On Minimum Phase

Über Minimalphasigkeit

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Wir diskutieren Minimalphasigkeit skalarer semi-Hurwitz Transferfunktionen; letzteres sind rationale Funktionen, bei denen das Nennerpolynom Nullstellen in der abgeschlossenen linken komplexen Halbebene hat. Minimalphasigkeit wird hier mittels der Ableitung der Argumentfunktion der Transferfunktion definiert. Es wird dann mit Hilfe der Hurwitz-Spiegelung gezeigt, daß jede semi-Hurwitz Transferfunktion eindeutig in ein Produkt von Allpass und minimalphasiger Funktion zerlegt werden kann. Das wesentliche Resultat ist, daß eine semi-Hurwitz Transferfunktion minimalphasig ist genau dann, wenn das Zählerpolynom der Transferfunktion semi-Hurwitz ist. Zur Charakterisierung der Nulldynamik verwenden wir im Zeitbereich die Byrnes-Isidori-Form und im Frequenzbereich die innere Schleifenform; beides sind kanonische Formen. Dies wird verwendet um zu zeigen, daß die Nulldynamik einer minimalen Realisation asymptotisch stabil ist genau dann, wenn das Zählerpolynom der Transferfunktion Hurwitz ist. Insbesondere folgt aus asymptotisch stabiler Nulldynamik die Minimalphasigkeit, aber keineswegs umgekehrt.

We discuss the concept of 'minimum phase' for scalar semi Hurwitz transfer functions. The latter are rational functions where the denominator polynomial has its roots in the closed left half complex plane. In the present note, minimum phase is defined in terms of the derivative of the argument function of the transfer function. The main tool to characterize minimum phase is the Hurwitz reflection. The factorization of a weakly stable transfer function into an all-pass and a minimum phase system leads to the result that any semi Hurwitz transfer function is minimum phase if, and only if, its numerator polynomial is semi Hurwitz. To characterize the zero dynamics, we use the Byrnes-Isidori form in the time domain and the internal loop form in the frequency domain. The uniqueness of both forms is shown. This is used to show in particular that asymptotic stable zero dynamics of a minimal realization of a transfer function yields minimum phase, but not vice versa.

Schlagwörter: Minimalphasigkeit, Allpassfilter, Hurwitz-Spiegelung, inner-outer-Faktorisierung, Byrnes-Isidori-Form, innere Schleifenform, Nulldynamik.

Keywords: Minimum phase, all-pass, Hurwitz reflection, inner-outer factorization, Byrnes-Isidori form, internal loop form, zero dynamics.

Nomenclature

$\mathbb{K}$	is either the field of real numbers $\mathbb{R}$ or of complex numbers $\mathbb{C}$
$\mathbb{N}$, $\mathbb{N}_0$	set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\mathbb{Z}$	the set of all integers
$\mathbb{C}_+ (\overline{\mathbb{C}}_+)$	open (closed) set of complex numbers with positive (non-negative) real part, resp.
$\mathbb{C}_- (\overline{\mathbb{C}}_-)$	open (closed) set of complex numbers with negative (non-positive) real part, resp.
$\imath$	imaginary unit in $\mathbb{C}$
$\bar{\alpha} :=$	$a - \imath b$, where $\alpha = a + \imath b$, $a, b \in \mathbb{R}$
$\mathbf{Gl}_n(\mathbb{K})$	the group of invertible $n \times n$ matrices with entries in $\mathbb{K}$
$\mathbb{K}[s]$	the ring of polynomials with coefficients in $\mathbb{K}$
monic $p(s)$	the leading coef. of $p(s) \in \mathbb{K}[s]$ is 1
$\mathbb{K}(s)$	the quotient field of $\mathbb{K}[s]$, i.e. the set of rational functions
$\mathbb{K}[s]^*$, $\mathbb{K}(s)^*$	$\mathbb{K}[s]\setminus\{0\}$, $\mathbb{K}(s)\setminus\{0\}$, resp.
$\mathbb{K}^{n\times m}$	the set of $n \times m$ matrices with entries in $\mathbb{K}$

Furthermore, a polynomial $p(s) \in \mathbb{C}[s]$ is called

$$\begin{aligned} \textit{Hurwitz} \quad &:\Longleftrightarrow \quad p(s_0) = 0 \;\Rightarrow\; \operatorname{Re} s_0 < 0\,, \\ \textit{anti Hurwitz} \quad &:\Longleftrightarrow \quad p(s_0) = 0 \;\Rightarrow\; \operatorname{Re} s_0 > 0\,, \\ \textit{semi Hurwitz} \quad &:\Longleftrightarrow \quad p(s_0) = 0 \;\Rightarrow\; \operatorname{Re} s_0 \le 0\,, \end{aligned}$$

and $g(s) = \frac{p(s)}{q(s)} \in \mathbb{C}(s)$ is called *semi Hurwitz* (*stable*) if, and only if, the denominator polynomial $q(s)$ is semi Hurwitz (Hurwitz), resp. We stress that the notion 'semi Hurwitz' allows multiple zeros on the imaginary axis, and hence a minimal realization may be unstable. We use the notation

$$\mathbb{C}[s]_{\mathrm{sH}} := \left\{ \frac{p(s)}{q(s)} \;\middle|\; \begin{array}{l} p(s), q(s) \in \mathbb{C}[s] \text{ are coprime} \\ q(s) \neq 0 \text{ is semi Hurwitz} \end{array} \right\}.$$

1 Introduction

We consider linear single-input single-output systems of the form

$$y(s) \;=\; g(s)\, u(s) \qquad (1.1)$$

with semi Hurwitz transfer function $g(s) = \frac{p(s)}{q(s)} \in \mathbb{C}(s)^*$, i.e., $q(s)$ is semi Hurwitz.

For systems (1.1), we are interested in the property of minimum phase which is not consistently used in the literature.

(i) It was Bode [2] who introduced for the first time this notion with the terse words "If the circuit includes no surplus lines or all-pass sections, it will have at every frequency the least phase shift (algebraically) which can be obtained from any physical structure having the given attenuation characteristic." In less quaint words: within the set of transfer functions $g(s) \in \mathbb{C}(s)$ of equal gain $|g(\imath\omega)|$ along the imaginary axis, we wish to single out the one transfer function minimizing the magnitude of the change of the argument $\arg g(\imath\omega_2) - \arg g(\imath\omega_1)$ for all $\omega_2 > \omega_1$.

Bode came up with this notion since it was observed in network synthesis problems (see his reference in [2] to Lee's paper from 1932 and also in [3, Ch. XIV] to Norbert Wiener and his students) that the gain characteristic does not uniquely characterize the phase. In order to eliminate this ambiguity, Bode introduced the minimum phase condition. This led to the famous Bode gain-phase relation; see [3, Ch. XIV] and for a modern exposition [16, Sec: 2.3.2].

In most classical textbooks on control systems, minimum phase concepts for transfer functions $\frac{p(s)}{q(s)} \in \mathbb{C}(s)^*$ are discussed briefly but their definitions vary:

(ii) Many authors call a transfer function minimum phase if, and only if, $p(s)$ and $q(s)$ are Hurwitz; see [11, 15, 18, 19], to name but a few. This definition avoids problems with poles and zeros on the imaginary axis and is often used in filter design (in communication) since the inverse filter is then Hurwitz, too.

(iii) A weaker definition is to require $q(s)$ Hurwitz as above and to allow for $p(s)$ to be semi Hurwitz; see, for example, [4, 5]. Additionally, $q(s)$ may have zeros in $s = 0$, see [3, 14].

(iv) Today one frequently finds as the defining property that $p(s)$ is semi Hurwitz; see, for example, [7, 8, 16]. This definition seems to be motivated from investigations of performance limitations. It is also of interest because it guarantees stability of the zero dynamics.

(v) In all of the above definitions the definiendum (minimum phase) and definiens (some location of the zeros of a polynomial) do not have a term in common. One might justify such a definition, if at least it were possible to prove the equivalence of the two statements. However, this is only the case if we assume from the outset that the transfer function does not have any zeros on the imaginary axis, as we will see below. We thus advocate (see Definition 1.2) a distinction in terms: minimum phase systems can be characterized in terms of the phase – the desired property of stability of the zero dynamics, however, is not an equivalent property. This definition is, as far as we are aware, not in the literature; we only know of [18, p. 483] where it is described in a prosaic way and not characterized.

(vi) Finally, we note in passing that another frequent slip of tongue is to associate properties of the step response with minimum phase. For instance, we may frequently read that initial undershoot in the step response is a "non-minimum phase characteristic". However, the results in [10] show that initial undershoot is not characterized equivalently by stability of the zero dynamics or for that matter by minimum phase.

We aim to define "minimum phase" in such a form that zeros of the numerator polynomial $p(s)$ in (1.1) are encompassed and the phase (where defined) of a minimum phase system is minimal under all gain equivalent transfer functions. The latter is stated precisely as follows.

Definition 1.1 *Two transfer functions* $g_1(s), g_2(s) \in \mathbb{C}(s)$ *are called* gain equivalent *(written* $g_1(s) \overset{\text{ge}}{\sim} g_2(s)$*) if, and only if,*

$$|g_1(\imath\omega)| = |g_2(\imath\omega)| \qquad \textit{for almost all } \omega > 0\,. \quad (1.2)$$

We note that the definition does not depend on properness or on the condition that there are no poles on $\imath\mathbb{R}$; indeed, the requirement that the equality only holds almost everywhere accounts for poles on $\imath\mathbb{R}$. If such poles do not exist, then continuity immediately implies that the equation holds everywhere.

Note that $\overset{\text{ge}}{\sim}$ defines an equivalence relation on $\mathbb{C}(s) \times \mathbb{C}(s)$ and we denote the *equivalence classes* by

$$[g(s)]_{\text{ge}} \;:=\; \{\tilde{g}(s) \in \mathbb{C}(s) |\; \tilde{g}(s) \overset{\text{ge}}{\sim} g(s)\}\,, \quad g(s) \in \mathbb{C}(s)\,.$$

The notion of minimum phase relies crucially on the concept of the phase or in mathematical terms of the argument of a transfer function. We recall (see e.g. [9, Proposition A.2.3]) that for any transfer function $\frac{p(s)}{q(s)} \in \mathbb{C}(s)^*$ with $q(s)$ semi Hurwitz there exists a differentiable function

$$\arg g(\imath\cdot) \;:\; \mathbb{R}\backslash\{\omega \in \mathbb{R}\,|p(\imath\omega)\,q(\imath\omega) = 0\,\} \to \mathbb{R}$$

such that

$$g(\imath\omega) = |g(\imath\omega)|\; \mathrm{e}^{\imath \arg g(\imath\omega)} \quad \forall\, \omega \in \mathbb{R} \;:\; p(\imath\omega)\, q(\imath\omega) \neq 0.$$

On every interval of definition, the derivative of $g(\imath\cdot)$ is unique since every argument function is unique up to a constant $2k\pi$, $k \in \mathbb{Z}$. In the following we will always consider argument functions that are differentiable on any open interval contained in the domain of definition.

