

Decentralized tracking of interconnected systems

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Abstract. Decentralized funnel controllers are applied to finitely many interacting single-input single-output, minimum phase, relative degree one systems in order to track reference signals of each system within a prespecified performance funnel. The reference signals as well as the systems belong to a fairly large class. The result is a generalization of the work by [2].

1 Introduction

We generalize the early work by Helmke, Prätzel-Wolters, and Schmidt [2] who exploited the standard high-gain adaptive controller $u(t) = -k(t)y(t)$, $\dot{k}(t) = y(t)^2$ (for linear minimum phase systems with relative degree one and positive high-frequency gain) to track reference signals of N systems which are interconnected. This approach, including the class of systems, the class of reference signals and internal models, the control objective, and the control strategy, is briefly summarized in Section 2.

In the present note we generalize Helmke's approach by the high-gain "funnel controller" as follows: We consider the **class of systems** described by $i = 1, \dots, N$ interconnected single-input single-output controlled functional differential systems of the form

$$\dot{y}_i(t) = T_i(y_1(\cdot), \dots, y_N(\cdot))(t) + \gamma_i v_i(t), \quad y_i|_{[-h,0]} = y_i^0 \in C^\infty([-h,0], \mathbb{R}) \quad (1)$$

where, loosely speaking, $h \geq 0$ quantifies the "memory" of the system, $\gamma_i > 0$, and the nonlinear causal operators T_i belong to the operator class $\mathcal{T}_h^{N,1}$; see Definition 2. Note that interconnections without any structure are incorporated since every T_i depends on all $y_1(\cdot), \dots, y_N(\cdot)$.

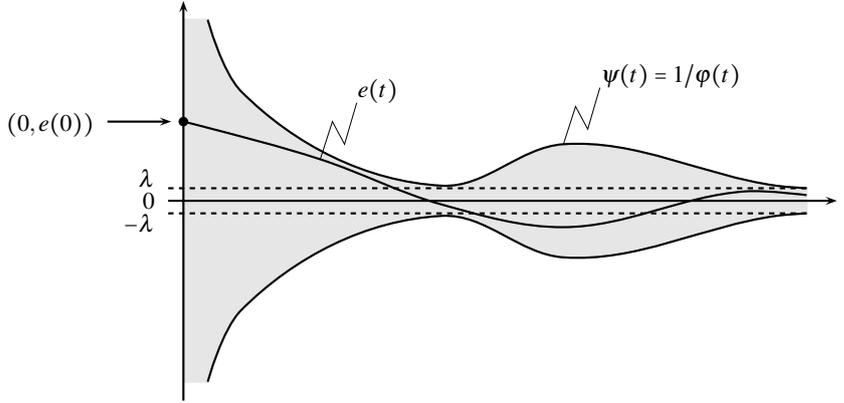
The **class of reference signals** \mathcal{Y}_{ref} , we allow for, are all absolutely continuous functions which are bounded with essentially bounded derivative

$$\mathcal{Y}_{\text{ref}} := W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}) := \{y_{\text{ref}}: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R} \text{ is abs. cont.} \mid y_{\text{ref}}, \dot{y}_{\text{ref}} \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})\} \quad (2)$$

where $L_{\text{loc}}^\infty(I, \mathbb{R})$ (resp. $L_{\text{loc}}^1(I, \mathbb{R})$) denote the space of measurable, locally essentially bounded (resp. locally integrable) functions $I \rightarrow \mathbb{R}$.

For the concept of "funnel control", we prespecify admissible functions φ belonging to

$$\Phi := \left\{ \varphi \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}) \left| \begin{array}{l} \forall t > 0 : \varphi(t) > 0, \liminf_{t \rightarrow \infty} \varphi(t) > 0, \\ \forall \delta > 0 : \varphi|_{[\delta, \infty)}(\cdot)^{-1} \text{ is globally Lipschitz} \end{array} \right. \right\} \quad (3)$$



“Infinite” funnel, that is the funnel defined on $(0, \infty)$ with pole at $t = 0$.

Figure 1: Error evolution in a funnel \mathcal{F}_φ with boundary $\psi(t) = 1/\varphi(t)$ for $t > 0$.

so that φ describes the reciprocal of the funnel boundary of the funnel

$$\mathcal{F}_\varphi := \{(t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \mid \varphi(t) |e| < 1\}. \quad (4)$$

See Figure 1, and Section 3.2 for a variety of funnels.

We will show that the simple **funnel controllers**

$$v_i(t) = \frac{-\varphi_i(t)}{1 - \varphi_i(t) |e_i(t)|} e_i(t), \quad e_i(\cdot) = y_i(\cdot) - y_{\text{ref},i}(\cdot), \quad i = 1, \dots, N, \quad (5)$$

achieve the **control objective**: for N prespecified performance funnels \mathcal{F}_{φ_i} , the N proportional output error feedback laws (5) applied to (1) yield a closed-loop system which has only bounded trajectories and, most importantly, each error $e_i(\cdot)$ evolves within the performance funnel \mathcal{F}_{φ_i} , for $i = 1, \dots, N$; see Figure 1 and Figure 2.

Funnel control seems advantageous when compared to high-gain adaptive control: the gain is no longer monotone but increases if necessary to exploit the high-gain property of the system and decreases if a high gain is not necessary. Most importantly, prespecified transient behaviour of the output error is addressed. Although asymptotic tracking of the reference signals is not guaranteed, the error is forced into an arbitrarily small strip; therefore, from a practical point of view this difference is negligible since the width of the funnel (see (23)) may be chosen arbitrarily small. Moreover, funnel control allows for much more general system classes and reference classes than in [2] and the interconnection between the subsystems is not limited as in [2]. If an identical reference trajectory is chosen for every subsystem, our control strategy could be called synchronization of interconnected systems. Decentralized funnel control for interconnected systems is the main contribution of the present note and it is treated in Section 3. We finalize the paper with some illustrative simulation in Section 4.

2 The approach by Uwe Helmke and coworkers

In the present section, the approach by Helmke, Prätzel-Wolters, and Schmidt [2] is summarized; the generalized approach will then be related to the latter in Section 3.

