SIAM J. CONTROL OPTIM.
Vol. 54, No. 3, pp. 1504–1534

THE BYRNES–ISIDORI FORM FOR INFINITE-DIMENSIONAL SYSTEMS*

ACHIM ILCHMANN†, TILMAN SELIG†, AND CARSTEN TRUNK†

Abstract. We define a Byrnes–Isidori form for a class of infinite-dimensional systems with relative degree r and show that any system belonging to this class can be transformed into this form. We also analyze the concept of (stable) zero dynamics and show that it is, together with the Byrnes–Isidori form, instrumental for static proportional high-gain output feedback stabilization. Moreover, we show that funnel control is feasible for any system with relative degree one and with exponentially stable zero dynamics; a funnel controller is a time-varying proportional output feedback controller which ensures, for a large class of reference signals, that the error between the output and the reference signal evolves within a prespecified funnel. Therefore transient behavior of the error is obeyed.

Key words. Byrnes–Isidori form, relative degree, infinite-dimensional systems, high-gain stabilizability, funnel control

AMS subject classifications. 93B10, 93C20, 93D21

DOI. 10.1137/130942413

NOTATION.

$\mathbb{R}_{\geq 0}$, $\mathbb{C}_{\geq\alpha}$	$[0,\infty)$, $\{\lambda\in\mathbb{C}\,\vert\,\mathrm{Re}(\lambda)\geq\alpha\}$
$0_{\mathbb{R}^r}$	$(0,\dots,0)^\top\in\mathbb{R}^r$
H (or $H,(\cdot,\cdot)$)	Hilbert space H with inner product $(\cdot,\cdot)$
$\mathcal{B}(H,X)$, $\mathcal{B}(H)$	The set of bounded linear operators from H to a Hilbert space X or to H, resp.
$L^\infty(\mathbb{R}_{\geq 0};H)$	The set of equivalence classes of essentially bounded, strongly measurable functions $f:\mathbb{R}_{\geq 0}\to H$
$L^1(\mathbb{R}_{\geq 0};H)$	The set of equivalence classes of integrable functions $f:\mathbb{R}_{\geq 0}\to H$
$L^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0},H)$	The set of locally integrable functions $f:\mathbb{R}_{\geq 0}\to H$
$\mathcal{C}^\ell(\mathbb{R}_{\geq 0};H)$	The set of ℓ-times continuously differentiable functions $f:\mathbb{R}_{\geq 0}\to H$
$\mathcal{C}(\mathbb{R}_{\geq 0};H)$	The set of continuous functions $f:\mathbb{R}_{\geq 0}\to H$
$W^{1,\infty}(\mathbb{R}_{\geq 0},H)$	The set of functions which together with their first (distributional) derivative belong to $L^\infty(\mathbb{R}_{\geq 0};H)$
$\mathrm{dom}\,A$	The domain of the linear operator A
$A^{-*}:=(A^{-1})^*$	The adjoint of the inverse of a closed, bijective operator A in H; note that $A^{-*}=(A^*)^{-1}$
$M\dot{+}N$	The direct sum of two (linear) subspaces M,N. Here, a subspace is always closed
$\mathrm{ls}\,\{x\}$	The linear span of $x\in H$
π_{X_i}	The projector onto the ith component of $X_1\times X_2$, $i=1,2$

*Received by the editors October 22, 2013; accepted for publication (in revised form) March 11, 2016; published electronically June 7, 2016. This work was supported by the Klaus Tschira Stiftung (Heidelberg, Germany).

http://www.siam.org/journals/sicon/54-3/94241.html

†Institut für Mathematik, Technische Universität Ilmenau, Weimarer Straße 25, D-98693 Ilmenau, Germany (achim.ilchmann@tu-ilmenau.de, tilman.selig@tu-ilmenau.de, carsten.trunk@tu-ilmenau.de).

1. Introduction. We consider the class of linear infinite-dimensional systems

$$\dot{x}(t) = Ax(t) + bu(t), \qquad t \geq 0, \tag{1.1a}$$

$$y(t) = (x(t), c)\,, \tag{1.1b}$$

where (A, b, c) satisfy, for some $r \in \mathbb{N}$, the assumptions

(A1) $A : \operatorname{dom} A \subset H \to H$ is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a real Hilbert space $(H, (\cdot,\cdot))$;

(A2) $b \in \operatorname{dom} A^r$ and $c \in \operatorname{dom} {A^*}^r$;

(A3) system (A, b, c) has *relative degree* r: $(A^{r-1}b, c) \neq 0$ and $(A^j b, c) = 0 \;\forall\, j = 0, 1, \ldots, r-2$;

this class of systems is denoted by Σ_r and we write $(A, b, c) \in \Sigma_r$.

For finite-dimensional systems, assumptions (A1) and (A2) are superfluous, and assumption (A3) means that (A, b, c) has relative degree r; see, e.g., [13, p. 137]. If the system (1.1) is finite dimensional and satisfies (A3), then it is easy to see that the relative degree r is the minimal number so that u appears explicitly for the first time in the rth derivative of y,

$$y^{(r)}(t) = cA^r x + cA^{r-1} b\, u \qquad \text{and} \qquad y^{(j)}(t) = cA^j\, x(t) \quad \forall\, j = 0, 1, \ldots, r-1. \tag{1.2}$$

For infinite-dimensional systems, assumption (A1) is standard in systems theory (see, e.g., [7]); assumption (A2) is restrictive and excludes several practical examples as will be discussed in Remark 6.1. However, the class of infinite dimensional systems considered in the present note allows for the Byrnes–Isidori form; this has a practical value as illustrated by funnel control of the heat equation in section 6, moreover, it elucidates some geometry behind the Byrnes–Isidori form.

Assumption (A2) together with (A3) implies, by [17, Lemma 2.9], that the transfer function of the system fulfills

$$\lim_{s\to\infty, s\in\mathbb{R}} s^r \left(c, (s-A)^{-1}b\right) \neq 0 \qquad \text{and} \qquad \lim_{s\to\infty, s\in\mathbb{R}} s^{r-1} \left(c, (s-A)^{-1}b\right) = 0. \tag{1.3}$$

This property is called *relative degree* r in [17, Definition 1.3]. A partial converse is contained in [17, Lemma 2.9], namely, that (1.3) together with (A2) implies (A3).

The guiding research idea of the present paper is whether it is possible to extend the following well-known results from finite-dimensional systems to the class Σ_r of infinite-dimensional systems. For linear, finite-dimensional systems the Byrnes–Isidori form is well understood; see [1, 12, 13]. This form together with the concept of stable zero dynamics is instrumental for various high-gain stabilization and tracking results; see again [13]. If the nominal system has relative degree one and stable zero dynamics (often called minimum phase), then it is high-gain stabilizable by a proportional output feedback $u(t) = -ky(t)$, provided the gain k is sufficiently large. The drawback is that it is unknown how large the gain has to be chosen. It can be resolved by an adaptive controller of the form $u(t) = -k(t)y(t)$, $\dot{k}(t) = y(t)^2$; see, for example, [4]. However, the drawback of this adaptive controller is that the gain $k(\cdot)$ is, although monotone and bounded, may become too large, whence additive noise corrupting the output may lead to instability and, more importantly, transient behavior is not obeyed at all. This drawback was resolved by the funnel controller introduced by [12].