The following property of the argument of polynomials is well known, see [9, Prop. 3.4.3]: Given a polynomial $p(s)$ without any zeros on the imaginary axis, we have that the total change of the argument along the imaginary axis satisfies

$$\begin{aligned} \Delta_{-\infty}^{\infty}\; p(\imath\omega) &:= \lim_{\omega\to\infty} \left[\arg p(\imath\omega) - \arg p(-\imath\omega)\right] \\ &= \left[\deg p(s) - 2\nu\right]\pi\,, \end{aligned}$$

where ν is the number of zeros of $p(s)$ in $\mathbb{C}_+$ (counting multiplicities). For a proper stable transfer function $g(s) = p(s)/q(s)$ we then obtain immediately (using $\arg 1/q(s) = -\arg q(s)$) that

$$\Delta_{-\infty}^{\infty} g(\imath\omega) = \left[\deg p(s) - 2\nu_p - \deg q(s)\right]\pi < 0\,,$$

where now ν_p denotes the number of zeros of p in $\mathbb{C}_+$. It is now obvious that the absolute value of the total change of the argument of $g(s)$ along the imaginary axis is minimal if $\nu_p = 0$.

This approach to a definition of minimum phase is unsatisfactory, as it excludes the possibility of zeros on the imaginary axis (in which case the necessary quantities are not well-defined). Also it does not capture the full strength of the minimum phase property, as the intermediate behaviour is not really taken into account. We learn however that the total phase change is a negative quantity and thus to keep the absolute value of this quantity small, we should aim at a derivative, which is as large as possible. This allows to suggest the following definition of minimum phase.

Definition 1.2 *A semi Hurwitz transfer function* $\widehat{g}(s) \in \mathbb{C}(s)^*$ *is said to be* minimum phase *if, and only if, all other gain equivalent semi Hurwitz transfer functions* $g(s) \in \mathbb{C}(s)$ *satisfy*

$$\frac{\mathrm{d}}{\mathrm{d}\omega} \arg g(\imath\omega) \leq \frac{\mathrm{d}}{\mathrm{d}\omega} \arg \widehat{g}(\imath\omega) \quad \textit{for almost all}\, \omega \in \mathbb{R}.$$

The definition says that within the equivalence class $[\widehat{g}(s)]_{\text{ge}}$ of all gain equivalent transfer functions with semi Hurwitz denominator function, the transfer function $\widehat{g}(s)$ is minimum phase if, and only if, the change of its argument function is at almost each $\omega \in \mathbb{R}$ smaller or equal than the change of the argument function of any other function belonging to the same equivalence class. As a prototypical picture of this scenario, we have plotted in Figure 1.1.1 the phase of two simple examples: $g(s)$ is a non-minimum phase transfer function which is gain equivalent to $\widehat{g}(s)$, a minimum phase transfer function. Please note that the derivative of the argument of $g(\imath\cdot)$ is strictly smaller than the derivative of the argument of $\widehat{g}(\imath\cdot)$. This confirms Definition 1.2 together with the result in Theorem 2.7.

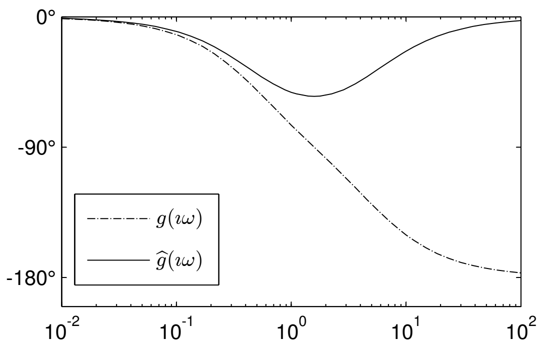

Bild 1.1.1: Phase of the minimum phase transfer function $\widehat{g}(s) = (0.2s+1)/(2s+1)$ and non-minimum phase transfer functions $g(s) = (-0.2s+1)/(2s+1)$ for $\omega \in [0, 100]$.

Another example of minimum phase transfer functions is

$$g(s) = s - \imath\omega_0 \qquad \text{for some } \;\omega_0 \in \mathbb{R}\,.$$

Then, for arbitrary $k, \ell \in \mathbb{Z}$,

$$g(\imath\omega) \;=\; \begin{cases} (\omega-\omega_0)\ \mathrm{e}^{\imath\left(\frac{\pi}{2}+2k\pi\right)} & , \omega > \omega_0 \\ 0 & , \omega = \omega_0 \\ (\omega_0-\omega)\ \mathrm{e}^{\imath\left(\frac{3\pi}{2}+2\ell\pi\right)} & , \omega < \omega_0 \end{cases} \tag{1.3}$$

and the argument of $\omega \mapsto g(\imath\omega)$ is constantly $\pi/2$ on (ω_0, ∞), $3\pi/2$ on $(-\infty, \omega_0)$, and has a discontinuity at $\omega = \omega_0$.

The paper is organized as follows: In Section 2 we factorize, in terms of the Hurwitz reflection and all-passes, all semi Hurwitz transfer functions. The main result is to show that a semi Hurwitz transfer functions is minimum phase if, and only if, its numerator polynomial is semi Hurwitz. As a side result we show that a proper and stable rational function is inner (outer) if, and only if, it is an all-pass (minimum phase), resp. In Section 3 we first consider the Byrnes-Isidori form, then introduce the internal loop form, and finally discuss the close relationship between asymptotic stable zero dynamics and minimum phase. It is shown that the former implies the latter but not vice versa.

2 Minimum phase

The main result of this section is the characterization of the minimum phase property of semi Hurwitz transfer functions $\frac{p(s)}{q(s)} \in \mathbb{C}(s)$ in Section 2.3. First, we collect basic properties of all-passes in Section 2.1 and show certain factorizations of semi Hurwitz transfer functions in Section 2.2.

2.1 All-pass and Hurwitz reflection

In this section we characterize an all-pass in terms of Hurwitz reflections.

Definition 2.1 *A transfer function $g(s) \in \mathbb{C}(s)$ is called an* all-pass *if, and only if, $g(s) \overset{\text{ge}}{\approx} 1$, that is*

$$|g(\imath\omega)| = 1 \qquad \text{for almost all } \ \omega \in \mathbb{R}\,. \tag{2.1}$$

Again, the notion "almost all" means finitely many as only the poles of $g(s)$ are affected by this.

A useful technical tool to investigate all-passes is the following mapping of polynomials.

Definition 2.2 *[9, Definition 3.4.4]*
The mapping

$$\begin{gathered} {}^\star\ :\ \mathbb{C}[s] \to \mathbb{C}[s], \\ \sum_{k=0}^{n} p_k s^k = p(s) \mapsto p^\star(s) := \overline{p}(-s) = \sum_{k=0}^{n} (-1)^k \overline{p}_k s^k\,, \end{gathered} \tag{2.2}$$

is called Hurwitz reflection.

The Hurwitz reflection is an involutive, $\mathbb{R}$-linear, degree preserving bijection and satisfies, for all $\alpha \in \mathbb{C}$ and all $p(s), q(s) \in \mathbb{C}[s]$,

$$(\alpha\, p(s))^\star = \overline{\alpha}\, p^\star(s) \quad \text{and} \quad (p(s)q(s))^\star = p^\star(s) q^\star(s)\,.$$

In particular, it follows for $p(s) = \gamma_p \prod_{k=1}^{n} (s - s_k) \in \mathbb{C}[s]$ that

$$p^\star(s) = (-1)^n \overline{\gamma}_p \prod_{k=1}^{n} (s + \bar{s}_k)\,. \tag{2.3}$$

Therefore, $s_0 \in \mathbb{C}$ is a zero of $p(s)$ if, and only if, $-\overline{s}_0$, i.e. the reflection of s_0 at the imaginary axis, is a zero of $p^\star(s)$. If $p(s) = p_0 \in \mathbb{C}$, then $p^\star(s) = \bar{p}_0$. The mapping $\,^\star$ maps any Hurwitz polynomial $p(s)$ to the *anti-Hurwitz polynomial* $p^\star(s)$, i.e. all zeros of $p^\star(s)$ lie in the open right half complex plane $\mathbb{C}_+$.

Proposition 2.3 *Any $r(s) \in \mathbb{C}[s]$ satisfies*

$$|r(\imath\omega)| \;=\; |r^\star(\imath\omega)| \qquad \forall\, \omega \in \mathbb{R}\,.$$

Proof We conclude, for $r(s)$ factorized as in (2.3), that

$$\begin{aligned} |r(\imath\omega)| &= \left|\gamma \prod_{k=1}^{n} (\imath\omega - s_k)\right| = \left|\bar{\gamma} \prod_{k=1}^{n} \overline{(\imath\omega - s_k)}\right| \\ &= \left|\bar{\gamma} \prod_{k=1}^{n} (\imath\omega + \overline{s}_k)\right| \;=\; |r^\star(\imath\omega)|\,. \quad \square \end{aligned}$$

We will now show that every all-pass has a simple representation.

Proposition 2.4 *For any $g(s) \in \mathbb{C}(s)$ we have*

$$g(s) \overset{\text{ge}}{\approx} 1 \iff g(s) \;=\; \mathrm{e}^{\imath\varphi}\, \frac{p(s)}{p^\star(s)}$$

for unique monic $p(s) \in \mathbb{C}[s]$ such that
$p(\imath\mathbb{R}) \cap \{0\} = \emptyset$ and $\varphi \in [0, 2\pi)$.