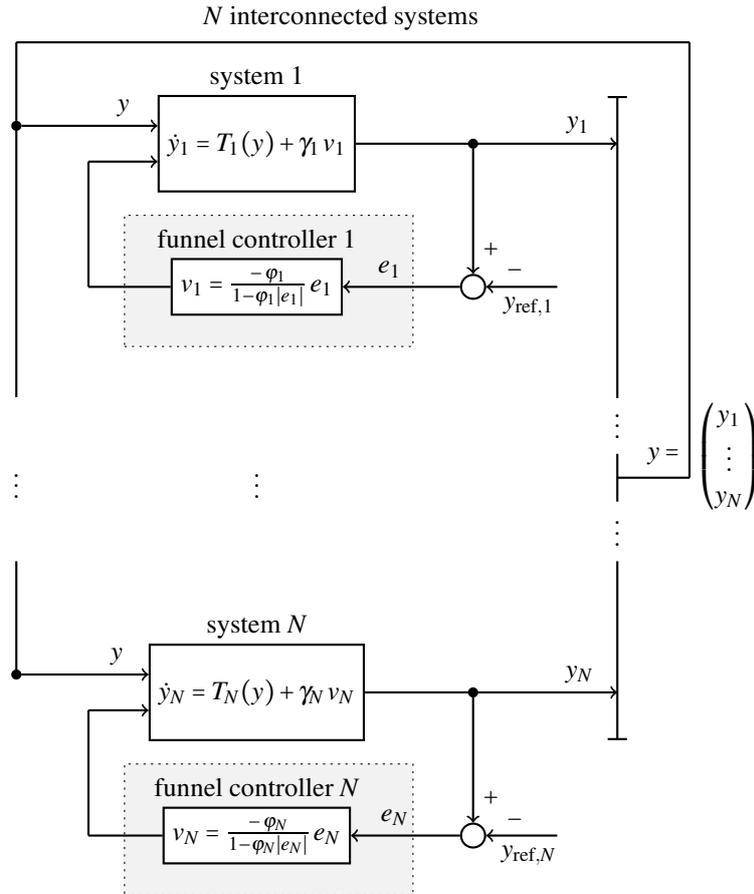


Figure 2: Decentralized funnel control of N interconnected systems

Roughly speaking, the underlying idea is to combine adaptive high-gain controllers and internal models (generating the signals to be tracked) to interconnected high-gain stabilizable, relative degree one systems; then tracking of reference signals of each subsystem is achieved if the interconnection has a certain structure which preserves for the interconnected system the minimum phase property inherited from the subsystems.

2.1 Class of linear systems

Consider $i = 1, \dots, N$ interconnected single-input single-output systems of the form

$$\begin{cases} \dot{x}_i(t) = A_i x_i(t) + b_i u_i(t) \\ y_i(t) = c_i x_i(t) \end{cases} \quad (6)$$

which all satisfy, for (unknown) $A_i \in \mathbb{R}^{n_i \times n_i}$, $b_i, c_i^\top \in \mathbb{R}^{n_i}$, the structural properties

$$\text{positive high-frequency gain and relative degree one, i.e. } c_i b_i > 0 \quad (7)$$

$$\text{minimum phase, i.e. } \det \begin{bmatrix} sI_n - A_i & b_i \\ c_i & 0 \end{bmatrix} \neq 0 \quad \forall s \in \overline{\mathbb{C}}_+ \quad (8)$$

$$u(t) = Fy(t) + v(t), \quad \text{for some } F \in \mathbb{R}^{N \times N} \\ \text{with interconnection structure } f_{ij} \ker c_j \subset \text{im } b_i \text{ for } i \neq j, \quad (9)$$

where $u(t) = (u_1(t), \dots, u_N(t))^\top$, $y(t) = (y_1(t), \dots, y_N(t))^\top$, $v(t) = (v_1(t), \dots, v_N(t))^\top$ and v denotes the N -dimensional input of the interconnected system.

It was well-known in the high-gain adaptive control community that the structural properties (7) and (8) allow for a simple adaptive high-gain controller

$$u_i(t) = -k_i(t)y_i(t), \quad \dot{k}_i(t) = y_i(t)^2, \quad (10)$$

which, if applied to (6) for arbitrary initial data $x_i(0) = x_i^0 \in \mathbb{R}^{n_i}$, $k_i(0) = k_i^0 \in \mathbb{R}$, yields in a closed-loop system (6), (10), and this system has a unique global solution and satisfies

$$\lim_{t \rightarrow \infty} y_i(t) = 0, \quad \lim_{t \rightarrow \infty} k_i(t) = k_i^\infty \in \mathbb{R}, \quad x_i(\cdot) \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^{n_i});$$

see, for example, [7] or [12]. One important issue of this approach is that no information on the system entries of (6) are incorporated in the feedback controller. However, one drawback is monotonically increasing gain functions $t \mapsto k_i(t)$ which may have a large limit and so possible noise in the output measurement is amplified.

2.2 Control objective

Let $y_{\text{ref},i} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ denote N reference signals which are periodic and satisfy a linear differential equation

$$y_{\text{ref},i}(\cdot) \in \ker P_i \left(\frac{d}{dt} \right) := \left\{ \zeta(\cdot) \in C^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}) \mid P_i \left(\frac{d}{dt} \right) \zeta = 0 \right\}$$

for given $P_i(s) \in \mathbb{R}[s]$, $i = 1, \dots, N$. The control objective is to find N decentralized adaptive controllers depending on the tracking error

$$e_i(\cdot) := y_i(\cdot) - y_{\text{ref},i}(\cdot) \mapsto v_i(\cdot)$$

in combination with an internal model (depending on $P_1(s), \dots, P_N(s)$) so that the closed-loop system has only bounded trajectories and the tracking errors satisfy, for any initial conditions,

$$\lim_{t \rightarrow \infty} e_i(t) = 0 \quad \forall i = 1, \dots, N.$$

2.3 Adaptive high-gain controller

Before we state the main result of [2] which is the following Theorem 1, we stress the underlying ideas of this result:

- The N systems in (6) may be written as one system with N inputs u and N outputs y , and the latter has strict relative degree one with high-frequency gain matrix $\text{diag}\{c_1 b_1, \dots, c_n b_n\}$ and it inherits the minimum phase property.
- The polynomials $P_i(s)$ allow to design an internal model so that the reference signals are, for suitable initial values, the output of the internal model.
- The special interconnection structure by F in (9) preserves the strict relative degree one and minimum phase property of the multi-input multi-output system $v \mapsto y$.
- The adaptive high-gain controllers in (10) are applicable.

