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ROBUSTNESS OF λ-TRACKING IN THE GAP METRIC*

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Abstract. For m-input, m-output, finite-dimensional, linear systems satisfying the classical assumptions of adaptive control (i.e., (i) minimum phase, (ii) relative degree one, and (iii) "positive" high-frequency gain), it is well known that the adaptive λ-tracker "$u = -k\,e$, $\dot{k} = \max\{0, |e| - \lambda\}|e|$" achieves λ-tracking of the tracking error e if applied to such a system: all states of the closed-loop system are bounded, and $|e|$ is ultimately bounded by λ, where $\lambda > 0$ is prespecified and may be arbitrarily small. Invoking the conceptual framework of the nonlinear gap metric, we show that the λ-tracker is robust. In the present setup this means in particular that the λ-tracker copes with bounded input and output disturbances, and, more importantly, it may even be applied to a system not satisfying any of the classical conditions (i)–(iii) as long as the initial conditions and the disturbances are "small" and the system is "close" (in terms of "small" gap) to a system satisfying (i)–(iii).

Key words. adaptive control, λ-tracking, gap metric, robust stability

AMS subject classifications. 93D21, 93D09, 93C40, 93D25

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Nomenclature

$\mathbb{C}_+$, $\mathbb{C}_-$	=	$\{s \in \mathbb{C} \,|\, \operatorname{Re} s > 0\}$, $\{s \in \mathbb{C} \,|\, \operatorname{Re} s < 0\}$, respectively
$A > 0$		if and only if $x^T A x > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$, where $A = A^T \in \mathbb{R}^{n\times n}$
$\lvert x\rvert$	=	$\sqrt{x^T x}$, the Euclidean norm of $x \in \mathbb{R}^n$
$\lvert A\rvert$	=	$\max\{\lvert A\,x\rvert \,\big\vert\, x \in \mathbb{R}^m, \lvert x\rvert = 1\}$, the induced matrix norm for $A \in \mathbb{R}^{n\times m}$
$\lVert v\rVert_{\mathcal{V}}$		the norm of $v \in \mathcal{V}$ for any normed vector space $\mathcal{V}$
$L^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)$		the space of p-integrable functions $y : \mathbb{R}_{\geq 0} \to \mathbb{R}^\ell$, $1 \leq p < \infty$ with norm
$\lVert y\rVert_{L^p(\mathbb{R}_{\geq 0}\to\mathbb{R}^\ell)}$	=	$\left(\int_0^\infty \lvert y(t)\rvert^p \,\mathrm{d}t\right)^{\frac{1}{p}}$
$L^p_{\mathrm{loc}}(I \to \mathbb{R}^\ell)$		the space of locally p-integrable functions $y : I \to \mathbb{R}^\ell$, with $\int_K \lvert y(t)\rvert^p \,\mathrm{d}t < \infty$ for all compact $K \subset I$, where $1 \leq p < \infty$ and $I \subset \mathbb{R}_{\geq 0}$ is an interval
$L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)$		the space of essentially bounded functions $y : \mathbb{R}_{\geq 0} \to \mathbb{R}^\ell$ with norm
$\lVert y\rVert_{L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^\ell)}$	=	$\operatorname{ess\,sup}_{t\geq 0} \lvert y(t)\rvert$
$L^\infty_{\mathrm{loc}}(I \to \mathbb{R}^\ell)$		the space of locally bounded functions $y : I \to \mathbb{R}^\ell$, with $\operatorname{ess\,sup}_{t\in K} \lvert y(t)\rvert < \infty$ for all compact $K \subset I$, where $I \subset \mathbb{R}_{\geq 0}$ is an interval

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$W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)$	the Sobolev space of absolutely continuous functions $y : \mathbb{R}_{\geq 0} \to \mathbb{R}^\ell$ with $y, \dot{y} \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)$ and norm
$\|y\|_{W^{1,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}^\ell)} =$	$\|y\|_{L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^\ell)} + \|\dot{y}\|_{L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^\ell)}$
$W^{1,\infty}_{\text{loc}}(I \to \mathbb{R}^\ell)$	the space of absolutely continuous functions $y : I \to \mathbb{R}^\ell$, with $y, \dot{y} \in L^\infty_{\text{loc}}(I \to \mathbb{R}^\ell)$, where $I \subset \mathbb{R}_{\geq 0}$ is an interval
$\operatorname{dist}(e, [-\lambda, \lambda])$	$= \max\{0, \|e\| - \lambda\}$ for $e \in \mathbb{R}^m$ and $\lambda > 0$
$d_\lambda(e)$	$= \max\{0, \|e\| - \lambda\}$ for $e \in \mathbb{R}^m$ and $\lambda > 0$

1. Introduction. In this paper we show robustness of λ-stabilization and λ-tracking (i.e., stabilization and tracking with a final accuracy of prespecified $\lambda > 0$) for linear n-dimensional, m-input, m-output systems of the form

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,u_1(t), \qquad x(0) = x^0, \\ y_1(t) = C\,x(t), \end{cases} \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B, C^T \in \mathbb{R}^{n\times m}$, $x^0 \in \mathbb{R}^n$, subject to additive input/output disturbances u_0, y_0, respectively,

$$u_0 \;=\; u_1 + u_2, \qquad y_0 \;=\; y_1 + y_2, \tag{1.2}$$

as depicted in Figure 1, where the *plant* P maps the interior input signal u_1 to the interior output signal y_1 and the *controller* C maps the interior output signal y_2 to the interior input signal u_2. In our setup, P will always be a linear initial value problem of the form (1.1) and the controller C will be a dynamical system, specified in due course. In case of zero disturbances $u_0 \equiv y_0 \equiv 0$, it is well known that (1.1) can be stabilized by proportional high-gain ($k \gg 0$) output feedback

$$u_2(t) = -k\,y_2(t), \tag{1.3}$$

provided (1.1) is *minimum phase*, i.e.,

$$\forall\, s \in \overline{\mathbb{C}}_+ \;:\; \det \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} \neq 0,$$

and its transfer function $C(sI-A)^{-1}B$ has *strict relative degree* one with "positive" high-frequency gain, i.e., $CB + (CB)^T > 0$. This system class is denoted, for $n, m \in \mathbb{N}$ with $n \geq m$, as

$$\widetilde{\mathcal{M}}_{n,m} := \left\{ \begin{array}{l} (A,B,C) \\ \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \end{array} \;\middle|\; \begin{array}{l} CB + (CB)^T > 0 \\ \forall\, s \in \overline{\mathbb{C}}_+ \,:\, \det \begin{bmatrix} sI_n - A & B \\ C & 0 \end{bmatrix} \neq 0 \end{array} \right\}.$$

Note that the state space dimension $n \in \mathbb{N}$ need not be known but only the input/output dimension $m \in \mathbb{N}$, and, most importantly, only structural assumptions are required, but the system entries are completely unknown. A sufficiently high-gain k in (1.3) can be found adaptively. More precisely, any system $(A,B,C) \in \widetilde{\mathcal{M}}_n$ can be stabilized adaptively, in the presence of L^2 input/output disturbances, by the controller (ubiquitous in the adaptive control literature)

$$\begin{cases} \dot{k}(t) = |y_2(t)|^2, \qquad k(0) = k^0 \in \mathbb{R}, \\ u_2(t) = -k(t)\,y_2(t) \end{cases} \tag{1.4}$$

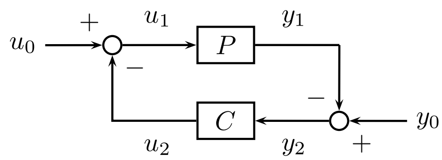

Fig. 1. *The closed-loop system* $[P, C]$.

in the sense that all states of the closed-loop system (1.1), (1.2), (1.4) are bounded and $\lim_{t\to\infty} y_1(t) = 0$. This approach has been introduced by the seminal work of [14, 15, 17]; see also the survey [6].

The surprising property of the controller (1.4) is not only its simplicity but also its robustness: it is also applicable in the presence of additive L^2 input/output disturbances, and it may stabilize systems (1.1) not belonging to $\widetilde{\mathcal{M}}_{n,m}$ but sufficiently "close"—in terms of the *gap metric* defined in section 3—to some (A, B, C) in normal form and belonging to $\widetilde{\mathcal{M}}_{n,m}$. This has been proved in [4].

However, the controller (1.4) has the shortcomings that, if tracking is the control objective, it needs to be combined with an internal model (thus becoming more involved) and, more importantly, fails for stabilizing nonlinear systems or in the presence of additive arbitrarily small input or output L^∞-disturbances. To overcome these shortcomings, the so-called λ-tracker

$$\text{(1.5)} \qquad \begin{cases} \dot{k}(t) = \operatorname{dist}\left(y_2(t), [-\lambda, \lambda]\right) \cdot |y_2(t)|, \quad k(0) = k^0, \\ u_2(t) = -k(t) y_2(t) \end{cases}$$

for $\lambda > 0$ and $k^0 \in \mathbb{R}$ has been introduced by [9]. The application of the λ-tracker (1.5) to any system (1.1) belonging to $\widetilde{\mathcal{M}}_{n,m}$, via (1.2), satisfies, in the presence of arbitrary input/output disturbances u_0, y_0 which are bounded with essentially bounded derivative, arbitrary initial conditions $x^0 \in \mathbb{R}^n$, $k^0 \in \mathbb{R}$ and any arbitrarily small design parameter $\lambda > 0$, the control objectives of λ-*tracking*:

- all signals and their derivatives of the closed-loop system (1.1), (1.2), (1.5) are bounded;
- $\limsup_{t\to\infty} \operatorname{dist}(y_2(t), [-\lambda, \lambda]) = 0$.

This result has been generalized to nonlinear and infinite-dimensional systems [10] and applied, to name but a few, to regulate biogas tower reactors [8], chemical reactors [12], and insulin delivery for diabetic patients [2] by preserving the simplicity of the control strategy. Note also that it is a tracking result without invoking an internal model: set $y_0(\cdot) \equiv y_{\mathrm{ref}}(\cdot)$ as a prespecified reference signal.

