathbb{R}^{N-M}$ be such that $\ker L = \operatorname{im} B$. It follows that the transformation $T:\mathbb{R}^N \to \mathbb{R}^N$, $x \mapsto (w, z) = (Cx, Lx)$ is invertible. Under this coordinate transformation and writing $e(t) = y(t) - y_{\mathrm{ref}}(t)$ (the output tracking error), system (1) can be expressed as

$$\left.\begin{aligned} \dot{e}(t) &= A_1 e(t) + A_2 z(t) + CB[u(t) + f_1(t, e(t), z(t))] \\ &\quad + g_1(t, e(t), z(t)), \\ \dot{z}(t) &= A_3 e(t) + A_4 z(t) + g_2(t, e(t), z(t)), \\ (e(t_0), z(t_0)) &= (e^0, z^0), \end{aligned}\right\} \quad (6)$$

where

$$f_1:(t, e, z) \mapsto f(t, T^{-1}(e - n(t) + y_{\mathrm{ref}}(t), z)),$$
$$g_1:(t, e, z) \mapsto Cg(t, T^{-1}(e - n(t) + y_{\mathrm{ref}}(t), z)) + \dot{n}(t) - \dot{y}_{\mathrm{ref}}(t) - A_1[n(t) - y_{\mathrm{ref}}(t)],$$
$$g_2:(t, e, z) \mapsto Lg(t, T^{-1}(e - n(t) + y_{\mathrm{ref}}(t), z)) - A_3[n(t) - y_{\mathrm{ref}}(t)].$$

By Assumption 3 and since $n(\cdot), y_{\mathrm{ref}}(\cdot) \in W^{1,\infty}(\mathbb{R}, \mathbb{R}^M)$, we have, for almost all t,

$$\|g_1(t, e, z)\| \le \gamma_1[1 + \|e\|], \quad \|g_2(t, e, z)\| \le \gamma_2[1 + \|e\|],$$

with

$$\gamma_1 := \gamma\, \|C\|\, [1 + \|y_{\mathrm{ref}}\|_\infty + \|n\|_\infty] + [1 + \|A_1\|][\|y_{\mathrm{ref}}\|_{1,\infty} + \|n\|_{1,\infty}]$$

and

$$\gamma_2 := \gamma\, \|L\|\, [1 + \|y_{\mathrm{ref}}\|_\infty + \|n\|_\infty] + \|A_3\|\, [\|y_{\mathrm{ref}}\|_\infty + \|n\|_\infty].$$

We now state three preliminary results (the proofs are contained in the Appendix), the first of which is a direct consequence of the minimum phase condition in Assumption 1.

Proposition 1. $\sigma(A_4) \subset \mathbb{C}_-$, that is, A_4 is asymptotically stable.

Lemma 1. Let $(\tau, z^\tau) \in \mathbb{R} \times \mathbb{R}^{N-M}$, $\omega > \tau$ and $I = [\tau, \omega)$. For every pair of locally integrable functions

$$\theta: I \to \mathbb{R}^M, \quad h: I \to \mathbb{R}^{N-M},$$

satisfying $\|h(t)\| \le \hat{h}[1 + \|\theta(t)\|]$ a.e., for some $\hat{h} \ge 0$,

there exists scalar c such that the solution $z(\cdot): I \to \mathbb{R}^{N-M}$ of the initial-value problem

$$\dot{z}(t) = A_4 z(t) + A_3\theta(t) + h(t), \quad z(\tau) = z^\tau, \quad \sigma(A_4) \subset \mathbb{C}_-$$

satisfies

$$\int_\tau^t \|\theta(s)\|\, \|z(s)\|\, ds \le c \int_\tau^t [\|\theta(s)\| + \|\theta(s)\|^2]\, ds \quad \forall t \in I.$$

The final lemma is a technicality which we require in later proofs.

Let $P \in \mathbb{R}^{M \times M}$ be positive-definite and symmetric. Define distance functions D_Λ, $d_\lambda: \mathbb{R}^M \to [0, \infty)$ (parameterized by $\Lambda, \lambda \ge 0$, respectively) by

$$D_\Lambda(\xi) := \begin{cases} \|\xi\|_P - \Lambda, & \text{if } \|\xi\|_P \ge \Lambda \\ 0, & \text{if } \|\xi\|_P < \Lambda \end{cases}$$

and

$$d_\lambda(\xi) := \begin{cases} \|\xi\| - \lambda, & \text{if } \|\xi\| \ge \lambda \\ 0, & \text{if } \|\xi\| < \lambda. \end{cases} \quad (7)$$

Writing $p = \sqrt{\sigma_{\min}(P)}$ and $q = \sqrt{\sigma_{\max}(P)} = \|P\|^{1/2}$, where $\sigma_{\min}(P)$ (respectively, $\sigma_{\max}(P)$) denotes the minimum (respectively, maximum) eigenvalue of P, we have the following:

Lemma 2. For all $\xi \in \mathbb{R}^M$, (i) $D_{p\lambda}(\xi) \ge p d_\lambda(\xi)$ and (ii) $D_{q\lambda}(\xi) \le q d_\lambda(\xi)$.

3. *The linear case*

In this section, we shall prove that the simple adaptive strategy (3) is a universal adaptive λ-tracking controller for a class of linear systems (a subclass of $\mathcal{S}$). Although this result is subsumed by the general results of subsequent sections, we present a separate proof here, which is free of the many technicalities intrinsic to the general proof, in order to illustrate the underlying structure of the general argument.

Specifically, for our immediate purpose we restrict attention to the class of linear, single-input, single-output, minimum-phase systems with positive high-frequency gain, and noise-corrupted output:

$$\left.\begin{aligned} &\dot{x}(t) = Ax(t) + bu(t), \quad x(t_0) = x^0 \\ &y(t) = cx(t) + n(t) \\ &e(t) = y(t) - y_{\mathrm{ref}}(t) \\ &n(\cdot), y_{\mathrm{ref}}(\cdot) \in W^{1,\infty}(\mathbb{R}, \mathbb{R}) \\ &(A, b, c) \in \mathbb{R}^{N \times N} \times \mathbb{R}^N \times \mathbb{R}^{1 \times N} \text{ minimum phase}, \\ &\qquad N \in \mathbb{N}, \quad cb > 0. \end{aligned}\right\} \quad (8)$$

The proposed adaptive strategy is

$$u(t) = -k(t)e(t), \quad \dot{k}(t) = d_\lambda(e(t))\,|e(t)|, \quad k(t_0) = k^0, \quad (9)$$

where $d_\lambda(\cdot)$ is defined in (7). By the classical theory of ordinary differential equations, for every $(t_0, x^0, k^0) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$, every output-corrupting noise signal $n \in W^{1,\infty}(\mathbb{R})$, and every reference signal $y_{\text{ref}} \in W^{1,\infty}(\mathbb{R})$, the initial-value problem defined by (8, 9) has a unique maximal solution, that is, there exists a function $(x(\cdot), k(\cdot)): I \to \mathbb{R}^{N+1}$, with $(x(t_0), k(t_0)) = (x^0, k^0)$ and maximally extended over $I = [t_0, \omega)$ (the maximal interval of existence), which is absolutely continuous on compact subintervals and which satisfies the differential equations in (8 and 9) almost everywhere.

Theorem 1. Fix $\lambda > 0$ and let $(x, k): [t_0, \omega) \to \mathbb{R}^{N+1}$ be the maximal solution of the initial-value problem (8, 9). Then

(i) $\omega = \infty$;

(ii) $(x(\cdot), k(\cdot))$ is bounded;

(iii) $\lim_{t\to\infty} k(t)$ exists and is finite;

(iv) The error $e(\cdot)$ approaches the closed interval $[-\lambda, +\lambda] \subset \mathbb{R}$ as $t \to \infty$.

Proof. In terms of the transformed system representation (6), the closed-loop system (8) and (9) is equivalent to

$$\left.\begin{aligned} &\dot{e}(t) = [A_1 - k(t)cb]e(t) + A_2 z(t) + h_1(t), \\ &(e(t_0), z(t_0)) = (Cx^0 + n(t_0) - y_{\text{ref}}(t_0), Lx^0) \\ &\dot{z}(t) = A_3 e(t) + A_4 z(t) + h_2(t), \\ &\dot{k}(t) = d_\lambda(e(t))\,|e(t)|, \quad k(t_0) = k^0, \end{aligned}\right\} \quad (10)$$

where

$$h_1(\cdot) := \dot{n}(\cdot) - \dot{y}_{\text{ref}}(\cdot) - A_1[n(\cdot) - y_{\text{ref}}(\cdot)] \in L^\infty(\mathbb{R}, \mathbb{R}),$$
$$h_2(\cdot) := -A_3[n(\cdot) - y_{\text{ref}}(\cdot)] \in L^\infty(\mathbb{R}, \mathbb{R}^{N-1}).$$

Differentiation of the following C^1 function (a Lyapunov-like candidate)

$$V_\lambda(e) := \tfrac{1}{2}d_\lambda^2(e) = \begin{cases} \tfrac{1}{2}(|e| - \lambda)^2, & \text{if} \quad |e| \geq \lambda \\ 0, & \text{if} \quad |e| < \lambda \end{cases}$$

along the solution component $e(\cdot) = y(\cdot) - y_{\text{ref}}(\cdot)$ yields,

$$\frac{d}{dt}V_\lambda(e(t)) = \theta(t)\dot{e}(t) \quad \text{for almost all } t \in [t_0, \omega),$$

where for notational convenience, we have introduced the continuous function

$$\theta: t \mapsto \theta(t) := \begin{cases} \dfrac{|e(t)| - \lambda}{|e(t)|}e(t), & \text{if} \quad |e(t)| \geq \lambda \\ 0, & \text{if} \quad |e(t)| < \lambda. \end{cases}$$

