Ilchmann, Achim ; Townley, Stuart :

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Zuerst erschienen in:
Systems & Control Letters, 36 (1999) S. 285-293
DOI: 10.1016/S0167-6911(98)00101-7

ELSEVIER

Systems & Control Letters 36 (1999) 285–293

SYSTEMS & CONTROL LETTERS

Adaptive high-gain λ-tracking with variable sampling rate [1]

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Received 31 March 1998; received in revised form 29 July 1998; accepted 5 October 1998

Abstract

It is well known that proportional output feedback control can stabilize any relative-degree one, minimum-phase system if the sign of the feedback is correct and the proportional gain is high enough. Moreover, there exists simple adaptation laws for tuning the proportional gain (the so-called high-gain adaptive controllers) which are not based on system identification or plant parameter estimation algorithms or injection of probing signals. If tracking of signals is desired, then these simple controllers are also applicable without invoking an internal model if the tracking error is not necessarily supposed to converge to zero but towards a ball around zero of arbitrarily small but prespecified radius $\lambda > 0$. In this note we consider a sampled version of the high-gain adaptive λ-tracking controller. The motivation for sampling arises from the possibility that the output of a system may not be available continuously, but only at discrete time instants. The problem is that the stiffness of the system increases as the proportional gain is increased. Our result shows that adaptive sampling tracking is possible if the product hk of the decreasing sampling rate h and the increasing proportional gain k decreases at a rate proportional to $1/\log k$. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Adaptive control; Minimum-phase systems; Proportional control; Robust tracking; Sampled-data control

1. Introduction and main result

Willems and Byrnes [10] and Morse [7] initiated the research of an area in adaptive control where the adaptive controller is of striking simplicity. It is not based on system identification or plant parameter estimation algorithms or injection of probing signals. They showed that the continuous-time, high-gain adaptive controller

$$u(t) = -k(t)y(t), \qquad \dot{k}(t) = y^2(t)$$

stabilizes any single-input, single-output, minimum-phase system with positive high-frequency gain. From then on similar results were obtained for various classes of minimum-phase system. For surveys see [1, 2, 5]. However, the simplicity of the adaptive controller is lost if reference signals are to be tracked asymptotically. Then an internal model has to be invoked, see [6]. But simplicity of the controller can be preserved if the control objective is weakened slightly so as to require that the tracking error converges asymptotically towards a ball around zero of arbitrarily small but prespecified radius $\lambda > 0$. This so-called λ-tracking is achieved by invoking a dead-zone in the gain adaptation and was introduced for linear high-gain stabilizable systems by Ilchmann and Ryan [3]. λ-tracking is very robust, it can cope with output noise and the class of allowed reference signals, consisting of bounded absolutely continuous

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[1] This work was supported by the Human Capital and Mobility programme (Project number CHRX-CT93-0402) and the University of Exeter, Small Grants Committee.

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0167-6911/99/$ – see front matter
PII: S0167-6911(98)00101-7

functions whose derivative is essentially bounded, is large. A different route, but also in the spirit of high-gain adaptive stabilization, was initiated by Owens [9] who considers the problem of stabilizing the system if the output is only available at sampled times. The *main novelty*, which distinguishes this problem from either continuous or discrete-time adaptive control, is the need to develop suitable mechanisms for adjusting a variable sampling rate. In [4] we introduce a simple adaptation so that the product hk of the sampling rate h and the proportional gain k decreases at a rate proportional to $1/\log k$ so that stabilization for minimum-phase, multi-output systems with positive high-frequency gain is achieved.

In the present note, we combine variable sampling with λ-tracking and prove that "practical" tracking of a fairly large class of reference signals can be achieved, that is the tracking error converges towards a ball around zero of arbitrarily small but prespecified radius $\lambda > 0$.

We focus on adaptive control of minimum-phase, multi-input, multi-output systems with the spectrum of the high-frequency in the open left half plane. These systems are high-gain stabilizable. Whilst this might be considered a restriction, it is presicely in this high-gain case that the conflict between gain adaptation and sampling is most apparent.

More precisely, suppose the system to be controlled is described by

$$\dot{x}(t) = Ax(t) + Bu(t),$$

$$y(t) = Cx(t), \quad x(0) = x_0, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$, B, $C^{\mathrm{T}} \in \mathbb{R}^{n\times m}$, $x_0 \in \mathbb{R}^n$ and n are all unknown. The assumption that Eq. (1.1) is *minimum phase* means that

$$\det \begin{bmatrix} sI_n - A & B \\ C & 0 \end{bmatrix} \neq 0 \quad \text{for all } s \in \overline{\mathbb{C}_+}. \tag{1.2}$$