The purpose of the present paper is to show robustness properties of the λ-tracker in terms of the gap metric. For example, we consider

$$\text{(1.6)} \qquad \begin{cases} \dot{x} = \widetilde{A}\, x + \widetilde{b}\, u_1, \quad x(0) = \widetilde{x}^0, \\ y_1 = \widetilde{c}\, x, \end{cases}$$

with $\widetilde{x}^0 \in \mathbb{R}^3$ and where, for $\alpha, N, M > 0$,

$$\widetilde{A} := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha NM & -NM + \alpha N + \alpha M & \alpha - N - M \end{bmatrix}, \quad \widetilde{b} := \begin{bmatrix} 0 \\ 0 \\ N \end{bmatrix}, \quad \widetilde{c} := [M, -1, 0].$$

This system does not belong to the class $\widetilde{\mathcal{M}}_{3,1}$; its transfer function $\frac{N(M-s)}{(s-\alpha)(s+N)(s+M)}$ does not satisfy any of the classical structural assumptions in adaptive control:

(1.7)
$$\begin{cases} \bullet & \text{it is not minimum phase;} \\ \bullet & \text{it has relative degree two;} \\ \bullet & \text{and its high-frequency gain } -N<0 \text{ has the "wrong" sign.} \end{cases}$$

However, defining, for $n, m \in \mathbb{N}$ with $n \geq m$, the system class

$$\mathcal{P}_{n,m} := \left\{ (A,B,C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \mid (A,B,C) \text{ is stabilizable and detectable} \right\},$$

(1.6) belongs to $\mathcal{P}_{3,1}$, and we will show in subsection 3.5 and Example 4.6 that (1.6) is close (in terms of the gap metric) to a system belonging to $\widetilde{\mathcal{M}}_{1,1}$ for N, M sufficiently large and $\widetilde{x}^0$ sufficiently small, and thus (1.5) applied to (1.6) achieves λ-tracking.

Instead of systems $(A,B,C) \in \widetilde{\mathcal{M}}_{n,m}$ we restrict our attention to systems in Byrnes–Isidori normal form; see, for example, [13, section 4]. That is, for each $(A, B, C) \in \widetilde{\mathcal{M}}_{n,m}$ the matrix

$$T = \left[B(CB)^{-1}, V \right], \text{ where } V \in \mathbb{R}^{n\times(n-m)} \text{ with } \operatorname{rk} V = n-m \text{ and } \operatorname{im} V = \ker C,$$

converts (1.1) via the coordinate transformation $\binom{y_1}{z} = T^{-1}x$ into

(1.8)
$$\begin{cases} \dot{y}_1 = A_1 y_1 + A_2 z + CB\, u_1, & y_1(0) = y_1^0 \in \mathbb{R}, \\ \dot{z} = A_3 y_1 + A_4 z, & z(0) = z^0 \in \mathbb{R}^{n-m}, \end{cases} \qquad \begin{pmatrix} y_1^0 \\ z^0 \end{pmatrix} = T^{-1}x^0,$$

where

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} := T^{-1}AT, \quad \begin{bmatrix} B_1 \\ 0_{(n-m)\times m} \end{bmatrix} := \begin{bmatrix} CB \\ 0 \end{bmatrix} = T^{-1}B, \quad \begin{bmatrix} I_m & 0_{m\times(n-m)} \end{bmatrix} = CT.$$

By the minimum-phase property, A_4 has spectrum in the open left half complex plane $\mathbb{C}_-$. Therefore, we introduce, for $n, m \in \mathbb{N}$ with $n \geq m$, the system class

$$\mathcal{M}_{n,m} := \left\{ (A,B,C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \;\middle|\; \begin{array}{l} A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},\ B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \\ C = \begin{bmatrix} I_m & 0 \end{bmatrix},\ B_1, A_1 \in \mathbb{R}^{m\times m}, \\ \operatorname{spec}(A_4) \subset \mathbb{C}_-,\ B_1 + B_1^T > 0. \end{array} \right\}.$$

We will study properties of the closed-loop system generated by the application of the λ-tracker (1.5) to systems (1.1) of class $\mathcal{M}_{n,m}$ or of class $\mathcal{P}_{n,m}$ in the presence of disturbances $(u_0, y_0) \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ satisfying the interconnection equations (1.2). The closed-loop system (1.8), (1.5), (1.2) is depicted in Figure 2.

The paper is organized as follows. In section 2 we show that λ-tracking is possible for all linear systems (1.1) belonging to class $\mathcal{M}_{n,m}$ in the presence of $W^{1,\infty}(\mathbb{R}_{\geq 0} \to$

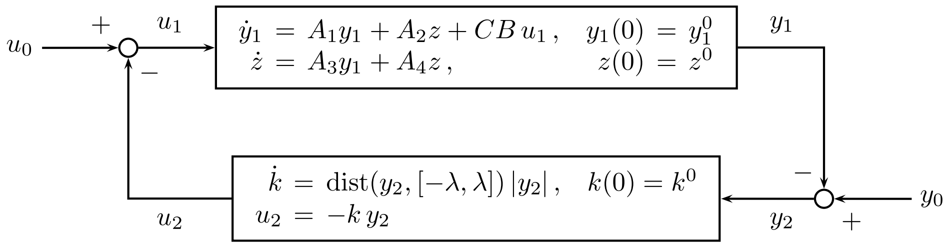

FIG. 2. *The adaptive closed-loop system.*

$\mathbb{R}^m$) input/output disturbances; see Figure 2. In section 3, we collect the basics of the framework of gap metric and graph topology from [5, 3, 4] necessary for our setup. Section 4 contains our main result, i.e., robustness of λ-tracking. We show that if the initial conditions, the input/output disturbances, and the gap between a nominal system belonging to class $\mathcal{P}_{q,m}$ and a system belonging to class $\mathcal{M}_{n,m}$ (for $m, q, n \in \mathbb{N}$ with $q, n \geq m$) are sufficiently small, then the controller (1.5) achieves λ-tracking for the nominal system.

2. $\boldsymbol{\lambda}$-tracking. In this section we show that the control strategy given by (1.5) applied to any linear system of class $\mathcal{M}_{n,m}$ achieves λ-tracking in the presence of $W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ input/output disturbances; see Figure 2. Set, for $n, m \in \mathbb{N}$ with $n \geq m$,

$$\mathcal{D}_{n,m} := \mathcal{M}_{n,m} \times (\mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m).$$

PROPOSITION 2.1. *Let $m, n \in \mathbb{N}$ with $n \geq m$ and $\lambda > 0$. Then there exists a continuous map $\nu\colon \mathcal{D}_{n,m} \to \mathbb{R}_{\geq 0}$ such that, for all $d = \left(\left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right], B, C, (y_1^0, z^0, k^0), u_0, y_0\right) \in \mathcal{D}_{n,m}$, the associated closed-loop initial value problem* (1.8), (1.2), (1.5) *satisfies*

$$\|(u_2, y_2, z, k)\|_{W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^{m+n+1})} \leq \nu(d) \tag{2.1}$$

and

$$\limsup_{t \to \infty} |y_2(t)| \leq \lambda. \tag{2.2}$$

The result that λ-tracking works for the class of systems $\mathcal{M}_n$ goes back to [9], and input disturbances are also considered in [7]. However, to prove the robustness of the λ-tracker in section 4, the existence of a continuous function $\nu(\cdot)$ satisfying (2.1) is crucial. Therefore, we had to find a new proof showing (2.1) which easily shows (2.2).

Proof of Proposition 2.1. Let $d = \left(\left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right], B, C, (y_1^0, z^0, k^0), u_0, y_0\right) \in \mathcal{D}_{n,m}$ and set, for notational convenience,

$$\begin{aligned} h(\cdot) &:= \dot y_0(\cdot) - A_1\, y_0(\cdot) - CB\, u_0(\cdot), \\ e(\cdot) &:= y_2(\cdot). \end{aligned}$$

The closed-loop initial value problem (1.8), (1.2), (1.5) is then given by

$$\left\{ \begin{array}{lllllll} \dot e &=& A_1\, e - A_2\, z - k\, CB\, e + h, & e(0) &=& e^0 := y_0(0) - y_1^0, \\ \dot z &=& -A_3\, e + A_4\, z + A_3\, y_0, & z(0) &=& z^0, \\ \dot k &=& d_\lambda(e)\, |e|, & k(0) &=& k^0, \end{array} \right. \tag{2.3}$$

where d_λ is defined in the Nomenclature. We divide the proof into 10 steps.

Step 1. Since the right-hand side of (2.3) is continuous and locally Lipschitz, it follows from the theory of ordinary differential equations that (2.3) has a solution

$$(e,z,k)\colon [0,\omega) \to \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}_{\geq 0}$$

on a maximal interval of existence $[0,\omega)$ for some $\omega \in (0,\infty]$. This solution is unique.

Step 2. We define some constants that are used in the following steps of the proof.

Since $\operatorname{spec}(A_4) \subset \mathbb{C}_-$ we have

$$\exists\, M_1, \mu > 0 \quad \forall\, t \geq 0 \ : \ |\exp(A_4\, t)| \leq M_1 \exp(-\mu t). \tag{2.4}$$

Set

$$\begin{aligned}
\sigma_1 &:= \min \operatorname{spec}\left(CB + (CB)^T\right)/2,\\
M_2 &:= M_1 + M_1|A_3|\big(\|y_0\|_{L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} + \lambda + \mu\big)/\mu,\\
M_3 &:= M_2\left(1 + |z^0|\right)/\lambda + M_2\left(1 + 1/\mu\right),\\
M_4 &:= |A_1| + |A_2| + \|h\|_{L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)}/\lambda,\\
M_5 &:= |k^0| + 2\left(M_4 + M_3 M_4 + 1\right)/\sigma_1,\\
M_6 &:= M_5 + |k^0| + |e^0|^2/2,\\
M_7 &:= \left(d_\lambda(e^0)^2 + 2(M_6 + |k^0|)\left[\sigma_1\left(M_6 + |k^0|\right)/2 + M_4 + M_3 M_4\right]\right)^{\frac{1}{2}} + \lambda,\\
M_8 &:= M_2\left(1 + |z^0| + M_7/\mu\right).
\end{aligned}$$

Step 3. We estimate the z-dynamics in the form

$$\forall\, t \in [0,\omega) \ : \ \int_0^t d_\lambda(e(\tau))\,|z(\tau)|\,\mathrm{d}\tau \leq\ M_3\ \left[k(t) - k^0\right]. \tag{2.5}$$

Applying variation of constants to the second equation in (2.3) and invoking (2.4) gives, for all $t \in [0,\omega)$,

$$\begin{aligned}
|z(t)| &\leq M_1\, e^{-\mu t}|z^0| + \int_0^t M_1\, e^{-\mu(t-\tau)}|A_3|\left(|e(\tau)| + |y_0(\tau)|\right)\,\mathrm{d}\tau\\
&\leq M_1\, e^{-\mu t}|z^0| + M_1|A_3|\int_0^t e^{-\mu(t-\tau)}\left(d_\lambda(e(\tau)) + \|y_0\|_{L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} + \lambda\right)\,\mathrm{d}\tau\\
&\leq M_1\, e^{-\mu t}|z^0| + M_1|A_3|\int_0^t e^{-\mu(t-\tau)}\|y_0\|_{L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)}\,\mathrm{d}\tau\\
&\quad + M_1|A_3|\int_0^t e^{-\mu(t-\tau)}\lambda\,\mathrm{d}\tau + M_1|A_3|\int_0^t e^{-\mu(t-\tau)}d_\lambda(e(\tau))\,\mathrm{d}\tau\\
&\leq M_1\,|z^0| + \frac{M_1|A_3|}{\mu}\left(\|y_0\|_{L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} + \lambda\right) + M_1|A_3|\int_0^t e^{-\mu(t-\tau)}d_\lambda(e(\tau))\,\mathrm{d}\tau\\
&\leq M_2\left[1 + |z^0| + \int_0^t e^{-\mu(t-\tau)}d_\lambda(e(\tau))\,\mathrm{d}\tau\right]. \qquad (2.6)
\end{aligned}$$