Observe that

$$d_\lambda(e(t)) = |\theta(t)| \leq \lambda^{-1}|\theta(t)|\,|e(t)|$$

and

$$\theta(t)e(t) = |\theta(t)|\,|e(t)| = \dot{k}(t).$$

Therefore, using the first and third differential equations in (10), we have

$$\begin{aligned} \theta(t)\dot{e}(t) &\leq [A_1 - k(t)cb]\,|\theta(t)|\,|e(t)| + |\theta(t)|\,\|h_1(\cdot)\|_\infty \\ &\quad + \|A_2\|\,|\theta(t)|\,\|z(t)\| \\ &\leq [M_2 - k(t)cb]d_\lambda(e(t))\,|e(t)| + \|A_2\|\,|\theta(t)|\,\|z(t)\| \\ &\quad \text{for almost all } t \in [t_0, \omega), \end{aligned}$$

where $M_2 := A_1 + \lambda^{-1}\|h_1(\cdot)\|_\infty$. By equation (9), we now conclude

$$\frac{d}{dt}V_\lambda(e(t)) \leq [M_2 - k(t)cb]\dot{k}(t) + \|A_2\|\,|\theta(t)|\,\|z(t)\|$$
$$\text{for almost all } t \in [t_0, \omega). \quad (11)$$

Now,

$$\begin{aligned} |\theta(t)| + \theta^2(t) &\leq [1 + \lambda^{-1}]\,|\theta(t)|\,|e(t)| \\ &\leq [1 + \lambda^{-1}]d_\lambda(e(t))\,|e(t)| \quad \text{for all } t \in [t_0, \omega). \end{aligned} \quad (12)$$

Defining

$$h(\cdot) := A_3(e(\cdot) - \theta(\cdot)) + h_2(\cdot),$$

then the second differential equation in (10) can be rearranged as

$$\dot{z}(t) = A_4 z(t) + A_3\theta(t) + h(t).$$

It is readily verified that $h(\cdot) \in L^\infty$ and so, an application of Lemma 1, together with equations (9) and (12), yields, for some scalar $c > 0$,

$$\begin{aligned} \int_{t_0}^{t} |\theta(s)|\,\|z(s)\|\,ds &\leq c\int_{t_0}^{t}[|\theta(s)| + \theta^2(s)]\,ds \\ &\leq c[1 + \lambda^{-1}][k(t) - k(t_0)] \end{aligned}$$

which is valid for all $t \in [t_0, \omega)$. Integration of equation (11) therefore gives

$$\begin{aligned} 0 \leq V_\lambda(e(t)) &\leq V_\lambda(e(t_0)) + \int_{t_0}^{t}[M_2 - k(s)cb]\dot{k}(s)\,ds \\ &\quad + \|A_2\|\,M_3[k(t) - k(t_0)] \\ &= V_\lambda(e(t_0)) + \int_{k(t_0)}^{k(t)}[M_2 - \tau cb]\,d\tau \\ &\quad + \|A_2\|\,M_3[k(t) - k(t_0)], \end{aligned} \quad (13)$$

where $M_3 = c[1 + \lambda^{-1}]$. If the monotone function $k(\cdot) \notin L^\infty(t_0, \omega)$, then there exists $\tau \in [t_0, \omega)$ such that the right-hand side of equation (13) is negative for all $t \in [\tau, \omega)$, a contradiction. Therefore $k(\cdot)$ is bounded, which, by (13), implies boundedness of $V_\lambda(e(\cdot))$, and hence $e(\cdot) \in L^\infty(t_0, \omega)$. By (10) and since A_4 is stable, we may conclude boundedness of $z(\cdot)$. We have now shown that the solution $(x(\cdot), k(\cdot))$ is bounded, and so $\omega = \infty$. This establishes assertions (i)–(ii). Assertion (iii) follows by boundedness and monotonicity of $k(\cdot)$.

It remains to prove (iv). Note that

$$\begin{aligned} |\theta(t)|\,\|z(t)\| &\leq \lambda^{-1}|\theta(t)|\,|e(t)|\,\|z(t)\| \\ &= \lambda^{-1}d_\lambda(e(t))\,|e(t)|\,\|z(t)\| \end{aligned}$$

and therefore (9) together with boundedness of $z(\cdot)$ yields, for some $M_4 > 0$,

$$|\theta(t)|\,\|z(t)\| \leq M_4\dot{k}(t) \quad \text{for almost all } t \geq t_0.$$

Defining $M_5 := M_2 - k^0cb + \|A_2\|\,M_4$, then, by (11),

$$\frac{d}{dt}V_\lambda(e(t)) \leq -\dot{k}(t) + [M_5 + 1]\dot{k}(t).$$

Therefore the (sign-indefinite) function

$$W: (e, k) \mapsto V_\lambda(e) - (M_5 + 1)k$$

is such that the following holds for almost all $t \geq t_0$

$$\frac{d}{dt}W(e(t), k(t)) \leq -\dot{k}(t) = -d_\lambda(e(t))\,|e(t)| \leq 0. \quad (14)$$

By boundedness of $(e(\cdot), z(\cdot), k(\cdot))$ and essential boundedness of $h_1(\cdot)$, there exists $R > 0$ such that

$$|\dot{e}(t)| < R \quad \text{for almost all } t \geq t_0.$$

Moreover, the bounded solution $t \mapsto (e(t), z(t), k(t))$ must tend, as $t \to \infty$, to its non-empty ω-limit set Ω. Hence, we will prove assertion (iv) (that is, $d_\lambda(e(t)) \to 0$ as $t \to \infty$) by showing that Ω is contained in the closed set $\Sigma = \{(e, z, k) \in \mathbb{R}^{N+1}\,|d_\lambda(e)\,|e| = 0\}$. Seeking a contradiction, suppose that $\Omega \not\subset \Sigma$. Then there exists $(\bar{e}, \bar{z}, \bar{k}) \in \Omega$ and $\epsilon > 0$ such that $d_\lambda(\bar{e})\,|\bar{e}| > 2\epsilon$. By continuity of $\xi \mapsto d_\lambda(\xi)\,|\xi|$, there exists $\delta > 0$ such that

$$|\xi - \bar{e}| < \delta \Rightarrow d_\lambda(\xi)\,|\xi| > \epsilon.$$

Since $(\bar{e}, \bar{z}, \bar{k})$ is an ω-limit point, there exists a sequence (t_j) with $t_j \to \infty$ and

$$(e(t_j), z(t_j), k(t_j)) \to (\bar{e}, \bar{z}, \bar{k})$$

as $j \to \infty$. By continuity of W,

$$W(e(t_j), k(t_j)) - W(\bar{e}, \bar{k}) < \frac{\epsilon\delta}{4R} \qquad (15)$$

for all j sufficiently large. Let j^* be such that $|e(t_j) - \bar{e}| < \frac{1}{2}\delta$ for all $j > j^*$. Observe that, for all $j > j^*$ and all $t \geq t_j$,

$$|e(t) - \bar{e}| \leq |e(t) - e(t_j)| + |e(t_j) - \bar{e}|$$
$$\leq \int_{t_j}^{t} |\dot{e}(s)|\,ds + |e(t_j) - \bar{e}| \leq R\,|t - t_j| + \tfrac{1}{2}\delta.$$

Therefore, for all $j > j^*$,

$$t \in [t_j, t_j + (\delta/3R)] \Rightarrow |e(t) - \bar{e}| < \delta \Rightarrow d_\lambda(e(t))\,|e(t)| > \epsilon.$$

By (14) we now have, for all $j > j^*$,

$$W(e(t_j), k(t_j)) - W(\bar{e}, \bar{k}) \geq \int_{t_j}^{t_j + (\delta/3R)} d_\lambda(e(t))\,|e(t)|\,dt \geq \frac{\epsilon\delta}{3R}$$

which contradicts (15). Therefore, $\Omega \subset \Sigma$ and so $d_\lambda(e(t)) \to 0$ as $t \to \infty$. Q.E.D.

Remarks.

(i) The above proof differs in an essential way from the 'standard' proof of the Byrnes–Willems result that $u = -ky$, $\dot{k} = y^2$ is a universal adaptive asymptotic stabilizer in the noise-free case: the latter proof is based on L^2-type arguments which do not extend in any obvious manner to our setting.

(ii) The extension of Theorem 1 to the unknown-sign case $cb \neq 0$ is straightforward by using (4). This extended result is simply a special case of the general result presented in Theorem 3 below.

4. *General case I: sign-definite spectrum $\sigma(CB) \subset \mathbb{C}_+$*

We turn to the general case with either Assumption 2(i) or 2(ii) in force. Without loss of generality, we impose Assumption 2(i). (If Assumption 2(ii) holds, then we simply change the sign of the control variable.) Then, there exists a unique, symmetric, positive-definite P (unknown to the controller) such that

$$PCB + (CB)^T P - 2I = 0.$$

For $\lambda > 0$, let $s_\lambda : \mathbb{R}^M \to \mathbb{R}^M$ be any continuous function with the properties:

(i) $0 < \langle \xi, s_\lambda(\xi) \rangle = \|\xi\|\,\|s_\lambda(\xi)\| \leq \|\xi\|\ \forall \xi \neq 0$,
(ii) $d_\lambda(\xi) \geq 0 \Rightarrow s_\lambda(\xi) = \|\xi\|^{-1}\xi$,

where, again, $d_\lambda(\cdot)$ is defined in (7). Note that (i) implies that the vector $s_\lambda(\xi)$ has the same 'direction' as ξ. A simple example of one such function s_λ is given by

$$s_\lambda(\xi) = \begin{cases} \|\xi\|^{-1}\xi, & \text{if } \|\xi\| \geq \lambda \\ \lambda^{-1}\xi, & \text{if } \|\xi\| < \lambda. \end{cases}$$

Our control strategy is of form (2) with

$$\left.\begin{aligned} u(t) &= -k(t)[e(t) + \bar{f}(e(t) + y_{\text{ref}}(t))s_\lambda(e(t))] \\ \dot{k}(t) &= d_\lambda(e(t))[\|e(t)\| + \bar{f}(e(t) + y_{\text{ref}}(t))], \quad k(t_0) = k^0. \end{aligned}\right\} \qquad (16)$$

Note that (16) simplifies to (9) if perturbation $f \equiv 0$ in (1).