Theorem 1. [2, Th. 2.4]

Consider N interconnected systems as in (6)-(9). Let $P_i(s) \in \mathbb{R}[s]$ such that $\ker P_i(\frac{d}{dt})$ contains periodic solutions only; $i = 1, \dots, N$. Choose a Hurwitz polynomial $Q(s) \in \mathbb{R}[s]$ such that

$$\ell := \deg Q(s) = \deg P(s) \quad \text{where } P(s) = \text{lcm}\{P_1(s), \dots, P_N(s)\}$$

and a minimal realization $(A_r, b_r, c_r) \in \mathbb{R}^{\ell \times \ell} \times \mathbb{R}^\ell \times \mathbb{R}^{1 \times \ell}$ such that

$$c_r (sI_\ell - A_r)^{-1} b_r + 1 = \frac{Q(s)}{P(s)}. \quad (11)$$

Then for any reference signals $y_{\text{ref},i}(\cdot) \in \ker P_i(\frac{d}{dt})$ and any initial conditions $x_i^0 \in \mathbb{R}^{n_i}$, $z_i^0 \in \mathbb{R}^\ell$, $k_i^0 \in \mathbb{R}$, the N decentralized high-gain controllers $e_i := y_i - y_{\text{ref},i} \mapsto v_i$ given by

$$\boxed{\begin{aligned} \dot{z}_i(t) &= A_r z_i(t) - b_r k_i(t) e_i(t), & z_i(0) &= z_i^0 \\ \dot{k}_i(t) &= e_i(t)^2, & k_i(0) &= k_i^0 \\ y_i(t) &= c_i x_i(t) \\ v_i(t) &= c_r z_i(t) - k_i(t) e_i(t), & e_i(\cdot) &= y_i(\cdot) - y_{\text{ref},i}(\cdot) \end{aligned}} \quad (12)$$

applied to (6), (9) yield a closed-loop system (6), (9), (12) which has solution, this solution is global and unique and satisfies, for $i = 1, \dots, N$,

$$\lim_{t \rightarrow \infty} e_i(t) = 0, \quad \lim_{t \rightarrow \infty} k_i(t) \in \mathbb{R}, \quad x_i(\cdot) \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^{n_i}), \quad z_i(\cdot) \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^\ell).$$

3 Main result

In this section we show how to generalize Theorem 1 in the following sense: The restriction of the interconnection (9) between the systems is superfluous. We allow for systems described by functional differential equations encompassing nonlinear

systems, infinite dimensional systems, systems with hysteresis such as relay or backlash. The class of reference signals are arbitrary signals which are bounded and have essentially bounded derivative; an internal model as in (11) is not needed. Furthermore, the control strategy does not involve a monotonically increasing gain $k_i(\cdot)$ as in (10) but a gain which is large if “necessary” and decreases thereafter. The control strategy will obey prespecified transient behaviour.

3.1 Class of systems

We consider $i = 1, \dots, N$ interconnected single-input single-output systems described by controlled functional differential equations of the form (1) where, loosely speaking, $h \geq 0$ quantifies the “memory” of the system, $\gamma_i > 0$, and the nonlinear causal operators T_i belong to the following operator class $\mathcal{T}_h^{N,q}$. Note that interconnections without any structure are incorporated since every T_i depends on all $y_1(\cdot), \dots, y_N(\cdot)$.

Definition 2 (Operator class $\mathcal{T}_h^{N,q}$). [3]

Let $h \geq 0$, $N, q \in \mathbb{N}$. An operator T is said to be of class $\mathcal{T}_h^{N,q}$ if, and only if, the following hold:

- (i) $T: C([-h, \infty), \mathbb{R}^N) \rightarrow L_{\text{loc}}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^q)$ is a causal operator.
- (ii) $\forall t \geq 0 \forall w \in C([-h, t], \mathbb{R}^N) \exists \tau > t, \exists \delta, \Delta > 0 \forall y, z \in C(w; h, t, \tau, \delta, N) :$

$$\text{ess-sup}_{s \in [t, \tau]} \|(Ty)(s) - (Tz)(s)\| \leq \Delta \cdot \max_{s \in [t, \tau]} \|y(s) - z(s)\|,$$

where $C(w; h, t, \tau, \delta, N)$ denotes the space of all continuous extensions z of $w \in C([-h, t], \mathbb{R}^N)$ to the interval $[-h, \tau]$ with the property that $\|z(s) - w(t)\| \leq \delta$.

- (iii) $\forall \delta > 0 \exists \Delta > 0 \forall y \in C([-h, \infty), \mathbb{R}^N)$ with $\sup_{s \in [-h, \infty)} \|y(s)\| \leq \delta :$

$$\|(Ty)(t)\| \leq \Delta \text{ for almost all } t \geq 0.$$

The crucial property is Property (iii): a bounded-input, bounded-output assumption on the operator T . Property (ii) is a technical assumption of local Lipschitz type which is used in establishing well-posedness of the closed-loop system. To interpret this assumption correctly, we need to give meaning to Ty for a function $y \in C(I, \mathbb{R}^N)$ on a bounded interval I of the form $[-h, \rho)$ or $[-h, \rho]$, where $0 < \rho < \infty$. This we do by showing that T “localizes” to an operator $\tilde{T}: C(I, \mathbb{R}^N) \rightarrow L_{\text{loc}}^\infty(J, \mathbb{R}^N)$, where $J := I \setminus [-h, 0)$. Let $y \in C(I)$. For each $\sigma \in J$, define $y_\sigma \in C([-h, \infty), \mathbb{R}^N)$ by

$$y_\sigma(t) := \begin{cases} y(t), & t \in [-h, \sigma], \\ y(\sigma), & t > \sigma. \end{cases}$$

By causality, we may define $\tilde{T}y \in L_{\text{loc}}^\infty(J, \mathbb{R}^N)$ by the property $\tilde{T}y|_{[0, \sigma]} = Ty_\sigma|_{[0, \sigma]}$ for all $\sigma \in J$. Henceforth, we will not distinguish notationally an operator T and its “localization” \tilde{T} : the correct interpretation being clear from context.

In the following we will show the wide range of system classes which can be written in the form (1) with an operator $T_i(y_1(\cdot), \dots, y_N(\cdot))$ belonging to the class $\mathcal{T}_h^{N,q}$.

3.1.1 Linear systems

We first study the linear prototype of systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t), & x(0) &= x^0 \\ y(t) &= cx(t) \end{aligned} \quad (13)$$

with $A \in \mathbb{R}^{n \times n}$, $b, c^\top \in \mathbb{R}^n$, $x^0 \in \mathbb{R}^n$ and relative degree one, i.e. $cb \neq 0$. We show that the interconnected systems (1) is a generalization of the interconnected system (6), (9). In our setup, with a slightly different control objective (see Section 2.2) than 1, the special assumption on F in (9) is superfluous.