Let

$$\forall\, t \in [0,\infty)\ \forall\, \varphi \in L^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R})\ :\ (L * \varphi)(t) := \int_0^t e^{-\mu(t-\tau)} \varphi(\tau)\, \mathrm{d}\tau.$$

Invoking the well-known inequality (see, for example, [16, p. 298])

$$\forall\, t \geq 0\ :\ \|L * \varphi\|_{L^2([0,t)\to\mathbb{R})} \leq \|e^{-\mu\,\cdot}\|_{L^1(\mathbb{R}_{\geq 0}\to\mathbb{R})} \|\varphi\|_{L^2([0,t)\to\mathbb{R})} = \tfrac{1}{\mu}\|\varphi\|_{L^2([0,t)\to\mathbb{R})}$$

and the fact that

$$\forall\, e \in \mathbb{R}\ :\ d_\lambda(e)^2 \leq d_\lambda(e)\,|e|$$

yields, by (2.3), (2.6), and the Cauchy–Schwarz inequality, for all $t \in [0,\omega)$,

$$\begin{aligned}
&\int_0^t d_\lambda(e(\tau))\, |z(\tau)|\, \mathrm{d}\tau \\
&\leq M_2 \int_0^t d_\lambda(e(\tau)) \left[1 + |z^0| + (L * d_\lambda(e))(\tau)\right] \mathrm{d}\tau \\
&\leq M_2\, \left[1 + |z^0|\right] \frac{1}{\lambda} \int_0^t d_\lambda(e(\tau))\, |e(\tau)|\, \mathrm{d}\tau \\
&\quad + M_2 \left[\|d_\lambda(e)\|^2_{L^2([0,t)\to\mathbb{R})} + \|L * d_\lambda(e)\|^2_{L^2([0,t)\to\mathbb{R})}\right] \\
&\leq M_2\, \left[1 + |z^0|\right] \frac{1}{\lambda} \int_0^t d_\lambda(e(\tau))\, |e(\tau)|\, \mathrm{d}\tau + M_2 \left(1 + \frac{1}{\mu}\right) \int_0^t d_\lambda(e(\tau))^2\, \mathrm{d}\tau.
\end{aligned}$$

This proves (2.5).

Step 4. We estimate the e-dynamics in the form

(2.7) $\quad \forall\, t \in [0,\omega)\ :$

$$\frac{1}{2} d_\lambda(e(t))^2 \leq \frac{1}{2} d_\lambda(e^0)^2 - \left(k(t) - k^0\right) \left[\frac{\sigma_1}{2}\left(k(t) + k^0\right) - M_4 - M_3 M_4\right].$$

By (2.3) and Step 2 we have, omitting the argument t,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2} d_\lambda(e(t))^2\right) &= d_\lambda(e)\, |e|^{-1}\, e^T\, \dot{e} \\
&= d_\lambda(e)\, |e|^{-1} e^T \left[A_1\, e - A_2\, z - k\, CB\, e + h\right] \\
&\leq d_\lambda(e)\, |e|\, |A_1| + d_\lambda(e)\, |z|\, |A_2| + d_\lambda(e)\, \|h\|_{L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} \\
&\quad - k\, d_\lambda(e)\, |e|^{-1}\, e^T \left(\tfrac{1}{2}(CB + (CB)^T\right) e \\
&\leq -(k\,\sigma_1 - |A_1|) d_\lambda(e)\, |e| + |A_2|\, d_\lambda(e)\, |z| + d_\lambda(e)\, \|h\|_{L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} \\
&\leq -(k\,\sigma_1 - |A_1|) d_\lambda(e)\, |e| + |A_2|\, d_\lambda(e)\, |z| + d_\lambda(e) \frac{|e|}{\lambda}\, \|h\|_{L^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^m)} \\
&\leq -(k\,\sigma_1 - M_4) d_\lambda(e)\, |e| + M_4\, d_\lambda(e)\, |z|,
\end{aligned}$$

and hence, by integration and invoking (2.5), we arrive at

$$\forall\, t \in [0,\omega) \;:\; \frac{1}{2} d_\lambda(e(t))^2 \le \frac{1}{2} d_\lambda(e^0)^2 - \int_0^t (k(\tau)\,\sigma_1 - M_4)\dot{k}(\tau)\,\mathrm{d}\tau + M_3 M_4 \left[k(t) - k^0\right],$$

which yields (2.7).

Step 5. We show boundedness of k in the form

$$\forall\, t \in [0,\omega) \;:\; k(t) \le M_6. \tag{2.8}$$

Suppose there exists $T \in [0,\omega)$ such that $k(T) = M_5$; otherwise (2.8) is obvious. Then, by monotonicity of k, it follows from (2.7) that, for all $t \in [T,\omega)$,

$$\begin{aligned} 0 \le \frac{1}{2} d_\lambda(e(t))^2 &\le \frac{1}{2} d_\lambda(e^0)^2 - \frac{\sigma_1}{2}\left(k(t) - k^0\right)\left[k(t) + k^0 - \frac{2}{\sigma_1}(M_4 + M_3 M_4)\right] \\ &\le \frac{1}{2} d_\lambda(e^0)^2 - \frac{\sigma_1}{2}\left(k(t) - k^0\right)\left[M_5 + k^0 - \frac{2}{\sigma_1}(M_4 + M_3 M_4)\right] \\ &= \frac{1}{2} d_\lambda(e^0)^2 - \frac{\sigma_1}{2}\left(k(t) - k^0\right)\left[|k^0| + k^0 + \frac{2}{\sigma_1}\right] \\ &\le \frac{1}{2} d_\lambda(e^0)^2 - \left(k(t) - k^0\right), \end{aligned}$$

and thus

$$\forall\, t \in [T,\omega) \;:\; k(t) - k^0 \le \frac{1}{2} d_\lambda(e^0)^2 \le \frac{1}{2}|e^0|^2$$

and

$$\forall\, t \in [0,T) \;:\; k(t) - k^0 \le M_5 - k^0,$$

whence (2.8).

Step 6. We show boundedness of e in the form

$$\forall\, t \in [0,\omega) \;:\; |e(t)| \le M_7. \tag{2.9}$$

An application of (2.8) to (2.7) gives, for all $t \in [0,\omega)$,

$$\begin{aligned} |e(t)| &\le d_\lambda(e(t)) + \lambda \\ &\le \left(d_\lambda(e^0)^2 - 2\left(k(t) - k^0\right)\left[\frac{\sigma_1}{2}\left(k(t) + k^0\right) - M_4 - M_3 M_4\right]\right)^{\frac{1}{2}} + \lambda \\ &\le \left(d_\lambda(e^0)^2 + 2(M_6 + |k^0|)\left[\frac{\sigma_1}{2}\left(M_6 + |k^0|\right) + M_4 + M_3 M_4\right]\right)^{\frac{1}{2}} + \lambda. \end{aligned}$$

Note that the argument of the root in the second line is nonnegative; see Step 5. Now (2.9) follows from Step 2.

Step 7. Boundedness of z in the form

$$\forall\, t \in [0,\omega) \;:\; |z(t)| \le M_2\left[1 + |z^0| + \int_0^t e^{-\mu(t-\tau)} M_7\,\mathrm{d}\tau\right] \le M_8 \tag{2.10}$$

follows from applying (2.9) to (2.6).

Step 8. We show $\omega = \infty$.

Since ω was chosen maximal, (2.8)–(2.10) yield $\omega = \infty$.

Step 9. We show (2.1).

It follows from Steps 5–8 that (u_2, y_2, z, k) is uniformly bounded in terms of $d = \left(\left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right], B, C, (y_1^0, z^0, k^0), u_0, y_0\right)$. Moreover, applying Steps 5–8 again and invoking (2.3) yields uniform boundedness of $\left(\dot u_2, \dot y_2, \dot z, \dot k\right)$ in terms of d. Now the existence of a continuous function $\nu\colon \mathcal{D}_{n,m} \to \mathbb{R}_{\geq 0}$ such that (2.1) holds is straightforward by invoking the constants from Step 2.

Step 10. We show (2.2).

Since $k \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$ by (2.1) it follows from $\|d_\lambda(y_2)\,|y_2|\|_{L^1([0,t)\to\mathbb{R})} = k(t) - k^0$ that $d_\lambda(y_2)\,|y_2| \in L^1(\mathbb{R}_{\geq 0} \to \mathbb{R})$.

Since $y_2 \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ there exists $M > 0$ such that $\operatorname{ess\,sup}_{t \geq 0} |(\dot y_2)_i(t)| < M$ for all $i \in \{1.\dots,m\}$, which gives

$$\forall s \geq 0\ \forall i \in \{1.\dots,m\}\ \forall t \in [0,s)\ \exists \tau_i \in (t,s)\ :\ (\dot y_2)_i(\tau_i) = \frac{(y_2)_i(s) - (y_2)_i(t)}{s-t} < M,$$

and so

$$\forall i \in \{1.\dots,m\}\ \forall t \in [0,s)\ :\ |(y_2)_i(s) - (y_2)_i(t)| < M(s-t).$$

For $\delta = \frac{\varepsilon}{M}$ we arrive at

$$\forall i \in \{1.\dots,m\}\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall t,s \in \mathbb{R}_{\geq 0} \text{ with } |t-s| < \delta\ :\ |(y_2)_i(t) - (y_2)_i(s)| < \varepsilon;$$

i.e., y_2 is uniformly continuous. Boundedness and uniform continuity of y_2 and the continuity of $e \mapsto d_\lambda(e)\,|e|$ give uniform continuity of $t \mapsto d_\lambda(y_2(t))\,|y_2(t)|$. So Barbălat's lemma (see [1]) gives

$$\lim_{t\to\infty} d_\lambda(y_2(t))\,|y_2(t)| = 0,$$

which yields (2.2) and completes the proof. □

3. The concept of gap metric. The material in this section is based on [5, section II], [4, section 2], and [3, section 2] and contains the fundamental results necessary for proving robustness in section 4.