The overall closed-loop system is described by the coupled system of differential equations (on $\mathbb{R}^{N+1}$) in (6 and 16). This system of differential equations satisfies the classical Carathéodory conditions, and so, for each $(t_0, e^0, z^0, k^0) \in \mathbb{R}^{N+2}$, the initial-value problem (6 and 16) has a solution (not necessarily unique) and every solution can be extended into a maximal solution.

Theorem 2. Let $(e(\cdot), z(\cdot), k(\cdot)) : [t_0, \omega) \to \mathbb{R}^{N+1}$ be a maximal solution of the initial-value problem (6, 16). Then,

(i) $\omega = \infty$;
(ii) $(e(\cdot), z(\cdot), k(\cdot))$ is bounded;
(iii) $\lim_{t\to\infty} k(t)$ exists and is finite;
(iv) $e(\cdot)$ approaches the closed ball $\bar{B}_\lambda(0) \subset \mathbb{R}^M$, that is, $d_\lambda(e(t)) \to 0$ as $t \to \infty$.

Proof. Let $\theta_\Lambda : [t_0, \omega) \to \mathbb{R}^M$, parameterized by $\Lambda > 0$, be the continuous map

$$\theta_\Lambda : t \mapsto \begin{cases} D_\Lambda(e(t))\,\|e(t)\|_P^{-1}\,e(t), & \|e(t)\|_P > \Lambda \\ 0, & \|e(t)\|_P \leq \Lambda \end{cases},$$

where $D_\Lambda(\cdot)$ is defined in equation (7). We remark, in passing, that $\|\theta_\Lambda(t)\|_P = \|P^{1/2}\theta_\Lambda(t)\| = d_\Lambda(e(t))$ and so, in the notation of Lemma 2,

$$p\,\|\theta_\Lambda(t)\| \leq D_\Lambda(e(t)) \leq q\,\|\theta_\Lambda(t)\| \quad \forall t.$$

Define a C^1 function $V_\Lambda : \mathbb{R}^M \to [0, \infty)$ (again parameterized by $\Lambda > 0$) by

$$\xi \mapsto V_\Lambda(\xi) := \tfrac{1}{2}D_\Lambda^2(\xi).$$

Note that $\nabla V_\Lambda(e(t)) = P\theta_\Lambda(t)$ and so, for almost all $t \in [t_0, \omega)$, we have

$$\frac{d}{dt}V_\Lambda(e(t))$$
$$\leq qD_\Lambda(e(t))[(\|A_1\| + \gamma_1)\,\|e(t)\| + \|A_2\|\,\|z(t)\| + \gamma_1]$$
$$+ \langle \theta_\Lambda(t), PCB[u(t) + f_1(t, e(t), z(t))] \rangle,$$

$$\langle \theta_\Lambda(t), PCBu(t) \rangle$$
$$\leq -D_\Lambda(e(t))\kappa(t)[\|e(t)\| + \bar{f}(e(t) + y_{\text{ref}}(t))]\,\|s_\lambda(e(t))\|,$$

where

$$\kappa(t) := \begin{cases} q^{-1}k(t), & k(t) \geq 0 \\ p^{-1}k(t), & k(t) < 0 \end{cases}$$

and, by Assumption 4, together with boundedness of $n(\cdot)$ and $y_{\text{ref}}(\cdot)$,

$$\langle \theta_\Lambda(t), PCBf_1(t, e(t), z(t)) \rangle$$
$$\leq \|P^{1/2}\theta_\Lambda(t)\|\,\|P^{1/2}CB\|\,\|f_1(t, e(t), z(t))\|$$
$$\leq \bar{\alpha}D_\Lambda(e(t))[\|e(t)\| + \|z(t)\| + 1 + \bar{f}(e(t) + y_{\text{ref}}(t))]$$

for some scalar $\bar{\alpha} > 0$. Now, $D_\Lambda(e(t)) \geq 0$ and $D_\Lambda(e(t)) > 0$ if, and only if, $\|e(t)\|_P > \Lambda$, whence

$$D_\Lambda(e(t)) \leq \Lambda^{-1}\,\|e(t)\|_P\,D_\Lambda(e(t)) \leq q\Lambda^{-1}\,\|e(t)\|\,D_\Lambda(e(t)). \qquad (17)$$

Therefore, we may conclude

$$\frac{d}{dt}V_\Lambda(e(t)) \leq \bar{\alpha}_1 D_\Lambda(e(t))\,\|z(t)\|$$
$$+ D_\Lambda(e(t))[\|e(t)\| + \bar{f}(e(t) + y_{\text{ref}}(t))]$$
$$\times [\bar{\alpha}_2 - q^{-1}\,\|s_\lambda(e(t))\|\,k(t)] \qquad (18)$$

for almost all $t \in [t_0, \omega)$, where $\bar{\alpha}_1 := \bar{\alpha} + q\,\|A_2\|$, $\bar{\alpha}_2 := q\Lambda^{-1}[\bar{\alpha} + q\gamma_1] + \bar{\alpha} + q[\gamma_1 + \|A_1\|]$. Inequality (18) holds for all $\Lambda > 0$. The proof proceeds in two basic steps, each using a different choice of Λ in (18).

(a) *We first prove that $k(\cdot)$ is bounded.* Set $\Lambda = p\lambda$. By properties of the function s_λ, we have

$$0 < \sigma_\lambda := \inf_{\|\xi\| \geq \lambda pq^{-1}} \|s_\lambda(\xi)\| = \min_{\lambda pq^{-1} \leq \|\xi\| \leq \lambda} \|s_\lambda(\xi)\|.$$

Noting that

$$D_\Lambda(\xi) \geq 0 \Leftrightarrow \|\xi\|_P \geq \Lambda = p\lambda \Rightarrow \|\xi\| \geq (p/q)\lambda,$$

we have $D_\Lambda(e(t))\,\|s_\lambda(e(t))\| \geq \sigma_\lambda D_\Lambda(e(t))$. Seeking a contradiction, suppose $k(\cdot)$ is unbounded. Then, by monotonicity, there exists $\tau \in [t_0, \omega)$ such that $k(t) > 1\ \forall t \in [\tau, \omega)$. By

(18), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}V_\Lambda(e(t)) \le \tilde{\alpha}_1 D_\Lambda(e(t))\,\|z(t)\| + D_\Lambda(e(t))[\|e(t)\| + \tilde{f}(e(t)+y_{\mathrm{ref}}(t))] \times [\tilde{\alpha}_2 - q^{-1}\sigma_\lambda k(t)] \quad (19)$$

for almost all $t \in [\tau, \omega)$.

It is easily verified that $t \mapsto (e(t) - \theta_\Lambda(t))$ is bounded on $[\tau, \omega)$. Defining

$$h : t \mapsto A_3[e(t) - \theta_\Lambda(t)] + g_2(t, e(t), z(t)),$$

we see that, for some $\hat{h} > 0$,

$$\begin{aligned}\|h(t)\| &\le \|A_3\|\,\|e(t) - \theta_\Lambda(t)\| + \gamma_2[1 + \|e(t)\|]\\ &\le \|A_3\|\,\|e(t) - \theta_\Lambda(t)\| + \gamma_2 \times [1 + \|e(t) - \theta_\Lambda(t)\| + \|\theta_\Lambda(t)\|]\\ &\le \hat{h}[1 + \|\theta_\Lambda(t)\|] \quad \text{almost everywhere.}\end{aligned}$$

Therefore, by Lemma 1 with $\theta(\cdot) = \theta_\Lambda(\cdot)$, there exists a scalar c_Λ such that

$$\tilde{\alpha}_1 \int_\tau^t D_\Lambda(s)\,\|z(s)\|\,\mathrm{d}s \le q\tilde{\alpha}_1 \int_\tau^t \|\theta_\Lambda(s)\|\,\|z(s)\|\,\mathrm{d}s \le c_\Lambda \int_\tau^t \|\theta_\Lambda(s)\|\,[1 + \|\theta_\Lambda(s)\|]\,\mathrm{d}s$$

for all $t \in [\tau, \omega)$. Now,

$$\begin{aligned}\|\theta_\Lambda(s)\|\,[1 + \|\theta_\Lambda(s)\|] &\le p^{-2}D_\Lambda(e(s))[p + D_\Lambda(e(s))]\\ &\le p^{-2}D_\Lambda(e(s))[p + \|e(s)\|_P]\\ &\le p^{-2}D_\Lambda(e(s))[p + q\,\|e(s)\|]\\ &\le p^{-2}q[1 + \lambda^{-1}]D_\Lambda(e(s))\,\|e(s)\|,\end{aligned}$$

(where we have used equation (17) to obtain the last inequality). Integrating equation (19), we obtain

$$0 \le V_\Lambda(e(t)) \le V_\Lambda(e(T)) + \int_T^t D_\Lambda(e(s))[\|e(s)\| + \tilde{f}(e(s)+y_{\mathrm{ref}}(s))][\tilde{\alpha}_3 - q^{-1}\sigma_\lambda k(s)]\,\mathrm{d}s \quad (20)$$

for all $T, t \in [\tau, \omega)$ with $t \ge T$, where $\tilde{\alpha}_3 := \tilde{\alpha}_2 + c_\Lambda p^{-2}q[1 + \lambda^{-1}]$.