The sign of the high-frequency gain is called *positive* if, and only if,

$$\sigma(CB) \subset \mathbb{C}_+. \tag{1.3}$$

The *control objectives* are described as follows: Suppose that the reference signals $y_{\mathrm{ref}}(\cdot)$ belong to $W^{1,\infty}$, i.e. the Sobolev space of bounded functions which are absolutely continuous on compact intervals and have essentially bounded derivatives. This space is equipped with the norm $\|\zeta(\cdot)\|_{1,\infty} := \|\zeta(\cdot)\|_\infty + \|\dot{\zeta}(\cdot)\|_\infty$. Let $\lambda > 0$ be fixed. Then design a simple scalar adaptation law

$$k_{j+1} = f(k_j, e_j), \quad t_{j+1} = g(t_j, k_j), \tag{1.4}$$

so that the proportional sampled-data output feedback

$$u(t) = -k_j e_j,$$

$$e_j := e(t_j) := y(t_j) - y_{\mathrm{ref}}(t_j), \quad t \in [t_j, t_{j+1}) \tag{1.5}$$

which uses only sampled output information $y_j := y(t_j)$, when applied to a system (1.1) satisfying Eqs. (1.2) and (1.3), yields a closed-loop system (1.1), (1.4), (1.5) with convergent gain adaptation, positive sampling interval length, and $\lim_{j\to\infty} \mathrm{dist}\{\|e_j\| \,|[0,\lambda]\} = 0$, the so-called sampled λ-tracking. A solution for designing the adaptation law which meets the control objective is described as follows:

Theorem 1.1. *Let* $\lambda, \gamma > 0$, $y_{\mathrm{ref}}(\cdot) \in W^{1,\infty}$, $\bar{u} \in \mathbb{R}^m$. *Suppose system* (1.1) *is minimum phase, so that Eq.* (1.2) *holds, and the "sign" of the high-frequency gain is known, so that Eq.* (1.3) *holds. Then the adaptive sampling output feedback law*

$$u(t) = -k_i e_i + \bar{u},$$

$$e_i = y_i - y_{\mathrm{ref}}(t_i), \qquad t \in [t_i, t_{i+1}), \tag{1.6}$$

where $\{k_i\}_{i\in\mathbb{N}_0}$ *and* $\{t_i\}_{i\in\mathbb{N}_0}$ *are generated by the gain and sampling-time adaptation mechanism*

$$h_i = \frac{1}{k_i \log k_i}, \quad t_{i+1} = t_i + h_i, \quad i \in \mathbb{N}_0$$

$$k_{i+1} = k_i + \gamma\delta(e_i)k_i h_i \|e_i\|^2, \tag{1.7}$$

$$\delta(e_i) := \begin{cases} 1 & \text{if } \|e_i\| \geq \lambda, \\ 0 & \text{if } \|e_i\| < \lambda \end{cases}$$

with $t_0 = 0$, $k_0 > 1$, *when applied to Eq.* (1.1) *results in a closed-loop system which admits a unique solution* $x(\cdot)$ *on* $[0,\infty)$. *Moreover,*

(i) $\lim_{i\to\infty} k_i = k_\infty \in \mathbb{R}$,
(ii) $\lim_{i\to\infty} h_i = h_\infty > 0$,
(iii) $\lim_{i\to\infty} \mathrm{dist}\,(\|y(t_i) - y_{\mathrm{ref}}(t_i)\|, [0,\lambda]) = 0$.

We present the proof of Theorem 1.1 in Section 2 and illustrate the controller by some simulations in Section 3. Before this we conclude this section with some remarks concerning Theorem 1.1.

In Eqs. (1.6) and (1.7), $\bar{u} \in \mathbb{R}^m$ and $\gamma > 0$ are design parameters which might be used to improve the performance considerably if more information abo[ut]

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The version in [3]:

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the real process is known. The speed of the adaptation is adjusted by changing γ. A sensible choice of γ is the order of magnitude of the inverse of the dominant time constant of the plant. The parameter $\bar{u}$ could be chosen as the estimated steady state input value at the main operating point.

The gain adaptation in Eq. (1.7) is a discrete time version of the λ-tracker with a dead-zone as introduced in [3]:

$$\dot{k}(t) = \begin{cases} |y(t)|(|y(t)| - \lambda), & |y(t)| \geq \lambda, \\ 0, & |y(t)| < \lambda. \end{cases}$$

It might be argued that, in meeting the objectives of λ-tracking, introducing a dead-zone is both standard practice (see e.g. [8]) and bound to work. However, the technical details arising in the proof, whilst familiar from high-gain adaptive control, are by no means obvious to us.

If $y_{\rm ref}(\cdot) \equiv 0$, then Eq. (1.7) coincides with the adaptation as in Ilchmann and Townley [4]. The sampling time h and the proportional gain k are adapted so that their product hk decreases at a rate proportional to $1/\log k$.

Note that Theorem 1.1 does not say anything about the inter-sampling behaviour. For arbitrary large $\alpha > 0$ one can construct $y_{\rm ref}(\cdot)$ satisfying the conditions of Theorem 1.1 and exhibiting an inter-sampling error of $\lim_{i\to\infty} \sup_{t\in[t_i,t_{i+1})} \|y(t) - y_{\rm ref}(t)\| = \alpha$. To construct such an example, suppose Theorem 1.1 has been applied and produced a sequence $\{t_i\}_{i\in\mathbb{N}_0}$ of sampling-times. Now start with identical data, but instead use the reference signal $y_{\rm ref}(\cdot) + \tilde{y}_{\rm ref}(\cdot)$, with

$$\tilde{y}_{\rm ref}(t) := -4\alpha \frac{(t-t_i)(t-t_{i+1})}{(t_{i+1}-t_i)^2}, \quad t \in [t_i, t_{i+1}),$$
$$i \in \mathbb{N}_0.$$

It is clear that this reference signal produces the same sampled output as $y_{\rm ref}(\cdot)$, but with inter-sample error α. However, this example is somewhat pathological since the zeros of the reference signal contain precisely the sampling times.