3.1. Terminology. Let $\mathcal{X}$ be a nonempty set and, for $0 < \omega \leq \infty$, let $\mathcal{S}_\omega$ denote the set of locally integrable maps $[0,\omega) \to \mathcal{X}$. For simplicity, we write $\mathcal{S} := \mathcal{S}_\infty$. For $0 < \tau < \omega \leq \infty$, $T_\tau : \mathcal{S}_\omega \to \mathcal{S}$ denotes the operator given by

$$T_\tau v\ :=\ \begin{cases} v(t), & t \in [0,\tau), \\ 0, & t \in [\tau,\infty). \end{cases}$$

With $\mathcal{V} \subset \mathcal{S}$ we associate spaces as follows:

$$\mathcal{V}_e = \left\{ v \in \mathcal{S} \,\middle|\, \forall \tau > 0\ :\ T_\tau v \in \mathcal{V} \right\}, \quad \textit{the extended space};$$

$$\mathcal{V}_\omega = \left\{ v \in \mathcal{S}_\omega \,\middle|\, \forall \tau \in (0,\omega)\ :\ T_\tau v \in \mathcal{V} \right\}, \qquad 0 < \omega \leq \infty;$$

$$\mathcal{V}_a = \bigcup_{\omega \in (0,\infty]} \mathcal{V}_\omega, \quad \textit{the ambient space.}$$

The ambient space $\mathcal{V}_a$ is a subset of the union of all $\mathcal{S}_\omega$, $\omega \in (0, \infty]$. Thus, for $v \in \mathcal{V}_a$ the domain of v, i.e., the set of values in $[0, \infty)$ where v is defined, denoted by $\mathrm{dom}(v)$, is not obvious. If $v, w \in \mathcal{V}_a$ with $v|_I = w|_I$ on $I = \mathrm{dom}(v) \cap \mathrm{dom}(w)$, then we write $v = w$. For $(u, y) \in \mathcal{V}_a \times \mathcal{V}_a$, the domains of u and y may be different; we adopt the convention

$$\mathrm{dom}(u, y) \;:=\; \mathrm{dom}(u) \cap \mathrm{dom}(y).$$

We say $\mathcal{V} \subset \mathcal{S}$ is a *signal space* if and only if it is a vector space and has the property that $\sup_{\tau \geq 0} \|T_\tau v\|_{\mathcal{V}} < \infty$ implies $v \in \mathcal{V}$. In our applications, $\mathcal{V}$ will frequently be the normed signal space $W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, in which case $\mathcal{V}_e = W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $\mathcal{V}_\omega = W^{1,\infty}_{\mathrm{loc}}([0, \omega) \to \mathbb{R}^m)$ for $\omega \in (0, \infty]$, and $\mathcal{V}_a = \cup_{0 < \omega \leq \infty} W^{1,\infty}_{\mathrm{loc}}([0, \omega) \to \mathbb{R}^m)$. It is important to note that $\mathcal{V}_\omega \supsetneq W^{1,\infty}([0, \omega) \to \mathbb{R}^m)$.

For a normed signal space $\mathcal{U}$ and the Euclidean space $\mathbb{R}^l$, $l \in \mathbb{N}$, we will also consider subsets of $\mathcal{V} = \mathbb{R}^l \times \mathcal{U}$, which, on identifying each $\theta \in \mathbb{R}^l$ with the constant signal $t \mapsto \theta$, can be thought of as a normed signal space with the norm given by $\|(\theta, x)\|_{\mathcal{V}} = \sqrt{|\theta|^2 + \|x\|^2_{\mathcal{U}}}$.

3.2. Well posedness. A mapping $Q\colon \mathcal{X}_1 \to \mathcal{X}_2$ between signal spaces is said to be *causal* if and only if, for all $\tau > 0$, $x, y \in \mathcal{X}_1$, $T_\tau x = T_\tau y$ implies $T_\tau Qx = T_\tau Qy$. Let $\mathcal{U}$ and $\mathcal{Y}$ be normed signal spaces and let $P\colon \mathcal{U}_a \to \mathcal{Y}_a$ and $C\colon \mathcal{Y}_a \to \mathcal{U}_a$ be causal mappings representing a plant and controller, respectively. Our central concern is the system of equations

$$[P, C] \;:\quad y_1 = Pu_1, \quad u_2 = Cy_2, \quad u_0 = u_1 + u_2, \quad y_0 = y_1 + y_2 \tag{3.1}$$

corresponding to the closed-loop feedback configuration as depicted in Figure 1; see section 1. By a solution of (3.1) we mean the following. For $w_0 = (u_0, y_0) \in \mathcal{W} := \mathcal{U} \times \mathcal{Y}$, a pair $(w_1, w_2) = \big((u_1, y_1), (u_2, y_2)\big) \in \mathcal{W}_a \times \mathcal{W}_a$, $\mathcal{W}_a := \mathcal{U}_a \times \mathcal{Y}_a$, is a *solution* of (3.1) if and only if (3.1) holds on $\mathrm{dom}(w_1, w_2)$. The (possibly empty) set of all solutions is denoted by

$$\mathcal{X}_{w_0} \;:=\; \big\{(w_1, w_2) \in \mathcal{W}_a \times \mathcal{W}_a \big|\; (w_1, w_2) \text{ solves (3.1)}\big\}.$$

The closed-loop system $[P, C]$, given by (3.1), is said to have

- the *existence property* if and only if $\mathcal{X}_{w_0} \neq \emptyset$,
- the *uniqueness property* if and only if

$$\forall\; w_0 \in \mathcal{W} \;:\; \Big[(\hat{w}_1, \hat{w}_2), (\tilde{w}_1, \tilde{w}_2) \in \mathcal{X}_{w_0} \\ \Longrightarrow (\hat{w}_1, \hat{w}_2) = (\tilde{w}_1, \tilde{w}_2) \quad \text{on} \quad \mathrm{dom}(\hat{w}_1, \hat{w}_2) \cap \mathrm{dom}\,(\tilde{w}_1, \tilde{w}_2)\Big].$$

Assume that $[P, C]$ has the existence and uniqueness properties. For each $w_0 \in \mathcal{W}$, define ω_{w_0}, $0 < \omega_{w_0} \leq \infty$, by the property

$$[0, \omega_{w_0}) := \bigcup_{(\hat{w}_1, \hat{w}_2) \in \mathcal{X}_{w_0}} \mathrm{dom}(\hat{w}_1, \hat{w}_2)$$

and define $(w_1, w_2) \in \mathcal{W}_a \times \mathcal{W}_a$, with $\mathrm{dom}(w_1, w_2) = [0, \omega_{w_0})$, by the property $(w_1, w_2)|_{[0,t)} \in \mathcal{X}_{w_0}$ for all $t \in [0, \omega_{w_0})$. This construction induces the operator

$$H_{P,C} : \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a, \qquad w_0 \mapsto (w_1, w_2).$$

The closed-loop system $[P, C]$, given by (3.1), is said to be

- *locally well posed* if and only if it has the existence and uniqueness properties and the operator $H_{P,C}\colon \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a,\ w_0 \mapsto (w_1, w_2)$, is causal,
- *globally well posed* if and only if it is locally well posed and $H_{P,C}(\mathcal{W}) \subset \mathcal{W}_e \times \mathcal{W}_e$,
- $\mathcal{W}$-*stable* if and only if it is locally well posed and $H_{P,C}(\mathcal{W}) \subset \mathcal{W} \times \mathcal{W}$,
- *regularly well posed* if and only if it is locally well posed and

$$\forall\, w_0 \in \mathcal{W}\ :\ \left[\ \omega_{w_0} < \infty \quad \Longrightarrow \quad T_{\omega_{w_0}} H_{P,C}(w_0) \notin \mathcal{W} \times \mathcal{W}\ \right]. \tag{3.2}$$

If $[P, C]$ is globally well posed, then for each $w_0 \in \mathcal{W}$ the solution $H_{P,C}(w_0)$ exists on the half-line $\mathbb{R}_{\geq 0}$. Regular well posedness means that if the closed-loop system has a finite escape time $\omega_{w_0} > 0$ for some disturbance $w_0 \in \mathcal{W}$, then at least one of the components u_1, u_2 or y_1, y_2 is not a restriction to $[0, \omega_{w_0})$ of a function in $\mathcal{U}$ or $\mathcal{Y}$, respectively. If $[P, C]$ is regularly well posed and satisfies

$$\forall\, w_0 \in \mathcal{W}\ :\ \left[\ \omega_{w_0} < \infty \quad \Longrightarrow \quad T_{\omega_{w_0}} H_{P,C}(w_0) \in \mathcal{W} \times \mathcal{W}\right],$$

there does not exist a solution of $[P, C]$ with a finite escape time, and therefore $[P, C]$ is globally well posed. However, global well posedness does not guarantee that each solution belongs to $\mathcal{W} \times \mathcal{W}$; the latter is ensured by $\mathcal{W}$-stability of $[P, C]$. Note also that neither regular nor global well posedness implies the other.

3.3. Graphs and gain-function stability. In our investigation of robustness of stability properties of a closed-loop system, the concept of graphs and gain-function stability will play a central role. Corresponding to a plant operator P (respectively, the controller operator C) is a subset of $\mathcal{W}$, called the *graph* of the plant $\mathcal{G}_P$ (respectively, the controller $\mathcal{G}_C$), defined as

$$\mathcal{G}_P = \left\{ \begin{pmatrix} u \\ Pu \end{pmatrix} \,\middle|\, u \in \mathcal{U},\ Pu \in \mathcal{Y} \right\} \subset \mathcal{W}, \qquad \mathcal{G}_C = \left\{ \begin{pmatrix} Cy \\ y \end{pmatrix} \,\middle|\, Cy \in \mathcal{U},\ y \in \mathcal{Y} \right\} \subset \mathcal{W}.$$

Note that we identify $\mathcal{G}_P \ni \left(\begin{smallmatrix} u \\ Pu \end{smallmatrix}\right) = (u, Pu) \in \mathcal{W}$, and analogously for $\mathcal{G}_C$.

A causal operator $F\colon \mathcal{X} \to \mathcal{V}_a$, where $\mathcal{X}, \mathcal{V}$ are subsets of normed signal spaces, is said to be *gain-function stable* if and only if $F(\mathcal{X}) \subset \mathcal{V}$ and the following nonlinear so-called *gain function* is well defined:

$$\begin{aligned} & g[F] : (r_0, \infty) \to \mathbb{R}_{\geq 0}, \\ & \qquad r \mapsto g[F](r) = \sup \left\{ \|T_\tau F x\|_{\mathcal{V}} \,\middle|\, x \in \mathcal{X},\ \|T_\tau x\|_{\mathcal{X}} \in (r_0, r],\ \tau > 0 \right\}, \end{aligned} \tag{3.3}$$

where $r_0 := \inf_{x \in \mathcal{X}} \|x\|_{\mathcal{X}} < \infty$. Observe that $\|T_\tau F x\|_{\mathcal{V}} \leq g[F](\|T_\tau x\|_{\mathcal{X}})$. A closed-loop system $[P, C]$ is said to be *gain-function stable* if and only if it is globally well posed and $H_{P,C}\colon \mathcal{W} \to \mathcal{W}_e \times \mathcal{W}_e$ is gain-function stable.

Note the following facts:

(i) global well posedness of $[P, C]$ implies that $\operatorname{im} H_{P,C} \subset \mathcal{W}_e \times \mathcal{W}_e$;
(ii) gain-function stability of $[P, C]$ implies $\mathcal{W}$-stability of $[P, C]$;
(iii) if $[P, C]$ is $\mathcal{W}$-stable, then $H_{P,C} : \mathcal{W} \to \mathcal{G}_P \times \mathcal{G}_C$ is a bijective operator with inverse $H_{P,C}^{-1} : (w_1, w_2) \mapsto w_1 + w_2$.