Proof
$\Longleftarrow$ The claim follows from Proposition 2.3.
$\Longrightarrow$ We proceed in several steps.
First we show that, for any $g(s) = \frac{p(s)}{q(s)}$ with $p(s) \in \mathbb{C}[s]$ and $q(s) \in \mathbb{C}[s]^*$, we have

$$g(s) \overset{\text{ge}}{\approx} 1 \iff \left[\forall\, z \in \mathbb{C}\ :\ p(z)p^\star(z) = q(z)q^\star(z)\right]. \tag{2.4}$$

As $\overline{p(\imath\omega)} = \overline{p}(-\imath\omega)$, we have, for almost all $\omega > 0$,

$$\begin{aligned} \frac{p(s)}{q(s)} \overset{\text{ge}}{\approx} 1 \quad &\overset{(1.2)}{\iff} \quad p(\imath\omega)\overline{p}(-\imath\omega) = q(\imath\omega)\overline{q}(-\imath\omega) \\ &\overset{(2.2)}{\iff} \quad p(\imath\omega)p^\star(\imath\omega) = q(\imath\omega)q^\star(\imath\omega)\,, \end{aligned}$$

and the identity property of analytic functions, see for example [6], yields

$$\forall\, z \in \mathbb{C}\ :\ p(z)p^\star(z) = q(z)q^\star(z)\,.$$

Next we show, for any $g(s) = \frac{p(s)}{q(s)}$ with coprime $p(s), q(s) \in \mathbb{C}[s]$ and $g(s) \overset{\text{ge}}{\approx} 1$, the two properties

$$p(z) = 0 \quad \Longleftrightarrow \quad q(-\bar{z}) = 0 \qquad \forall z \in \mathbb{C} \tag{2.5}$$

and

$$\{z \in \mathbb{C} |\, p(z) = p(-\bar{z}) = 0\} = \emptyset\,. \tag{2.6}$$

Note that $p(s), q(s) \in \mathbb{C}[s]$ are coprime (or, equivalently, solving the Bézout equation, there exist $\alpha(s), \beta(s) \in \mathbb{C}[s]$ such that $\alpha(s)p(s) + \beta(s)q(s) = 1$) if, and only if, $p^\star(s), q^\star(s)$ are coprime. Therefore, for any $z \in \mathbb{C}$,

$$p(z) = 0 \overset{(2.4)}{\Leftrightarrow} q^\star(z) = 0 \overset{(2.2)}{\Leftrightarrow} \overline{q(-z)} = 0 \Leftrightarrow \quad q(-\bar{z}) = 0\,.$$

This proves (2.5). To show Assertion (2.6), assume that $p(z) = 0 = p(-\bar{z})$ for some $z \in \mathbb{C}$. Then Assertion (2.5) yields that $q(z) = 0$, and this contradicts coprimeness of $p(s)$, $q(s)$; hence Assertion (2.6) follows.

Finally, assume that $g(s)$ is factorized as

$$g(s) = r\,\mathrm{e}^{\imath\varphi}\,\frac{p(s)}{q(s)} \quad \text{for} \quad \begin{array}{l} p(s), q(s) \in \mathbb{C}[s] \;\; \text{monic}, \\ \varphi \in [0, 2\pi),\ r > 0. \end{array}$$

Then $r = 1$ by $g(s) \overset{\text{ge}}{\approx} 1$, and (2.5) yields $q(s) = p^\star(s)$, and (2.6) gives $p(\imath\mathbb{R}) \cap \{0\} = \emptyset$.

This completes the proof of the proposition. □

2.2 Factorization

Before we prove a general result on the factorization of transfer function, consider a stable transfer function $g(s) = \dfrac{p(s)}{q_{\mathrm{H}}(s)}$ and assume that $p(s), q_{\mathrm{H}}(s) \in \mathbb{C}[s]$ are coprime and $p(\imath\mathbb{R}) \cap \{0\} = \emptyset$. Let $p(s)$ be uniquely factorized into

$p(s) = \gamma\, p_{\mathrm{H}}(s)\, p_{\mathrm{aH}}(s)$, where $\gamma \in \mathbb{C}$,
$p_{\mathrm{H}}(s)$ is monic and Hurwitz,
$p_{\mathrm{aH}}(s)$ is monic and anti-Hurwitz.

Then we may factorize

$$\frac{p(s)}{q_{\mathrm{H}}(s)} \;=\; \gamma\; \frac{p_{\mathrm{H}}(s)\; p^\star_{\mathrm{aH}}(s)}{q_{\mathrm{H}}(s)} \cdot \frac{p_{\mathrm{aH}}(s)}{p^\star_{\mathrm{aH}}(s)}\,. \tag{2.7}$$

Note that, in view Proposition 2.4, $p_{\mathrm{aH}}(s)/p^\star_{\mathrm{aH}}(s)$ is an all-pass and the numerator and denominator polynomials of the remaining factor $\gamma\, p_{\mathrm{H}}(s)\; p^\star_{\mathrm{aH}}(s)/q_{\mathrm{H}}(s)$ are both Hurwitz polynomials. Moreover, it is easy to see that the factorization in (2.7) is indeed unique in the following sense: Suppose

$$\frac{p(s)}{q_{\mathrm{H}}(s)} \;=\; \tilde{\gamma} \cdot \frac{\tilde{p}_{\mathrm{H}}(s)}{\tilde{q}_{\mathrm{H}}(s)} \cdot \frac{\pi(s)}{\pi^\star(s)},$$

where $\tilde{\gamma} \in \mathbb{C}$, $\tilde{p}_{\mathrm{H}}(s)$, $\tilde{q}_{\mathrm{H}}(s)$, $\pi^\star(s) \in \mathbb{C}[s]$ are monic and Hurwitz. Then

$$\underbrace{\gamma\; p_{\mathrm{aH}}(s)\;\;[\;p_{\mathrm{H}}(s)}_{=p(s)}\;\; \tilde{q}_{\mathrm{H}}(s)\, \pi^\star(s)\,\big] \;=\; \big[\;\tilde{\gamma}\; q_{\mathrm{H}}(s)\, \tilde{p}_{\mathrm{H}}(s)\;\big]\; \pi(s)$$

and since all polynomials are monic it follows that $\gamma = \tilde{\gamma}$; since the polynomials in parenthesis are Hurwitz and the remaining are anti-Hurwitz it follows that $\pi(s) = p_{\mathrm{aH}}(s)$ is uniquely determined; finally, coprimeness of $p(s)$ and $q(s)$ together with $\frac{\tilde{p}_{\mathrm{H}}(s)}{\tilde{q}_{\mathrm{H}}(s)} = \frac{p(s)\,\pi^\star(s)}{q_{\mathrm{H}}(s)\,\pi(s)}\gamma^{-1}$ shows that $\tilde{p}_{\mathrm{H}}(s)$ and $\tilde{q}_{\mathrm{H}}(s)$ are uniquely defined.

In the following theorem we will show that the above observation holds in a general context. The factorization is well-known; however, uniqueness is only considered in more restrictive cases, see for example [5, Lemma 6.2].

Theorem 2.5 *For any semi Hurwitz transfer function $\frac{p(s)}{q(s)} \in \mathbb{C}(s)^*$ with coprime $p(s)$ and $q(s)$, there exists a unique factorization*

$$\boxed{\frac{p(s)}{q(s)} \;=\; \gamma \cdot \frac{p_\imath(s)}{q_\imath(s)} \cdot \frac{p_{\mathrm{H}}(s)}{q_{\mathrm{H}}(s)} \cdot \frac{\pi(s)}{\pi^\star(s)}} \tag{2.8}$$

such that $\gamma \in \mathbb{C}$ *and the following polynomials are mutually coprime and satisfy*
$p_\imath(s), q_\imath(s), p_{\mathrm{H}}(s), q_{\mathrm{H}}(s), \pi(s) \in \mathbb{C}[s]^*$ *are monic,*
$p_\imath(s), q_\imath(s)$ *have zeros on* $\imath\mathbb{R}$ *only,*
$p_{\mathrm{H}}(s), q_{\mathrm{H}}(s)$ *are Hurwitz,*
$\pi^\star(s)$ *is Hurwitz.*

The unique factorization in (2.8) *can be constructed as follows: Factorize* $p(s)$ *and* $q(s)$ *uniquely into*
$p(s) \;=\; \gamma_p\, p_\imath(s)\, p_{\mathrm{s}}(s)\, p_{\mathrm{u}}(s), \quad q(s) \;=\; \gamma_q\, q_\imath(s)\, q_{\mathrm{s}}(s)\,,$
where $\gamma_p, \gamma_q \in \mathbb{C}$ *and*
$p_\imath(s), q_\imath(s), p_{\mathrm{s}}(s), q_{\mathrm{s}}(s), p_{\mathrm{u}}(s) \in \mathbb{C}[s]^*$ *are monic*
$p_\imath(s), q_\imath(s)$ *have zeros on* $\imath\mathbb{R}$ *only,*
$p_{\mathrm{s}}(s), q_{\mathrm{s}}(s), p^\star_{\mathrm{u}}(s)$ *are Hurwitz*

Then the factorization in (2.8) *is given by*

$$\begin{aligned} p_{\mathrm{H}}(s) &:= \frac{p_{\mathrm{s}}(s)p^\star_{\mathrm{u}}(s)}{\gcd\big(p^\star_{\mathrm{u}}(s), q_{\mathrm{s}}(s)\big)}, \\ q_{\mathrm{H}}(s) &:= \frac{q_{\mathrm{s}}(s)}{\gcd\big(p^\star_{\mathrm{u}}(s), q_{\mathrm{s}}(s)\big)}\,, \\ \pi(s) &:= p_{\mathrm{u}}(s)\,, \qquad\qquad \gamma := \gamma_p/\gamma_q\,, \end{aligned}$$

where $\gcd\big(r(s), \hat{r}(s)\big) \in \mathbb{C}[s]$ *denotes the monic greatest common divisor of* $r(s), \hat{r}(s) \in \mathbb{C}[s]$.

Proof Existence of a factorization (2.8) with the required properties for all polynomials follows from the definitions in the statement of the theorem.

We show uniqueness of the factorization (2.8):
Substituting the factorized polynomials $p(s)$ and $q(s)$ into (2.8), we obtain

$$\begin{aligned} &\gamma_p\, p_\imath(s)\, p_{\mathrm{s}}(s)\, p_{\mathrm{u}}(s)\;\cdot\; q_\imath(s)\, q_{\mathrm{H}}(s)\, \pi^\star(s) \\ &\qquad = \;\gamma_q\, q_\imath(s)\, q_{\mathrm{s}}(s)\;\cdot\; \gamma\; p_\imath(s)\, p_{\mathrm{H}}(s)\, \pi(s)\,, \end{aligned}$$

and since all polynomials are monic, we conclude $\gamma = \gamma_p/\gamma_q$. Now cancelation gives

$$p_{\mathrm{u}}(s)\,\big[p_{\mathrm{s}}(s)\, q_{\mathrm{H}}(s)\, \pi^\star(s)\big] \;=\; \big[q_{\mathrm{s}}(s)\, p_{\mathrm{H}}(s)\big]\, \pi(s)\,.$$

Since the polynomials in parenthesis are Hurwitz and $p_{\rm u}(s)$, $\pi(s)$ are anti-Hurwitz, we conclude that $\pi(s) = p_{\rm u}(s)$ is unique and thus

$$p_{\rm s}(s)\, q_{\rm H}(s)\, \pi^\star(s) \;=\; q(s)\, p_{\rm H}(s)\,.$$

This gives

$$\frac{\frac{p_{\rm s}(s)p_{\rm u}^\star(s)}{\gcd\left(p_{\rm u}^\star(s),q_{\rm s}(s)\right)}}{\frac{q_{\rm s}(s)}{\gcd\left(p_{\rm u}^\star(s),q_{\rm s}(s)\right)}} \;=\; \frac{p_{\rm H}(s)}{q_{\rm H}(s)}\,,$$

and since the quotients on the left hand side are coprime, uniqueness of $p_{\rm H}(s)$ and $q_{\rm H}(s)$ follows. This completes the proof of the theorem. □

We stress that in view of Proposition 2.4, the last factor in (2.8) satisfies $\frac{\pi(s)}{\pi^\star(s)} \overset{\rm ge}{\approx} 1$; therefore, this factor only changes the phase of $\omega \mapsto \frac{p(\imath\omega)}{q(\imath\omega)}$ and not the gain.