In a practical situation there is a possibility of model uncertainties at high frequencies that are ignored in the controller design. These might become excited and cause the sampling period to become very small.

2. Proof of Theorem 1.1

In order to rewrite closed-loop system (1.1), (1.6), (1.7), we make use of the minimum-phase assumption and $\det CB \neq 0$ which allows us to decompose the state space into $\operatorname{im} B \oplus \ker C$. Hence, without loss of generality (see e.g. [2, p. 11]), we can assume A, B, C and x are of the form

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} CB \\ 0 \end{bmatrix},$$

$$C = [I_m, \ 0], \quad x = \begin{pmatrix} y \\ z \end{pmatrix}.$$

Here $\sigma(A_4) \subset \mathbb{C}_-$ and the blocks are structured according to $y \in \mathbb{R}^m$, $z \in \mathbb{R}^{n-m}$. With this notation and substitution of Eq. (1.6) into Eq. (1.1) we have, for all $t \in [t_i, t_{i+1})$,

$$\dot{e}(t) = A_1 e(t) + A_2 z(t) + A_1 y_{\rm ref}(t) - \dot{y}_{\rm ref}(t) - k_i CB e_i + CB\bar{u},$$
$$\dot{z}(t) = A_3 e(t) + A_4 z(t) + A_3 y_{\rm ref}(t). \tag{2.1}$$

Existence and uniqueness of the solution of closed-loop system (2.1), Eq. (1.7) on some maximal interval $[0, \omega)$, $\omega \in (0, \infty]$, follows from the theory of ordinary differential equations. It is clear that $\omega = \infty$ if $\{k_i\}_{i\in\mathbb{N}_0}$ is bounded.

In order to establish the boundedness of $\{k_i\}_{i\in\mathbb{N}_0}$, we proceed in several steps. Seeking a contradiction, suppose that

$$\lim_{i\to\infty} k_i = \infty. \tag{2.2}$$

Step 1: We restrict our attention to the discrete time system induced by Eq. (2.1) at the time instants t_i. More precisely, we set

$$\eta_i := \begin{pmatrix} e_i \\ z_i \end{pmatrix} := \begin{pmatrix} e(t_i) \\ z(t_i) \end{pmatrix}$$

and

$$g(t) := \begin{pmatrix} A_1 y_{\rm ref}(t) - \dot{y}_{\rm ref}(t) + CB\bar{u} \\ A_3 y_{\rm ref}(t) \end{pmatrix}$$

and apply variation-of-constants to Eq. (2.1) to obtain, for all $i \in \mathbb{N}_0$,

$$\eta_{i+1} = e^{Ah_i}\eta_i - k_i \int_0^{h_i} e^{As}\,ds \begin{bmatrix} CB & 0 \\ 0 & 0 \end{bmatrix} \eta_i + \int_0^{h_i} e^{As} g(t_{i+1} - s)\,ds. \tag{2.3}$$

To simplify Eq. (2.3), we use the power-series expansion

$$e^{Ah} - k\int_0^h e^{As}\,ds \begin{bmatrix} CB & 0 \\ 0 & 0 \end{bmatrix} = \left[I_n + hA + \frac{1}{2!}h^2A^2 + \cdots \right] - k\left[hI_n + \frac{h^2}{2!}A + \cdots \right] \begin{bmatrix} CB & 0 \\ 0 & 0 \end{bmatrix}$$

and the notation

$$T_h(A) := \tfrac{1}{2!}A + \tfrac{1}{3!}hA^2 + \cdots,$$

$$U_{h,k} := T_h(A)[A - kBC],$$

$$\Psi_{h,k} := I_n - kh \begin{bmatrix} CB & 0 \\ 0 & 0 \end{bmatrix} + h \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

$$L_{h_i} := \int_0^{h_i} e^{As} g(t_{i+1} - s)\,ds,$$

so that Eq. (2.3) becomes

$$\eta_{i+1} = (\Psi_{h_i,k_i} + h_i^2 U_{h_i,k_i})\eta_i + L_{h_i}. \tag{2.4}$$

Note that by boundedness of $g(\cdot)$ there exists $\bar{g} > 0$ and $M_1 > 0$ so that, for all $h_i \leqslant h_0$,

$$\|L_{h_i}\| \leqslant \int_0^{h_i} e^{\|A\|s}\bar{g}\,ds \leqslant \bar{g}\frac{e^{\|A\|h_i} - 1}{\|A\|} \leqslant h_i\bar{g}e^{\|A\|h_i} \leqslant M_1 h_i \tag{2.5}$$

and since $h = 1/(k \log k)$, there exists $M_2 > 0$, so that, for all $k \geqslant k_0$,

$$\|\Psi_{h,k}\| + \frac{1}{k\sqrt{\log k}}\|U_{h,k}\| \leqslant M_2. \tag{2.6}$$