To see (iii), note that $H_{P,C}(\mathcal{W}) \subset \mathcal{W} \times \mathcal{W}$ implies that $H_{P,C}(\mathcal{W}) \subset \mathcal{G}_P \times \mathcal{G}_C$, and since for any $w_1 \in \mathcal{G}_P \subset \mathcal{W}$, $w_2 \in \mathcal{G}_C \subset \mathcal{W}$ we have $w_1 + w_2 \in \mathcal{W}$, it follows that $H_{P,C}(\mathcal{W}) \supset \mathcal{G}_P \times \mathcal{G}_C$. Therefore, we can think of a gain-function stable $H_{P,C}$ as a surjective operator $H_{P,C} \colon \mathcal{W} \to \mathcal{G}_P \times \mathcal{G}_C$. The inverse of $H_{P,C} : \mathcal{W} \to \mathcal{G}_P \times \mathcal{G}_C$ is obviously $H_{P,C}^{-1} : (w_1, w_2) \mapsto w_1 + w_2$.

Finally, with a closed-loop system $[P, C]$, we associate the following two parallel projection operators: $\Pi_{P//C} : \mathcal{W} \to \mathcal{W}_a$, $w_0 \mapsto w_1$, and $\Pi_{C//P} : \mathcal{W} \to \mathcal{W}_a$, $w_0 \mapsto w_2$. Clearly, $H_{P,C} = \big(\Pi_{P//C}, \Pi_{C//P}\big)$ and $\Pi_{P//C} + \Pi_{C//P} = I$. Therefore, gain-function stability of one of the operators $\Pi_{P//C}$ and $\Pi_{C//P}$ implies the gain-function stability of the other, and so gain-function stability of either operator implies gain-function stability of the closed-loop system $[P, C]$.

3.4. The nonlinear gap. The essence of the paper is a study of robust stability in a specific adaptive control context. Robust stability is the property that the stability properties of a globally well-posed closed-loop system $[P, C]$ persists under "sufficiently small" perturbations of the plant. In other words, robust stability is the property that $[P_1, C]$ inherits the stability properties of $[P, C]$ when the plant P is replaced by any plant P_1 sufficiently "close" to P. In the context of this paper, plants P and P_1 are deemed to be close if and only if their respective graphs are *close* in the gap sense of [5]. The nonlinear gap is defined as follows:

Let, for signal spaces $\mathcal{U}$ and $\mathcal{Y}$,

$$\Gamma := \big\{ P : \mathcal{U}_a \to \mathcal{Y}_a \;\big|\; P \text{ is causal} \big\}$$

and, for $P_1, P_2 \in \Gamma$, define the (possibly empty) set

$$\mathcal{O}_{P_1,P_2} := \big\{ \Phi : \mathcal{G}_{P_1} \to \mathcal{G}_{P_2} \;\big|\; \Phi \text{ is causal, surjective, and } \Phi(0) = 0 \big\}.$$

The *directed nonlinear gap* is given by

$$\vec{\delta} : \Gamma \times \Gamma \to [0, \infty],$$

$$(P_1, P_2) \mapsto \vec{\delta}(P_1, P_2) := \inf_{\Phi \in \mathcal{O}_{P_1,P_2}} \sup_{x \in \mathcal{G}_{P_1} \setminus \{0\},\, \tau > 0} \left(\frac{\|T_\tau (\Phi - I)(x)\|_{\mathcal{U} \times \mathcal{Y}}}{\|T_\tau x\|_{\mathcal{U} \times \mathcal{Y}}} \right),$$

with the convention that $\vec{\delta}(P_1, P_2) := \infty$ if $\mathcal{O}_{P_1,P_2} = \emptyset$, and the *nonlinear gap* δ is

$$\delta : \Gamma \times \Gamma \to [0, \infty], \qquad (P_1, P_2) \mapsto \delta(P_1, P_2) := \max\{\vec{\delta}(P_1, P_2), \vec{\delta}(P_2, P_1)\}.$$

3.5. Example. In this subsection we illustrate the previous graph and gap concepts by two operators $P_\alpha, P_{N,M,\alpha}$ induced by state space systems

$$\left\{ \begin{array}{llll} P_\alpha & : & \dot{x} = \alpha x + u_1, & x(0) = x^0, \\ & & y_1 = x, & \end{array} \right. \tag{3.4}$$

$$\left\{ \begin{array}{llll} P_{N,M,\alpha} & : & \dot{x} = \widetilde{A}\, x + \widetilde{b}\, u_1, & x(0) = \widetilde{x}^0, \\ & & y_1 = \widetilde{c}\, x & \end{array} \right. \tag{3.5}$$

for $\alpha > 0$, $x^0 \in \mathbb{R}$, and $\big(\widetilde{A}, \widetilde{b}, \widetilde{c}\big)$ as in (1.6), with $\widetilde{x}^0 \in \mathbb{R}^3$. Throughout this example assume that $x^0 = 0$, $\widetilde{x}^0 = 0$. The second purpose of this example is to show that P_α is close to $P_{N,M,\alpha}$ in the sense that

$$\limsup_{M \to \infty} \vec{\delta}(P_\alpha, P_{2M,M,\alpha}) = 0. \tag{3.6}$$

First, recall that $(\widetilde{A}, \widetilde{b}, \widetilde{c}) \in \mathcal{P}_{3,1} \setminus \widetilde{\mathcal{M}}_{3,1}$ and $(\alpha, 1, 1) \in \mathcal{M}_{1,1}$.

Second, recall that the graphs of P_α and $P_{N,M,\alpha}$ are given, respectively, by

$$\mathcal{G}_{P_\alpha} = \left\{ \begin{pmatrix} u_1 \\ y_1 \end{pmatrix} \middle| \; u_1, y_1 \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}),\; y_1 \text{ solves } (3.4) \right\},$$

$$\mathcal{G}_{P_{N,M,\alpha}} = \left\{ \begin{pmatrix} u_1 \\ y_1 \end{pmatrix} \middle| \; u_1, y_1 \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}),\; y_1 \text{ solves } (3.5) \right\}.$$

To determine an upper bound for the *gap* between P_α and $P_{N,M,\alpha}$, consider the bijective mapping Φ from graph $\mathcal{G}_{P_\alpha}$ to graph $\mathcal{G}_{P_{N,M,\alpha}}$ given by

$$\Phi \,:\, \mathcal{G}_{P_\alpha} \to \mathcal{G}_{P_{N,M,\alpha}}, \quad \begin{pmatrix} u \\ \int_0^{\cdot} e^{\alpha(\cdot - s)}\, u(s)\,\mathrm{d}s \end{pmatrix} \mapsto \begin{pmatrix} u \\ \widetilde{c} \int_0^{\cdot} e^{\widetilde{A}(\cdot - s)} \widetilde{b}\, u(s)\,\mathrm{d}s \end{pmatrix}.$$

By the definition of the nonlinear gap (see section 3.4), we obtain

$$\vec{\delta}(P_\alpha, P_{N,M,\alpha}) \leq \sup_{w \in \mathcal{G}_{P_\alpha} \setminus \{0\}} \frac{\|(\Phi - I)(w)\|_{\mathcal{W}}}{\|w\|_{\mathcal{W}}},$$

where $\mathcal{W} := W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ and, for $w = (u, y) \in \mathcal{W}$, the norm is defined by

$$\|(u, y)\|_{\mathcal{W}} := \|u\|_{W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})} + \|y\|_{W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})}.$$

To estimate

$$|(\Phi - I)(w)(t)| \quad \text{for} \quad w := \begin{pmatrix} u \\ \int_0^{\cdot} e^{\alpha(\cdot - s)}\, u(s)\,\mathrm{d}s \end{pmatrix} \in \mathcal{G}_{P_\alpha}$$

we calculate that the output y_1 of (3.5) is given, for all $t \geq 0$, by

$$\begin{aligned} y_1(t) = \widetilde{c} \int_0^t e^{\widetilde{A}(t-s)} \widetilde{b}\, u_1(s)\,\mathrm{d}s = & \int_0^t \frac{N(M - \alpha)}{(\alpha + N)(\alpha + M)} e^{\alpha(t-s)}\, u_1(s)\,\mathrm{d}s \\ & + \int_0^t \frac{N(N + M)}{(N - M)(\alpha + N)} e^{-N(t-s)}\, u_1(s)\,\mathrm{d}s \\ & + \int_0^t \frac{-2NM}{(N - M)(\alpha + M)} e^{-M(t-s)}\, u_1(s)\,\mathrm{d}s, \end{aligned}$$

and thus, for all $t \geq 0$,

$$\begin{aligned}
|(\Phi - I)(w)(t)| \leq & \left| \left(\tfrac{N(M-\alpha)}{(\alpha+N)(\alpha+M)} - 1 \right) \int_0^t e^{\alpha(t-s)}\, u(s)\, \mathrm{d}s \right| \\
& + \left| \tfrac{N(N+M)}{(N-M)(\alpha+N)} \int_0^t e^{-N(t-s)}\, u(s)\, \mathrm{d}s \right| \\
& + \left| \tfrac{-2NM}{(N-M)(\alpha+M)} \int_0^t e^{-M(t-s)}\, u(s)\, \mathrm{d}s \right| \\
\leq & \left| \tfrac{N(M-\alpha)}{(\alpha+N)(\alpha+M)} - 1 \right| \left| \int_0^t e^{\alpha(t-s)}\, u(s)\, \mathrm{d}s \right| \\
& + \left(\left| \tfrac{N(N+M)}{(N-M)(\alpha+N)} \int_0^t e^{-N(t-s)}\, \mathrm{d}s \right| \right. \\
& \left. + \left| \tfrac{-2NM}{(N-M)(\alpha+M)} \int_0^t e^{-M(t-s)}\, \mathrm{d}s \right| \right) \|u\|_{L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})} \\
\leq & \left| \tfrac{N(M-\alpha)}{(\alpha+N)(\alpha+M)} - 1 \right| \left\| \int_0^\cdot e^{\alpha(\cdot-s)}\, u(s)\, \mathrm{d}s \right\|_{W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})} \\
& + \left(\left| \tfrac{N+M}{(N-M)(\alpha+N)} \right| + \left| \tfrac{2N}{(N-M)(\alpha+M)} \right| \right) \|u\|_{W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})}.
\end{aligned}$$

Hence

$$\vec{\delta}(P_\alpha, P_{N,M,\alpha}) \leq \left| \frac{N(M-\alpha)}{(\alpha+N)(\alpha+M)} - 1 \right| + \left| \frac{N+M}{(N-M)(\alpha+N)} \right| + \left| \frac{2N}{(N-M)(\alpha+M)} \right|,$$

which yields (3.6).