Clearly, (13) has relative degree one if, and only if, $\mathbb{R}^n = \text{im } b \oplus \ker c$. If this is the case, then there exists $V \in \mathbb{R}^{n \times (n-1)}$ with $\text{im } V = \ker C$ such that the coordinate transformation

$$x \mapsto \begin{bmatrix} y \\ z \end{bmatrix} := S^{-1}x \quad \text{where } S := [b(cb)^{-1}, V]$$

takes (13) into the equivalent form

$$\begin{aligned} \dot{y}(t) &= A_1 y(t) + A_2 z(t) + cbu(t), & y(0) &= y^0 \\ \dot{z}(t) &= A_3 y(t) + A_4 z(t), & z(0) &= z^0, \end{aligned} \quad (14)$$

with $z(t) \in \mathbb{R}^{n-1}$ and real matrices A_1, A_2, A_3, A_4 of conforming formats. This allows to rewrite (13) in terms (14) and the linear and causal operator,

$$\begin{aligned} T^{z^0} : C(\mathbb{R}_{\geq 0}, \mathbb{R}) &\rightarrow C(\mathbb{R}_{\geq 0}, \mathbb{R}) \\ y(\cdot) &\mapsto \left(t \mapsto A_1 y(t) + A_2 \left[e^{A_4 t} z^0 + \int_0^t e^{A_4(t-\tau)} A_3 y(\tau) d\tau \right] \right), \end{aligned} \quad (15)$$

parametrized by $z^0 \in \mathbb{R}^{n-1}$, as a functional differential equation in $y(\cdot)$ only:

$$\dot{y}(t) = T^{z^0} y(\cdot)(t) + cbu(t), \quad y(0) = y^0. \quad (16)$$

If (13) is minimum phase (see (8)), then equivalently $\sigma(A_4) \subset \mathbb{C}_-$; and hence the operator T^{z^0} has the crucial property

$$\forall \delta > 0 \exists \Delta > 0 \forall y(\cdot) \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}) \text{ with } \|y\|_\infty < \delta : \|T^{z^0} y\|_\infty < \Delta, \quad (17)$$

and it is readily checked that T^{z^0} belongs to the class $\mathcal{T}_0^{1,1}$. Therefore, each minimum phase system (6) with positive high-frequency gain $c_i b_i > 0$ can be equivalently written in the form (1).

Next we consider the class of systems (6) which satisfy the structural properties (7) and (8), write them in the form

$$\dot{y}_i(t) = T_i^{z_i^0} (y_i(\cdot))(t) + c_i b_i u_i(t), \quad y_i = c x_i^0 \quad (18)$$

and interconnect them with the feedback (9). Writing $F = \begin{bmatrix} f^1 \\ \dots \\ f^N \end{bmatrix}$, this results in

$$\dot{y}_i(t) = \underbrace{T_i^{z_i^0}(y_i(\cdot))(t) + c_i b_i [f^i y(t)] + c_i b_i v_i(t)}_{=: T_i^{z_i^0}(y(\cdot))(t)}, \quad y_i = c x_i^0$$

and the so defined operator $y(\cdot) \mapsto T_i^{z_i^0}(y)(\cdot)$ also belongs to class $\mathcal{T}_0^{1,1}$ and we arrive at the structure of (1).

3.1.2 Infinite dimensional linear systems

The finite-dimensional class of systems of the form (13) can be extended to infinite dimensions by reinterpreting the operators A_j in (14) as the generating operators of a regular linear system (regular in the sense of [11]). In the infinite-dimensional setting, A_4 is assumed to be the generator of a strongly continuous semigroup $\mathbf{S} = (\mathbf{S}_t)_{t \geq 0}$ of bounded linear operators and a Hilbert space X with norm $\|\cdot\|_X$. Let X_1 denote the space $\text{dom}(A_4)$ endowed with the graph norm and let X_{-1} denote the completion of X with respect to the norm $\|z\|_{-1} = \|(s_0 I - A_4)^{-1} z\|_X$, where s_0 is any fixed element of the resolvent set of A_4 . Then A_3 is assumed to be a bounded linear operator from \mathbb{R} to X_{-1} and A_2 is assumed to be a bounded linear operator from X_1 to \mathbb{R} . Assuming that the semigroup \mathbf{A}_4 is exponentially stable and that \mathbf{A}_4 extends to a bounded linear operator (again denoted by \mathbf{A}_4) from X to \mathbb{R} , then the operator T given by

$$(Ty)(t) := A_1(t)y(t) + A_2 \left[\mathbf{S}_t z^0 + \int_0^t \mathbf{S}_{t-\tau} A_3 y(\tau) d\tau \right]$$

is of class $\mathcal{T}_0^{1,1}$ and we arrive at the structure of (1). For more details see [8], and for a similar but more general approach see [3, Appendix A.2].

3.1.3 Nonlinear systems

Consider the following nonlinear generalization of (14):

$$\begin{aligned} \dot{y}(t) &= f(p(t), y(t), z(t)) + g(y(t), z(t), u(t)), & y(0) &= y^0 \in \mathbb{R} \\ \dot{z}(t) &= h(t, y(t), z(t)), & z(0) &= z^0 \in \mathbb{R}^{n-1} \end{aligned} \quad (19)$$

with continuous

$$f: \mathbb{R}^P \times \mathbb{R} \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}, \quad g: \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}, \quad h: \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$$

having the properties: $h(\cdot, y, z)$ measurable for all $(y, z) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and

$$\begin{aligned} \forall \text{ compact } \mathcal{C} \subset \mathbb{R} \times \mathbb{R}^{n-1} \exists \kappa \in L_{\text{loc}}^1(\mathbb{R}_{\geq 0}, \mathbb{R}) \text{ for a.a. } t \geq 0 \forall (y, z), (\bar{y}, \bar{z}) \in \mathcal{C} \\ : \|h(t, y, z) - h(t, \bar{y}, \bar{z})\| \leq \kappa(t) \|(y, z) - (\bar{y}, \bar{z})\|. \end{aligned}$$

Then, viewing the second of the differential equations in (19) in isolation (with input y), it follows that, for each $(z^0, y) \in \mathbb{R}^{n-1} \times L_{\text{loc}}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})$, the initial-value

problem $\dot{z}(t) = h(t, y(t), z(t))$, $z(0) = z^0 \in \mathbb{R}^{n-1}$, has unique maximal solution, which we denote by $[0, \omega) \rightarrow \mathbb{R}^{n-1}$, $t \mapsto z(t; z^0, y)$.

In addition, we assume

$$\exists c_0 > 0 \exists q > 1 \forall (u, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-1} : u \cdot g(y, z, u) \geq c_0 |u|^q \quad (20)$$

and

$$\begin{aligned} \exists \theta \in C(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}) \exists c > 0 \forall y \in L_{\text{loc}}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}) \forall t \in [0, \omega) \\ : \|z(t, z^0, y)\| \leq c [1 + \text{ess-sup}_{s \in [0, t]} \theta(|y(s)|)] \end{aligned} \quad (21)$$

which, in turn, implies that $\omega = \infty$. Note that this is akin to, but weaker than, Sontag's [9] concept of input-to-state stability. Now fix $z^0 \in \mathbb{R}^{n-1}$ arbitrarily, and define the operator

$$T : C(\mathbb{R}_{\geq 0}, \mathbb{R}) \rightarrow L_{\text{loc}}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R} \times \mathbb{R}^{n-1}), \quad y \mapsto Ty = (y(\cdot), z(\cdot, z^0, y)).$$