Theorem 2.5 allows to parameterize each equivalence class $\left[\frac{p(s)}{q(s)}\right]_{\rm ge}$ of a semi Hurwitz transfer function $\frac{p(s)}{q(s)} \in \mathbb{C}(s)^*$ by a number of modulus 1 and an all-pass as follows.

Proposition 2.6 *Let $\frac{p(s)}{q(s)} \in \mathbb{C}(s)^*$ be a semi Hurwitz coprime transfer function factorized as in (2.8). Then $\tilde g(s)$ is a gain equivalent semi Hurwitz coprime transfer function if, and only if, it has the form*

$$\tilde g(s) = {\rm e}^{\imath\varphi}\gamma \cdot \frac{p_\imath(s)}{q_\imath(s)} \cdot \frac{p_{\rm H}(s)}{q_{\rm H}(s)} \cdot \frac{\tilde\pi(s)}{\tilde\pi^\star(s)} \tag{2.9}$$

for some $\varphi \in [0, 2\pi)$ and some monic Hurwitz $\tilde\pi^\star(s) \in \mathbb{C}[s]$.

Proof If $\tilde g(s)$ has the form (2.9), then it is semi Hurwitz and gain equivalent by Proposition 2.4.

Conversely, consider a gain equivalent semi Hurwitz transfer function. In view of Theorem 2.5, we may assume that

$$\frac{p(s)}{q(s)} \overset{\rm ge}{\approx} \tilde\gamma \cdot \frac{\tilde p_\imath(s)}{\tilde q_\imath(s)} \cdot \frac{\tilde p_{\rm H}(s)}{\tilde q_{\rm H}(s)} \cdot \frac{\tilde\pi(s)}{\tilde\pi^\star(s)}$$

where $\tilde\gamma \in \mathbb{C}$ and

$\tilde p_\imath(s), \tilde q_\imath(s), \tilde p_{\rm H}(s), \tilde q_{\rm H}(s), \tilde\pi(s) \in \mathbb{C}[s]^*$ are monic
$\tilde p_{\rm H}(s), \tilde q_{\rm H}(s)$ are coprime and Hurwitz
$\tilde p_\imath(s), \tilde q_\imath(s)$ are coprime and have zeros on $\imath\mathbb{R}$ only,
$\tilde\pi^\star(s)$ is Hurwitz.

Then gain equivalence together with Proposition 2.3 yields, for almost all $\omega > 0$,

$$\left|\gamma\, \frac{p_\imath(\imath\omega)}{q_\imath(\imath\omega)}\, \frac{p_{\rm H}(\imath\omega)}{q_{\rm H}(\imath\omega)}\right| \;=\; \left|\tilde\gamma\, \frac{\tilde p_\imath(\imath\omega)}{\tilde q_\imath(\imath\omega)}\, \frac{\tilde p_{\rm H}(\imath\omega)}{\tilde q_{\rm H}(\imath\omega)}\right|. \tag{2.10}$$

Since all polynomials in (2.10) are monic, we conclude that $|\gamma| = |\tilde\gamma|$ and hence there exists $\varphi \in [0, 2\pi)$ such that $\tilde\gamma = {\rm e}^{\imath\varphi}\gamma$. Now the identity property of analytic functions, see for example [6], gives, for almost all $s \in \mathbb{C}$,

$$\left|\frac{p_\imath(s)}{q_\imath(s)}\, \frac{p_{\rm H}(s)}{q_{\rm H}(s)}\right| \;=\; \left|\frac{\tilde p_\imath(s)}{\tilde q_\imath(s)}\, \frac{\tilde p_{\rm H}(s)}{\tilde q_{\rm H}(s)}\right|. \tag{2.11}$$

Since $p_\imath(s)$, $q_\imath(s)$, $\tilde p_\imath(s)$, $\tilde q_\imath(s)$ have zeros on $\imath\mathbb{R}$ only, and $p_{\rm H}(s)$, $q_{\rm H}(s)$, $\tilde p_{\rm H}(s)$, $\tilde q_{\rm H}(s)$ are Hurwitz, and the fractions are all coprime, it follows from (2.11) that

$$p_\imath(s) = \tilde p_\imath(s), q_\imath(s) = \tilde q_\imath(s), p_{\rm H}(s) = \tilde p_{\rm H}(s), q_{\rm H}(s) = \tilde q_{\rm H}(s).$$

This completes the proof of the proposition. □

2.3 Minimum phase characterization

We now characterize minimum phase systems.

Theorem 2.7 *For any semi Hurwitz coprime transfer function $\frac{p(s)}{q(s)} \in \mathbb{C}(s)^*$ we have:*

$$\begin{aligned} \frac{p(s)}{q(s)} \text{ is min. phase } &\iff p(s) \text{ is semi Hurwitz} \\ &\iff \pi(s) = 1 \quad \text{in } (2.8)\,. \end{aligned}$$

Proof Let $\frac{p(s)}{q(s)}$ be factorized as in (2.8). Denote the set of all roots of $p_\imath(\imath\cdot)$ and $q_\imath(\imath\cdot)$ on $\imath\mathbb{R}$ by

$$\mathcal{R}(p_\imath q_\imath) \;:=\; \{\omega \in \mathbb{R} \,|\, p_\imath(\imath\omega) q_\imath(\imath\omega) = 0\}$$

and choose argument functions

$$\begin{aligned} \arg(p/q)(\imath\cdot) &: \mathbb{R}\backslash\mathcal{R}(p_\imath q_\imath) \to \mathbb{R} \\ \arg p_\imath(\imath\cdot) &: \mathbb{R}\backslash\mathcal{R}(p_\imath q_\imath) \to \mathbb{R} \\ \arg(p_{\rm H}/q_{\rm H})(\imath\cdot) &: \mathbb{R} \to \mathbb{R} \\ \arg \pi(\imath\cdot) &: \mathbb{R} \to \mathbb{R} \\ \arg \pi^\star(\imath\cdot) &: \mathbb{R} \to \mathbb{R}\,. \end{aligned}$$

We proceed in several steps.
Step 1: If $s - s_0$ is Hurwitz, then an elementary geometric argument yields that $\frac{\rm d}{{\rm d}\omega} \arg(\imath\omega - s_0) > 0$ for all $\omega \in \mathbb{R}$. This is the "phase increasing property" which also holds for any Hurwitz polynomial, see [9, Proposition 3.4.5]. Therefore, if $\deg \pi(s) > 0$ for $\pi(s)$ as in (2.8), then the Hurwitz polynomial $\pi^\star(s)$ satisfies

$$\frac{\rm d}{{\rm d}\omega} \arg \pi^\star(\imath\omega) \;>\; 0 \qquad \forall\, \omega \in \mathbb{R}\,,$$

and since $\pi(s)$ is anti-Hurwitz, we may derive similarly that

$$\frac{\rm d}{{\rm d}\omega} \arg \pi(\imath\omega) \;<\; 0 \qquad \forall\, \omega \in \mathbb{R}\,.$$

Therefore we arrive, if $\deg \pi(s) > 0$, at

$$\frac{\rm d}{{\rm d}\omega} \arg \frac{\pi(\imath\omega)}{\pi^\star(\imath\omega)} < 0 \qquad \forall\, \omega \in \mathbb{R}\,. \tag{2.12}$$

Step 2: Note that (1.3) yields

$$\frac{\rm d}{{\rm d}\omega} \arg \frac{p_\imath(\imath\omega)}{q_\imath(\imath\omega)} = 0 \qquad \forall\, \omega \in \mathbb{R}\backslash\mathcal{R}(p_\imath q_\imath)\,. \tag{2.13}$$

Step 3: Now we conclude, for all $\omega \in \mathbb{R}\backslash\mathcal{R}(p_\imath q_\imath)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\omega}\arg\frac{p(\imath\omega)}{q(\imath\omega)} &\overset{(2.8)}{=} \frac{\mathrm{d}}{\mathrm{d}\omega}\arg\left(\gamma\,\frac{p_\imath(\imath\omega)}{q_\imath(\imath\omega)}\,\frac{p_{\mathrm{H}}(\imath\omega)}{q_{\mathrm{H}}(\imath\omega)}\,\frac{\pi(\imath\omega)}{\pi^\star(\imath\omega)}\right)\\ &\overset{(2.13)}{=} \frac{\mathrm{d}}{\mathrm{d}\omega}\arg\left(\gamma\,\frac{p_{\mathrm{H}}(\imath\omega)}{q_{\mathrm{H}}(\imath\omega)}\right) + \frac{\mathrm{d}}{\mathrm{d}\omega}\arg\left(\frac{\pi(\imath\omega)}{\pi^\star(\imath\omega)}\right) \quad (2.14)\end{aligned}$$

Step 4: We are now ready to finalize the proof. The equivalence of $\frac{p(s)}{q(s)}$ being minimum phase and $\pi(s) = 1$ is a consequence of Proposition 2.6 together with (2.12) and (2.14).
The equivalence of $p(s)$ having zeros in the closed left half complex plane only and $\pi(s) = 1$ is a consequence of the definition of $\pi(s)$ in Theorem 2.5. This completes the proof of the theorem. ☐

Theorem 2.7 allows to show that the factorization (2.8) is closely related to the so called inner-outer factorization, see [4, Sect. 2.8]. However, their setup is more restrictive since only stable and proper transfer functions are allowed. To explain this concept in more detail, we need to generalize the Hurwitz reflection to rational functions as follows:

$$g^\star(s) \;:=\; \frac{p^\star(s)}{q^\star(s)} \qquad \text{for any } g(s) = \frac{p(s)}{q(s)} \in \mathbb{C}(s)^*\,.$$

Definition 2.8 *Let* $g(s) = \frac{p(s)}{q(s)} \in \mathbb{C}(s)^*$ *be a stable and proper transfer function, i.e.* $q(s)$ *is Hurwitz and* $\deg p(s) \le \deg q(s)$. *Then* $g(s)$ *is called*

$$\begin{aligned}&\text{inner} &&:\Longleftrightarrow\; g^\star(s)\,g(s) = 1\\ &\text{outer} &&:\Longleftrightarrow\; \exists\, h(s) \in \mathbb{C}(s) \text{ analytic in } \mathbb{C}_+\\ & && \qquad\;\; \text{such that } g(s)\,h(s) = 1\,.\end{aligned}$$

Note that the function $h(\cdot)$ in the definition of outer is semi Hurwitz. In the literature, quite often the notation $g^\sim(s)$ is used instead of $g^\star(s)$, see for example [4] .

For the convenience of the reader, we include a proof of the fact that inner (outer) is the same as a stable all-pass (minimum phase), resp.