A generalization of these finite-dimensional results to infinite-dimensional systems cannot be expected in full generality. However, we show that the class Σ_r is an appropriate class to allow for a Byrnes–Isidori form and subsequently for control theoretic consequences such as funnel control.

We describe the literature on infinite-dimensional systems related to our results. A popular approach to identify the zero dynamics is to determine the largest controlled invariant subspace in $c^\perp$. Many authors have studied existence and geometric and invariance properties of this subspace; see, for example, [5, 6, 17, 22]. It is well known that under our conditions (A1)–(A3) the largest feedback invariant subspace of $c^\perp$ is

$$H_{c,A}^\perp := c^\perp \cap (A^*c)^\perp \cap \cdots \cap (A^{*^{r-1}}c)^\perp,$$

and that it is closed-loop invariant; see [22, section 4] and [17, Theorem 2.10]. Furthermore, this gives rise to the decomposition of H into

$$H = c^\perp \cap (A^*c)^\perp \cap \cdots \cap (A^{*^{r-1}}c)^\perp \dot{+} \operatorname{ls}\{b\} \dot{+} \operatorname{ls}\{Ab\} \dot{+} \cdots \dot{+} \operatorname{ls}\{A^{r-1}b\}. \tag{1.4}$$

For the class Σ_1 of systems with relative degree one this means $H = c^\perp \dot{+} \operatorname{ls}\{b\}$, which has been used in [2, 3, 16] for the purpose of high-gain control.

However, the Byrnes–Isidori form presented here is more sophisticated and is based on the decomposition

$$H = \underbrace{\operatorname{ls}\{c\} \dot{+} \operatorname{ls}\{A^*c\} \dot{+} \cdots \dot{+} \operatorname{ls}\{A^{*^{r-1}}c\}}_{=:H_{c,A}} \dot{+} \underbrace{\{b\}^\perp \cap \{Ab\}^\perp \cap \cdots \cap \{A^{r-1}b\}^\perp}_{=:H_{A,b}}. \tag{1.5}$$

Compared with (1.4) we have

$$H = H_{c,A}^\perp \dot{+} H_{A,b}^\perp.$$

This fact is often hidden behind matrix computations for finite-dimensional systems; cf. [12]. The difference between the two decompositions (1.4) and (1.5) is essentially in taking orthogonal complements. Apart from that, (1.5) can also be interpreted as the decomposition (1.4) of the adjoint system (A^*, c, b).

Similarly, in [15] a multi-input multioutput system is called of generalized degree one if and only if its transfer-function matrix $G(\cdot)$ is meromorphic on $\mathbb{C}_0 := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and admits a representation as

$$G^{-1}(s) = sD^{-1} + H(s), \tag{1.6}$$

where D is an invertible matrix and $H(\cdot)$ a bounded analytic function defined on $\mathbb{C}_0$. If $\sigma(D) \subset \mathbb{C}_0$, then [14] show that the plant described by $G(\cdot)$ can be stabilized by static output feedback of the form $u(t) = -ky(t)$, provided the feedback gain k is sufficiently large.

Clearly, the results presented here give an alternative proof of the finite-dimensional results on the Byrnes–Isidori form and the funnel controller as presented in [12, 13].

The present paper is organized as follows. In section 2 we define a Byrnes–Isidori form for systems Σ_r of relative degree r and show that any system belonging to Σ_r can be transformed into Byrnes–Isidori form. Furthermore, an internal loop form is derived; this form is instrumental for proving regulation results later. In section 3, we define the concept of (stable) zero dynamics and characterize it in terms of the Byrnes–Isidori form. From section 4 onwards we restrict our attention to relative degree one systems belonging to Σ_1. The previous results allow us to show in section 4 that any system of class Σ_1 with exponentially stable zero dynamics satisfies the high-gain property, that means it is high-gain stabilizable by a static proportional output feedback if the gain is sufficiently large. Finally, in section 5 we show that the well-known funnel controller from finite-dimensional systems is also feasible for systems belonging to Σ_1 and having exponentially stable zero dynamics. This means, tracking of a large class of reference signals within a prespecified funnel is possible with a time-varying nonmonotone gain.

The above theoretical results are verified for the one-dimensional heat equation in section 6. To make the presentation more readable, we have delegated some of the lemmas and proofs to Appendix A.

2. The Byrnes–Isidori form.

We start with the definition of the Byrnes–Isidori form.

DEFINITION 2.1. *Let H be a Hilbert space, A be the generator of a strongly continuous semigroup in H, and $b, c \in H$. Then the system (A, b, c) is said to be in Byrnes–Isidori form if and only if $H = \mathbb{R}^r \times V$ for some Hilbert space V and (A, b, c) satisfies the following four conditions:*

(i) *There exists an operator $Q : \operatorname{dom} Q \subset V \to V$ which generates a strongly continuous semigroup in V.*

(ii) *The operator A has the domain*

$$\operatorname{dom} A = \mathbb{R}^r \times \operatorname{dom} Q$$

and a representation of the form

$$A \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{pmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & 1 & 0 & 0 \\ 0 & 0 & & 0 & 1 & 0 \\ P_0 & P_1 & \cdots & P_{r-2} & P_{r-1} & S \\ R & 0 & \cdots & 0 & 0 & Q \end{bmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{pmatrix} \quad \text{for} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{pmatrix} \in \operatorname{dom} A, \tag{2.1}$$

where

(a) *the operators $P_0, \dots, P_{r-1} : \mathbb{R} \to \mathbb{R}$, $S : V \to \mathbb{R}$, $R : \mathbb{R} \to V$ are bounded;*

(b) *the vectors b and c have the form*

$$b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b_r \\ \hline 0 \end{pmatrix} \in \mathbb{R}^r \times V \quad \text{with } b_r \in \mathbb{R} \setminus \{0\} \quad \text{and} \quad c = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \hline 0 \end{pmatrix} \in \mathbb{R}^r \times V. \quad \square$$

It is possible to define a Byrnes–Isidori form for a generalization of the class Σ_r which includes MIMO systems; this is similar to the finite-dimensional case as in [12, Lemma 3.5]. However, in the present paper we restrict ourselves for technical reasons to the SISO case.