In view of (21), Property (ii) of Definition 2 holds; setting $h = 0$, we see that Property (iii) of Definition 2 also holds. Arguing as in [8, Sect. 3.2.3], via an application of Gronwall's Lemma, it can be shown that Property (iii)(b) holds. Therefore, this construction yields a family (parameterized by the initial data z^0) of operators T of class $\mathcal{T}_0^{1,n}$. Therefore, (19) is equivalent to

$$\dot{y}(t) = f(p(t), (Ty)(t)) + g((Ty)(t), u(t)). \quad (22)$$

Clearly, (22) is not of the form (1). However, the nonlinear function $g((Ty)(t), u(t))$ compared to $\gamma u(t)$ allows for high-gain stabilization since assumption (20) yields, for any compact set $\mathcal{C} \subset \mathbb{R}^P \times \mathbb{R}^{M \times L}$,

$$\forall u \in \mathbb{R} : \min_{(v,w) \in \mathcal{C}} \frac{u [f(v,w) + g(w,u)]}{|u|} \geq - \max_{(v,w) \in \mathcal{C}} |f(v,w)| + c_0 |u|^{q-1},$$

and further, this gives the following condition (akin to radial unboundedness or weak coercivity)

$$\forall (u_n) \in (\mathbb{R}^*)^{\mathbb{N}} \text{ with } \lim_{n \rightarrow \infty} |u_n| = \infty : \lim_{n \rightarrow \infty} \min_{(v,w) \in \mathcal{C}} \frac{u_n [f(v,w) + g(w, u_n)]}{|u_n|} = \infty.$$

Now our general result Theorem 3 can be shown if condition (20) holds, but we omit this to keep the presentation simple; for details see [5, Remark 4(iv)]. The other reason why (22) is not of the form (1) is the first summand in (22). Again, for technical reasons we omit to show how to incorporate this more general form but refer to Step 1 of the proof of Theorem 3: the arguments used there indicate how the right hand side of (1) could be generalized. Under the assumption that N systems of the form (19) can be written in a feasible form, we may interconnect them via

$$u(t) = F(y(t)) + v(t), \quad \text{for some continuous } F: \mathbb{R}^N \rightarrow \mathbb{R}^N$$

and we arrive at the structure of (1).

3.1.4 Nonlinear delay systems

Let functions

$$\mathcal{G}_i: \mathbb{R} \times \mathbb{R}^\ell \rightarrow \mathbb{R}^q : (t, \zeta) \mapsto \mathcal{G}_i(t, \zeta), \quad i = 0, \dots, n,$$

be measurable in t and locally Lipschitz in ζ uniformly with respect to t : precisely,

- (i) $\forall \zeta \in \mathbb{R}^\ell : \mathcal{G}_i(\cdot, \zeta)$ is measurable;
- (ii) \forall compact $\mathcal{K} \subset \mathbb{R}^l \exists c > 0$ for a.a. $t \geq 0 \forall \zeta, \psi \in \mathcal{K}$
 $: \|\mathcal{G}_i(t, \zeta) - \mathcal{G}_i(t, \psi)\| \leq c \|\zeta - \psi\|$.

For $i = 0, \dots, n$, let $h_i \geq 0$ and define $h := \max_i h_i$. The operator T , defined for $\zeta \in C([-h, \infty), \mathbb{R}^l)$ by

$$(T\zeta)(t) := \int_{-h_0}^0 \mathcal{G}_0(s, \zeta(t+s)) ds + \sum_{i=1}^n \mathcal{G}_i(t, \zeta(t-h_i)) \quad \forall t \geq 0.$$

is of class $\mathcal{T}_h^{\ell, q}$; for details see [8].

3.1.5 Systems with hysteresis

A general class of hysteresis operators, which includes many physically motivated hysteretic effects, is discussed in [6]. Examples of such operators include backlash hysteresis, elastic-plastic hysteresis, and Preisach operators. In [4], it is pointed out that these operators are of class $\mathcal{T}_0^{1,1}$. For illustration, we describe two particular examples of a hysteresis operators.

Relay hysteresis. Let $a_1 < a_2$ and let $\rho_1 : [a_1, \infty) \rightarrow \mathbb{R}$, $\rho_2 : (-\infty, a_2] \rightarrow \mathbb{R}$ be continuous, globally Lipschitz and satisfy $\rho_1(a_1) = \rho_2(a_1)$ and $\rho_1(a_2) = \rho_2(a_2)$. For a given input $y \in C(\mathbb{R}_{\geq 0}, \mathbb{R})$ to the hysteresis element, the output w is such that $(y(t), w(t)) \in \text{graph}(\rho_1) \cup \text{graph}(\rho_2)$ for all $t \geq 0$: the value $w(t)$ of the output at $t \geq 0$ is either $\rho_1(y(t))$ or $\rho_2(y(t))$, depending on which of the threshold values a_2 or a_1 was "last" attained by the input y . When suitably initialized, such a hysteresis element has the property that, to each input $y \in C(\mathbb{R}_{\geq 0}, \mathbb{R})$, there corresponds a unique output $w = Ty \in C(\mathbb{R}_{\geq 0}, \mathbb{R})$: the operator T , so defined, is of class $\mathcal{T}_0^{1,1}$.

Backlash hysteresis with a *backlash* or *play* operator of class $\mathcal{T}_0^{1,1}$ is also feasible: see [5, Sect. 4.5.2].

3.2 Control objective: funnel control

The *class of reference signals* \mathcal{Y}_{ref} is all absolutely continuous functions which are bounded, see (2). Obviously, the class \mathcal{Y}_{ref} is considerably larger than the class of periodic functions solving a time-invariant linear differential equation as in Section 2.2.

The *control objective* is met by decentralized funnel control (see Figure 2) as follows: The N decentralized proportional output error feedback **funnel controllers** (5) applied to (1) yield, for N prespecified performance funnels \mathcal{F}_{φ_i} determined by $\varphi_i \in \Phi$

(see (3)) and arbitrary N reference signals $y_{\text{ref},i}(\cdot) \in \mathcal{Y}_{\text{ref}}$ (see (2)), a closed-loop system which has only bounded trajectories and, most importantly, each error $e_i(\cdot)$ evolves within the performance funnel \mathcal{F}_{φ_i} , for $i = 1, \dots, N$; see Figure 1.

Note that, by assumption,

$$\lambda_{\varphi} := \inf_{t>0} \varphi(t)^{-1} = \frac{1}{\|\varphi\|_{\infty}} > 0, \quad \forall \varphi \in \Phi; \quad (23)$$

and λ_{φ} describes the minimal width of the funnel bounded away from zero. If $\varphi(0) = 0$, then the width of the funnel is infinity at $t = 0$; see Figure 1. In the following we only treat “infinite” funnels for technical reasons; if the funnel is finite, i.e. $\varphi(0) > 0$, then we certainly need to assume that the initial error is within the funnel at $t = 0$, i.e. $\varphi(0)|Cx^0 - y_{\text{ref}}(0)| < 1$, and this assumption suffices.