4. Robustness of the λ-tracker.

4.1. Well posedness of the closed-loop system.

For $m, n \in \mathbb{N}$ with $n \geq m$, consider $\mathcal{P}_{n,m}$ as a subspace of the Euclidean space $\mathbb{R}^{n^2+2mn}$ by identifying a plant $\theta = (A, B, C)$ with a vector θ consisting of the elements of the plant matrices, ordered lexicographically. With normed signal spaces $\mathcal{U}$ and $\mathcal{Y}$ and $(\theta, x^0) \in \mathcal{P}_{n,m} \times \mathbb{R}^n$, where $x^0 \in \mathbb{R}^n$ is the initial value of a linear system (1.1), we associate the causal plant operator

$$\widetilde{P}(\theta, x^0) \; : \; \mathcal{U}_a \to \mathcal{Y}_a, \qquad u_1 \mapsto \widetilde{P}(\theta, x^0)(u_1) := y_1, \tag{4.1}$$

where, for $u_1 \in \mathcal{U}_a$ with $\mathrm{dom}(u_1) = [0, \omega)$, we have $y_1 = cx$, x being the unique solution of (1.1) on $[0, \omega)$. Note that $\widetilde{P}$ is a map from $\cup_{n \geq m}(\mathcal{P}_{n,m} \times \mathbb{R}^n)$ to the space of maps $\mathcal{U}_a \to \mathcal{Y}_a$. Consider, for $\lambda > 0$, the adaptive strategy (1.5) and associate the causal control operator, parameterized by λ and the initial value $k^0 \in \mathbb{R}$, i.e.,

$$\widetilde{C}(\lambda, k^0) \ : \ \mathcal{Y}_a \to \mathcal{U}_a, \qquad y_2 \mapsto \widetilde{C}(\lambda, k^0)(y_2) := u_2. \tag{4.2}$$

Note that $\widetilde{C}$ is a map from $\mathbb{R}_{>0} \times \mathbb{R}$ to the space of causal maps $\mathcal{Y}_a \to \mathcal{U}_a$.

In this subsection we show that, for $\mathcal{U} = \mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, the closed-loop system $[\widetilde{P}(\theta, x^0), \widetilde{C}(\lambda, k^0)]$ of any plant of the form (1.1) (with associated operator $\widetilde{P}(\theta, x^0)$) and adaptive controller (1.5) (with associated operator $\widetilde{C}(\lambda, k^0)$), where $(\theta, x^0) \in \mathcal{P}_{n,m} \times \mathbb{R}^n$ and $(\lambda, k^0) \in \mathbb{R}_{>0} \times \mathbb{R}$, is regularly well posed. Furthermore, we show that, for $\theta \in \mathcal{M}_{n,m}$, the closed-loop system $[\widetilde{P}(\theta, x^0), \widetilde{C}(\lambda, k^0)]$ is globally well posed and $(\mathcal{U} \times \mathcal{Y})$-stable.

PROPOSITION 4.1. *Let $m, n \in \mathbb{N}$ with $n \geq m$, $\lambda > 0$, $(\theta, x^0, k^0) \in \mathcal{M}_{n,m} \times \mathbb{R}^n \times \mathbb{R}$, and $u_0, y_0 \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$. Then, for plant operator $\widetilde{P}(\theta, x^0)$ and control operator $\widetilde{C}(\lambda, k^0)$, given by* (4.1) *and* (4.2)*, respectively, the closed-loop initial value problem $[\widetilde{P}(\theta, x^0), \widetilde{C}(\lambda, k^0)]$, given by* (1.8), (1.2), (1.5)*, is globally well posed and $\big(W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)\big)$-stable.*

Proof. The proposition is a direct consequence of Proposition 2.1. ∎

Note that, for $(A, B, C) \in \mathcal{P}_{n,m}$, $x^0 \in \mathbb{R}^n$, $\lambda > 0$, and $k^0 \in \mathbb{R}$, the closed-loop initial value problem (1.1), (1.2), (1.5) may be written as

$$\left\{ \begin{array}{rcll} \dot{x}(t) & = & Ax(t) + B[u_0(t) - u_2(t)], & x(0) = x^0 \in \mathbb{R}^n, \\ \dot{k}(t) & = & d_\lambda(y_2(t))\,|y_2(t)|, & k(0) = k^0 \in \mathbb{R}, \\ y_2(t) & = & y_0(t) - C\,x(t), & \\ u_2(t) & = & -k(t)y_2(t), & \end{array} \right. \tag{4.3}$$

where d_λ is defined in the Nomenclature.

PROPOSITION 4.2. *Let $m, n \in \mathbb{N}$ with $n \geq m$, $\lambda > 0$, $(\theta, x^0, k^0) \in \mathcal{P}_{n,m} \times \mathbb{R}^n \times \mathbb{R}$, and $u_0, y_0 \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$. Then, for plant operator $\widetilde{P}(\theta, x^0)$ and control operator $\widetilde{C}(\lambda, k^0)$, given by* (4.1) *and* (4.2)*, respectively, the closed-loop initial value problem $[\widetilde{P}(\theta, x^0), \widetilde{C}(\lambda, k^0)]$, given by* (4.3)*, has the following properties:*

(i) *there exists a unique maximal solution $(x, k) \ : \ [0, \omega) \to \mathbb{R}^n \times \mathbb{R}$ for some $\omega \in (0, \infty]$;*
(ii) *if $k \in W^{1,\infty}([0, \omega) \to \mathbb{R})$, then $\omega = \infty$;*
(iii) *if $y_2 \in W^{1,\infty}([0, \omega) \to \mathbb{R}^m)$, then $\omega = \infty$;*
(iv) *$[\widetilde{P}(\theta, x^0), \widetilde{C}(\lambda, k^0)]$ is regularly well posed.*

Proof. (i) Since the right-hand side of (4.3) is continuous and locally Lipschitz, the statement follows from the theory of ordinary differential equations.

(ii) Suppose $k \in W^{1,\infty}([0, \omega) \to \mathbb{R})$ and, for contradiction, $\omega < \infty$. Since $d_\lambda(y_2)^2 \leq d_\lambda(y_2)\,|y_2| = \dot{k} \in L^\infty([0, \omega) \to \mathbb{R}_{\geq 0})$, we have $d_\lambda(y_2) \in L^\infty([0, \omega) \to \mathbb{R}_{\geq 0})$ and $d_\lambda(y_2) + \lambda \in L^\infty([0, \omega) \to \mathbb{R}_{\geq 0})$. Thus $y_2 \in L^\infty([0, \omega) \to \mathbb{R}^m)$.

Since $k \in L^\infty([0, \omega) \to \mathbb{R})$, variation of constants applied to (4.3) yields the existence of constants $c_0, c_1 > 0$ such that

$$\forall\, t \in [0, \omega) \ : \ |x(t)| \leq c_0 \left(e^{c_1 \omega} + \int_0^\omega e^{c_1(\omega - s)} \left(|u_0(s)| + |y_2(s)|\right) \mathrm{d}s \right). \tag{4.4}$$

Since $y_2 \in L^\infty([0, \omega) \to \mathbb{R}^m)$ and $u_0 \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, it follows from the convolution in (4.4) that the right-hand side of (4.4) is bounded on $[0, \omega)$, which contradicts the maximality of the solution x. Hence $\omega = \infty$.

(iii) Suppose $y_2 \in W^{1,\infty}([0,\omega) \to \mathbb{R}^m)$ and, for contradiction, $\omega < \infty$. Then $\dot{k} = d_\lambda(y_2)\,|y_2| \in L^\infty([0,\omega) \to \mathbb{R})$, and, combined with

$$\forall\, t \in [0,\omega)\ :$$

$$k(t) - k^0 = \int_0^t d_\lambda(y_2(s))\,|y_2(s)|\,\mathrm{d}s \le \int_0^t \|y_2\|^2_{L^\infty([0,\omega)\to\mathbb{R}^m)}\,\mathrm{d}s = \omega\,\|y_2\|^2_{L^\infty([0,\omega)\to\mathbb{R}^m)},$$

we arrive at $k \in W^{1,\infty}([0,\omega) \to \mathbb{R})$. Now (ii) yields that $\omega = \infty$. This is a contradiction, and so $\omega = \infty$.

(iv) Let $\mathcal{W} = W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$. By (i), the closed-loop $[\widetilde{P}(\theta,x^0), \widetilde{C}(\lambda,k^0)]$ is locally well posed. To prove that $[\widetilde{P}(\theta,x^0), \widetilde{C}(\lambda,k^0)]$ is regularly well posed, it suffices to show that (3.2) holds. For arbitrary $w_0 = (u_0,y_0) \in \mathcal{W}$ consider $(w_1,w_2) = H_{\widetilde{P}(\theta,x^0),\widetilde{C}(\lambda,k^0)}(w_0)$, where $\mathrm{dom}(w_1,w_2) = [0,\omega)$ is maximal. Suppose, contrary to the right-hand side of (3.2), that $T_\omega(w_1,w_2) \in \mathcal{W} \times \mathcal{W}$. Then $y_2 \in W^{1,\infty}([0,\omega) \to \mathbb{R}^m)$, which, in view of (iii), yields $\omega = \infty$, i.e., the contrary of the left-hand side of (3.2). Hence the closed-loop system is regularly well posed. □

4.2. Robustness. In Propositions 4.1 and 4.2 we have established that, for $(\theta,x^0,k^0) \in \mathcal{M}_{n,m} \times \mathbb{R}^n \times \mathbb{R}$ and $m,n \in \mathbb{N}$ with $n \ge m$, $\lambda > 0$, $u_0,y_0 \in W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$, the closed-loop system $[\widetilde{P}(\theta,x^0), \widetilde{C}(\lambda,k^0)]$ is globally well posed and has certain stability properties. Furthermore, in Proposition 2.1 λ-tracking is shown for linear systems belonging to class $\mathcal{M}_{n,m}$.

The purpose of this subsection is to determine conditions under which these properties are maintained when the plant $\widetilde{P}(\theta,x^0)$ is perturbed to a plant $\widetilde{P}(\widetilde{\theta},\widetilde{x}^0)$, where $(\widetilde{\theta},\widetilde{x}^0) \in \mathcal{P}_{q,m} \times \mathbb{R}^q$ for some $q \in \mathbb{N}$, in particular when $\widetilde{\theta} \notin \mathcal{M}_{q,m}$. The main result, Theorem 4.5, shows that stability properties and λ-tracking persist if (a) the plants $\widetilde{P}(\widetilde{\theta},0)$ and $\widetilde{P}(\theta,0)$ are sufficiently close (in the gap sense) and (b) the initial data $\widetilde{x}^0$ and disturbance $w_0 = (u_0,y_0)$ are sufficiently small.