PROPOSITION 2.2. *Every system that is in Byrnes–Isidori form belongs to the class Σ_r.*

Proof. Let (A, b, c) be a system in Byrnes–Isidori form. Then

$$\Big\{ \begin{pmatrix}\alpha_0 & \dots & \alpha_{r-1} & 0\end{pmatrix}^\top \in \mathbb{R}^r \times V \mid \alpha_0 = \cdots = \alpha_{l-2} = 0 \Big\} \subset \operatorname{dom} A^l \qquad \forall l \in \{2, \dots, r\}$$

and $b \in \operatorname{dom} A^r$. Let $\begin{pmatrix}\alpha_0 & \dots & \alpha_{r-1} & \eta\end{pmatrix}^\top \in \operatorname{dom} A^r$. The special structure of c and A yields

$$(c, A^l x) = \alpha_l \quad \forall\, l \in \{0, \dots, r-1\} \qquad \text{and} \quad (c, A^r x) = \sum_{i=0}^{r-1} \alpha_i P_i + S\eta.$$

This shows that $(c, A^r x)$ depends continuously on x and therefore c is in $\operatorname{dom} A^{*^r}$. Since we have already shown $b \in \operatorname{dom} A^r$, we may replace x by b in the above equations and conclude

$$(c, A^l b) = 0 \quad \forall\; l \in \{0, \dots, r-2\} \qquad \text{and} \qquad (c, A^{r-1} b) = b_r \neq 0.$$

This completes the proof of the proposition. ∎

The main result of the present section is that every system $(A, b, c) \in \Sigma_r$ can be transformed into Byrnes–Isidori form by a boundedly invertible transformation. Since the class Σ_r is invariant under such transformations, Proposition 2.2 shows that the assumption $(A, b, c) \in \Sigma_r$ is necessary to achieve this. In order to construct the appropriate transformation we use the following three lemmas.

A simple, but for our analysis instrumental, observation is that the Hilbert space H may be decomposed, for any $(A, b, c) \in \Sigma_r$, into the direct sum

$$H = \underbrace{\operatorname{ls}\{c\} \dotplus \operatorname{ls}\{A^* c\} \dotplus \cdots \dotplus \operatorname{ls}\{A^{*^{r-1}} c\}}_{=: H_{c,A}} \dotplus \underbrace{\{b\}^\perp \cap \{Ab\}^\perp \cap \cdots \cap \{A^{r-1} b\}^\perp}_{=: H_{A,b}} . \tag{2.2}$$

This follows immediately from assumptions (A2) and (A3): first, $c, A^* c, \dots, A^{*^{r-1}} c$ are linearly independent, and second

$$\forall\, i \in \{0, \dots, r-1\} \;:\; H_{A,b} \cap \operatorname{ls}\{A^{*^i} c\} = \{0\};$$

hence the sum $H_{c,A} \dotplus H_{A,b}$ is direct and since $H_{A,b}$ has by definition at most codimension r, (2.2) follows.

LEMMA 2.3. *Let $(A, b, c) \in \Sigma_r$ and define the operators*

$$P^m : H \to \mathbb{R}, \quad x \mapsto P^m x := P^m_{m+1} x - \sum_{j=m+2}^{r} P^m_j x\,, \qquad m = 0, \dots, r-1\,, \tag{2.3}$$

where

$$P^m_{m+1} : \quad H \to \mathbb{R}, \quad x \mapsto P^m_{m+1} x := \frac{\big(x, A^{r-(m+1)} b\big)}{(c, A^{r-1} b)}\,, \quad m = 0, 1, \dots, r-1,$$

$$P^m_j : \quad H \to \mathbb{R}, \quad x \mapsto P^m_j x := \left(P^m_{m+1} A^{*^{j-1}} c - \sum_{k=m+2}^{j-1} P^m_k A^{*^{j-1}} c \right) \frac{\big(x, A^{r-j} b\big)}{(c, A^{r-1} b)}\,, \quad j = m+2, \dots, r.$$

Then for any $\ell, m \in \{0, \dots, r-1\}$ *we have*

$$P^m H_{A,b} = \{0\}, \tag{2.4}$$

$$P^m {A^*}^{\ell} c = \begin{cases} 1 & \text{if } \ell = m, \\ 0 & \text{if } \ell \neq m\,. \end{cases} \tag{2.5}$$

Proof. Assertion (2.4) follows from assumption (A3) and the definition of P^m. Similarly, assertion (2.5) follows for $\ell = m$ and $\ell = 0, \dots, m-1$. It remains to show (2.5) for $\ell \in \{m+1, \dots, r-1\}$. By the definition of P_j^m and assumption (A3) we have

$$P_j^m {A^*}^{\ell} c = 0 \quad \text{for all } j = \ell+2, \dots, r\,, \tag{2.6}$$

and therefore

$$\begin{aligned} P^m {A^*}^{\ell} c &= P_{m+1}^m {A^*}^{\ell} c - \textstyle\sum_{j=m+2}^{r} P_j^m {A^*}^{\ell} c \\ &= P_{m+1}^m {A^*}^{\ell} c - P_{\ell+1}^m {A^*}^{\ell} c - \textstyle\sum_{j=m+2}^{\ell} P_j^m {A^*}^{\ell} c \\ &= P_{m+1}^m {A^*}^{\ell} c - \textstyle\sum_{j=m+2}^{\ell} P_j^m {A^*}^{\ell} c \\ &\quad - \Big(P_{m+1}^m {A^*}^{\ell} c - \textstyle\sum_{k=m+2}^{\ell} P_k^m {A^*}^{\ell} c\Big) \underbrace{\frac{\big({A^*}^{\ell} c, A^{r-1-\ell} b\big)}{(c, A^{r-1} b)}}_{=1} \\ &= 0\,. \end{aligned}$$

This completes the proof. ∎

LEMMA 2.4. *Let* $(A, b, c) \in \Sigma_r$ *and use the notation as in Lemma* 2.3. *Then the operator*

$$P_{A,b} : H \to H\,, \quad x \mapsto P_{A,b}\, x := \left(I - \sum_{j=0}^{r-1} {A^*}^{j} c P^j \right) x \tag{2.7}$$

is a projection onto $H_{A,b}$, *and every* $x \in H$ *has a unique decomposition with respect to* (2.2) *of the form*

$$x = (P^0 x) c + (P^1 x) A^* c + \dots + (P^{r-1} x) {A^*}^{r-1} c + P_{A,b} x\,. \tag{2.8}$$

Proof. By the definition of $P_{A,b}$ and (2.4) we have

$$P_{A,b} x = x \qquad \text{for all } x \in H_{A,b}$$

and, by (2.5), we have $\operatorname{ls}\{c\} \dotplus \operatorname{ls}\{A^* c\} \dotplus \cdots \dotplus \operatorname{ls}\{{A^*}^{r-1} c\} = H_{c,A} \subset \ker P_{A,b}$. Hence, in view of (2.2), $P_{A,b}$ is a projection. Finally, (2.8) is a direct consequence of the definition of $P_{A,b}$. ∎