As indicated in Figure 1, we do not assume that the funnel boundary decreases monotonically; whilst in most situation the control designer will choose a monotone funnel, there are situations where widening the funnel at some later time might be beneficial: e.g., when it is known that the reference signal changes strongly or the system is perturbed by some calibration so that a large error would enforce a large control action.

A variety of funnels are possible; we describe some of them here.

- 1) For $a \in (0, 1)$ and $b > 0$, the function

$$t \mapsto \varphi(t) = \begin{cases} \frac{1}{1-at} & , t \in [0, \frac{1-b}{a}] \\ \frac{1}{b} & , t \geq \frac{1-b}{a} \end{cases} \quad (24)$$

determines the funnel boundary $t \mapsto \varphi(t)^{-1} := \max\{1-at, b\}$, which is defined on the whole of $\mathbb{R}_{\geq 0}$; hence \mathcal{F}_{φ} is a “finite” funnel.

- 2) For $a > 0$ and $b \in (0, 1)$, the function $t \mapsto \varphi(t) := \min\{at, b^{-1}\}$ determines the “infinite” funnel \mathcal{F}_{φ} and the funnel boundary $t \mapsto \varphi(t)^{-1} = \max\{\frac{1}{at}, b\}$ is defined for all $t > 0$. The funnel boundary decays strictly monotonically in the transient phase on the interval $[0, (ab)^{-1}]$ and is equal to the constant value $b^{-1} > 0$ thereafter.
- 3) Let $M, \mu, \lambda > 0$ with $M > \lambda$. Then the function $t \mapsto \varphi(t)^{-1} := \max\{Me^{-\mu t}, \lambda\}$ determines a “finite” funnel and ensures error evolution with prescribed exponential decay in the transient phase $[0, T]$, $T = \ln(M/\lambda)/\mu$, and tracking accuracy $\lambda > 0$ thereafter. Note that with this choice we may capture the control objective of “practical (M, μ) -stability”.
- 4) The choice $t \mapsto \varphi(t) = \min\{t/\tau, 1\}/\lambda$ with $\tau, \lambda > 0$, ensures that the modulus of the error decays at rate $\tau\lambda/t$ in the “initial (transient) phase” $(0, \tau]$, and, is bounded by λ in the “terminal phase” $[\tau, \infty)$.

The above examples are only given to illustrate the shape of the funnel boundary in the initial phase; it need not be constant or monotone in the terminal phase.

3.3 Funnel control

We are now in a position to state the main result; see Figure 2 for illustration. Note that in comparison to Theorem 1, we address prespecified transient behaviour, the gain is no longer monotone, and the class of reference signals as well as the class of systems is much larger. However, funnel control does not guarantee that the output errors $e_i(t)$ tend to zero asymptotically as t tends to infinity; but from a practical point of view this difference is negligible since the width of the funnel (see (23)) may be chosen arbitrarily small.

Theorem 3. Consider N interconnected systems (1) for $T_i \in \mathcal{T}_h^{N,1}$ and $\gamma_i > 0$ and let, for $\varphi_i \in \Phi$, associated performance funnels \mathcal{F}_{φ_i} be given, where $i = 1, \dots, N$. Then for any reference signals and initial data

$$y_{\text{ref},i}(\cdot) \in \mathcal{Y}_{\text{ref}}, \quad y_i|_{[-h,0]} = y_i^0 \in C^\infty([-h,0], \mathbb{R}), \quad i = 1, \dots, N,$$

the N decentralized funnel controllers (5) applied to (1) yield, for $i = 1, \dots, N$, a closed-loop initial value problem which has a solution, every solution can be maximally extended, and every maximal solution $y: [-h, \omega) \rightarrow \mathbb{R}^N$ has the following properties:

- (i) $\omega = \infty$, i.e. no finite escape time;
- (ii) The gains $\frac{\varphi_i(\cdot)}{1 - \varphi_i(\cdot)|e_i(\cdot)|}$, the outputs $y_i(\cdot)$, and the inputs $v_i(\cdot)$ are all bounded on $\mathbb{R}_{\geq 0}$ for all $i = 1, \dots, N$;
- (iii) every tracking error $e_i(\cdot)$ evolves within the funnel \mathcal{F}_{φ_i} and is uniformly bounded away from the funnel boundary in the sense:

$$\forall i = 1, \dots, N \exists \varepsilon_i > 0 \forall t > 0 : |e_i(t)| \leq \varphi_i(t)^{-1} - \varepsilon_i.$$

Proof. Step 1: We use the notation

$$y = (y_1, \dots, y_N)^\top, \quad y_{\text{ref}} = (y_{\text{ref},1}, \dots, y_{\text{ref},N})^\top, \quad Ty := (T_1 y, \dots, T_N y)^\top.$$

In view of the potential singularity in the feedback (5), some care is required in formulation of the closed-loop initial-value problem (1), (5). We therefore define

$$\mathcal{D} := \{(t, \zeta) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^N \mid \forall i = 1, \dots, N : (t, \zeta_i - y_{\text{ref},i}(t)) \in \mathcal{F}_{\varphi_i}\}$$

and

$$F: \mathcal{D} \times \mathbb{R}^N \rightarrow \mathbb{R}^N, \quad ((t, \zeta), w) \mapsto F((t, \zeta), w) = (F_1((t, \zeta), w), \dots, F_N((t, \zeta), w))^\top$$

where

$$F_i((t, \zeta), w) := w_i - \gamma_i \frac{\varphi_i(t) [\zeta_i - y_{\text{ref},i}(t)]}{1 - \varphi_i(t) |\zeta_i - y_{\text{ref},i}(t)|}, \quad i = 1, \dots, N.$$

In this case, the closed-loop, initial-value problem (1), (5) is formulated as

$$\dot{y}(t) = F((t, y(t)), (Ty)(t)), \quad y|_{[-h,0]} = y^0. \quad (25)$$

Since $T_i \in \mathcal{T}_h^{N,1}$, it follows immediately from the definition of T that $T \in \mathcal{T}_h^{N,N}$; and since the function F is a Carathéodory function¹, we may apply [3, Theorem B.1]² to conclude that the closed-loop initial-value problem (25) has a *solution* (a function $y \in C([-h, \omega], \mathbb{R}^N)$ where $\omega \in (0, \infty]$ such that $y|_{[-h,0]} = y^0$, $y|_{[0,\omega]}$ is locally absolutely continuous, with $(t, y(t)) \in \mathcal{D}$ for all $t \in [0, \omega]$ and (25) holds for almost all $t \in [0, \omega]$) and every solution can be extended to a maximal solution (that means it has no proper right extension that is also a solution); moreover, noting that F is locally essentially bounded, if $y: [-h, \omega) \rightarrow \mathbb{R}^N$ is a maximal solution, then the closure of $\text{graph}(y|_{[0,\omega)})$ is not a compact subset of \mathcal{D} .