By supposition, the monotone function $k(\cdot)$ is unbounded and so there exists $T \in [\tau, \omega)$ such that

$$\tilde{\alpha}_3 - q^{-1}\sigma_\lambda k(s) < 0 \quad \forall s \in [T, \omega).$$

Recalling that $\Lambda = p\lambda$, we have, by Lemma 2(i), $D_\Lambda(e(s)) \ge pd_\lambda(e(s))$ and so, from (20), we deduce

$$\begin{aligned}0 &\le V_\Lambda(e(t))\\ &\le V_\Lambda(e(T)) + \int_T^t pd_\lambda(e(s))[\|e(s)\| + \tilde{f}(e(s)+y_{\mathrm{ref}}(s))] \times [\tilde{\alpha}_3 - q^{-1}\sigma_\lambda k(s)]\,\mathrm{d}s\\ &= V_\Lambda(e(T)) + \int_{k(T)}^{k(t)} p[\tilde{\alpha}_3 - q^{-1}\sigma_\lambda \mu]\,\mathrm{d}\mu\\ &= M_0 + p\tilde{\alpha}_3 k(t) - \tfrac{1}{2}pq^{-1}\sigma_\lambda k^2(t) \quad \text{for all } t \in [T, \omega),\end{aligned} \quad (21)$$

where $M_0 = V_\Lambda(e(T)) - p\tilde{\alpha}_3 k(T) + \frac{1}{2}pq^{-1}\sigma_\lambda k^2(T)$.

Clearly, (21) contradicts the supposition of unboundedness of $k(\cdot)$. Therefore, $k(\cdot)$ is bounded. It immediately follows, by (21), that $V_\Lambda(e(\cdot))$ and hence $e(\cdot)$ is bounded. Boundedness of $z(\cdot)$ is now a direct consequence of the second differential equation in (6), together with Proposition 1 (that is, $\sigma(A_4) \subset \mathbb{C}_-$), boundedness of $e(\cdot)$ and essential boundedness of $t \mapsto g_2(t, e(t), z(t))$ (the latter holding by virtue of boundedness of $e(\cdot)$).

Thus, we may conclude assertions (i), (ii) and (iii) of the Theorem.

(b) *It remains to prove assertion (iv).* Now set $\Lambda = q\lambda$.

By boundedness of $z(\cdot)$ and (17), there exists scalar $\tilde{\alpha}_\Lambda$ such that

$$\tilde{\alpha}_1 D_\Lambda(e(t))\,\|z(t)\| \le \tilde{\alpha}_\Lambda D_\Lambda(e(t))\,\|e(t)\|.$$

Define

$$\tilde{\alpha}_4 := \sup\{t \ge t_0 \mid \tilde{\alpha}_\Lambda + \tilde{\alpha}_2 - q^{-1}\|s_\lambda(e(t))\|\,k(t)\} < \infty,$$
$$\tilde{\alpha}_5 := \tilde{\alpha}_4 q + 1,$$

(the former being finite by virtue of boundedness of $s_\lambda(e(\cdot))$ and of $k(\cdot)$).

From (18) and using Lemma 2(ii), we deduce

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_\Lambda(e(t)) &\le \tilde{\alpha}_4 D_\Lambda(e(t))[\|e(t)\| + \tilde{f}(e(t)+y_{\mathrm{ref}}(t))]\\ &\le \tilde{\alpha}_4 q d_\lambda(e(t))[\|e(t)\| + \tilde{f}(e(t)+y_{\mathrm{ref}}(t))]\\ &\le -d_\lambda(e(t))\,\|e(t)\| + \tilde{\alpha}_5 \dot{k}(t),\end{aligned}$$

which holds for almost all $t \ge t_0$.

Introduce a function $\mathbb{R}^{N+1} \to \mathbb{R}$ given by

$$W : (e, k) \mapsto V_\Lambda(e) - \tilde{\alpha}_5 k.$$

For almost all $t \ge t_0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}W(e(t), k(t)) \le -d_\lambda(e(t))\,\|e(t)\| \le 0 \quad (22)$$

and so $W(e(\cdot), k(\cdot))$ is monotone.

By boundedness of $(e(\cdot), z(\cdot), k(\cdot))$ together with Assumptions 3 and 4, it follows that

$$t \mapsto CB[u(t) + f_1(t, e(t), z(t))] + g_2(t, e(t), z(t))$$

is essentially bounded and so, by (6), we may conclude the existence of $R > 0$ such that

$$\|\dot{e}(t)\| \le R \quad \text{for almost all } t \ge t_0.$$

We can now employ the same argument as that following (14) in the proof of Theorem 1 to establish Assertion (iv).

5. *General case II: p known, $PCB + (CB)^TP$ sign-definite of unknown sign*

In this section, we consider the general case with Assumption 2(iii) in force, that is, we assume that a symmetric, positive-definite P is known such that $PCB + (CB)^TP$ is sign definite, but of unknown sign. (We reiterate that, in the single-variable case $M = 1$, this is simply the weak assumption that the high-frequency gain is non-zero.) Specifically, a positive-definite symmetric P is available to the controller with the property that, for some positive-definite symmetric Q and scalar $\beta \in \{-1, +1\}$,

$$PCB + (CB)^TP = \beta Q.$$

Fix $\lambda > 0$ and define

$$\tilde{\lambda} := pq^{-1}\lambda, \quad \Lambda := q\tilde{\lambda} = p\lambda.$$

Similar to the previous section, let $s_{\tilde{\lambda}} : \mathbb{R}^M \to \mathbb{R}^M$ be any continuous function with the properties:

(i) $0 < \langle \xi, s_{\tilde{\lambda}}(\xi)\rangle = \|\xi\|\,\|s_{\tilde{\lambda}}(\xi)\| \le \|\xi\|\ \forall \xi \ne 0$,
(ii) $d_{\tilde{\lambda}}(\xi) \ge 0 \Rightarrow s_{\tilde{\lambda}}(\xi) = \|\xi\|^{-1}\xi$.

Let $\nu(\cdot)$ be any continuous function of Nussbaum type, precisely, any continuous function $\mathbb{R} \to \mathbb{R}$ with properties (5), with the additional property that, if $M \ge 2$, then ν is *scaling invariant* (this concept has been introduced by Logemann and Owens (1988)) in the following sense: for each pair $\mu = (\mu_1, \mu_2)$ of positive scalars $\mu_1, \mu_2 > 0$, the function

$$\nu_\mu : k \mapsto \nu_\mu(k) := \begin{cases}\mu_1\nu(k), & \text{if } \nu(k) \ge 0\\ \mu_2\nu(k), & \text{if } \nu(k) < 0\end{cases},$$

also has the Nussbaum properties, viz.

$$\limsup_{k\to\infty} \frac{1}{k}\int_0^k \nu_\mu(s)\,\mathrm{d}s = +\infty \quad \text{and} \quad \liminf_{k\to\infty} \frac{1}{k}\int_0^k \nu_\mu(s)\,\mathrm{d}s = -\infty. \quad (23)$$

For example, it has been shown by Logemann and Owens (1988) that the function $v: \tau \mapsto \exp(\tau^2)\cos(\frac{1}{2}\pi\tau)$ (originally introduced by Nussbaum, 1983) is scaling invariant.

Any such Nussbaum function v (scaling invariant if $M \geq 2$) can be adopted to compensate for the lack of knowledge of which half plane, $\mathbb{C}_-$ or $\mathbb{C}_+$, actually contains the spectrum $\sigma(CB)$. Specifically, our control strategy is of form (2) with

$$\left.\begin{aligned} u(t) &= v(k(t))[e(t) + \bar{f}(e(t) + y_{\text{ref}}(t))s_{\bar{\lambda}}(e(t))] \\ \dot{k}(t) &= D_\Lambda(e(t))[\|e(t)\| + \bar{f}(e(t) + y_{\text{ref}}(t))], \quad k(t_0) = k^0 \end{aligned}\right\} \quad (24)$$

where again, $D_\Lambda(\cdot)$ is defined in (7). The closed-loop system representation is given by (6) and (24), for which the Carathéodory conditions hold and so, for each $(t_0, e^0, z^0, k^0) \in \mathbb{R}^{N+2}$, the initial-value problem (6) and (24) has a solution and every solution can be extended into a maximal solution. We now arrive at the main result of the paper, namely, that the output feedback and gain adaptation (24) is a universal λ-regulator solving the servomechanism problem for the class of systems (1) under Assumptions 1, 2(iii), 3 and 4.

Theorem 3. Let $(e(\cdot), z(\cdot), k(\cdot)): [t_0, \omega) \to \mathbb{R}^{N+1}$ be a maximal solution of the initial-value problem (6, 24). Then,

(i) $\omega = \infty$;

(ii) $(e(\cdot), z(\cdot), k(\cdot))$ is bounded;

(iii) $\lim_{t\to\infty} k(t)$ exists and is finite;

(iv) $e(\cdot)$ approaches the closed ball $\bar{B}_\lambda(0) \subset \mathbb{R}^M$, that is, $d_\lambda(e(t)) \to 0$ as $t \to \infty$.