LEMMA 2.5. *The operator*

$$U : H \to \mathbb{R}^r \times H_{A,b}\,, \qquad x \mapsto Ux := \begin{pmatrix} P^0 x \\ P^1 x \\ \vdots \\ P^{r-1} x \\ P_{A,b} x \end{pmatrix} \tag{2.9}$$

is bounded and bijective with inverse

$$U^{-1} : \mathbb{R}^r \times H_{A,b} \to H, \qquad \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{pmatrix} \mapsto \sum_{j=0}^{r-1} \alpha_j {A^*}^j c \;+\; \eta. \tag{2.10}$$

Furthermore, with the orthogonal projector $P^\perp : H \to H$ *onto* $H_{A,b}$*, we have*

$$U^{-*} : H \to \mathbb{R}^r \times H_{A,b}, \qquad x \mapsto \begin{pmatrix} (x, c) \\ (x, A^*c) \\ \vdots \\ (x, {A^*}^{r-1} c) \\ P^\perp x \end{pmatrix}. \tag{2.11}$$

Moreover, U^{-*} *maps* $H_{c,A}^\perp$ *bijectively onto* $\{0\} \times H_{A,b}$*.*

Proof. The assertions about U and its inverse are a direct consequence of Lemma 2.4. The formula for U^{-*} follows since for all $(\alpha_0, \dots, \alpha_{r-1}, \eta)^\top \in \mathbb{R}^r \times H_{A,b}$ and all $x \in H$ we have

$$\begin{aligned} \left(x,\; U^{-1} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{pmatrix} \right)_H &= \left(x,\; \sum_{j=0}^{r-1} \alpha_j {A^*}^j c + \eta \right)_H \\ &= \left(\begin{pmatrix} (x, c) \\ (x, A^*c) \\ \vdots \\ (x, {A^*}^{r-1} c) \\ P^\perp x \end{pmatrix}, \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{pmatrix} \right)_{\mathbb{R}^r \times H_{A,b}}. \end{aligned}$$

The last statement on U^{-*} follows from the fact that U is bijective together with formula (2.11). ∎

We are now in a position to state the main result of this section and show that any system $(A, b, c) \in \Sigma_r$ may be transformed into Byrnes–Isidori form.

THEOREM 2.6 (Byrnes–Isidori form). *Let* $(A, b, c) \in \Sigma_r$*. Then the bijective and bounded operator* $U : H \to \mathbb{R}^r \times H_{A,b}$ *defined in* (2.9) *converts the system* (A, b, c) *into the system*

$$(\widehat{A}, \widehat{b}, \widehat{c}) := \left(U^{-*} A U^*, U^{-*} b, U c \right) \qquad \text{with} \qquad \operatorname{dom} \widehat{A} := U^{-*} \operatorname{dom} A\,, \tag{2.12}$$

which is in Byrnes–Isidori form. More precisely, we have

$$(2.13)\qquad (\widehat{A},\widehat{b},\widehat{c}) = \left(\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ P_0 & P_1 & \cdots & P_{r-2} & P_{r-1} & S \\ R & 0 & \cdots & 0 & 0 & Q \end{bmatrix}, \begin{pmatrix} 0 \\ \\ \vdots \\ \\ 0 \\ (A^{r-1}b,c) \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \\ \vdots \\ \\ \\ 0 \end{pmatrix} \right),$$

where $P^{\perp}: H \to H$ denotes the orthogonal projector onto $H_{A,b}$ and

$$(2.14)\qquad P_i = P^i A^{*^r} c \qquad \forall\, i \in \{0,\dots,r-1\},$$

$$(2.15)\qquad R: \mathbb{R} \to H_{A,b}, \qquad \alpha \mapsto R\alpha = \frac{\alpha}{(c, A^{r-1}b)} P^{\perp} A^r b,$$

$$(2.16)\qquad S: H_{A,b} \to \mathbb{R}, \qquad \eta \mapsto S\eta = \left(P_{A,b} A^{*^r} c, \eta \right),$$

$$(2.17)\qquad Q\eta = P^{\perp} A\eta - R(c,\eta) \qquad \forall\, \eta \in \operatorname{dom} Q = H_{A,b} \cap \operatorname{dom} A\,,$$

$$(2.18)\qquad \operatorname{dom} \widehat{A} = \mathbb{R}^r \times (H_{A,b} \cap \operatorname{dom} A) = \mathbb{R}^r \times \operatorname{dom} Q,$$

and Q generates a strongly continuous semigroup $(T_Q(t))_{t\geq 0}$ in $H_{A,b}$.

The proof is in Appendix A.

Remark 2.7. Let $(A,b,c) \in \Sigma_r$ be in Byrnes–Isidori form as in Definition 2.1. Then A has a block operator structure of the form

$$(2.19)\qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & Q \end{bmatrix},$$

where $A_{11}: \mathbb{R}^r \to \mathbb{R}^r$, $A_{12}: H_{A,b} \to \mathbb{R}^r$, and $A_{21}: \mathbb{R}^r \to H_{A,b}$ are bounded operators, and only Q is (possibly) unbounded.

Corollary 2.8. *Let $(A,b,c) \in \Sigma_r$. The transformation U^* from Theorem* 2.6 *induces the bijective mapping*

$$\widehat{U}^*: H_{A,b} \to H_{c,A}^{\perp}, \quad \eta \mapsto U^* \begin{pmatrix} 0_{\mathbb{R}^r} \\ \eta \end{pmatrix}$$

$$\text{with inverse} \qquad \widehat{U}^{-*}: H_{c,A}^{\perp} \to H_{A,b}, \quad x \mapsto P^{\perp} x.$$

Define the bounded operator $K: H \to \mathbb{R}$ by

$$(2.20)\qquad x \mapsto Kx := \frac{(x, A^{*^r} c)}{(b, A^{*^{r-1}} c)}.$$

Then $A - bK$ maps $H_{c,A}^{\perp}$ into itself and the operator Q from (2.17)*, defined on $H_{A,b}$, is similar to $A - bK$ on the largest controlled invariant subspace $H_{c,A}^{\perp}$ in $c^{\perp}$. It satisfies*

$$(2.21)\qquad Q = \widehat{U}^{-*} (A - bK) \widehat{U}^*.$$

Proof. The bijectivity of $\widehat{U}^*$ is immediate from Lemma 2.5. By [17, Theorem 2.10], the operator $A - bK$ maps $H_{c,A}^{\perp}$ into itself. Hence we have, for all $\eta \in \operatorname{dom} Q \subset H_{A,b}$,

$$\begin{aligned}
&\widehat{U}^{-*}\left(A\widehat{U}^*\eta - b\frac{(\widehat{U}^*\eta, A^{*^r}c)}{(A^{*^{r-1}}c, b)}\right) \\
&\quad = P^{\perp}\left(A\widehat{U}^*\eta - b\frac{(\widehat{U}^*\eta, A^{*^r}c)}{(A^{*^{r-1}}c, b)}\right) = P^{\perp}AU^*\begin{pmatrix}0_{\mathbb{R}^r}\\ \eta\end{pmatrix} = \begin{pmatrix}0_{\mathbb{R}^r}^{\top} & I\end{pmatrix}\widehat{A}\begin{pmatrix}0_{\mathbb{R}^r}\\ \eta\end{pmatrix} = Q\eta. \qquad \square
\end{aligned}$$

The operators K and $A - bK$ are often used for the analysis of the dynamics in $c^{\perp}$; see, e.g., [16, 17].