Proposition 2.9 *Any stable and proper* $g(s) = \frac{p(s)}{q(s)} \in \mathbb{C}(s)^*$ *satisfies:*

(i) $g(s)$ *is inner* $\Longleftrightarrow$ $g(s)$ *is an all-pass.*
(ii) $g(s)$ *is outer* $\Longleftrightarrow$ $g(s)$ *is minimum phase.*

Proof (i) Write

$$g(s) = \gamma\frac{\hat p(s)}{\hat q(s)} \quad \text{for } \gamma \in \mathbb{C} \text{ and monic } \hat p(s),\, \hat q(s) \in \mathbb{C}[s].$$

If $g(s)$ is inner, then

$$\left|\gamma\,\frac{\hat p(s)}{\hat q(s)}\;\cdot\;\bar\gamma\,\frac{\hat p^\star(s)}{\hat q^\star(s)}\right| = 1 \qquad \text{for almost all } s \in \mathbb{C}$$

and since $\hat p(s)$, $\hat q(s)$ are monic, it follows that $|\gamma\bar\gamma| = 1$ and so $|\gamma| = 1$. Therefore, Proposition 2.3 yields, for almost all $s \in \mathbb{C}$, $|p(s)/q(s)| = 1$ and thus $g(s)$ is an all-pass.

If $g(s)$ is an all-pass, then Proposition 2.3 yields that $g^\star(s)$ is an all-pass; this gives $|g(\imath\omega)\,g^\star(\imath\omega)| = 1$ for almost all $\omega > 0$. Now the identity property of analytic functions yields that $g(s)$ is inner.
(ii) This equivalence is immediate from the definition of outer and Theorem 2.7. In particular, if $g(s)$ is minimum phase, then $h(s) = q(s)/p(s)$. ☐

Finally, we show that the factorization (2.8) leads in fact to a canonical form for gain equivalent semi Hurwitz transfer functions. First recall the definition of a canonical form.

Definition 2.10 *[1] Let* $\mathcal{M}$ *be a nonempty set and* $\sim$ *an equivalence relation on* $\mathcal{M}\times\mathcal{M}$. *A map* $\Gamma : \mathcal{M} \to \mathcal{M}$ *is called a* canonical form *for the equivalence relation* $\sim$ *if, and only if,*

$$\forall\, m, m' \in \mathcal{M} \,:\, \Gamma(m){\sim}m \;\wedge\; [m{\sim}m' \Leftrightarrow \Gamma(m) = \Gamma(m')]\,.$$

This means that the set $\mathcal{M}$ is divided into disjoint equivalence classes, which gives a partition $\widehat{\mathcal{M}}$ of $\mathcal{M}$, so that for all components of the partition the mapping Γ assigns a unique element of the component.

Proposition 2.11 *The map*

$$\begin{aligned}\Gamma :\quad & \mathbb{C}(s)_{\mathrm{sH}} && \to && \mathbb{C}(s)_{\mathrm{sH}}\\ & \gamma\,\frac{p_\imath(s)}{q_\imath(s)}\,\frac{p_{\mathrm{H}}(s)}{q_{\mathrm{H}}(s)}\,\frac{\pi(s)}{\pi^\star(s)} && \mapsto && |\gamma|\,\frac{p_\imath(s)}{q_\imath(s)}\,\frac{p_{\mathrm{H}}(s)}{q_{\mathrm{H}}(s)}\,,\end{aligned}$$

where we use the unique factorization (2.8)*, is a canonical form for* $\overset{\mathrm{ge}}{\approx}$ *on* $\mathbb{C}(s)_{\mathrm{sH}}$.

Proof It follows from the definition of Γ and Proposition 2.4 that $g(s) \overset{\mathrm{ge}}{\approx} \Gamma(g(s))$. Furthermore, by Proposition 2.6 we have for any $\tilde g(s) \in \mathbb{C}(s)_{\mathrm{sH}}$ that

$$\gamma\,\frac{p_\imath(s)}{q_\imath(s)}\,\frac{p_{\mathrm{H}}(s)}{q_{\mathrm{H}}(s)}\,\frac{\pi(s)}{\pi^\star(s)} \quad \overset{\mathrm{ge}}{\approx} \quad \tilde g(s)$$

if, and only if,

$$\tilde g(s) = \mathrm{e}^{\imath\varphi}\gamma\cdot\frac{p_\imath(s)}{q_\imath(s)}\cdot\frac{p_{\mathrm{H}}(s)}{q_{\mathrm{H}}(s)}\cdot\frac{\tilde\pi(s)}{\tilde\pi^\star(s)}$$

with $\tilde\pi^\star(s)$ Hurwitz. This is the case if, and only if, $\Gamma(g(s)) = \Gamma(\tilde g(s))$ and the proof is complete. ☐

Proposition 2.11 may be described as follows: Within each equivalence class $[g(s)]_{\mathrm{ge}}$ of a semi Hurwitz transfer function $g(s) = \frac{p(s)}{q(s)} \in \mathbb{C}(s)$ we may single out a transfer function which is minimum phase. In other words, the minimum phase system is the canonical representative within its equivalence class. In view of the factorization (2.8), a possible canonical representative is $\widehat{g}(s) := |\gamma| \cdot \frac{p_\imath(s)}{q_\imath(s)} \cdot \frac{p_{\mathrm{H}}(s)}{q_{\mathrm{H}}(s)}$, but could equally be chosen as $\mathrm{e}^{\imath\varphi}\gamma \cdot \frac{p_\imath(s)}{q_\imath(s)} \cdot \frac{p_{\mathrm{H}}(s)}{q_{\mathrm{H}}(s)}$ for any $\varphi \in [0, 2\pi)$.

3 Zero dynamics

In this section we show how asymptotically stable zero dynamics – a concept in the time domain – is closely related to minimum phase – a concept in the frequency domain.

If one defines a minimum phase system as in (ii) in the Introduction, namely that numerator and denominator polynomial of the transfer function are Hurwitz, then the above two concept do in fact coincide. This has led to the abuse of terminology that a nonlinear systems is called "minimum phase" if the zero dynamics is asymptotically stable. However, for other definitions and in particular for Definition 1.2, asymptotic stable zero dynamics implies minimum phase but not vice versa.

Compared to the previous section, we now allow only for strictly proper transfer functions since the state space results in this subsection are on systems with no feedthrough.

First, we recall some basic notions. Consider the system class

$$\Sigma_n(\mathbb{K}) := \left\{(A, b, c) \in \mathbb{K}^{n\times n} \times \mathbb{K}^n \times \mathbb{K}^{1\times n} \,|\, c(sI_n - A)^{-1}b \neq 0\right\},$$

and associate to any $(A, b, c) \in \Sigma_n(\mathbb{K})$ the state space system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t), \\ y(t) &= cx(t), \end{aligned} \tag{3.1}$$

where the space of trajectories is a subset of $\mathcal{X} := \mathcal{AC}(\mathbb{R}_{\geq 0}; \mathbb{R}^n)$, i.e. the set of absolutely continuous functions $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, see [9, Def. A.3.12]; and the input space is $\mathcal{U} := L^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$, i.e. the set of locally integrable functions.

We do not assume that the system (3.1) is minimal, that means controllable and observable.

Definition 3.1 *The relative degree of a transfer function $\frac{p(s)}{q(s)} \in \mathbb{C}(s)^*$ is the number $\rho := \deg q(s) - \deg p(s)$.*

The relative degree can be characterized for state space systems as follows.

Remark 3.2 *It is straightforward to show that the relative degree of the transfer function $c(sI_n - A)^{-1}b$ of system (3.1) is equal to the smallest $\rho \in \mathbb{N}$ such that*

$$cA^{\rho-1}b \neq 0 \text{ and, if } \rho > 1, \quad \forall\, k = 0, \dots, \rho - 2 \;:\; cA^k b = 0\,. \tag{3.2}$$

The relationship between the time and frequency domain explained in the next subsections is depicted in Figure 1.3.1.

3.1 The Byrnes-Isidori form

The Byrnes-Isidori form is well-known for systems belonging to $\Sigma_n(\mathbb{K})$; see for example [12, Sec. 4.1]. We will show that the entries in the Byrnes-Isidori form are, to a certain extent, uniquely defined and uniquely related to minimum phase transfer functions. To the best of the authors' knowledge, uniqueness of the Byrnes-Isidori form, and furthermore the question as to whether it is a canonical form, has not been discussed in the literature.

Definition 3.3 *System* (3.1) *is said to be in* Byrnes-Isidori form *if, and only if, for some $\rho \in \mathbb{N}$, the matrices are of the form*

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ R_1 & R_2 & \cdots & R_{\rho-1} & R_\rho & S \\ P & 0 & \cdots & 0 & 0 & Q \end{bmatrix}, b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \gamma \\ 0_{n-\rho} \end{pmatrix}, c = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix}^\top \tag{3.3}$$

where

$$R_1, \dots, R_\rho \in \mathbb{K}, \quad \gamma \in \mathbb{K}^*, \quad Q \in \mathbb{K}^{(n-\rho)\times(n-\rho)}, \quad P, S^\top \in \mathbb{K}^{(n-\rho)\times 1}.$$

Note that if (3.1) is in Byrnes-Isidori form (3.3), then it has relative degree ρ because $\gamma \neq 0$. We are now in a position to derive for any $(A, b, c) \in \Sigma_n(\mathbb{K})$, not necessarily minimal, a Byrnes-Isidori form.

Theorem 3.4 *For any linear system* (3.1) *with relative degree $\rho \in \mathbb{N}^*$ there exists a coordinate transformation $U \in \mathbf{Gl}_n(\mathbb{K})$ such that*

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} := \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_\rho \\ \eta \end{pmatrix} := \begin{pmatrix} cx \\ cAx \\ \vdots \\ cA^{(\rho-1)}x \\ \eta \end{pmatrix} = \begin{pmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(\rho-1)} \\ \eta \end{pmatrix} = U\,x\,, \tag{3.4}$$

where $\xi_1, \dots, \xi_\rho \in \mathbb{K}$, $\eta \in \mathbb{K}^{n-\rho}$, transforms (3.1) *into Byrnes-Isidori form* (3.3).
As for uniqueness, if $\widetilde{U}, \widehat{U} \in \mathbf{Gl}_n(\mathbb{K})$ transform (3.1)

into $(\widetilde{A}, \widetilde{b}, \widetilde{c})$, $(\widehat{A}, \widehat{b}, \widehat{c})$, *resp., both in Byrnes-Isidori form* (3.3) *and the entries are written with* $\widetilde{\ }$ *and* $\widehat{\ }$, *then*

(i) $\widetilde{\gamma} = \widehat{\gamma} = cA^{\rho-1}b, \quad \widetilde{R}_i = \widehat{R}_i$ *for all* $i = 1, \ldots, \rho$,

(ii) $\exists\, Z \in \mathbf{Gl}_{n-\rho}(\mathbb{R}) \;:\; (\tilde{Q}, \tilde{P}, \tilde{S}) = (Z\widehat{Q}\,Z^{-1}, Z\widehat{P}, \widehat{S}Z^{-1})$.