Step 2: To investigate the evolution of Eq. (2.3), respectively, Eq. (2.4), consider the Lyapunov–function candidate

$$V(e,z) := \begin{pmatrix} e \\ z \end{pmatrix}^{\mathrm{T}} R \begin{pmatrix} e \\ z \end{pmatrix},$$

where

$$R = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$$

and $P = P^{\mathrm{T}} \in \mathbb{R}^{m\times m}$, $Q = Q^{\mathrm{T}} \in \mathbb{R}^{(n-m)\times(n-m)}$ denote the positive–definite solutions of

$$(CB)^{\mathrm{T}}P + P(CB) = I_m \quad \text{and}$$

$$A_4^{\mathrm{T}}Q + QA_4 = -I_{n-m}. \tag{2.7}$$

We prove that if Eq. (2.2) is satisfied, then for i_0 sufficiently large, we have for all $i \geqslant i_0$,

$$\Delta V_i := V(e_{i+1}, z_{i+1}) - V(e_i, z_i) \leqslant \alpha_i(e_i, z_i), \tag{2.8}$$

where, for some $M_3 > 0$,

$$\alpha_i(e,z) := -\frac{h_i}{4}(k_i\|e\| - 4M_3)\|e\| - \frac{h_i}{4}(\|z\| - 4M_3)\|z\| + M_3h_i^2.$$

To see this, compute ΔV_i along the solution of Eq. (2.4) and use Eqs. (2.5) and (2.6) to obtain, for some $M_4 > 0$,

$$\begin{aligned} \Delta V_i &= \eta_i^{\mathrm{T}}\Psi_{h_i,k_i}^{\mathrm{T}}R\Psi_{h_i,k_i}\eta_i - \eta_i^{\mathrm{T}}R\eta_i \\ &\quad + 2h_i^2\eta_i^{\mathrm{T}}U_{h_i,k_i}^{\mathrm{T}}R\Psi_{h_i,k_i}\eta_i + h_i^4\eta_i^{\mathrm{T}}U_{h_i,k_i}^{\mathrm{T}}RU_{h_i,k_i}\eta_i \\ &\quad + 2\eta_i^{\mathrm{T}}[\Psi_{h_i,k_i}^{\mathrm{T}} + h_i^2U_{h_i,k_i}^{\mathrm{T}}]RL_{h_i} + L_{h_i}^{\mathrm{T}}RL_{h_i} \\ &\leqslant \eta_i^{\mathrm{T}}[\Psi_{h_i,k_i}^{\mathrm{T}}R\Psi_{h_i,k_i} - R]\eta_i \\ &\quad + M_4\left[\frac{h_i}{\sqrt{\log k_i}}\|\eta_i\|^2 + h_i\|\eta_i\| + h_i^2\right]. \end{aligned}$$

Since

$$\Psi_{h,k}^{\mathrm{T}}R\Psi_{h,k} = \left(I + h\begin{bmatrix} -kCB + A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}^{\mathrm{T}}\right) \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \left(I + h\begin{bmatrix} -kCB + A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\right).$$

we have

$$\Psi_{h,k}^{\mathrm{T}} R \Psi_{h,k} - R = -h \begin{bmatrix} kI_m & 0 \\ 0 & I_{n-m} \end{bmatrix}$$

$$+h \begin{bmatrix} A_1^{\mathrm{T}} P + PA_1 & A_3^{\mathrm{T}} Q + PA_2 \\ A_2^{\mathrm{T}} P + QA_3 & 0 \end{bmatrix}$$

$$+h^2 \begin{bmatrix} -kCB + A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}^{\mathrm{T}}$$

$$R \begin{bmatrix} -kCB + A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

and so there exists $M_5 > 0$ such that

$$\begin{aligned} \Delta V_i \leqslant {} & -hk\|e_i\|^2 - h\|z_i\|^2 \\ & +hM_5(\|e_i\|^2 + \|e_i\|\|z_i\|) + h^2k^2M_5\|e_i\|^2 \\ & +M_5 h_i^2[\|e_i\|^2 + \|z_i\|^2] \\ & +M_5 \Bigg[\frac{h_i}{\sqrt{\log k_i}}(\|e_i\|^2 + \|z_i\|^2) \\ & \quad +h_i(\|e_i\| + \|z_i\|) + h_i^2 \Bigg] \end{aligned}$$

and hence, by using $\|e\|\|z\| \leqslant 2M_3\|e\|^2 + (2M_3)^{-1}\|z\|^2$, we conclude for all $i \in \mathbb{N}_0$,

$$\begin{aligned} \Delta V_i \leqslant {} & -h_ik_i\Bigg[1 - M_5\Bigg(\frac{1}{k_i} + \frac{2M_5}{k_i} + h_ik_i + h_i \\ & + \frac{1}{k_i\sqrt{\log k_i}}\Bigg)\Bigg]\|e_i\|^2 \\ & -h_i\left[1 - \frac{1}{2} - M_5h_ik_i - M_5\frac{1}{\sqrt{\log k_i}}\right]\|z_i\|^2 \\ & +M_5h_i[\|e_i\| + \|z_i\| + h_i]. \end{aligned}$$

Since we assumed Eq. (2.2), there exists i_0 sufficiently large so that for all $i \geqslant i_0$,

$$\begin{aligned} \Delta V_i \leqslant {} & -\frac{h_i}{4}k_i\|e_i\|^2 - \frac{h_i}{4}\|z_i\|^2 \\ & + M_5h_i[\|e_i\| + \|z_i\| + h_i]. \end{aligned}$$

After a little manipulation, the right-hand side of the above inequality turns out to be $\alpha_i(e_i, z_i)$ for $M_3 := M_5$. This proves Eq. (2.8).