Theorem 2.6 together with Corollary 2.8 allows a simple description of the invariant zeros of (A, b, c) in terms of the eigenvalues of Q. Recall (cf. [18]) that $s \in \mathbb{C}$ is called an *invariant zero of* (A, b, c) if and only if there exist $x \in \operatorname{dom} A \setminus \{0\}$ and $u \in \mathbb{R}$ such that $(s - A)x + bu = 0$ and $(c, x) = 0$. By [18, Theorem 2.3] and [17, Theorem 2.10], the invariant zeros of (A, b, c) are the eigenvalues of $A - bK$. Hence, (2.21) shows the following corollary.

COROLLARY 2.9. *Let* $(A, b, c) \in \Sigma_r$ *with Byrnes–Isidori form* (2.12)*. Then the eigenvalues of* Q *are the invariant zeros of* (A, b, c)*.*

We are now in a position to describe the transfer function of the system (A, b, c) in terms of the entries of the Byrnes–Isidori form (2.13).

PROPOSITION 2.10. *Let* $(A, b, c) \in \Sigma_r$ *with Byrnes–Isidori form* (2.13)*. Then*

$$\rho(A) \cap \rho(Q) = \left\{ \lambda \in \rho(Q) \;\middle|\; \lambda^r - \sum_{k=0}^{r-1} P_k\lambda^k - S(\lambda - Q)^{-1}R \neq 0 \right\}$$

and the transfer function G *of* (A, b, c) *is given by*

$$G(\lambda) := (c, (\lambda - A)^{-1}b) = \frac{(A^{r-1}b, c)}{\lambda^r - \sum_{k=0}^{r-1} P_k\lambda^k - S(\lambda - Q)^{-1}R} \qquad \forall\, \lambda \in \rho(A) \cap \rho(Q). \tag{2.22}$$

Proof. Since the transformation to Byrnes–Isidori form $(\widehat{A}, \widehat{b}, \widehat{c})$ in Theorem 2.6 is boundedly invertible, we have $\rho(A) = \rho(\widehat{A})$ and

$$G(\lambda) = (c, (\lambda - A)^{-1}b) = (\widehat{c}, (\lambda - \widehat{A})^{-1}\widehat{b}).$$

Let $\lambda \in \rho(Q)$. Then $\lambda - \widehat{A}$ is boundedly invertible if and only if the Schur complement

$$\begin{bmatrix} \lambda & -1 & \cdots & 0 & 0 \\ 0 & \lambda & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & -1 & 0 \\ 0 & 0 & & \lambda & -1 \\ -P_0 & -P_1 & \cdots & -P_{r-2} & \lambda - P_{r-1} \end{bmatrix} - \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -S \end{pmatrix} (\lambda - Q)^{-1} \begin{pmatrix} -R & 0 & \cdots & 0 & 0 \end{pmatrix}$$

is boundedly invertible; see, e.g., [20, Theorem 2.3.3]. The latter is equivalent to

$$\lambda^r - \sum_{k=0}^{r-1} P_k\lambda^k - S(\lambda - Q)^{-1}R \neq 0. \tag{2.23}$$

Hence, the first claim follows from $\rho(A) = \rho(\widehat{A})$. Now let $\lambda \in \rho(Q) \cap \rho(A)$. Then (2.23) holds and we may define

$$\alpha := \frac{(A^{r-1}b, c)}{\lambda^r - \sum_{k=0}^{r-1} P_k \lambda^k - S(\lambda - Q)^{-1}R} \qquad \text{and}$$
$$x := \begin{pmatrix}1, & \lambda, & \ldots, & \lambda^{r-1}, & (\lambda - Q)^{-1}R\end{pmatrix}^\top \alpha.$$

It is easily verified that

$$(\lambda - \widehat{A})x = \begin{bmatrix} \lambda & -1 & \cdots & 0 & 0 & 0 \\ 0 & \lambda & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & -1 & 0 & 0 \\ 0 & 0 & & \lambda & -1 & 0 \\ -P_0 & -P_1 & \cdots & -P_{r-2} & \lambda - P_{r-1} & -S \\ -R & 0 & \cdots & 0 & 0 & \lambda - Q \end{bmatrix} \begin{pmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{r-1} \\ (\lambda - Q)^{-1}R \end{pmatrix},$$
$$\alpha = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ (A^{r-1}b, c) \\ 0 \end{pmatrix} = \widehat{b}.$$

Hence, $x = (\lambda - \widehat{A})^{-1}\widehat{b}$ and exploiting the simple structure of $\widehat{c}$ we obtain

$$(\widehat{c}, (\lambda - \widehat{A})^{-1}\widehat{b}) = (\widehat{c}, x) = \alpha.$$

This completes the proof of the proposition. ∎

Remark 2.11. Let $(A, b, c) \in \Sigma_r$ with Byrnes–Isidori form (2.13).

(i) Since A and Q generate strongly continuous semigroups, there exists some $\alpha \in \mathbb{R}$ such that $\rho(A) \cap \rho(Q) \subset \mathbb{C}_{\geq \alpha}$. Therefore, the representation (2.22) shows that the frequency domain relative degree condition (1.3) holds.

(ii) The inverse of the transfer function is given by

$$G(s)^{-1} = (A^{r-1}b, c)^{-1} \left(s^r - \sum_{k=0}^{r-1} P_k s^k - S(s - Q)^{-1}R \right).$$

If Q is exponentially stable, then $S(s - Q)^{-1}R$ is in $H^\infty(\mathbb{C}_{\geq 0})$, and we get a higher relative degree version of the condition (1.6) from [15].