Proof. Let function V_Λ be defined as in the proof of Theorem 2. Noting that

$$D_\Lambda(e) = D_{q\bar{\lambda}}(e) > 0 \Rightarrow q\bar{\lambda} < \|e\|_P \leq q\,\|e\| \Rightarrow \|s_{\bar{\lambda}}(e)\| = 1,$$

we have

$$D_\Lambda(e(t))\|s_{\bar{\lambda}}(e(t))\| = D_\Lambda(e(t)) \quad \forall t \in [t_0, \omega).$$

Define

$$m_1 := \tfrac{1}{2}\sigma_{\max}(Q)/\sqrt{\sigma_{\min}(P)}$$

and

$$m_2 := \tfrac{1}{2}\sigma_{\min}(Q)/\sqrt{\sigma_{\max}(P)},$$

where $\sigma_{\max}(R) > 0$ (respectively, $\sigma_{\min}(R) > 0$) denotes the maximum (respectively, minimum) eigenvalue of a positive-definite symmetric matrix R. Writing $\mu = (\mu_1, \mu_2)$, with

$$\mu_1 := \begin{cases} m_1, & \text{if } \beta = +1 \\ m_2, & \text{if } \beta = -1 \end{cases} \quad \text{and} \quad \mu_2 := \begin{cases} m_2, & \text{if } \beta = +1 \\ m_1, & \text{if } \beta = -1 \end{cases},$$

we find that

$$\langle \nabla V_\Lambda(e(t)), CBu(t) \rangle \leq \beta v_\mu(k(t)) D_\Lambda(e(t))[\|e(t)\| + \bar{f}(e(t) + y_{\text{ref}}(t))].$$

Therefore, analogous to (18),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V_\Lambda(e(t)) &\leq \alpha_1 D_\Lambda(e(t))\,\|z(t)\| \\ &\quad + D_\Lambda(e(t))[\|e(t)\| + \bar{f}(e(t) + y_{\text{ref}}(t))] \\ &\quad \times [\alpha_2 + \beta v_\mu(k(t))] \end{aligned} \quad (25)$$

for some positive scalars α_1 and α_2.

We will first show that $k(\cdot)$ is bounded. Seeking a contradiction, suppose that $k(\cdot)$ is unbounded. Then, by monotonicity, there exists $\tau \in [t_0, \omega)$ such that $k(t) \geq 1$ for all $t \in [\tau, \omega)$. By Lemma 1 and arguing as in part (b) of the proof of Theorem 2, there exists a scalar c_Λ such that

$$\int_\tau^t D_\Lambda(e(s))\,\|z(s)\|\,\mathrm{d}s \leq c_\Lambda \int_\tau^t D_\Lambda(e(s))\,\|e(s)\|\,\mathrm{d}s \quad \forall t \in [\tau, \omega).$$

Integrating (25),

$$\begin{aligned} 0 \leq V_\Lambda(e(t)) &\leq V_\Lambda(e(\tau)) + \int_\tau^t D_\Lambda(e(s))[\|e(s)\| + \bar{f}(e(t) \\ &\quad + y_{\text{ref}}(t))][\alpha_3 + \beta v_\mu(k(s))]\,\mathrm{d}s \\ &= V_\Lambda(e(\tau)) + \alpha_3[k(t) - k(\tau)] + \beta \int_{k(\tau)}^{k(t)} v_\mu(s)\,\mathrm{d}s, \end{aligned}$$

for all $t \in [\tau, \omega)$, where $\alpha_3 = \alpha_1 c_\Lambda + \alpha_2$. Dividing by $k(t) \geq 1$ and taking limit inferior as $t \to \omega$ $(k(t) \to \infty)$ yields

$$0 \leq \text{constant} + \liminf_{\xi\to\infty} \frac{\beta}{\xi} \int_{k(\tau)}^{\xi} v_\mu(s)\,\mathrm{d}s,$$

which depending on the value of $\beta \in \{-1, +1\}$, contradicts one or the other of properties (23) (or of properties (5) in the case $M = 1$). Therefore, $k(\cdot)$ is bounded. As in the proof of Theorem 2, Assertions (i), (ii) and (iii) immediately follow.

It remains to prove assertion (iv). By boundedness of $z(\cdot)$ and (17), there exists scalar α_Λ such that

$$\alpha_1 D_\Lambda(e(t))\,\|z(t)\| \leq \alpha_\Lambda D_\Lambda(e(t))\,\|e(t)\| \quad \forall t \geq t_0.$$

Therefore, by (25), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V_\Lambda(e(t)) &\leq -D_\Lambda(e(t))\,\|e(t)\| \\ &\quad + [1 + \alpha_\Lambda + \alpha_2 + \beta v_\mu(k(t))]\dot{k}(t) \end{aligned}$$

which is valid for almost all $t \geq t_0$.

Define

$$\alpha_4 := \sup\{t \geq t_0 \mid 1 + \alpha_\Lambda + \alpha_2 + \beta v_\mu(k(t))\} < \infty,$$

(finite by virtue of boundedness of $k(\cdot)$ and continuity of v_μ).

Now introduce the function

$$W: (e, k) \mapsto V_\Lambda(e) - \alpha_4 k.$$

For almost all $t \geq t_0$, we have, by Lemma 2(i),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} W(e(t), k(t)) &\leq -D_\Lambda(e(t))\,\|e(t)\| \\ &\leq -p d_\lambda(e(t))\,\|e(t)\| \leq 0. \end{aligned}$$

We now find ourselves in precisely the same situation as that following equation (22) in the proof of Theorem 2 and so assertion (iv) follows. Q.E.D.

6. *Example*

Let $N = 2$ and $M = 1$ and let

$$\hat{f} = \bar{f}: \mathbb{R} \to \mathbb{R}, \quad y \mapsto 1 + y^3.$$

Consider the following system* (A, B, C, f, g) of form (1) with

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad B = \begin{bmatrix} b \\ 0 \end{bmatrix}, \quad C = [1 \quad 0],$$

$$f(t, x) = \alpha_1 x_1^{1/3} + \alpha_2 \cos(x_1 x_2) + \alpha_3 x_1^3,$$

$$g(t, x) = \begin{bmatrix} \alpha_4 \sin(x_2)\sqrt{1 + x_1^2} + \alpha_5 x_1^{1/3} \\ 0 \end{bmatrix},$$

with *unknown* real parameters a_i, b, and α_i, with noise-corrupted output

$$y(t) = Cx(t) + n(t), \qquad n(\cdot) \in W^{1,\infty}(\mathbb{R}),$$

and with initial condition $x(t_0) = x^0$.

If

$$a_4 < 0 \quad \text{and} \quad b \neq 0,$$

* We do not claim any engineering interpretation for this particular example: the nonlinearities are chosen in an *ad hoc* manner, solely with the intention of illustrating the wide class of admissible perturbations.

we see that Assumptions 1, 3, 4 hold and Assumption 2(iii) holds with $P = 1$. Therefore, $(A, B, C, f, g) \in \mathcal{S}$ and so, by Theorem 3, for each fixed $\lambda > 0$, the continuous adaptive output feedback strategy

$$u(t) = \nu(k(t))[y(t) - y_{\rm ref}(t) + (1 + y^3(t))s_\lambda(y(t) - y_{\rm ref}(t))]$$
$$\dot{k}(t) = d_\lambda(y(t) - y_{\rm ref}(t))[y(t) - y_{\rm ref}(t) + (1 + y^3(t))],$$
$$k(t_0) = k^0,$$

guarantees λ-tracking of every reference signal $y_{\rm ref}(\cdot)$ of class $W^{1,\infty}$ in the sense that, for each (t_0, x^0, k^0) and every realization of noise $n(\cdot) \in W^{1,\infty}$, the error $y(t) - y_{\rm ref}(t)$ tends, as $t \to \infty$, to the compact interval $[-\lambda, \lambda]$.

By way of illustration, let $y_{\rm ref}(\cdot)$ be defined as follows:

$$t \mapsto y_{\rm ref}(t) = \begin{cases} q_1^0, & -\infty < t < 0 \\ q_1(t), & 0 \le t < \infty \end{cases},$$

where $q_1(\cdot)$ is the first component of the solution of the initial-value problem for the Lorenz equations:

$$\dot{q}_1(t) = 10(q_2(t) - q_1(t))$$
$$\dot{q}_2(t) = 28q_1(t) - q_2(t) - 10q_1(t)q_3(t)$$
$$\dot{q}_3(t) = 10q_1(t)q_2(t) - 8q_3(t)/3$$
$$(q_1(0), q_2(0), q_3(0)) = (q_1^0, q_2^0, q_3^0).$$

For the chosen parameter values, this system of equations exhibits chaotic behaviour and so the reference signal $y_{\rm ref}(\cdot)$ might be regarded as a stringent test for the tracking controller (one such reference signal is depicted in Fig. 2). That $y_{\rm ref}(\cdot)$ is of class $W^{1,\infty}$ is an immediate consequence of boundedness (as studied in Sparrow (1982), Appendix C) of the solution $(q_1(\cdot), q_2(\cdot), q_3(\cdot))$.

Again for purposes of illustration, we will assume that the output noise is also generated via the above Lorenz equations but now operating on a faster time scale (one such realization of noise is depicted in Fig. 5).

As Nussbaum function, we use

$$\nu : \eta \mapsto \eta^2 \cos \eta$$

and, for the function $s_\lambda(\cdot)$, we take

$$s_\lambda : \xi \mapsto \begin{cases} \operatorname{sgn}(\xi), & \text{if } |\xi| \ge \lambda \\ \lambda^{-1}\xi, & \text{if } |\xi| < \lambda \end{cases}$$

parameterized by $\lambda > 0$. For the parameter values (unknown to the controller) $a_1 = 0$, $a_2 = 1 = a_3$, $a_4 = -1$, $b = 1 = \alpha_i$, $i = 1, 2, 3, 4, 5$, and adopting a controller parameter value $\lambda = 0.2$, Figs 2–4 depict the behaviour of the system with initial data

$$(t_0, x^0, k^0) = (0, 0, 0, 0), \quad (q_1^0, q_2^0, q_3^0) = (1, 0, 3),$$

in the *absence* of noise $n(\cdot) = 0$.