To establish gap margin results, we will need to construct the augmented plant and controller operators as in [4]. Note that $0 \notin \mathcal{M}_{n,m}$. Define $\widetilde{\mathcal{U}} := \mathbb{R}^{n^2+2n} \times W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$ and let $\widetilde{\mathcal{W}} := \widetilde{\mathcal{U}} \times W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$, which can be considered as signal spaces by identifying $\theta \in \mathbb{R}^{n^2+2mn}$ with the constant function $t \mapsto \theta$ and endowing $\widetilde{\mathcal{U}}$ with the norm $\|(\theta,u)\|_{\widetilde{\mathcal{U}}} := \sqrt{|\theta|^2 + \|u\|^2_{W^{1,\infty}(\mathbb{R}_{\ge 0}\to\mathbb{R}^m)}}$. For given $\widetilde{P}(\theta,0)$ as in (4.1), we define the (augmented) plant operator as

$$(4.5)\quad P\ :\ \widetilde{\mathcal{U}}_a \to W_a^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m), \qquad (\theta,u_1) = \widetilde{u}_1 \mapsto y_1 = P(\widetilde{u}_1) := \widetilde{P}(\theta,0)(u_1).$$

Fix $\lambda > 0$ and $k^0 \in \mathbb{R}$ and define, for $\widetilde{C}(\lambda,k^0)$ as in (4.2), the (augmented) controller operator as

(4.6)

$$C\ :\ W_a^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m) \to \widetilde{\mathcal{U}}_a, \qquad y_2 \mapsto \widetilde{u}_2 = C(y_2) := \left(0, \widetilde{C}(\lambda,k^0)(y_2)\right) = (0,u_2)\,.$$

For each nonempty $\Omega \subset \mathcal{M}_{n,m}$, define

(4.7)

$$\mathcal{W}^\Omega := \left(\Omega \times W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)\right) \times W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m) \quad \text{and} \quad H^\Omega_{P,C} := H_{P,C}|_{\mathcal{W}^\Omega}.$$

It follows from Proposition 4.1 that $H^{\Omega}_{P,C} : \mathcal{W}^{\Omega} \to \widetilde{\mathcal{W}} \times \widetilde{\mathcal{W}}$ is a causal operator for any $\Omega \subset \mathcal{M}_{n,m}$. In Proposition 4.3 we show gain-function stability of $H^{\Omega}_{P,C}$. This is a supposition of Theorem 5.2 in [3], the latter being used to show Proposition 4.4 and thus the main result, Theorem 4.5.

PROPOSITION 4.3. *Let $m, n \in \mathbb{N}$ with $n \geq m$, $k^0 \in \mathbb{R}$, and $\lambda > 0$ and assume $\Omega \subset \mathcal{M}_{n,m}$ is closed. Then, for the closed-loop system $[P, C]$ given by* (3.1), (4.5), *and* (4.6), *the operator $H^{\Omega}_{P,C}$ given by* (4.7) *is gain-function stable.*

Proof. Note that $((\theta, u_1), y_1) = ((\theta, u_0), y_0) - ((0, u_2), y_2)$. For $\nu : \mathcal{D}_{n,m} \to \mathbb{R}_{\geq 0}$ as in Proposition 2.1 and $\mathcal{W}^{\Omega}$ given by (4.7), we have

$$\forall\ ((\theta, u_0), y_0) \in \mathcal{W}^{\Omega} \ :$$

$$\begin{aligned} \left\| H^{\Omega}_{P,C}((\theta, u_0), y_0) \right\|_{\widetilde{\mathcal{W}} \times \widetilde{\mathcal{W}}} &= \left\| \left(((\theta, u_1), y_1), ((0, u_2), y_2) \right) \right\|_{\widetilde{\mathcal{W}} \times \widetilde{\mathcal{W}}} \\ &\leq \| ((\theta, u_0), y_0) \|_{\widetilde{\mathcal{W}}} + 2 \| ((0, u_2), y_2) \|_{\widetilde{\mathcal{W}}} \\ &\leq \| (u_0, y_0) \|_{\mathcal{W}} + |\theta| + 2\, \nu(\theta, (0, k^0), u_0, y_0), \end{aligned}$$

and so, for $r_0 := \inf_{w \in \mathcal{W}^{\Omega}} \|w\|_{\widetilde{\mathcal{W}}}$ and $r \in (r_0, \infty)$, closedness of Ω yields

$$g\left[H^{\Omega}_{P,C}\right](r)$$

$$:= \sup \left\{ \| (u_0, y_0) \|_{\mathcal{W}} + |\theta| + 2\, \nu(\theta, (0, k^0), u_0, y_0) \;\middle|\; \begin{array}{c} (\theta, u_0, y_0) \in \mathcal{W}^{\Omega}, \\ \| (\theta, u_0, y_0) \|_{\widetilde{\mathcal{W}}} \leq r \end{array} \right\} < \infty.$$

Thus, a gain function for $H^{\Omega}_{P,C}$ exists, and the proof is complete. ∎

The following proposition establishes $\big(W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)\big)$-stability of the closed-loop system $[\widetilde{P}(\widetilde{\theta}, \widetilde{x}^0), \widetilde{C}(\lambda, k^0)]$ for a system $\widetilde{\theta}$ belonging to the system class $\mathcal{P}_{q,m}$ if, for a system θ belonging to $\mathcal{M}_{n,m}$, the gap between $\widetilde{P}(\widetilde{\theta}, 0)$ and $\widetilde{P}(\theta, 0)$, the initial value $\widetilde{x}^0 \in \mathbb{R}^q$ and the input/output disturbances $w_0 = (u_0, y_0)$ are sufficiently small. The proof is based on results from [3].

PROPOSITION 4.4. *Let $m, n, q \in \mathbb{N}$ with $n, q \geq m$, $\mathcal{U} = \mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$, and $\theta \in \mathcal{M}_{n,m}$. For $(\widetilde{\theta}, \widetilde{x}^0, k^0) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \mathbb{R}$ and $\lambda > 0$, consider $\widetilde{P}(\widetilde{\theta}, \widetilde{x}^0) \colon \mathcal{U}_a \to \mathcal{Y}_a$ and $\widetilde{C}(\lambda, k^0) \colon \mathcal{Y}_a \to \mathcal{U}_a$ defined by* (4.1) *and* (4.2), *respectively. Then there exist a continuous function $\eta \colon (0, \infty) \to (0, \infty)$ and a function $\psi \colon \mathcal{P}_{q,m} \to (0, \infty)$ such that the following holds:*

$$\text{(4.8)} \quad \forall\ \left(\widetilde{\theta}, \widetilde{x}^0, w_0, r\right) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \mathcal{W} \times (0, \infty) \ :$$

$$\left. \begin{array}{l} \psi(\widetilde{\theta}) |\widetilde{x}^0| + \|w_0\|_{\mathcal{W}} \leq r, \\ \vec{\delta}\left(\widetilde{P}(\theta, 0), \widetilde{P}(\widetilde{\theta}, 0)\right) \leq \eta(r) \end{array} \right\} \quad \Longrightarrow \quad H_{\widetilde{P}(\widetilde{\theta}, \widetilde{x}^0), \widetilde{C}(\lambda, k^0)}(w_0) \in \mathcal{W} \times \mathcal{W}.$$

Proof. We need to show how the gain-function stability of the augmented closed loop $[P, C]$, given by (4.5) and (4.6), yields the robustness property (4.8) for the unaugmented closed-loop system $[\widetilde{P}(\widetilde{\theta}, \widetilde{x}^0), \widetilde{C}(\lambda, k^0)]$.

By Proposition 4.2 the closed-loop system $[\widetilde{P}(\widetilde{\theta}, \widetilde{x}^0), \widetilde{C}(\lambda, k^0)]$ is regularly well posed for all $\widetilde{\theta} \in \mathcal{P}_{q,m}$. Consider the augmented operators defined by (4.5) and (4.6), i.e.,

$$\begin{aligned} P\colon \mathcal{P}_{n,m}\times\mathcal{U}_a\to\mathcal{Y}_a, \qquad (\widetilde{\theta},u_1)\mapsto P(\widetilde{\theta},u_1) &= \widetilde{P}(\widetilde{\theta},0)(u_1),\\ C\colon \mathcal{Y}_a\to\mathcal{P}_{n,m}\times\mathcal{U}_a, \qquad y_2\mapsto C(y_2) &= (0,\widetilde{C}(\lambda,k^0)(y_2)). \end{aligned}$$

For $\theta\in\mathcal{M}_{n,m}$ set $\Omega=\{\theta\}$. By Proposition 4.3, $H^{\Omega}_{P,C}=H_{P,C}|_{\mathcal{W}^\Omega}$, given by (4.7), is gain-function stable. By, for example, the proof of Theorem 4.D in [18], $T_\tau\Pi_{\widetilde{P}(\theta,0)//\widetilde{C}(\lambda,k^0)}$ is continuous for all $\tau>0$, and so $T_\tau\Pi_{P//C}|_{\mathcal{W}^\Omega}$ is continuous for all $\tau>0$.

Then [3, Theorem 5.2] gives the existence of a continuous function $\mu\colon(0,\infty)\times\Omega\to(0,\infty)$ such that

$$\forall\ (\theta,\widetilde{\theta},w_0,r)\in\Omega\times\mathcal{P}_{q,m}\times\mathcal{W}\times(0,\infty)\ :$$

$$\left[\|w_0\|_{\mathcal{W}}\le r \quad\wedge\quad \vec{\delta}\left(\widetilde{P}(\theta,0),\widetilde{P}(\widetilde{\theta},0)\right)\le\mu(r,\theta)\right] \implies H_{\widetilde{P}(\widetilde{\theta},0),\widetilde{C}(\lambda,k^0)}(w_0)\in\mathcal{W}\times\mathcal{W}.$$

Note that the proof of [3, Theorem 5.2] holds also for the signal space $W^{1,\infty}(\mathbb{R}_{\ge0}\to\mathbb{R}^m)$, although it is proved in [3] for $\mathcal{U}=\mathcal{Y}=L^p(\mathbb{R}_{\ge0}\to\mathbb{R}^m)$, $1\le p\le\infty$.

To prove (4.8) we will use [3, Theorem 5.3]. The statement of [3, Theorem 5.3] has been proved for $\mathcal{U}=\mathcal{Y}=L^p(\mathbb{R}_{\ge0}\to\mathbb{R}^m)$, $1\le p\le\infty$. The statement holds also for $\mathcal{U}=\mathcal{Y}=W^{1,\infty}(\mathbb{R}_{\ge0}\to\mathbb{R}^m)$. To see this, invoke the fact that for any Hurwitz matrix $M\in\mathbb{R}^{n\times n}$ it is $\big(t\mapsto\exp(M\,t)\big)$, $\big(t\mapsto\frac{\mathrm{d}}{\mathrm{d}t}\exp(M\,t)\big)\in L^\infty(\mathbb{R}_{\ge0}\to\mathbb{R}^{n\times n})$. Now the statement of [3, Theorem 5.3] for $\mathcal{U}=\mathcal{Y}=W^{1,\infty}(\mathbb{R}_{\ge0}\to\mathbb{R}^m)$ yields the existence of a continuous function $\mu\colon(0,\infty)\times\Omega\to(0,\infty)$ and a function $\psi\colon\mathcal{P}_{q,m}\to(0,\infty)$ such that

$$\text{(4.9)}\quad \forall\ (\widetilde{\theta},\theta,\widetilde{x}^0,w_0,r)\in\mathcal{P}_{q,m}\times\mathcal{M}_{n,m}\times\mathbb{R}^q\times\mathcal{W}\times(0,\infty)\ :$$

$$\left.\begin{array}{l}\psi(\widetilde{\theta})|\widetilde{x}^0|+\|w_0\|_{\mathcal{W}}\le r,\\ \vec{\delta}\left(\widetilde{P}(\theta,0),\widetilde{P}(\widetilde{\theta},0)\right)\le\mu\,(r,\theta)\end{array}\right\}\quad\implies\quad H_{\widetilde{P}(\widetilde{\theta},\widetilde{x}^0),\widetilde{C}(\lambda,k^0)}(w_0)\in\mathcal{W}\times\mathcal{W}.$$

Finally, statement (4.8) follows on setting $\eta(\cdot)=\mu(\cdot,\theta)$. ◻

Note that [3, Theorem 5.3] requires stabilizability of system $\widetilde{\theta}\in\mathcal{P}_{q,m}$.