Step 2: In the following let $y: [-h, \omega) \rightarrow \mathbb{R}^N$ for $\omega \in (0, \infty]$ be a maximal solution of the closed-loop, initial-value problem (25).

Then $e := y - y_{\text{ref}}$ evolves on $[0, \omega)$ within the funnel and is therefore bounded. Also, by definition of \mathcal{D} ,

$$\forall i = 1, \dots, N \quad \forall t \in [0, \omega) : \varphi_i(t) |e_i(t)| < 1.$$

The initial-value problem (25) is equivalent to the system of $i = 1, \dots, N$ functional initial-value problems

$$\dot{e}_i(t) = T_i(e(\cdot) - y_{\text{ref},i}(\cdot))(t) - \dot{y}_{\text{ref},i}(t) - \gamma_i \frac{\varphi_i(t) e_i(t)}{1 - \varphi_i(t) |e_i(t)|}, \quad e|_{[-h,0]} = y^0 - y_{\text{ref}}(0). \quad (26)$$

Now define, for arbitrary but fixed $\delta \in (0, \omega)$ and $i = 1, \dots, N$,

$$\begin{aligned} \hat{f}_i &:= \sup_{t \in [0, \omega)} |(T_i y)(t) - \dot{y}_{\text{ref},i}(t)| \\ \lambda_i &:= \inf_{t \in (0, \omega)} \varphi_i(t)^{-1} \\ L_i &> 0 \quad \text{Lipschitz bound of } \varphi_i|_{[\delta, \infty)}(\cdot)^{-1} \\ k_i(t) &:= \frac{\varphi_i(t)}{1 - \varphi_i(t) |e_i(t)|} \quad \forall t \in [0, \omega) \\ \varepsilon_i &:= \min \left\{ \frac{\lambda_i}{2}, \frac{\gamma_i \lambda_i}{2[L_i + \hat{f}_i]}, \min_{t \in [0, \delta]} \{ \varphi_i(t)^{-1} - |e_i(t)| \} \right\}. \end{aligned} \quad (27)$$

We show that

$$\forall i = 1, \dots, N \quad \forall t \in (0, \omega) : \varphi_i(t)^{-1} - |e_i(t)| \geq \varepsilon_i. \quad (28)$$

¹Let \mathcal{D} be a *domain* in $\mathbb{R}_+ \times \mathbb{R}$ (that is, a non-empty, connected, relatively open subset of $\mathbb{R}_+ \times \mathbb{R}$). A function $F: \mathcal{D} \times \mathbb{R}^q \rightarrow \mathbb{R}$, is deemed to be a *Carathéodory function* if, for every "rectangle" $[a, b] \times [c, d] \subset \mathcal{D}$ and every compact set $K \subset \mathbb{R}^q$, the following hold: (i) $F(t, \cdot, \cdot): [c, d] \times K \rightarrow \mathbb{R}$ is continuous for all $t \in [a, b]$; (ii) $F(\cdot, x, w): [a, b] \rightarrow \mathbb{R}$ is measurable for each fixed $(x, w) \in [c, d] \times K$; (iii) there exists an integrable function $\gamma: [a, b] \rightarrow \mathbb{R}_+$ such that $|F(t, x, w)| \leq \gamma(t)$ for almost all $t \in \mathbb{R}_+$ and all $(x, w) \in [c, d] \times K$.

²In [3, Theorem B.1] only the class $\mathcal{T}_h^{1,q}$ is considered. However, it is only a technicality to show the same result for the class $\mathcal{T}_h^{N,q}$.

The inequalities in (28) hold on $(0, \delta]$ by definition of ε_i . Seeking a contradiction, suppose that

$$\exists i \in \{1, \dots, N\} \exists t_1 \in [\delta, \omega) : \varphi_i(t_1)^{-1} - |e_i(t_1)| < \varepsilon_i.$$

Then there exists

$$t_0 := \max \{t \in [\delta, t_1] \mid \varphi_i(t)^{-1} - |e_i(t)| = \varepsilon_i\}$$

and we readily conclude that, for all $t \in [t_0, t_1]$,

$$\varphi_i(t)^{-1} - |e_i(t)| \leq \varepsilon_i \quad \text{and} \quad |e_i(t)| \geq \varphi_i(t)^{-1} - \varepsilon_i \geq \lambda_i - \varepsilon_i \stackrel{(27)}{\geq} \lambda_i/2$$

and

$$k(t)|e_i(t)| = \frac{|e_i(t)|}{\varphi_i(t)^{-1} - |e_i(t)|} \geq \frac{\lambda_i}{2\varepsilon_i}$$

so that

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} e_i(t)^2 &= e_i(t) [(Ty)(t) - \dot{y}_{\text{ref},i}(t) - \gamma k(t)e_i(t)] \\ &\leq -\gamma k(t)e_i(t)^2 + \hat{f}_i |e_i(t)| \leq \left[-\gamma \frac{\lambda_i}{2\varepsilon_i} + \hat{f}_i \right] |e_i(t)| \leq -L_i |e_i(t)| \end{aligned} \quad (29)$$

and therefore

$$\begin{aligned} |e_i(t_1)| - |e_i(t_0)| &= \int_{t_0}^{t_1} \frac{e_i(\tau) \dot{e}_i(\tau)}{|e_i(\tau)|} d\tau \\ &\leq -L_i(t_1 - t_0) \leq -|\varphi_i(t_1)^{-1} - \varphi_i(t_0)^{-1}| \leq \varphi_i(t_1)^{-1} - \varphi_i(t_0)^{-1} \end{aligned}$$

and we arrive at the contradiction

$$\varepsilon_i = \varphi_i(t_0)^{-1} - |e_i(t_0)| \leq \varphi_i(t_1)^{-1} - |e_i(t_1)| < \varepsilon_i.$$

This proves (28).

Step 3: (28) is equivalent to $k(\cdot) \in L^\infty([0, \omega), \mathbb{R})$. Since the errors $e_i(\cdot)$ evolve within the funnels, they are bounded on $[0, \omega)$ and also the input functions satisfy $v_i(\cdot) \in L^\infty([0, \omega), \mathbb{R})$; since the reference signals $y_{\text{ref},i}(\cdot)$ are bounded, it follows that $y_i(\cdot) \in L^\infty([0, \omega), \mathbb{R})$. Finally, boundedness of all functions and maximality of $[0, \omega)$ yields that $\omega = \infty$, whence Assertion (i) and Assertion (iii); and Assertion (ii) is a consequence of (28). This completes the proof of the theorem. \square

Step 2 of the proof of Theorem 3 is “compact”; a more intuitive, but slightly more technical, alternative would go as follows:

Suppose, after the definition of t_0 , that $e_i(t_0) > 0$. Then $e_i(t) > 0$ for all $t \in [t_0, t_1]$ and (29) may be replaced by

$$\frac{d}{dt} e_i(t) \leq -L_i \leq \frac{d}{dt} \varphi_i(t)^{-1} \quad \forall t \in [t_0, t_1].$$

This shows that the increase of $e_i(t)$ is smaller than the increase of the funnel boundary $\varphi_i(t)^{-1}$ at each $t \in [t_0, t_1]$; hence the error evolution cannot hit the funnel boundary on $[t_0, t_1]$; this violates the definition of t_1 . The case $e_i(t_0) < 0$ is then treated analogously.