We now investigate the behavior of a system $(A, b, c) \in \Sigma_r$, i.e., the solutions of (1.1). A *mild solution* to the inhomogeneous Cauchy problem (1.1a) is given, for any $u \in L^1_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R})$ and any $x^0 \in H$, by

$$x(t) = T(t)x^0 + \int_0^t T(t - s)\, b\, u(s)\, \mathrm{d}s, \qquad t \geq 0\,. \tag{2.24}$$

It is well known (see, for example, [7, Lemma 3.1.5]) that x and y are continuous functions. Therefore, we may define the *(mild) behavior* of (1.1) by

$$\mathfrak{B}_{(A,b,c)} := \left\{ (x,u,y) \in \mathcal{C}(\mathbb{R}_{\geq 0}; H) \times L^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}) \times \mathcal{C}(\mathbb{R}_{\geq 0}; H) \;\middle|\; \begin{array}{l} x(t) = T(t)x(0) + \int_0^t T(t-s)\, bu(s)\,\mathrm{d}s \\ \text{and} \quad y(t) = (x(t), c), \qquad t \geq 0 \end{array} \right\}.$$

Let $(A,b,c) \in \Sigma_r$ be transformed into Byrnes–Isidori form $(\widehat{A}, \widehat{b}, \widehat{c})$ as in (2.12). Then U^{-*} applied to the mild solution (2.24) yields

$$U^{-*}x(t) = U^{-*}x^0 + \int_0^t U^{-*}T(t-s)U^*\,\widehat{b}\,u(s)\,\mathrm{d}s, \qquad t \geq 0\,, \tag{2.25}$$

and since the semigroup $(\widehat{T}(t))_{t\geq 0}$ generated by $\widehat{A}$ is $\widehat{T}(t) = U^{-*}T(t)U^*$, we conclude

$$(x,u,y) \in \mathfrak{B}_{(A,b,c)} \qquad \Longleftrightarrow \qquad (U^{-*}x, u, y) \in \mathfrak{B}_{(\widehat{A},\widehat{b},\widehat{c})}\,. \tag{2.26}$$

Next we show that the mild solution (x,u,y) of a system in Byrnes–Isidori form is closely related to a solution (u,y) of a functional differential equation. The latter allows for a simpler input-output mapping; see Figure 1.

PROPOSITION 2.12 (internal loop form). *Let $(A,b,c) \in \Sigma_r$, $x^0 \in H$, $u \in L^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R})$, and consider the system* (1.1). *Then, with the notation as in Theorem* 2.6, *the following are equivalent:*

(i) $\exists\,(x,u,y) \in \mathfrak{B}_{(A,b,c)}$ *with* $x(0) = x^0$.

(ii) *The function y is $(r-1)$-times continuously differentiable and satisfies*

$$\begin{pmatrix} y(t) \\ \vdots \\ y^{(r-2)}(t) \\ y^{(r-1)}(t) \end{pmatrix} = \begin{pmatrix} (x^0, c) \\ \vdots \\ (x^0, {A^*}^{r-2} c) \\ (x^0, {A^*}^{r-1} c) \end{pmatrix} + \begin{pmatrix} \int_0^t y^{(1)}(s)\,\mathrm{d}s \\ \vdots \\ \int_0^t y^{(r-1)}(s)\,\mathrm{d}s \\ \int_0^t \sum_{i=0}^{r-1} P_i y^{(i)}(s) + S\eta(s) + (A^{r-1}b, c)u(s)\,\mathrm{d}s \end{pmatrix}, \tag{2.27}$$

$$\eta(t) = T_Q(t)P^{\perp}x^0 + \int_0^t T_Q(t-s)\,[R, 0\ldots, 0]\,(y(s), \ldots, y^{(r-1)}(s))^{\top}\mathrm{d}s \quad \forall\, t \geq 0. \tag{2.28}$$

(iii) *The function y is $(r-1)$-times continuously differentiable and its rth derivative satisfies*

$$y^{(r)}(t) = \sum_{i=0}^{r-1} P_i\, y^{(i)}(t) + (\mathbf{T}_{\eta^0} y)(t) + (A^{r-1}b, c)\,u(t) \qquad a.e.\,, \tag{2.29}$$

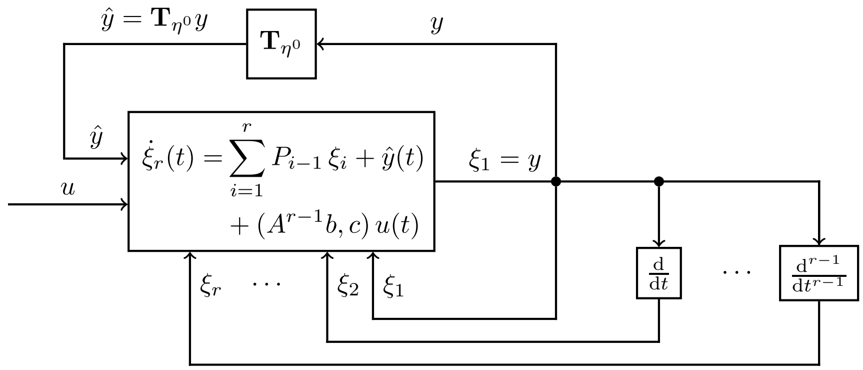


FIG. 1. *The internal loop form.*

and, for the orthogonal projector $P^{\perp} : H \to H$ *onto* $H_{A,b}$*,*

$$\begin{pmatrix} y(0) \\ \vdots \\ y^{(r-1)}(0) \\ \eta(0) \end{pmatrix} = U^{-*}x^0 = \begin{pmatrix} (x^0, c) \\ \vdots \\ (x^0, {A^*}^{r-1} c) \\ P^{\perp}x^0 \end{pmatrix}, \tag{2.30}$$

where for $\eta^0 = \eta(0)$ *the causal linear operator* $\mathbf{T}_{\eta^0}$ *is defined by*

$$\begin{array}{rcccl} \mathbf{T}_{\eta^0} & : & L^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}) & \to & \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}), \\ & & y & \mapsto & \left(t \mapsto ST_Q(t)\eta^0 + S\int_0^t T_Q(t-s)\, Ry(s)\mathrm{d}s\right). \end{array}$$

The functions x *and* η *in* (i) *and* (ii) *are related by*

$$x(t) = U^*(y(t), y^{(1)}(t), \ldots, y^{(r-1)}(t), \eta(t))^{\top}.$$

In the proof of Proposition 2.12 we make use of the following lemma which may not be surprising but we sketch its proof for the convenience of the reader.

LEMMA 2.13. *Let* U*,* X_1*, and* X_2 *be Hilbert spaces and let* A_{ii} *be an operator which generates the semigroup* $(S_{ii}(t))_{t\geq 0}$ *in* X_i *for* $i = 1, 2$*. Let* $A_{12} \in \mathcal{B}(X_2, X_1)$*,* $A_{21} \in \mathcal{B}(X_1, X_2)$*, and* $B_1 \in \mathcal{B}(U, X_1)$*,* $B_2 \in \mathcal{B}(U, X_2)$ *be bounded operators,* $x^0 \in X_1 \times X_2$*, and* $u \in L^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, U)$*. Then* x *is a mild solution of*

$$\dot{x}(t) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x(t) + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u(t), \qquad x(0) = x^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} \tag{2.31}$$

if and only if the projected functions $x_1(\cdot) = \pi_{X_1}x(\cdot)$ *and* $x_2(\cdot) = \pi_{X_2}x(\cdot)$ *satisfy*

$$x_1(t) = S_{11}(t)x_1^0 + \int_0^t S_{11}(t-s)\,(A_{12}x_2(s) + B_1u(s))\,\mathrm{d}s, \tag{2.32}$$

$$x_2(t) = S_{22}(t)x_2^0 + \int_0^t S_{22}(t-s)\,(A_{21}x_1(s) + B_2u(s))\,\mathrm{d}s. \tag{2.33}$$