Step 3: We will prove that

$$\text{if } \lim_{i\to\infty} k_i = \infty$$

$$\text{then } e(\cdot) \text{ and } z(\cdot) \in L_\infty(0,\infty). \tag{2.9}$$

Assume that we have already proved boundedness of $\{e_i\}_{i\in\mathbb{N}_0}$ and of $\{z_i\}_{i\in\mathbb{N}_0}$. An application of variation-of-constants to Eq. (2.1) yields, for all $t \in [t_i, t_{i+1})$ and suitable M_6, $M_7 > 0$,

$$\begin{aligned} \left\|\begin{pmatrix} e(t) \\ z(t) \end{pmatrix}\right\| \leqslant {} & \mathrm{e}^{\|A\|h_i}\left\|\begin{pmatrix} e_i \\ z_i \end{pmatrix}\right\| \\ & + \int_{t_i}^{t} \mathrm{e}^{\|A\|(t-s)}\left\|\begin{pmatrix} M_6 + k_i\|CBe_i\| \\ M_6 \end{pmatrix}\right\| \mathrm{d}s \\ \leqslant {} & \mathrm{e}^{\|A\|h_i}\left\|\begin{pmatrix} e_i \\ z_i \end{pmatrix}\right\| \\ & + \frac{\mathrm{e}^{\|A\|h_i} - 1}{\|A\|}(M_7 + M_7k_i\|e_i\|). \end{aligned}$$

Eq. (2.9) now follows since $\{((\mathrm{e}^{\|A\|h_i} - 1)/\|A\|)k_i\}_{i\in\mathbb{N}_0}$ is bounded if $\{k_i\}_{i\in\mathbb{N}_0}$ is unbounded. Thus it remains to prove that

$$\text{if } \lim_{i\to\infty} k_i = \infty, \text{ then}$$

$$\{e_i\}_{i\in\mathbb{N}_0}, \{z_i\}_{i\in\mathbb{N}_0} \in l^\infty. \tag{2.10}$$

Now positivity of $\alpha_i(e_i, z_i)$ is equivalent to

$$(k_i\|e_i\| - 4M_3)\|e_i\| + (\|z_i\| - 4M_3)\|z_i\| < 4M_3h_i.$$

Since the left-hand side defines a quadratic in $\|e_i\|$ and $\|z_i\|$, with positive leading coefficients, we can clearly find $M_8 > 0$ such that if $\alpha_i(e_i, z_i) > 0$ then $\|e_i\| + \|z_i\| \leqslant M_8$. Hence by Eq. (2.8) we conclude that

$$V(e_{i+1}, z_{i+1}) \leqslant \begin{cases} V(e_i, z_i) + \overline{\alpha} & \text{if } \alpha_i(e_i, z_i) > 0, \\ V(e_i, z_i) & \text{if } \alpha_i(e_i, z_i) \leqslant 0, \end{cases}$$

where

$$\overline{\alpha} := \sup\{\alpha_i(e, z) \mid i \in \mathbb{N}_0, \|e\| + \|z\| \leqslant M_8\}.$$

This proves that $\{e_i\}_{i\in\mathbb{N}_0}$ and $\{z_i\}_{i\in\mathbb{N}_0}$ are bounded.

Step 4: We will prove that

$$\text{if } \lim_{i\to\infty} k_i = \infty \text{ then } \liminf_{i\to\infty} \|e_i\| = 0. \tag{2.11}$$

To see this suppose not. Then there exist $i_0 \in \mathbb{N}_0$ and $\bar{\gamma} > 0$ such that

$$\|e_i\| \geqslant \bar{\gamma} \quad \text{for all } i \geqslant i_0.$$

Then Eqs. (2.8) and (2.10) imply, for some suitable $M_9 > 0$, i_0 sufficiently large and all $i \geqslant i_0$,

$$\begin{aligned}\Delta V_i &\leqslant -\frac{h_i}{4} k_i \|e_i\|^2 + M_9 h_i \\ &\leqslant -\left(\frac{h_i}{4} k_i - M_9 h_i \frac{1}{\bar{\gamma}^2}\right) \|e_i\|^2 \\ &\leqslant -\frac{1}{8} h_i k_i \|e_i\|^2.\end{aligned}$$

Hence, using Eq. (1.7) we have that

$$\begin{aligned}&V(e_{N+1}, z_{N+1}) - V(e_{i_0}, z_{i_0}) \\ &\quad = \sum_{i=i_0}^{N} \Delta V_i \leqslant -\frac{1}{8} \sum_{i=i_0}^{N} h_i k_i \|e_i\|^2 \\ &\quad \leqslant -\frac{1}{8\bar{\gamma}} \sum_{i=i_0}^{N} (k_{i+1} - k_i) = -\frac{k_{N+1} - k_{i_0}}{8\bar{\gamma}}.\end{aligned}$$

Since $\{k_i\}_{i\in\mathbb{N}_0}$ is unbounded and $\{V(e_{N+1}, z_{N+1})\}_{N\in\mathbb{N}_0}$ is bounded, the latter yields a contradiction. This proves the claim and Eq. (2.11) holds.