Finally, we are in the position to state and prove the main result of the present paper. Loosely speaking, we show that the λ-tracker also works for systems $\big(\widetilde{A},\widetilde{B},\widetilde{C}\big)\in\mathcal{P}_{q,m}$ which are not necessarily minimum phase, may have higher relative degree, and may have negative high-frequency gain. However, $\big(\widetilde{A},\widetilde{B},\widetilde{C}\big)$ has to be sufficiently close—in the terms of the gap metric—to a system $(A,B,C)\in\widetilde{\mathcal{M}}_{n,m}$, and the initial value $\widetilde{x}^0\in\mathbb{R}^q$ for $\big(\widetilde{A},\widetilde{B},\widetilde{C}\big)$ and the input/output disturbances (u_0,y_0) have to be sufficiently small.

Theorem 4.5. *Let $m,n,q\in\mathbb{N}$ with $n,q\ge m$, $\mathcal{U}=\mathcal{Y}=W^{1,\infty}(\mathbb{R}_{\ge0}\to\mathbb{R}^m)$, $\mathcal{W}=\mathcal{U}\times\mathcal{Y}$, $k^0\in\mathbb{R}$, $\lambda>0$, and $\theta\in\mathcal{M}_{n,m}$. For $(\widetilde{\theta},\widetilde{x}^0)\in\mathcal{P}_{q,m}\times\mathbb{R}^q$ consider the associated operators $\widetilde{P}(\widetilde{\theta},\widetilde{x}^0)\colon\mathcal{U}_a\to\mathcal{Y}_a$ and $\widetilde{C}(\lambda,k^0)\colon\mathcal{Y}_a\to\mathcal{U}_a$ defined by* (4.1) *and* (4.2)*, respectively, and the closed-loop initial value problem* (1.1), (1.2), (1.5). *Then there exist a continuous function $\eta\colon(0,\infty)\to(0,\infty)$ and a function $\psi\colon\mathcal{P}_{q.m}\to(0,\infty)$*

such that the following holds:

$$\text{(4.10)} \quad \forall \ \left(\widetilde{\theta}, \widetilde{x}^0, w_0, r\right) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \mathcal{W} \times (0,\infty) \ :$$

$$\left.\begin{array}{l} \psi(\widetilde{\theta})|\widetilde{x}^0| + \|w_0\|_{\mathcal{W}} \le r, \\ \vec{\delta}\left(\widetilde{P}(\theta,0), \widetilde{P}(\widetilde{\theta},0)\right) \le \eta(r) \end{array}\right\} \implies \left\{\begin{array}{l} \limsup_{t\to\infty} |y_2(t)| \le \lambda, \\ k \in W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}), \\ x \in W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^q), \end{array}\right.$$

where (x,k) *and* y_2 *satisfy* (4.3).

Proof. *Step* 1. We show that

$$((u_1,y_1),(u_2,y_2)) = H_{\widetilde{P}(\widetilde{\theta},\widetilde{x}^0),\widetilde{C}(\lambda,k^0)}(w_0) \in \mathcal{W} \times \mathcal{W}. \tag{4.11}$$

Choose functions $\eta\colon (0,\infty) \to (0,\infty)$ and $\psi\colon \mathcal{P}_{q,m} \to (0,\infty)$ from Proposition 4.4. Let

$$\left(\widetilde{\theta}, \widetilde{x}^0, w_0, r\right) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \mathcal{W} \times (0,\infty) \ : \quad \psi(\widetilde{\theta})|\widetilde{x}^0| + \|w_0\|_{\mathcal{W}} \le r \ \wedge \ \vec{\delta}\left(\widetilde{P}(\theta,0), \widetilde{P}(\widetilde{\theta},0)\right) \le \ \eta(r).$$

Then Proposition 4.4 gives (4.11).

Step 2. By Proposition 4.2 it follows that (4.3) has a unique solution

$$(x,k)\colon [0,\omega) \to \mathbb{R}^q \times \mathbb{R}$$

on a maximal interval of existence $[0,\omega)$ for some $\omega \in (0,\infty]$. Proposition 4.2(iii) yields $\omega = \infty$.

Step 3. We show that $\dot{k} \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R})$. Suppose, for contradiction, that $\dot{k} \notin L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R})$; i.e., there exists a sequence $(t_i) \in (\mathbb{R}_{\ge 0})^{\mathbb{N}}$ with $t_i > t_{i+1}$ and $\lim_{i\to\infty} \dot{k}(t_i) = \infty$. Then

$$\lim_{i\to\infty} d_\lambda(y_2(t_i))\, |y_2(t_i)| = \infty$$

and thus

$$\lim_{i\to\infty} |y_2(t_i)| = \infty,$$

a contradiction to Step 1.

Step 4. We show that $k \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R})$. Suppose, for contradiction, that $k \notin L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R})$; i.e., $\lim_{t\to\infty} k(t) = \infty$. Since $u_2 \in W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$, the fourth equation in (4.3) yields $\lim_{t\to\infty} y_2(t) = 0$, and thus

$$\exists\, T > 0 \quad \forall\, t \ge T \ : \ \dot{k}(t) = d_\lambda(y_2(t))\, |y_2(t)| = 0,$$

which contradicts the assumption on k.

Step 5. By Steps 3 and 4 we obtain $k \in W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R})$.

Step 6. By Proposition 4.4 we have in particular $y_2, \dot{y}_2 \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$. Similar to Step 10 of the proof of Proposition 2.1, we may establish that y_2 is uniformly continuous.

Step 7. By Step 6 and continuity of $e \mapsto d_\lambda(e)|e|$ we obtain that $t \mapsto d_\lambda(y_2(t))$ $|y_2(t)|$ is uniformly continuous. Hence, in view of $\dot{k} = d_\lambda(y_2)|y_2| \in L^1(\mathbb{R}_{\ge 0} \to$

$\mathbb{R}$), which is equivalent to $k \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$, and Barbălat's lemma (see [1]), $\lim_{t\to\infty} d_\lambda(y_2(t))|y_2(t)| = 0$ holds. This gives $\limsup_{t\to\infty} |y_2(t)| \leq \lambda$.

Step 8. It remains to show that $x \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$. Let $(\widetilde{A}, \widetilde{B}, \widetilde{C}) \in \mathcal{P}_{q,m}$ associated with (1.1). Detectability of $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ yields the existence of $F \in \mathbb{R}^{q\times m}$ such that $\operatorname{spec}(\widetilde{A}+F\widetilde{C}) \subset \mathbb{C}_-$. Setting $g := -[F + k\,\widetilde{B}]\,(y_0 - y_2) + \widetilde{B}\,u_0 + \widetilde{B}\,ky_0$ gives

$$\dot{x} = \left[\widetilde{A} - k\,\widetilde{B}\widetilde{C}\right] x + \widetilde{B}\,u_0 + \widetilde{B}\,ky_0 = \left[\widetilde{A} + F\widetilde{C}\right] x + g. \tag{4.12}$$

By Proposition 4.4 and Step 5 we have $y_2 \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ and $k \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$, and since $w_0 = (u_0, y_0) \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ it follows that $g \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$. Hence, by (4.12) we obtain $x \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$. The first equation in (4.3) then gives $\dot{x} \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$, which shows that $x \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$, and the proof is complete. □

Example 4.6. Finally, we revisit example (1.6).

In subsection 3.5 we have already shown that for zero initial conditions the gap between the system $(\widetilde{A}, \widetilde{b}, \widetilde{c}) \in \mathcal{P}_{3,1} \setminus \mathcal{M}_{3,1}$ and $(\alpha, 1, 1) \in \mathcal{M}_{1,1}$ tends to zero as $N = 2M$ and M tends to infinity; see (3.6). Now, in view of Theorem 4.5, there exist a continuous function $\eta\colon (0,\infty) \to (0,\infty)$ and a function $\psi\colon \mathcal{P}_{3,1} \to (0,\infty)$ such that

$$\forall\, (\widetilde{x}^0, w_0, r) \in \mathbb{R}^3 \times \mathcal{W} \times (0,\infty) \;:$$

$$\left.\begin{array}{l} \psi\big((\widetilde{A}, \widetilde{b}, \widetilde{c})\big)|\widetilde{x}^0| + \|w_0\|_{\mathcal{W}} \leq r, \\ \vec{\delta}\left(\widetilde{P}_1\big((\alpha,1,1),0\big), \widetilde{P}_2\big((\widetilde{A}, \widetilde{b}, \widetilde{c}),0\big)\right) \leq \eta(r) \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} k \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}), \\ \limsup_{t\to\infty} |y_0(t) - y_1(t)| \leq \lambda, \\ x \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^3), \end{array}\right.$$

where $\mathcal{W} = W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$.

This means in particular that λ-tracking is achieved by the adaptive control strategy (1.5) applied to system (1.6) despite the fact that it has unstable zero dynamics, has relative degree two, and has negative high-frequency gain. The only restrictions are that the zero is "far" in the right half complex plane, the initial condition $\widetilde{x}^0$ is "small," and the $W^{1,\infty}$ input/output disturbances u_0 and y_0 are "small," too.

5. Conclusions. We have shown the robustness of the λ-tracker (1.5) for a class of linear systems close in the gap metric to minumum phase systems with strict relative degree one; moreover, the λ-tracker copes with certain bounded input/output disturbances. Although this result may be worth knowing, the shortcoming of the control strategy (1.5) is that the gain $k(\cdot)$ is a monotone nondecreasing function which converges to a potentially "too high" gain, thus amplifying noise. Recently, a simple time-varying proportional output feedback law called "funnel control" has been introduced to achieve "practical" tracking with prespecified transient behavior [11]; this control law is applicable even to nonlinear minimum phase systems, and the gain is no longer monotone but may decrease again. The conceptional results of the present paper may indicate how to use the gap metric framework to show the robustness of the funnel controller.

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