Before we prove the above theorem at the end of the present subsection, two important observations are warranted.

Proposition 3.5 *Any system in Byrnes-Isidori form* (3.3) *satisfies:*
(A, b) *is stabilizable (controllable) if, and only if,* (Q, P) *is stabilizable (controllable).*
(A, c) *is detectable (observable) if, and only if,* (Q, S) *is detectable (observable).*

Proof Since the considered properties are invariant under state space transformation, the claim follows in a straightforward manner from the Hautus criteria; see, for example, [17, pp. 94, 215, 272, 318]. □

Remark 3.6 *Theorem 3.4 shows that the Byrnes-Isidori form* (3.4)*, consists of a unique loop of integrators and an internal loop of the system* (Q, P, S) *which is unique up to a state space transformation; see the south-west corner of Figure 1.3.1. The constant* γ *denotes the so-called high-frequency gain of the system. This decomposition is the core of the Byrnes-Isidori form and is extensively exploited to solve many regulation problems [12].*

Quite often the Byrnes-Isidori form (3.3) is called a canonical form. This is, strictly speaking not true since the entries are not unique, but can be said in the following sense.

Proposition 3.7 *The map*

$$\begin{aligned}\Gamma :\; \Sigma_n(\mathbb{C}) \;&\to \Sigma_n(\mathbb{C})\\ (A, b, c) &\mapsto (\hat{A}, \hat{b}, \hat{c}) \textit{ in Byrnes-Isidori form } (3.3)\\ &\qquad \textit{with } (Q, P, S) \textit{ in control canonical form}\end{aligned}$$

is a canonical form for the similarity action

$$\begin{aligned}&(A, b, c) \overset{\text{sim}}{\sim} (\hat{A}, \hat{b}, \hat{c})\\ :\Longleftrightarrow\; &\exists\, U \in \mathbf{Gl}_n(\mathbb{C}) \;:\; (\hat{A}, \hat{b}, \hat{c}) = (U^{-1}AU, U^{-1}b, cU)\,.\end{aligned}$$

Proof Define the map

$$\begin{aligned}\alpha \;:\; \mathbf{Gl}_n(\mathbb{C}) \times \Sigma_n(\mathbb{C}) &\to \Sigma_n(\mathbb{C})\\ (U, (A, b, c)) &\mapsto (U^{-1}AU, U^{-1}b, cU)\,.\end{aligned}$$

The map

$$\begin{aligned}\Sigma_n(\mathbb{C}) \to \Sigma_n(\mathbb{C}),\; (A, b, c) \mapsto (U^{-1}AU, U^{-1}b, cU)\\ \text{in Byrnes-Isidori form (3.3)}\end{aligned}$$

is in general not well-defined, since the subsystem (Q, P, S) in (3.3) is only unique up to similarity; see Theorem 3.4. However, if we require (Q, P, S) to be in control canonical form, then Γ is well-defined and it is easy to see that it is a canonical form. □

Proof of Theorem 3.4 We prove existence of the Byrnes-Isidori form first. The proof is constructive, we determine the coordinate transformation U in terms of (A, b, c) and use the following notation:

$$\mathcal{B} := [b, Ab, \ldots A^{\rho-1}b] \in \mathbb{K}^{n\times\rho}$$

$$\mathcal{C} := \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{\rho-1}\end{bmatrix} \in \mathbb{K}^{\rho\times n}\,, \qquad \gamma := cA^{\rho-1}b \in \mathbb{K}\,. \tag{3.5}$$

Now choose

$$V \in \mathbb{K}^{n\times(n-\rho)} \;\text{ such that }\; \mathrm{im\,V} = \ker\mathcal{C}\,, \tag{3.6}$$

and define

$$N := (V^\top V)^{-1}V^\top[I_n - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}] \in \mathbb{K}^{(n-\rho)\times n} \tag{3.7}$$

$$U := \begin{bmatrix}\mathcal{C}\\ N\end{bmatrix} \in \mathbb{K}^{n\times n} \tag{3.8}$$

Step 1: We show that

$$(\widehat{A}\,, \widehat{b}\,, \widehat{c}) := \big(UAU^{-1}, Ub\,, cU^{-1}\big) \tag{3.9}$$

is in Byrnes-Isidori form (3.3) where the entries are written with $\widehat{\ }$ and and

$$\begin{aligned}&[\widehat{R}_1, \ldots, \widehat{R}_\rho, \widehat{S}] = cA^\rho U^{-1} \in \mathbb{K}^{1\times n}\,,\\ &\widehat{P} = NA^\rho b\gamma^{-1} \in \mathbb{K}^{(n-\rho)\times 1}\,,\\ &\widehat{Q} = NAV \in \mathbb{K}^{(n-\rho)\times(n-\rho)}.\end{aligned} \tag{3.10}$$

First note that $\gamma \neq 0$ together with Remark 3.2 yields

$$\mathcal{C}\mathcal{B} = \begin{bmatrix} 0 & & \gamma \\ & \iddots & \\ \gamma & & * \end{bmatrix} \in \mathbf{Gl}_\rho(\mathbb{K})\,. \tag{3.11}$$

Therefore, $\operatorname{rank}\mathcal{C} = \rho$, hence $\operatorname{rank} V = n - \rho$ and $(V^\top V)^{-1}$ exists, and so N is well defined. Furthermore, $[I_n - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}]$ annihilates $\mathcal{B}$ and so $N\mathcal{B} = 0$. Since $U\,[\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}, V] = I_n$, U has inverse

$$U^{-1} = [\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}, V] \tag{3.12}$$

and a routine calculation yields that $\widehat{b} = Ub$ and $\widehat{c} = cU^{-1}$ have the structure given in (3.3). Now

$$U\,A = \widehat{A}\,U \tag{3.13}$$

implies that $\widehat{A}$ has the form

$$\widehat{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 \\ \widehat{R}_1 & \widehat{R}_2 & \dots & \widehat{R}_{\rho-1} & \widehat{R}_\rho & \widehat{S} \\ \widehat{P}_1 & \widehat{P}_2 & \dots & \widehat{P}_{\rho-1} & \widehat{P}_\rho & \widehat{Q} \end{bmatrix}, \tag{3.14}$$

with

$$\widehat{R}_1, \dots, \widehat{R}_\rho \in \mathbb{K},\ \widehat{P}_1, \dots, \widehat{P}_\rho \in \mathbb{K}^{(n-\rho)\times 1},\ \widehat{S} \in \mathbb{K}^{1\times(n-\rho)}$$

and, in view of (3.12),

$$\widehat{Q} = NAV \in \mathbb{K}^{(n-\rho)\times(n-\rho)},\ [\widehat{R}_1, \dots, \widehat{R}_\rho, \widehat{S}] = cA^\rho U^{-1}.$$

Recalling that $N\mathcal{B} = 0$, we see that

$$[\widehat{P}_1, \dots, \widehat{P}_\rho] = NA\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} = [\,0,\dots,0,NA^\rho b\,] \begin{bmatrix} * & & \gamma^{-1} \\ & \cdot^{\cdot^{\cdot}} & \\ \gamma^{-1} & & 0 \end{bmatrix},$$

and hence $\widehat{P}_2 = \dots = \widehat{P}_\rho = 0$. Writing $\widehat{P} = \widehat{P}_1$, it follows that $\widehat{A}$ takes the form (3.14) and $\widehat{P} = NA^\rho b\gamma^{-1}$. This proves that the coordinate transformation (3.4) converts (3.1) into Byrnes-Isidori form (3.3).

It remains to show the uniqueness properties. First note that by construction the relative degree of a Byrnes-Isidori form is one larger than the number of integrators represented by the off-diagonal identity matrices in the matrix $\widehat{A}$. As the relative degree is invariant under similarity transformation, it follows that any Byrnes-Isidori form of (A, b, c) has the structure as in (3.3). Therefore, $\gamma = cA^{\rho-1}b$ is the high-frequency gain which does not depend on the similarity transformation.

Suppose now that $\widetilde{U}, \widehat{U} \in \mathbf{Gl}_n(\mathbb{K})$ transform (3.1) into $(\widetilde{A}, \widetilde{b}, \widetilde{c})$, $(\widehat{A}, \widehat{b}, \widehat{c})$, resp., both in Byrnes-Isidori form (3.3) with entries are written with $\tilde{}$ and $\hat{}$. Define

$$\widetilde{U}\widehat{U}^{-1} =: Y =: \begin{bmatrix} Y^1 \\ \vdots \\ Y^{\rho+1} \end{bmatrix} =: [Y_1, \dots, Y_{\rho+1}] \tag{3.15}$$

for $(Y^i)^\top, Y_i \in \mathbb{K}^n,\ i = 1, \dots, \rho,$ and $Y^{\rho+1}, (Y_{\rho+1})^\top \in \mathbb{K}^{(n-\rho)\times n}$. Then

$$Y\widehat{A}Y^{-1} = \tilde{A} \tag{3.16}$$

$$Y\widehat{b} = \tilde{b} \tag{3.17}$$

$$\widehat{c} = \tilde{c}Y. \tag{3.18}$$

This gives

$$Y^1 \overset{(3.18)}{=} [\,1,\,0,\,\dots,\,0\,], \quad Y_\rho^\top \overset{(3.17)}{=} [\,0,\,\dots,\,0,\,1,\,0\,]^\top \tag{3.19}$$

and we proceed

$$\begin{aligned} [0,1,0,\dots,0] &\overset{(3.3)}{=} Y^1\widehat{A} \overset{(3.19)}{=} Y^1 Y\widehat{A} \overset{(3.16)}{=} Y^1\tilde{A}Y \\ &\overset{(3.3)}{=} [0,1,0,\dots,0]Y \overset{(3.15)}{=} Y^2 \end{aligned}$$

$$\begin{aligned} [0,0,1,0,\dots,0] &\overset{(3.3)}{=} Y^2\widehat{A} = Y^2 Y\widehat{A} \overset{(3.16)}{=} Y^2\tilde{A}Y \\ &\overset{(3.3)}{=} [0,0,1,0,\dots,0]Y \overset{(3.15)}{=} Y^3 \\ &\ \ \vdots \end{aligned}$$

$$\begin{aligned} [0,\dots,0,1,0] &\overset{(3.3)}{=} Y^{\rho-1}\widehat{A} = Y^{\rho-1} Y\widehat{A} \overset{(3.16)}{=} Y^{\rho-1}\tilde{A}Y \\ &\overset{(3.3)}{=} [0,\dots,0,1,0]Y \overset{(3.15)}{=} Y^\rho. \end{aligned}$$

Therefore, Y is, for some $Z \in \mathbf{Gl}_{n-\rho}(\mathbb{K})$, of the form

$$Y = \begin{bmatrix} 1 & 0 & \dots & & 0 & 0 \\ 0 & 1 & & & & 0 \\ \vdots & \ddots & & \ddots & & \vdots \\ 0 & \dots & & 0 & 1 & 0 \\ Y_{\rho+1,1} & \dots & & Y_{\rho+1,\rho-1} & 0 & Z \end{bmatrix}. \tag{3.20}$$