Step 5: We will prove that

if $\lim_{i\to\infty} k_i = \infty$ then

$$e_{i+1} = [I_m - k_i h_i CB] e_i + h_i w_i, \tag{2.12}$$

for some m-vector sequence $\{w_i\}_{i\in\mathbb{N}_0} \in l^\infty$. To see this we again apply variation-of-constants to the first equation in Eq. (2.1) to give (compare with Eq. (2.4))

$$e_{i+1} = (I_m - k_i h_i CB) e_i + h_i [A_1 + h_i \tilde{U}_{h_i,k_i}] e_i + \tilde{L}_{h_i},$$

where

$$\tilde{U}_{h,k} := T_h(A_1)(A_1 - khCB),$$

$$\tilde{L}_{h_i} := \int_0^{h_i} e^{A_1 s} \tilde{g}(t_{i+1} - s)\, \mathrm{d}s,$$

$$\tilde{g}(t) := A_2 z(t) + A_1 y_{\text{ref}}(t) + CB\bar{u}.$$

Since $\{A_1 + h_i \tilde{U}_{h_i,k_i}\}_{i\in\mathbb{N}_0}$, $\{e_i\}_{i\in\mathbb{N}_0}$ and $\tilde{g}(\cdot)$ are bounded, there exists $M_{10} > 0$, so that $\|\tilde{L}_{h_i}\| \leqslant M$ and hence we can conclude Eq. (2.12).

Step 6: We will prove that, for $W(e) := e^{\mathrm{T}} P e$, some $M_{11} > 0$, and $P = P^{\mathrm{T}} > 0$ as in Eq. (2.7),

if $\lim_{i\to\infty} k_i = \infty$ then

$$W(e_{i+1}) \leqslant \left(1 - \frac{k_i h_i}{2\|P\|}\right) W(e_i) + M_{11} h_i. \tag{2.13}$$

To see this, we use Eq. (2.12) to give

$$\begin{aligned}W(e_{i+1}) &= ([I - k_i h_i CB] e_i + h_i w_i)^{\mathrm{T}} \\ &\qquad P([I_m - k_i h_i CB] e_i + h_i w_i) \\ &= W(e_i) - k_i h_i e_i^{\mathrm{T}} [(CB)^{\mathrm{T}} P + PCB] e_i \\ &\quad + 2 e_i^{\mathrm{T}} P h_i w_i + k_i^2 h_i^2 e_i^{\mathrm{T}} (CB)^{\mathrm{T}} PCB e_i \\ &\quad - 2 k_i h_i^2 e_i^{\mathrm{T}} (CB)^{\mathrm{T}} P w_i + h_i^2 w_i^{\mathrm{T}} P w_i.\end{aligned}$$

Eq. (2.13) now follows from Eq. (2.7) and boundedness of $\{e_i\}_{i\in\mathbb{N}_0}$ and $\{w_i\}_{i\in\mathbb{N}_0}$ (see Steps 3 and 5).

Step 7: This is the crucial step in which we finally prove boundedness of $\{k_i\}_{i\in\mathbb{N}_0}$. Suppose not, then intuitively, using Eq. (2.11) we can be sure that there exists i_0 such that k_{i_0} is large enough and $\|e_{i_0}\|$ is small and less than λ. Thus $\|e_i\|$ cannot escape from $[0, \lambda)$ and k_i remains constant.

More precisely, suppose $\lim_{i\to\infty} k_i = \infty$. Define

$$\rho_i := 1 - \frac{k_i h_i}{2\|P\|}.$$

By Eq. (2.11) we may choose $i_0 \in \mathbb{N}$ so that $\rho_{i_0} > 0$ and

$$W(e_{i_0}) + M_{11} h_{i_0} \frac{1}{1 - \rho_{i_0}} < \sigma_{\min}(P) \lambda^2. \tag{2.14}$$

From Eq. (2.14) we deduce that $\|e_{i_0}\| < \lambda$, so that $k_{i_0+1} = k_{i_0}$ and Eq. (2.13) yields, for $i = i_0$,

$$W(e_{i+1}) \leqslant \rho_{i_0}^{i+1-i_0} W(e_{i_0}) + \sum_{j=i_0}^{i} \rho_{i_0}^{i-j} M_{11} h_j \tag{2.15}$$

and again

$$\sigma_{\min}(P)\|e_{i_0+1}\|^2 \leqslant W(e_{i_0+1})$$
$$\leqslant W(e_{i_0}) + M_{11} h_{i_0} \frac{1}{1-\rho_{i_0}}$$
$$< \sigma_{\min}(P)\lambda^2,$$

so that $\|e_{i_0+1}\| < \lambda$. Then, proceeding by induction, we can show that (2.15) and $\|e_i\| < \lambda$ hold true for all $i \geqslant i_0$. Therefore, $e(t_i)$ remains in the dead-zone and so $k_i = k_{i_0}$ for all $i \geqslant i_0$. This contradicts the unboundedness of $\{k_i\}_{i\in\mathbb{N}_0}$.