Now consider the same system in the presence of output noise. Figures 6–8 depict the behaviour of the system with a realization of noise as shown in Fig. 5. Note that the 'reference signal to noise ratio' is approximately 10, and the noise magnitude is commensurate with the chosen controller parameter value $\lambda = 0.2$.

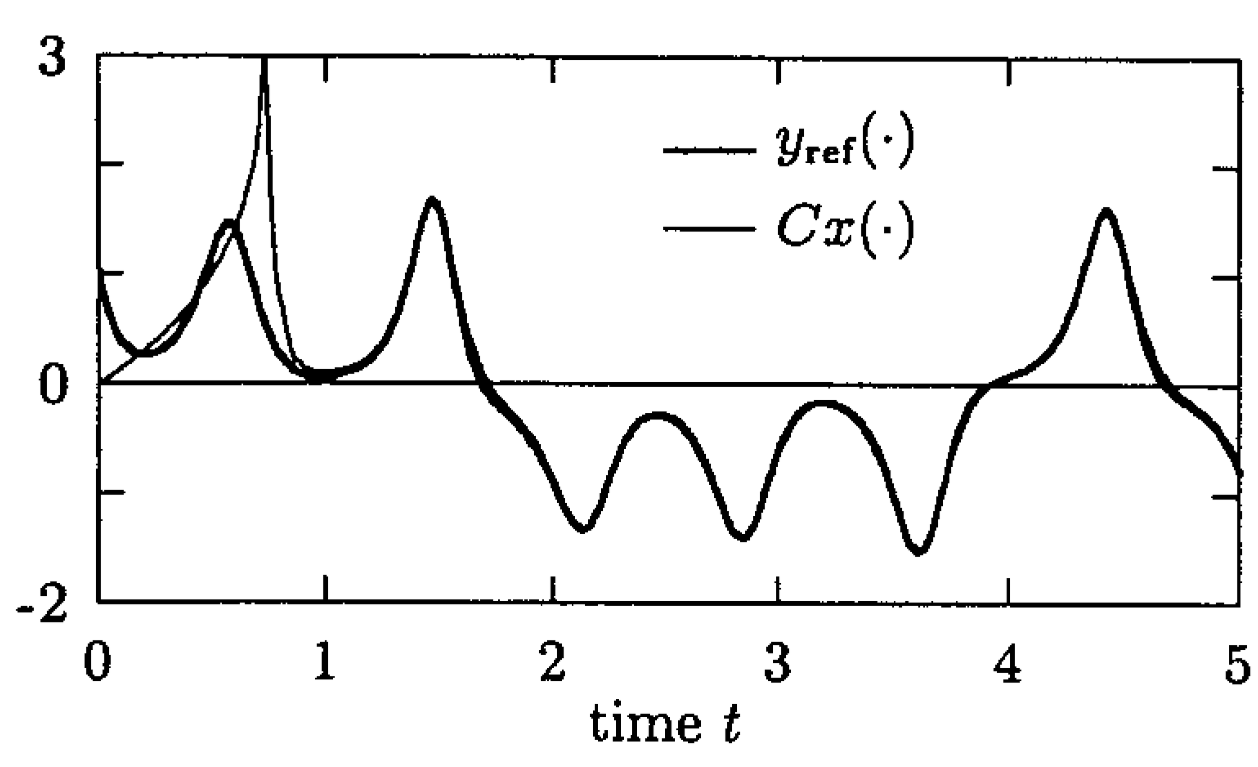

FIG. 2. Tracking behaviour in absence of noise.

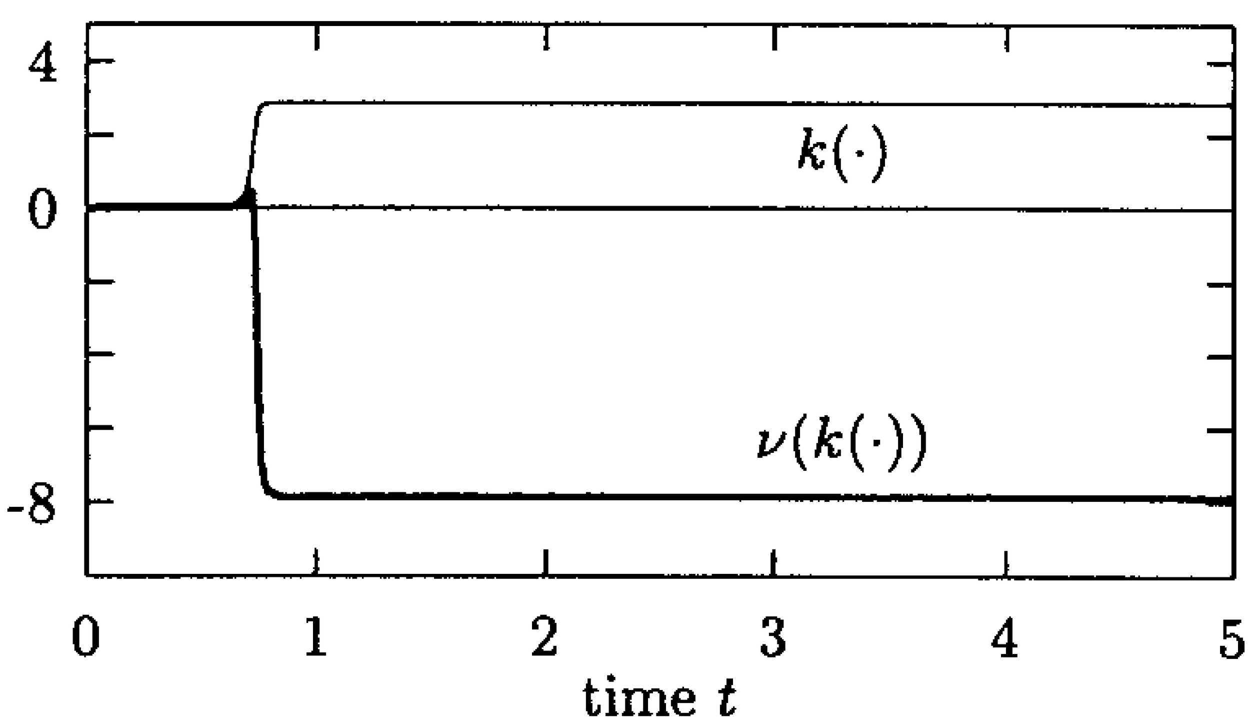

FIG. 3. Evolution of controller gain in absence of noise.

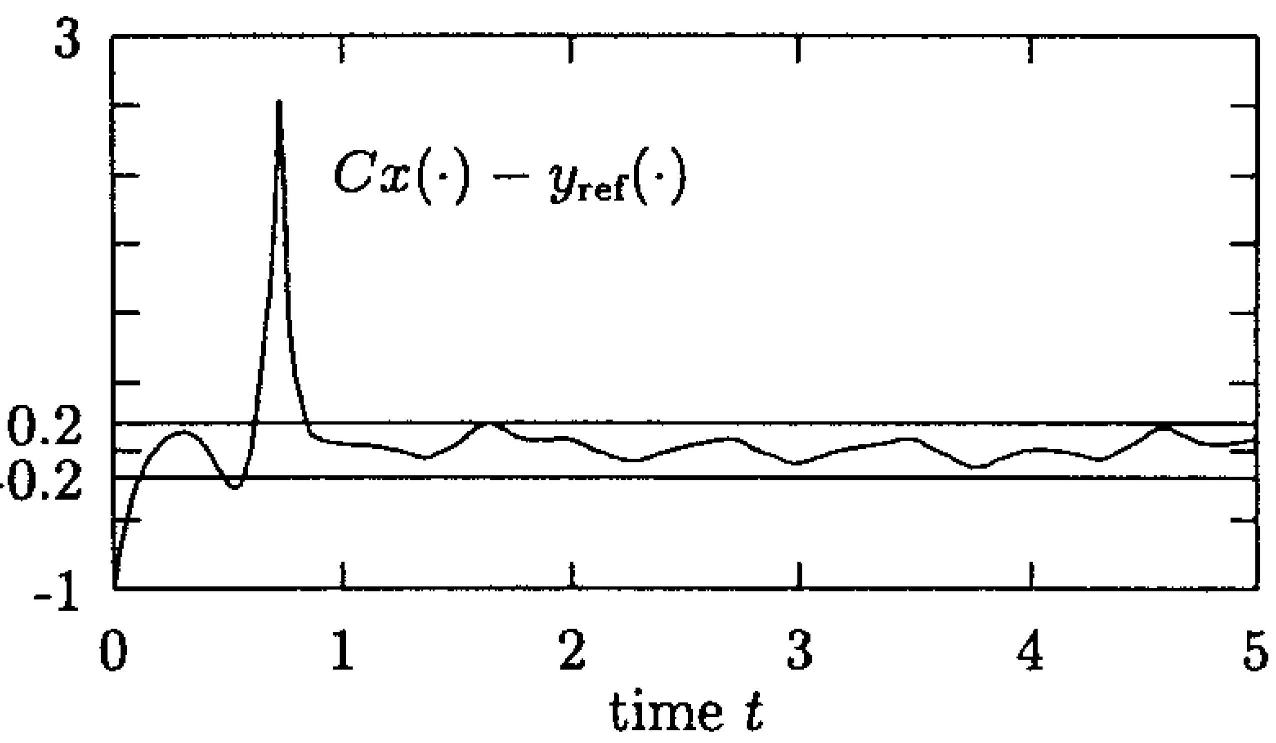

FIG. 4. Tracking error and prescribed error band.