This together with (3.16) already shows that $\tilde{S} = \widehat{S}Z^{-1}$. Now the special structure of $\widehat{A}$, $\tilde{A}$ in (3.3) and of Y in (3.20) yields

$$\begin{aligned} &[Z\widehat{P}, Y_{\rho+1,1}, \dots, Y_{\rho+1,\rho-1}, Z\widehat{Q}] \\ &= [\,0,\,\dots,\,0,\,I_{n-\rho}\,] \begin{bmatrix} 1 & 0 & \dots & & 0 & 0 \\ 0 & 1 & & & & 0 \\ \vdots & \ddots & & \ddots & & \vdots \\ 0 & \dots & & 0 & 1 & 0 \\ Y_{\rho+1,1} & \dots & & Y_{\rho+1,\rho-1} & 0 & Z \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ \widehat{R}_1 & \widehat{R}_2 & \cdots & \widehat{R}_{\rho-1} & \widehat{R}_\rho & \widehat{S} \\ \widehat{P} & 0 & \cdots & 0 & 0 & \widehat{Q} \end{bmatrix} \\ &= [\,0,\,\dots,\,0,\,I_{n-\rho}\,]\, Y\widehat{A} \\ &\overset{(3.16)}{=} [\,0,\,\dots,\,0,\,I_{n-\rho}\,]\, \tilde{A}Y \\ &= [\,0,\,\dots,\,0,\,I_{n-\rho}\,] \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ \tilde{R}_1 & \tilde{R}_2 & \cdots & \tilde{R}_{\rho-1} & \tilde{R}_\rho & \tilde{S} \\ \tilde{P} & 0 & \cdots & 0 & 0 & \tilde{Q} \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & & 0 & 0 \\ 0 & 1 & & & & 0 \\ \vdots & \ddots & & \ddots & & \vdots \\ 0 & \dots & & 0 & 1 & 0 \\ Y_{\rho+1,1} & \dots & & Y_{\rho+1,\rho-1} & 0 & Z \end{bmatrix} \\ &= \left[\tilde{P} + \tilde{Q}Y_{\rho+1,1}, \tilde{Q}Y_{\rho+1,2}, \dots, \tilde{Q}Y_{\rho+1,\rho-1},\ 0\ , \tilde{Q}Z\right]. \end{aligned}$$

By successively comparing the blocks in order from $(\rho-1)$th to first and by finally considering the $(\rho+1)$th block, we see that

$$Y_{\rho+1,\rho-1} = 0, \dots, Y_{\rho+1,1} = 0,\ \tilde{P} = Z\widehat{P},\ \tilde{Q} = Z\widehat{Q}\,Z^{-1}.$$

Finally,

$$Y = \operatorname{diag}(1, \dots, 1, Z) \tag{3.21}$$

and (3.16)–(3.18) give (i) and (ii). This completes the proof of the theorem. □

3.2 The internal loop form

This subsection is the counterpart to the previous one in the frequency domain.

Consider a strictly proper transfer function $\frac{p(s)}{q(s)} \in \mathbb{K}(s)^*$ and choose unique coprime and monic $\tilde{p}(s), \tilde{q}(s) \in \mathbb{K}[s]$ and $\gamma \in \mathbb{K}$ such that

$$y(s) = \frac{p(s)}{q(s)}\,u(s) \ = \ \gamma\,\frac{\tilde{p}(s)}{\tilde{q}(s)}\,u(s)\,. \tag{3.22}$$

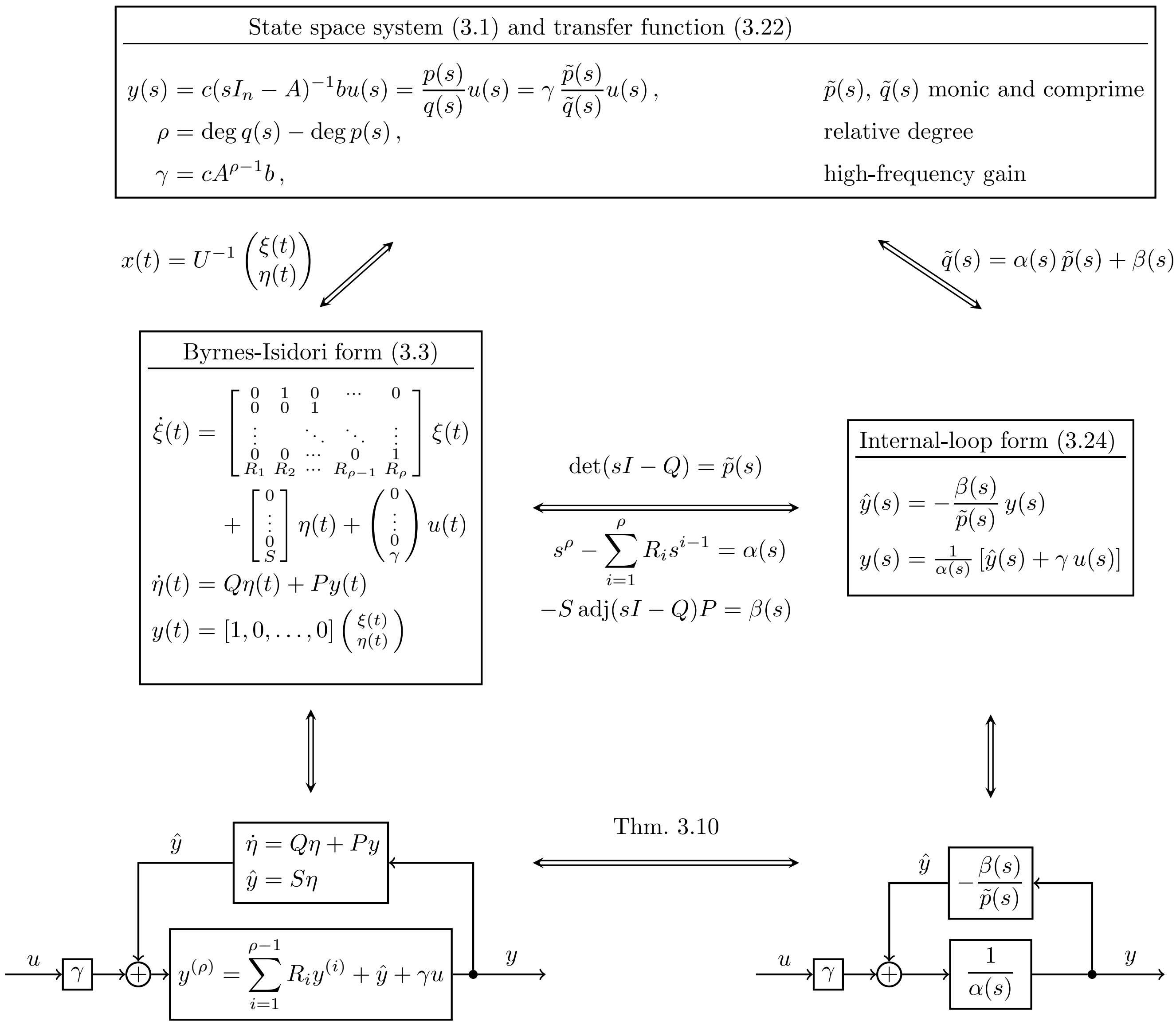

Bild 1.3.1: The relationship between zero dynamics and minimum phase via the canonical forms

Now long division yields

$$\tilde q(s) = \alpha(s)\,\tilde p(s) + \beta(s) \quad \text{for some } \alpha(s), \beta(s) \in \mathbb{K}[s] \quad \text{so that } \deg\beta(s) < \deg\tilde p(s)\,,$$

and little algebraic manipulation allows to write (3.22) as

$$\begin{aligned} y(s) &= \frac{p(s)}{q(s)}u(s) = \gamma\,\frac{\tilde p(s)}{\alpha(s)\;\tilde p(s) + \beta(s)}\,u(s)\\ &= \frac{1}{\alpha(s)}\Big[\underbrace{-\frac{\beta(s)}{\tilde p(s)}\,y(s)}_{=:\tilde y(s)} + \gamma\,u(s)\Big]\,. \end{aligned} \tag{3.23}$$

Note that γ is the high-frequency gain and $\deg\alpha(s) = \rho$ is the relative degree of $\frac{p(s)}{q(s)}$. The decomposition (3.23), depicted in the south-east corner of Figure 1.3.1, separates $y(s) = \gamma\,\frac{\tilde p(s)}{\tilde q(s)}\,u(s)$ into a ρ-times integrator in combination with an “internal loop” $y(s) \mapsto \hat y(s) = -\frac{\beta(s)}{\tilde p(s)}y(s)$. For various control design purposes, it is important that the zeros of $\tilde p(s)$ determine completely the internal stability properties of the system.

The decomposition gives rise to the following definition, the counterpart to the Byrnes-Isidori form.

Definition 3.8 *A strictly proper transfer function* $\frac{p(s)}{q(s)} \in \mathbb{K}(s)^*$ *is said to be in* internal-loop form *if, and only if,*

$$\frac{p(s)}{q(s)} \;=\; \gamma \, \frac{\tilde p(s)}{\alpha(s)\; \tilde p(s) + \beta(s)}\,, \tag{3.24a}$$

where

$$\left.\begin{array}{l}
\gamma = \lim\limits_{s\to\infty} \frac{s^\rho p(s)}{q(s)}\,, \quad \rho = \deg q(s) - \deg p(s) \\
\tilde p(s) = s^{n-\rho} - \sum_{i=1}^{n-\rho} \tilde p_i s^{i-1} \in \mathbb{K}[s] \\
\beta(s) = s^\ell - \sum_{i=1}^{\ell} \beta_i s^{i-1} \in \mathbb{K}[s],\;\; \ell < n-\rho \\
\alpha(s) = s^\rho - \sum_{i=1}^{\rho} \alpha_i s^{i-1} \in \mathbb{K}[s] \\
\tilde p(s),\;\; \alpha(s)\tilde p(s) + \beta(s) \quad \text{are coprime}\,.
\end{array}\right\} \tag{3.24b}$$

Finally, we record that the internal-loop form is also a canonical form on the equivalence of rational functions; this is the following counterpart to Theorem 3.4.

Proposition 3.9 *The map*

$$\begin{array}{rcl}
\Gamma : \mathbb{C}(s)_{\text{proper}} & \to & \mathbb{C}(s)_{\text{proper}} \\
g(s) & \mapsto & g(s) \text{ in internal-loop form (3.24)}
\end{array}$$

is a canonical form for the equivalence relation of rational functions

Proof It is easy to see that the polynomials and numbers in (3.24a) are uniquely determined by the properties in (3.24b). Therefore, any strictly proper transfer function $\frac{p(s)}{q(s)} \in \mathbb{K}(s)^*$ can uniquely be written in internal-loop form (3.24). This completes the proof. ☐

3.3 One-to-one correspondence between Byrnes-Isidori form and internal-loop form

There is an "almost" one-to-one correspondence between the state space Byrnes-Isidori form (3.3) and the frequency domain internal-loop form (3.24); this is made precise in the next theorem, see also Figure 1.3.1.