Step 8: Boundedness of $\{k_i\}_{i\in\mathbb{N}_0}$ immediately yields (i) and (ii). To prove (iii) note that

$$\frac{1}{\log k_\infty} \sum_{i=0}^{N-1} \delta(e_i)\gamma\|e_i\|^2 \leqslant \sum_{i=0}^{N-1} \delta(e_i)\gamma k_i h_i \|e_i\|^2$$
$$= k_N - k_0 \leqslant k_\infty - k_0.$$

Therefore $\lim_{i\to\infty} \delta(e_i)\|e_i\|^2 = 0$ and hence $\|e_i\| \to [0,\lambda]$ as $i \to \infty$. This proves (iii) and completes the proof. □

3. Simulations

For the purposes of simulation, we consider a single-input single-output system described by

$$y(s) = G(s)u(s)$$

where

$$G(s) = \frac{s^4 + 4s^3 + 7s^2 + 4s + 2}{s^5 - s^4 - 16s + 16}. \quad (3.1)$$

A state space realization of Eq. (3.1), in control canonical form, is given by

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -16 & 16 & 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t),$$

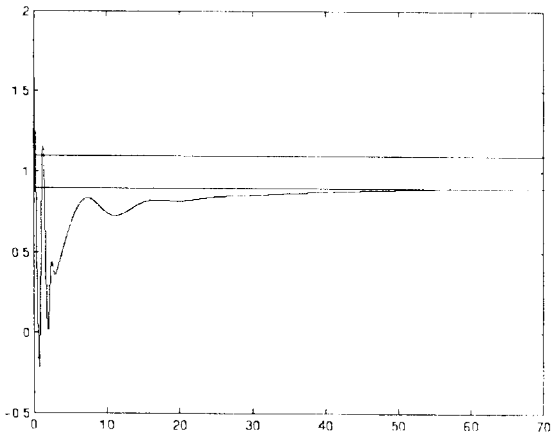

Fig. 1. Output $t_i \mapsto y(t_i)$ with tracking regulator (1.6)–(1.7) applied to Eq. (3.2) with $y_{\text{ref}}(\cdot) \equiv 1$, $\lambda = 0.1$.

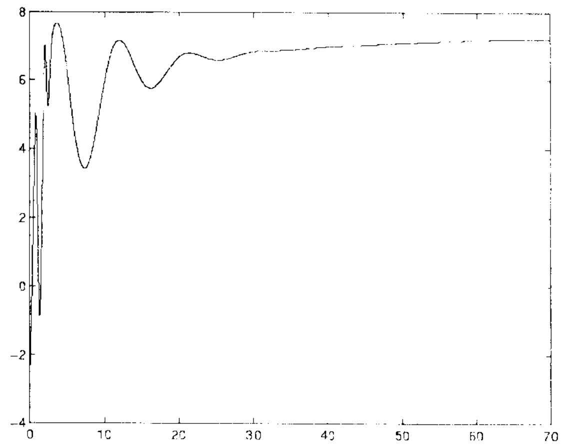

Fig. 2. Input $t_i \mapsto u(t_i)$ with tracking regulator (1.6)–(1.7) applied to Eq. (3.2) with $y_{\text{ref}}(\cdot) \equiv 1$, $\lambda = 0.1$.

$$y(t) = \begin{bmatrix} 2 \\ 4 \\ 7 \\ 4 \\ 1 \end{bmatrix}^{\mathrm{T}} x(t), \quad x_0 = 0.1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \quad (3.2)$$

It is easily checked that Eq. (3.1) satisfies the minimum phase condition (1.2) and the positive sign condition (1.3). In order to assess the performance of the controller (1.6), (1.7) and, in particular, unnecessary over-adaptation of the high-gain k, we first calculate

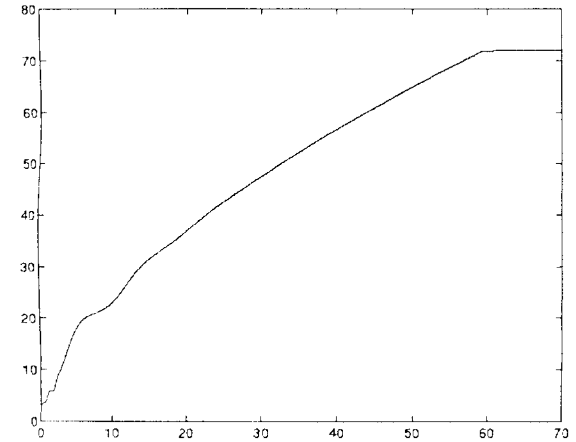

Fig. 3. Gain variation $t_i \mapsto k_i$ with tracking regulator (1.6)–(1.7) applied to Eq. (3.2) with $y_{ref}(\cdot) \equiv 1$, $\lambda = 0.1$.

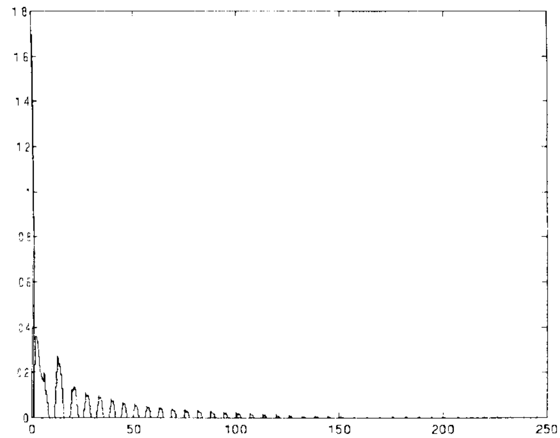

Fig. 4. Distance error $t_i \mapsto \mathrm{dist}(e(t_i), [-\lambda, \lambda])$ with tracking regulator (1.6)–(1.7) applied to Eq. (3.2) with $y_{ref}(t) = |\sin(0.5t)|$, $\lambda = 0.1$.

the smallest value of the gain-parameter k necessary for non-adaptive λ-tracking of constant reference signals.