7. *Conclusions*

In this paper, we have shown that the Willems–Byrnes adaptive controller, $u = -ky$, $\dot{k} = y^2$, (which assures attractivity of the zero state for every linear, single-input, single-output, minimum-phase system of relative degree one with positive high-frequency gain) has a natural extension for achieving λ-tracking in the presence of noise for a large class of multivariable nonlinear systems. By sacrificing asymptotic tracking for the weaker (but nonetheless practical) requirement of tracking with error asymptotic to a ball of arbitrarily small prescribed radius $\lambda > 0$, we have demonstrated that it is possible to λ-track reference signals in the presence of noise signals, provided that both signals belong to the class $W^{1,\infty}(\mathbb{R}, \mathbb{R}^M)$ of bounded functions that are

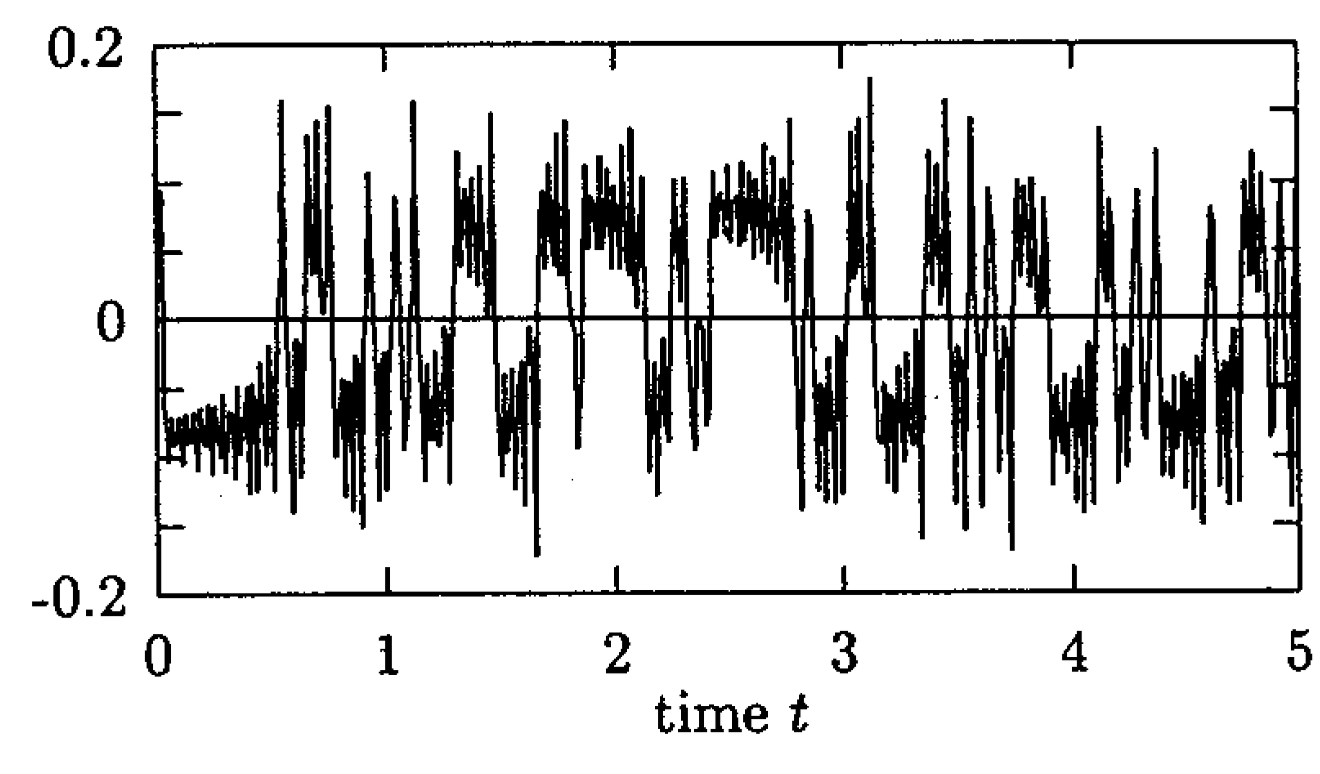

FIG. 5. Realization of output noise $n(\cdot)$.

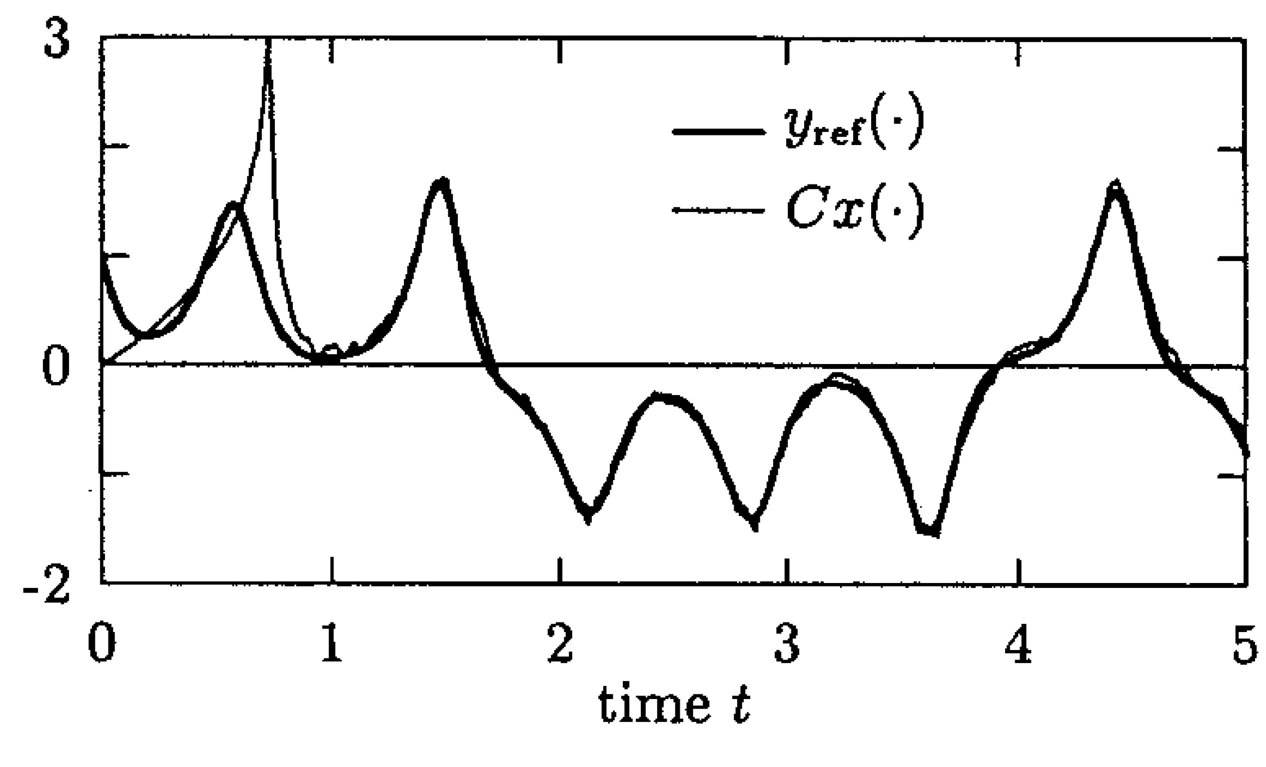

FIG. 6. Tracking behaviour in presence of noise.

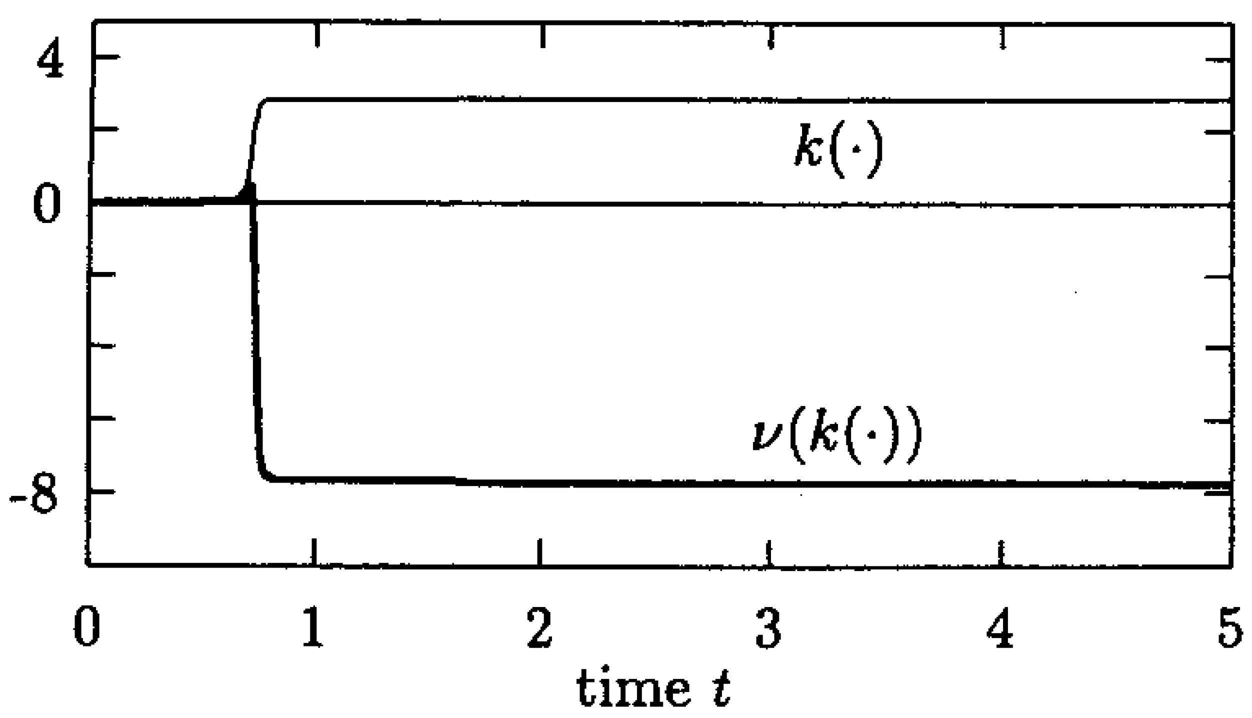

FIG. 7. Evolution of controller gain in presence of noise.