Theorem 3.10
(i) Given

$$\frac{p(s)}{q(s)} \;=\; \gamma \, \frac{\tilde p(s)}{\alpha(s)\; \tilde p(s) + \beta(s)} \in \mathbb{K}(s)^* \tag{3.25}$$

in internal-loop form (3.24)*. Then a controllable and observable realization of* $\frac{p(s)}{q(s)}$ *is given by the Byrnes-Isidori form* (3.3) *with*

$$R_i = \alpha_i \qquad \text{for} \quad i = 1,\ldots,\rho \tag{3.26a}$$

and (Q,P,S) *given by the triple*

$$\left(\begin{bmatrix} 0 & 0 & \cdots & 0 & \tilde p_1 \\ -1 & 0 & \cdots & 0 & \tilde p_2 \\ & \ddots & \ddots & \vdots & \vdots \\ & & -1 & 0 & \tilde p_{n-\rho-1} \\ 0 & \cdots & 0 & -1 & \tilde p_{n-\rho} \end{bmatrix}, \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_\ell \\ -1 \\ 0_{n-\rho-\ell-1} \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}^{\top} \right). \tag{3.26b}$$

(ii) The internal-loop form (3.24) *of a controllable and observable system* $(A,b,c) \in \Sigma_n(\mathbb{K})$ *in Byrnes-Isidori form* (3.3) *is given by*

$$\begin{pmatrix} \gamma \\ \alpha(s) \\ \beta(s) \\ \tilde p(s) \end{pmatrix} = \begin{pmatrix} cA^{\rho-1}b \\ s^\rho - \sum_{i=1}^{\rho} R_i s^{i-1} \\ -S\,\mathrm{adj}(sI_{n-\rho} - Q)P \\ \det(sI_{n-\rho} - Q) \end{pmatrix}. \tag{3.27}$$

Proof We first show that the transfer function of a system in Byrnes-Isidori form (3.3) satisfies

$$c(sI_n - A)^{-1}b \;=\; \frac{cA^{\rho-1}b \cdot \det(sI_{n-\rho} - Q)}{\left(s^\rho - \sum\limits_{i=1}^{\rho} R_i s^{i-1}\right) \det(sI_{n-\rho} - Q) - S\,\mathrm{adj}(sI_{n-\rho} - Q)P}\,. \tag{3.28}$$

Since the transfer function does not depend on a particular realization, we have

$$c(sI_n - A)^{-1}b \overset{(3.3)}{=} (1\;0\;\ldots\;0) \begin{bmatrix} s & -1 & & & \\ & \ddots & \ddots & & \\ & & s & -1 & \\ -R_1 & \ldots & -R_{\rho-1} & (s-R_\rho) & -S \\ -P & 0 & \ldots & 0 & sI_{n-\rho}-Q \end{bmatrix}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \gamma \\ 0 \end{pmatrix}.$$

To obtain our expression, we determine the solution $X(s) \in \mathbb{K}(s)^n$ of the linear equation

$$\begin{bmatrix} s & -1 & & & \\ & \ddots & \ddots & & \\ & & s & -1 & \\ -R_1 & \ldots & -R_{\rho-1} & (s-R_\rho) & -S \\ -P & 0 & \ldots & 0 & sI_{n-\rho}-Q \end{bmatrix} \begin{pmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_\rho(s) \\ X_{\rho+1}(s) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \gamma \\ 0 \end{pmatrix}.$$

A simple iterative calculation yields

$$\begin{aligned}
sX_i(s) &= X_{i+1}(s), \qquad \text{for } i = 1,\ldots \rho-1, \\
-\sum_{i=1}^{\rho-1} R_i X_i(s) &+ (s-R_\rho)X_\rho(s) - SX_{\rho+1}(s) = \gamma, \\
&-PX_1(s) + (sI_{n-\rho} - Q)X_{\rho+1}(s) = 0,
\end{aligned}$$

and this is equivalent to

$$\begin{aligned}
X(s) &= \big(X_1(s), sX_1(s), \ldots, s^{\rho-1}X_1(s), X_{\rho+1}(s)\big)^\top, \\
\gamma &= -\sum_{i=1}^{\rho-1} R_i s^{i-1} X_1(s) + (s-R_\rho)s^{\rho-1}X_1(s) - SX_{\rho+1}(s), \\
X_{\rho+1}(s) &= (sI_{n-\rho} - Q)^{-1}PX_1(s),
\end{aligned}$$

and thus

$$\begin{aligned}
c(sI_n - A)^{-1}b &= X_1(s) \\
&= \left[-\textstyle\sum_{i=1}^{\rho} R_i s^{i-1} + s^\rho - S(sI_{n-\rho} - Q)^{-1}P\right]^{-1} \gamma.
\end{aligned}$$

This proves (3.28).

Next we show (i). It is well-known and easy to verify that $S(sI_{n-\rho}-Q)^{-1}P = -\beta(s)/\tilde{p}(s)$. Then coprimeness of $\tilde{p}(s)$ and $\beta(s)$ yields (see, for example, [13, Th. 2.4-5]) that the system

$$\begin{aligned}\dot{\eta} &= Q\,\eta + P\,y \\ \hat{y} &= S\,\eta\end{aligned}$$

is a minimal realization with state space dimension $\deg \tilde{p}(s) = n-\rho$. Finally, the choice

$$x(t) := \Big(y(t), y^{(1)}(t), \ldots, y^{(\rho-1)}(t), \eta(t)^\top\Big)^\top$$

and a term by term comparison of (3.28) with (3.25) yields that (3.3) is a realization of (3.25). Controllability and observability follows from Proposition 3.5.

It remains to show (ii). In view of (3.28), we have to verify the conditions in (3.24b). The first four are obvious from (3.27). We prove the last condition: First note that (Q,P,S) is minimal by assumption and Proposition 3.5. Therefore, in terms of (3.27), $\tilde{p}(s)$ and $\beta(s)$ are coprime; see, for example, [13, Th. 2.4-5]. Now it is easy to see that the last condition in (3.24b) holds. This completes the proof of the theorem. □

Remark 3.11 *The design of the Byrnes-Isidori form for systems $(A,b,c) \in \Sigma_n(\mathbb{K})$ is quite involved; see the proof of Theorem 3.4. However, if we restrict the attention to minimal systems, then the internal-loop form provides a conceptually simple procedure: write the transfer function $c(sI_n - A)^{-1}b$ in internal-loop form* (3.24) *and simply read off from the coefficients of the polynomials in* (3.24a) *the Byrnes-Isidori form as in* (3.26).

Moreover, the above procedure induces a canonical form on the set of minimal systems in $\Sigma_n(\mathbb{K})$.

3.4 Zero dynamics

We are now in a position to show that if a minimal realization of a proper transfer function $\frac{p(s)}{q(s)} \in \mathbb{K}(s)^*$ has asymptotically stable zero dynamics, then $\frac{p(s)}{q(s)}$ is minimum phase; however, the converse implication does, in general, not hold true.

The zero dynamics and its stability is defined for systems, state and input spaces as in (3.1).

Definition 3.12 *The* zero dynamics *of system* (3.1) *is defined as the set of trajectories*

$$\mathcal{ZD} \;:=\; \left\{ (x,u) \in \mathcal{X} \times \mathcal{U} \;\middle|\; \begin{array}{l} (x,u) \textit{ solves } (3.1) \\ \textit{on } \mathbb{R}_{\geq 0} \textit{ a.e.} \\ \textit{such that } y \equiv 0 \end{array} \right\}.$$

The zero dynamics are called asymptotically stable *if, and only if,*

$$\forall\,(x,u) \in \mathcal{ZD} \;:\; \lim_{t\to\infty} \textit{ess-sup}_{[t,\infty)} \|(x(\cdot),u(\cdot))\| \;= 0\,.$$

By linearity of (3.1), the set $\mathcal{ZD}$ is a real vector space and it can be shown that it carries the structure of a linear dynamical system as, for example, defined in [9, Definition 2.1.1].

It also can be shown that the zero trajectory $(x,u)=0$ of $\mathcal{ZD}$ is attractive and stable if, and only if, $\mathcal{ZD}$ is asymptotically stable.

Loosely speaking, we will now show that the zeros of the (not necessarily stable) transfer function $\frac{p(s)}{q(s)} \in \mathbb{K}(s)^*$ coincide with the stability behaviour of the zero dynamics of a minimal realization (A,b,c) of the transfer function.

Theorem 3.13 *Let $(A,b,c) \in \mathbb{K}^{n\times n} \times \mathbb{K}^n \times \mathbb{K}^{1\times n}$ be a controllable and observable realization of a semi Hurwitz transfer function $\frac{p(s)}{q(s)} \in \mathbb{C}(s)^*$ with coprime $p(s)$, $q(s)$. Then the following are equivalent:*

(i) System (3.1) *has asymptotically stable zero dynamics.*

(ii) $p(s)$ is Hurwitz.

(iii) $\sigma(Q) \subset \mathbb{C}_-$, where Q is given in Theorem 3.4.

In fact, we have

$$\{s \in \mathbb{C} \,|\, p(s) = 0\} \;=\; \sigma(Q)\,. \tag{3.29}$$

Proof Since asymptotic stability of the zero dynamics and the spectrum of Q in (3.3) are invariant under state space transformation, we may assume, without restriction of generality, that the system (3.1) is in the form (3.3).

(i) ⇔ (ii) The zero dynamics of (3.3) are given by

$$\mathcal{ZD} = \left\{ \left(\left(\begin{smallmatrix} 0_\rho \\ \eta(\cdot) \end{smallmatrix}\right),\; -(cA^{\rho-1}b)^{-1}S\eta(\cdot) \right) \,\middle|\; \dot{\eta} = Q\eta \right\}.$$

Since (Q,P,S) is a minimal realization of $\frac{\beta(s)}{\tilde{p}(s)}$ and $\tilde{p}(s)$, $p(s)$ differ only by a nonzero constant, (3.29) holds.

(ii) ⇔ (iii) is a consequence of Theorem 3.10. □

An immediate consequence of Theorem 3.13 is that asymptotically stable zero dynamics yields that $\frac{p(s)}{q(s)}$ is minimum phase; and furthermore, that the converse implication is, in general, not true.

Conclusions We have defined and characterized the concept of minimum phase for semi Hurwitz transfer functions. This has led to the observation that asymptotic stability of the zero dynamics and minimum phase are distinct concepts. A main tool in the analysis are the two canonical forms: the Byrnes-Isidori form in the time domain and the internal loop form in the frequency domain.

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