If $y_{ref}(\cdot) \equiv y_{ref} \in \mathbb{R}$, then $u(t) = -ke(t)$, with $e(t) = y(t) - y_{ref}$, applied to Eq. (3.1) gives $e(s) = -(1 + kG(s))^{-1} y_{ref}$. Hence using the final-value-theorem, we have that

$$e_\infty := \lim_{t \to \infty} e(t) = -\lim_{s \to 0} s(1 + kG(s))^{-1} \frac{1}{s} y_{ref}$$

$$= -\frac{16}{16 + 2k} y_{ref}. \qquad (3.3)$$

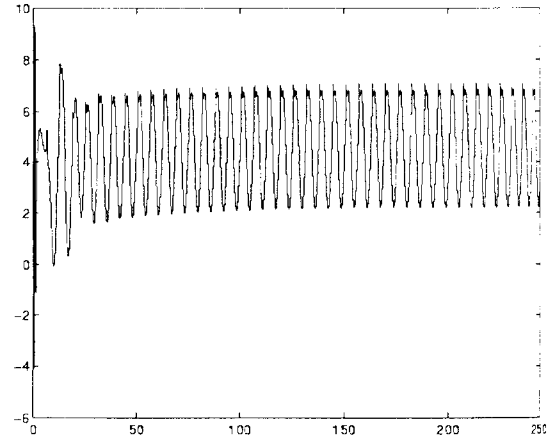

Fig. 5. Input $t_i \mapsto u(t_i)$ with tracking regulator (1.6)–(1.7) applied to Eq. (3.2) with $y_{ref}(t) = |\sin(0.5t)|$, $\lambda = 0.1$.

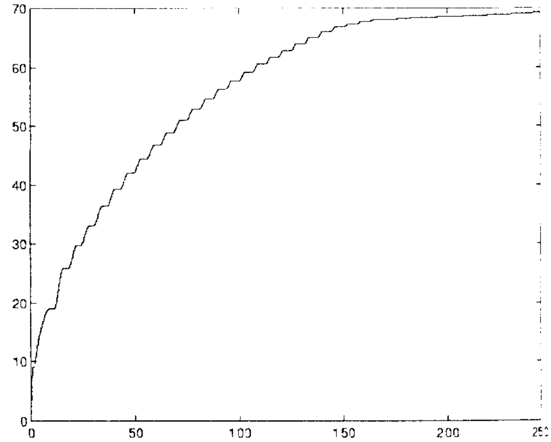

Fig. 6. Gain variation $t_i \mapsto k_i$ with tracking regulator (1.6)–(1.7) applied to Eq. (3.2) with $y_{ref}(t) = |\sin(0.5t)|$, $\lambda = 0.1$.

Thus if we require $|e_\infty| \leqslant \lambda$, then $k \geqslant 16(|y_{ref}| - \lambda)/\lambda$. If $\lambda = 0.1$ and $y_{ref} = 1$, then λ-tracking is achieved only if $k \geqslant 72$.

Suppose that we require system (3.2) to track the constant reference signal $y_{ref}(\cdot) \equiv 1$. If we use Eqs. (1.6) and (1.7) with design parameters $\lambda = 0.1$, $\gamma = 1$, $\bar{u} = 0$, $k_0 = 3$, then the transient behaviour is reasonable. Indeed, in 5 s the output is within 0.5 of the given reference signal $y_{ref}(\cdot) \equiv 1$, and converges to the prespecified 0.1-strip around this reference signal (Fig. 1). Notice that the input (Fig. 2) behaves well and the adaptive gain (Fig. 3) increases monotonically until $t = 60$, at which time the output reaches the 0.1-strip and adaptation terminates. Moreover, and quite remarkably, the limit gain is equal to 72.0531, which, by the calculation in Eq. (3.3), is within 1% of the constant gain necessary to achieve this tracking by continuous-time output feedback.

Of course, there always exist initial conditions and high-frequency gains so that performance is very poor. However, numerous simulations lead us to believe that, in some "generic" sense, performance is good. It would be an interesting topic for further research to formulate this problem precisely, perhaps specifying performance in terms of scaled plant data.

In a second run of simulations, with the same design parameters as above, we demonstrate λ-tracking for a time-varying reference signal $y_{\text{ref}}(t) = |\sin(0.5t)|$. As in the previous simulation, the tracking (Fig. 4) is very satisfactory and the input (Fig. 5) exhibits only a small transient before settling down to the necessary evolution. In Fig. 6 we have only shown the gain adaptation up to $t = 250$. Note that the gain is levelling out but still increasing slowly towards the value of 72 required for non-adaptive tracking of a constant reference of the same amplitude.

Acknowledgements

We are indebted to an anonymous reviewer for pointing out to us the example at the end of the introduction and to our colleague Joseph Obermaier (GSF, Munich) for helpful comments.

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