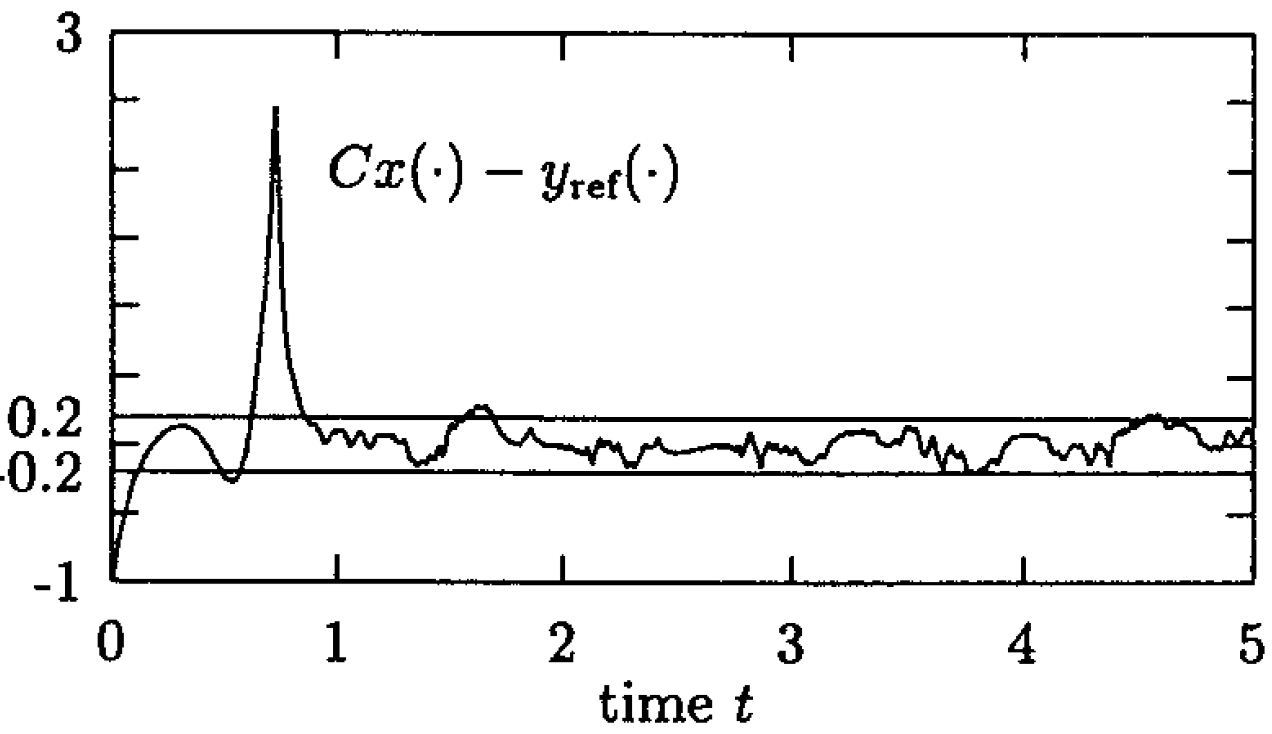

FIG. 8. Tracking error in presence of noise.

absolutely continuous on compact subintervals of $\mathbb{R}$ and have essentially bounded derivatives. This weakened concept of tracking obviates the necessity of an internal model principle in the controller design. The simple λ-tracking regulator can tolerate a large class of nonlinearities.

One important question remains open in relation to Assumption 2(iii): in the single-input, single-output case, this is simply a requirement that the high-frequency gain be non-zero (of unknown sign); in the multivariable case, it is a more restrictive. We conjecture that, in the latter case, the weaker requirement that it be known only that $\sigma(CB)$ lies either in $\mathbb{C}_+$ or in $\mathbb{C}_-$ (but which half plane is unknown to the controller), is sufficient for λ-tracking. However, we have been unable to prove this conjecture.

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Appendix: Proof of Propositions and Lemmas

A.1. *Proof of Proposition* 1. By Assumption 1 and (6), for all $s \in \bar{\mathbb{C}}_+$,

$$0 \neq \det \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} = \det \begin{bmatrix} sI - A_1 & -A_2 & CB \\ -A_3 & sI - A_4 & 0 \\ I & 0 & 0 \end{bmatrix} = \det[sI - A_4] \det[CB].$$

Therefore, $\sigma(A_4) \subset \mathbb{C}_-$.

A.2. *Proof of Lemma* 1. Since $\sigma(A_4) \subset \mathbb{C}_-$, there exist scalars M_0, $\mu > 0$ such that $\|\exp(A_4 t)\| \le M_0 e^{-\mu t}$. Now,

$$z(t) = (\exp(A_4(t-\tau)))z^\tau + \int_\tau^t (\exp(A_4(t-s)))[A_3\theta(s) + h(s)]\,ds$$

and so

$$\|z(t)\| \le M_0 e^{-\mu(t-\tau)} \|z^\tau\| + M_0[\|A_3\| + 2\hat{h}]L(1 + \|\theta(\cdot)\|)(t) \le M_1[1 + L(\|\theta(\cdot)\|)(t)],$$

where $M_1 := M_0\|z^\tau\| + M_0[1 + \mu^{-1}][\|A_3\| + 2\hat{h}]$ and L is the operator

$$L: \phi(\cdot) \mapsto \left(t \mapsto \int_\tau^t e^{-\mu(t-s)}\phi(s)\,ds\right).$$

Therefore, using Hölder's inequality,

$$\begin{aligned}\int_\tau^t \|\theta(s)\|\,\|z(s)\|\,ds &\le M_1 \int_\tau^t \|\theta(s)\|\,[1 + L(\|\theta(\cdot)\|)(s)]\,ds \\ &\le M_1 \int_\tau^t \|\theta(s)\|\,ds + M_1\left(\int_\tau^t [L(\|\theta(\cdot)\|)(s)]^2\,ds\right)^{1/2} \times \left(\int_\tau^t \|\theta(s)\|^2\,ds\right)^{1/2}.\end{aligned}$$

Integration by parts and a second application of Hölder's inequality now yields

$$\begin{aligned}\int_\tau^t [L(\|\theta(\cdot)\|)(s)]^2\,ds &\le -\frac{1}{2\mu}[L(\|\theta(\cdot)\|)(t)]^2 + \frac{1}{\mu}\int_\tau^t \|\theta(s)\|\,L(\|\theta(\cdot)\|)(s)\,ds \\ &\le \frac{1}{\mu}\left(\int_\tau^t \|\theta(s)\|^2\,ds\right)^{1/2} \times \left(\int_\tau^t [L(\|\theta(\cdot)\|)(s)]^2\,ds\right)^{1/2}\end{aligned}$$

Therefore,

$$\left(\int_\tau^t [L(\|\theta(\cdot)\|)(s)]^2\,ds\right)^{1/2} \le \frac{1}{\mu}\left(\int_\tau^t \|\theta(s)\|^2\,ds\right)^{1/2},$$

and the result follows, with $c = M_1[1 + \mu^{-1}]$.

A.3. *Proof. of Lemma* 2. Let $\bar{E}_\Lambda(0)$ denote the closed ellipsoid $\{\xi \mid \|\xi\|_P \le \Lambda\} \subset \mathbb{R}^M$ and let $\bar{B}_\lambda(0)$ denote the closed ball $\{\xi \mid \|\xi\| \le \lambda\} \subset \mathbb{R}^M$.

Let $\xi \in \mathbb{R}^M$ be arbitrary and define

$$\hat{k} := \begin{cases} p\lambda\,\|\xi\|_P^{-1}\xi, & \|\xi\|_P \ge p\lambda \\ \xi, & \|\xi\|_P < p\lambda \end{cases}$$

and

$$\tilde{k} := \begin{cases} \lambda\,\|\xi\|^{-1}\xi, & \|\xi\| \ge \lambda \\ \xi, & \|\xi\| < \lambda \end{cases}.$$

Then,

$$\begin{aligned} D_{p\lambda}(\xi) = \|\xi - \hat{k}\|_P \ge p\,\|\xi - \hat{k}\| &\ge p \min\{\|\xi - k\| \mid k \in \bar{E}_{p\lambda}(0)\} \\ &\qquad (\text{since } \hat{k} \in \bar{E}_{p\lambda}(0)) \\ &\ge p \min\{\|\xi - k\| \mid k \in \bar{B}_\lambda(0)\} \\ &\qquad (\text{since } \bar{E}_{p\lambda}(0) \subset \bar{B}_\lambda(0)) \\ &= p d_\lambda(\xi), \end{aligned}$$

and

$$\begin{aligned} q d_\lambda(\xi) = q\,\|\xi - \tilde{k}\| \ge \|\xi - \tilde{k}\|_P &\ge \min\{\|\xi - k\|_P \mid k \in \bar{B}_\lambda(0)\} \\ &\qquad (\text{since } \|\tilde{k}\| \le \lambda) \\ &\ge \min\{\|\xi - k\|_P \mid k \in \bar{E}_{q\lambda}(0)\} \\ &\qquad (\text{since } \bar{E}_{q\lambda}(0) \supset \bar{B}_\lambda(0)) \\ &= D_{q\lambda}(\xi). \end{aligned}$$

This completes the proof.

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