If (2.32) *holds and* A_{11} *is bounded, then* x_1 *is differentiable with respect to the norm of* X_1 *almost everywhere with*

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) \qquad \text{for a.a. } t \geq 0\,. \tag{2.34}$$

Proof. The difference of $A := \left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right]$ and $D := \left[\begin{smallmatrix} A_{11} & 0 \\ 0 & A_{22} \end{smallmatrix}\right]$ is a bounded operator in $X_1 \times X_2$ and hence the rigged spaces $X_{A,-1}$ for A and $X_{D,-1}$ for D coincide; see [19, section 3.6]. [19, Thm 3.8.2 (iv)] yields that x is a mild solution of (2.31) if and only if for all $t \geq 0$,

$$\begin{aligned} x(t) &= \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} + \int_0^t A|_H x(\tau) + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u(\tau)\mathrm{d}\tau \\ &= \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} + \int_0^t D|_H x(\tau) + \left[\begin{smallmatrix} 0 & A_{12} \\ A_{21} & 0 \end{smallmatrix}\right] x(\tau) + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u(\tau)\mathrm{d}\tau, \end{aligned}$$

where the integral is taken in $H_{A,-1}$. The same theorem shows that the latter is equivalent to (2.32)–(2.33). The additional claim follows from [19, Thm 3.8.2] and Cor. 2 of [9, Thm 3.8.5]. ◻

Proof of Proposition 2.12. (i) $\Rightarrow$ (ii): Let $(x, u, y) \in \mathfrak{B}_{(A,b,c)}$ with $x(0) = x^0$. Then x is a mild solution of (1.1a). By Lemma 2.5 and (2.26), the boundedly invertible transformation U^{-*} maps x onto the mild solution $(\alpha_0, \ldots, \alpha_{r-1}, \eta)^\top := U^{-*}x$ of (2.35)

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \alpha_0(t) \\ \vdots \\ \alpha_{r-1}(t) \\ \eta(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & & 1 & 0 \\ P_0 & P_1 & \cdots & P_{r-1} & S \\ R & 0 & \cdots & 0 & Q \end{bmatrix} \begin{pmatrix} \alpha_0(t) \\ \vdots \\ \alpha_{r-1}(t) \\ \eta(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (A^{r-1}b, c) \\ 0 \end{pmatrix} u(t).$$

Since

$$\begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & & 1 \\ P_0 & P_1 & \cdots & P_{r-1} \end{bmatrix}$$

is bounded, we may apply Lemma 2.13 to (2.35) to conclude from (2.34) that

(2.36)
$$\begin{pmatrix} \alpha_0(t) \\ \vdots \\ \alpha_{r-2}(t) \\ \alpha_{r-1}(t) \end{pmatrix} = \begin{pmatrix} \alpha_0(0) \\ \vdots \\ \alpha_{r-2}(0) \\ \alpha_{r-1}(0) \end{pmatrix} + \begin{pmatrix} \int_0^t \alpha_1(s)\,\mathrm{d}s \\ \vdots \\ \int_0^t \alpha_{r-1}(s)\,\mathrm{d}s \\ \int_0^t \sum_{i=0}^{r-1} P_i\alpha_i(s) + S\eta(s) + (A^{r-1}b, c)u(s)\,\mathrm{d}s \end{pmatrix},$$

(2.37)
$$\eta(t) = T_Q(t)\eta(0) + \int_0^t T_Q(t-s)\,[R, 0 \ldots, 0]\,(\alpha_0(s), \ldots, \alpha_{r-1}(s))^\top ds \quad \forall\, t \geq 0,$$

and

$$\begin{pmatrix} \alpha_0(0) \\ \vdots \\ \alpha_{r-1}(0) \\ \eta(0) \end{pmatrix} = U^{-*}x(0) = \begin{pmatrix} (x^0, c) \\ \vdots \\ (x^0, {A^*}^{r-1} c) \\ P^\perp x^0 \end{pmatrix}.$$

Since, by (2.12) and (2.13), $Uc = [1, 0, \ldots, 0]^\top$, we have

$$y(t) = (x(t), c) = (U^{-*}x, Uc) = \alpha_0(t) \quad \forall\, t \geq 0\,, \tag{2.38}$$

and we conclude from (2.36) that $y^{(i)} = \alpha_0^{(i)} = \alpha_i$ for all $i = 0, \ldots, r-1$. Hence, (ii) is shown.

(ii) $\Rightarrow$ (iii): If (ii) holds, then the lower line of (2.27) shows that the function $y^{(r-1)} = \alpha_{r-1}$ is absolutely continuous. Therefore, it is almost everywhere differentiable and its derivative satisfies (2.29).

(iii) $\Rightarrow$ (i): Assume y satisfies (iii). Define $(\alpha_0, \ldots, \alpha_{r-1}) := (y, \ldots, y^{(r-1)})$ and η by (2.28). Then (2.36) and (2.37) are fulfilled. Therefore, by (2.26) and Lemma 2.13, the function

$$x(\cdot) := U^*(\alpha_0(\cdot), \ldots, \alpha_{(r-1)}(\cdot), \eta(\cdot))^\top$$

is a mild solution of (1.1a) with initial value x^0. Finally, (2.38) yields (i). This completes the proof of the proposition. $\square$

Note that the right-hand side of (2.29) may be interpreted as an ordinary differential term $\sum_{i=0}^{r-1} P_i\, y^{(i)}(t) + (A^{r-1}b, c)u(t)$ which is perturbed by a functional term $(\mathbf{T}_{\eta^0} y)(t)$; see Figure 1. This structure will be exploited to control the system in section 5.

We close this section with a result on the uniqueness of the Byrnes–Isidori form. It shows in particular that all P_i in the entries of the representation of A in (2.1) are uniquely defined. In terms of the internal loop form in Figure 1, this means that the main block, i.e., the ordinary differential equation, is uniquely given. Moreover, the input-output behavior of the perturbation block $\mathbf{T}_{\eta^0}$ is unique, although the triple (Q, R, S) which determines the mapping $\mathbf{T}_{\eta^0}$ is internally only unique up to a bounded bijective invertible transformation. This is made precise in the following proposition.