4 Illustrative simulation

We consider the same set of $N = 4$ single-input, single-output minimum phase systems with high-frequency gain 1 as in [1, Sect. 4] given by transfer functions $u_i \mapsto y_i$:

$$\begin{aligned} g_1(s) &= \frac{s+1}{s^2-2s+1}, & g_2(s) &= \frac{s^3+4s^2+5s+2}{s^4-5s^3+3s^2+4s-1}, \\ g_3(s) &= \frac{1}{s-1}, & g_4(s) &= \frac{s^2+2+1}{s^3+2s^2+3s-2} \end{aligned} \quad (30)$$

and interconnection matrix

$$F = \begin{bmatrix} 0 & 2 & 1 & 1/2 \\ 1 & 0 & 1/3 & 1/4 \\ 1/2 & 1 & 0 & 1 \\ 1/4 & 3/4 & 3/2 & 0 \end{bmatrix} \quad (31)$$

for (9). In [1, Sect. 4], the reference signals $y_{\text{ref},i}(t) = \sin(t + (i-1)\pi/4)$ and the internal model $\frac{P(s)}{Q(s)} = \frac{s^2+16}{(s+\pi)^2}$ is chosen according to (11) for $i = 1, 2, 3, 4$, resp. We have confirmed, for applying the high-gain controllers (12) to (30), the same simulation results, but not depicted here. Instead, for purposes of illustration we have chosen a randomly generated matrix

$$F \approx \begin{pmatrix} 8.15 & 6.32 & 9.58 & 9.57 \\ 9.06 & 0.98 & 9.65 & 4.85 \\ 1.27 & 2.78 & 1.58 & 8 \\ 9.13 & 5.47 & 9.71 & 1.42 \end{pmatrix} \quad (32)$$

with no special structure as in (9), no internal model (11), and (chaotic) reference signals

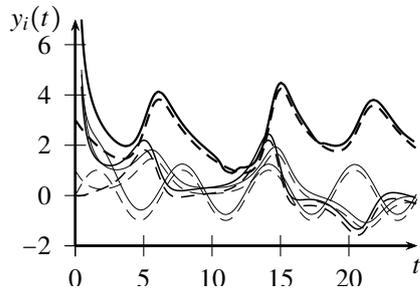
$$y_{\text{ref},1} = \xi_1, \quad y_{\text{ref},2} = \xi_2, \quad y_{\text{ref},3} = \xi_3, \quad y_{\text{ref},4}(t) = \sin(t\pi/4), \quad (33)$$

where (ξ_1, ξ_2, ξ_3) is the solution of initial-value problem for the following Lorenz system:

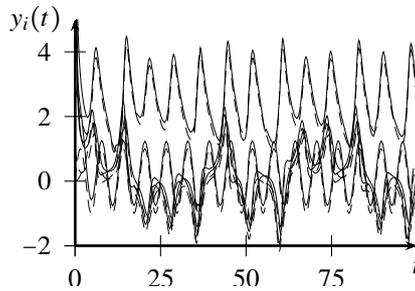
$$\begin{aligned} \dot{\xi}_1 &= \xi_2 - \xi_1, & \xi_1(0) &= 1 \\ \dot{\xi}_2 &= (28\xi_1/10) - (\xi_2/10) - \xi_1\xi_3, & \xi_2(0) &= 0 \\ \dot{\xi}_3 &= \xi_1\xi_2 - (8\xi_3/30), & \xi_3(0) &= 3. \end{aligned} \quad (34)$$

It is well known that the unique global solution of (34) is bounded with bounded derivative; see, for example, [10]. The function φ as in (24) with parameters $a = 0.5$ and $b = 0.25$ has been chosen to specify the performance \mathcal{F}_φ .

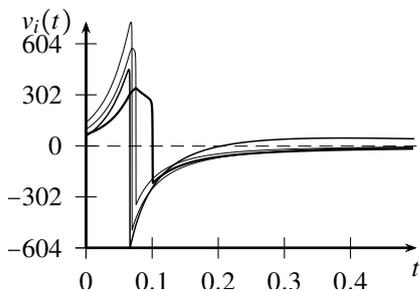
The results of Theorem 3 have been confirmed by the simulations depicted in Figure 3 on the next page. Due to the rapidly decreasing funnel in the transient phase $[0, 0.1]$, all errors tends to the funnel boundary, and hence the gain increases to preclude boundary contact; this makes the gain very large and yields $|u_i(t)| \approx 600$. After that the $u_i(t)$ take moderate values in $[-5, 5]$ and the errors stay away from the funnel boundary.



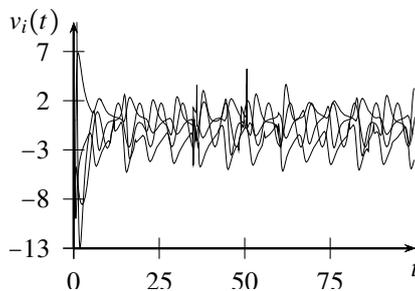
(a) Solutions and reference signals – short run



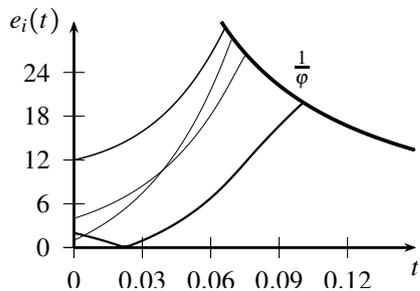
(b) Solutions and reference signals – long run



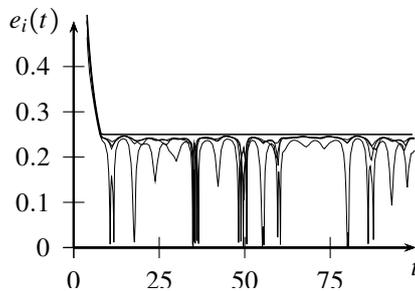
(c) Inputs $v_i(t)$ – short run



(d) Inputs $v_i(t)$ – long run



(e) Errors $|e_i(t)|$ and performance funnel \mathcal{F}_φ – short run



(f) Errors $|e_i(t)|$ and performance funnel \mathcal{F}_φ – long run

Figure 3: Simulation of solutions $y_i(t)$, reference signals $y_{\text{ref},i}(t)$, and errors $e_i(t)$ (from thickest to thinnest) with respect to $i = 4, 1, 2, 3$ and performance funnel \mathcal{F}_φ

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