PROPOSITION 2.14 (uniqueness of the Byrnes–Isidori form). *If $(A, b, c) \in \Sigma_r$ is transformed by a bijective and bounded operator $W : H \to \mathbb{R}^r \times \widetilde{V}$ into the Byrnes–Isidori form*

$$\big(W^{-*}AW^*, W^{-*}b, Wc\big) = \left(\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ \widetilde{P}_0 & \widetilde{P}_1 & \cdots & \widetilde{P}_{r-2} & \widetilde{P}_{r-1} & \widetilde{S} \\ \widetilde{R} & 0 & \cdots & 0 & 0 & \widetilde{Q} \end{bmatrix}, \begin{pmatrix} 0 \\ \\ \vdots \\ \\ 0 \\ \widetilde{b}_r \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \\ \vdots \\ \\ \\ 0 \end{pmatrix} \right), \tag{2.39}$$

then the entries of (2.13) *and* (2.39) *are related as follows:*

(i) $P_i = \widetilde{P}_i$ *for all* $i = 0, \ldots, r-1$*, and they are uniquely defined by* (A, b, c)*;*

(ii) $(\widetilde{Q}, \widetilde{R}, \widetilde{S}) = (\widetilde{Y} Q\, \widetilde{Y}^{-1}, \widetilde{Y} R, S \widetilde{Y}^{-1})$*, with* $\operatorname{dom} \widetilde{Q} = \widetilde{Y} \operatorname{dom} Q$ *for some bounded and bijective operator* $\widetilde{Y} : \widetilde{V} \to H_{A,b}$*;*

(iii) $\widetilde{b}_r = b_r = (A^{r-1}b, c)$.

The proof is in Appendix A.

3. Zero dynamics. In this section the zero dynamics of a system $(A, b, c) \in \Sigma_r$ are investigated. Roughly speaking, the zero dynamics are those dynamics of the system which are not visible at the output.

DEFINITION 3.1. *The* zero dynamics *of a system* $(A, b, c) \in \Sigma_r$ *is the real subvector space of the mild behavior defined by*

$$\mathcal{ZD}_{(A,b,c)} := \left\{ (x, u, y) \in \mathfrak{B}_{(A,b,c)} \middle|\ y \equiv 0 \right\} . \qquad \square$$

The Byrnes–Isidori form is instrumental for simplifying the presentation of the zero dynamics as will be shown in the following proposition.

PROPOSITION 3.2. *Let* $(A, b, c) \in \Sigma_r$*. Then, with the notation as in Theorem* 2.6 *and Corollary* 2.8, *the zero dynamics are given by*

$$\mathcal{ZD}_{(A,b,c)} = \left\{ \left(U^* \begin{pmatrix} 0_{\mathbb{R}^r} \\ T_Q(\cdot)\, P^\perp x^0 \end{pmatrix}, -\frac{S T_Q(\cdot)\, P^\perp x^0}{(A^{r-1} b, c)}\ , 0 \right) \middle|\ x^0 \in H \right\}, \tag{3.1}$$

$$= \left\{ \left(T_{A-bK}(\cdot)\, x^0, K T_{A-bK}(\cdot)\, x^0\ , 0 \right) \middle|\ x^0 \in H_{c,A}^\perp \right\}, \tag{3.2}$$

where K *is as in* (2.20) *and* $(T_Q(t))_{t \geq 0}$ *and* $(T_{A-bK}(t))_{t\geq 0}$ *denote the semigroups generated by* Q *and* $A - bK$*, resp.*

Proof. We first show (3.1): Let $(x, u, y) \in \mathcal{ZD}_{(A,b,c)}$ with $x(0) = x^0$. By Proposition 2.12, $(y, \ldots, y^{(r-1)})^\top$ and $\eta(\cdot) := P^\perp x(\cdot)$ satisfy (2.27) and (2.28). Since $y = 0$, we have $y^{(i)} = 0$ for all $i = 0, \ldots, r-1$, and solving (2.27) and (2.28) for u and η yields $\eta(t) = T_Q(t) P^\perp x^0$ and $u(t) = -(A^{r-1} b, c)^{-1} S \eta(t)$. Since Proposition 2.12 also states that $x(t) = U^*(y(t), \ldots, y^{(r-1)}(t), \eta(t))^\top$, the triple (x, u, y) belongs to the right-hand side of (3.1).

Conversely, let $\widetilde{x}^0 \in H$ be given and define $x^0 := U^* \begin{pmatrix} 0_{\mathbb{R}^r} \\ P^\perp \widetilde{x}^0 \end{pmatrix}$. Then (2.11) shows

$$\begin{pmatrix} (x^0, c) \\ \vdots \\ (x^0, {A^*}^{r-1} c) \\ P^\perp x^0 \end{pmatrix} = U^{-*} x^0 = \begin{pmatrix} 0_{\mathbb{R}^r} \\ P^\perp \widetilde{x}^0 \end{pmatrix}.$$

Using this equation, it can be seen that the functions

$$y(t) := 0, \qquad u(t) := -\frac{S T_Q(t) P^\perp \widetilde{x}^0}{(A^{r-1} b, c)}, \qquad \eta(t) = T_Q(t) P^\perp \widetilde{x}^0 \qquad \forall\, t \geq 0.$$

satisfy (ii) of Proposition 2.12 with x^0. Hence, this proposition implies that

$$\left(U^* \begin{pmatrix} 0_{\mathbb{R}^r} \\ T_Q(\cdot) P^\perp \widetilde{x}^0 \end{pmatrix}, -\frac{S T_Q(\cdot) P^\perp \widetilde{x}^0}{(A^{r-1} b, c)}, 0 \right) = \left(U^* \begin{pmatrix} 0_{\mathbb{R}^r} \\ \eta(\cdot) \end{pmatrix}, u(\cdot), y(\cdot) \right) \in \mathfrak{B}_{(A,b,c)}.$$

Since $y \equiv 0$, the left-hand side belongs to the zero dynamics and (3.1) is shown.

It remains to prove (3.2). Observe that the set in (3.1) is equal to

$$\left\{ \left(\widehat{U}^* T_Q(\cdot)\, x^0, -\frac{S T_Q(\cdot)\, x^0}{(A^{r-1} b, c)}\ , 0 \right) \middle|\ x^0 \in H_{A,b} \right\},$$

where $\widehat{U}^*$ is as in Corollary 2.8. Corollary 2.8 implies $\widehat{U}^* T_Q(t) x^0 = T_{A-bK}(t) \widehat{U}^* x^0$ for all $x^0 \in H_{A,b}$; see, e.g., [8, section II.2.1]. Since, by Corollary 2.8, $\widehat{U}^* : H_{A,b} \to H_{c,A}^\perp$

is bijective, the proof is finished with the calculation

$$\frac{S\widehat{U}^{-*}x}{(A^{r-1}b,c)} = \frac{(P_{A,b}A^{*^r}c,\widehat{U}^{-*}x)}{(A^{r-1}b,c)} = \frac{([0_{\mathbb{R}^r}^\top\ I]UA^{*^r}c,\widehat{U}^{-*}x)}{(A^{r-1}b,c)}$$
$$= \frac{(UA^{*^r}c,U^{-*}x)}{(A^{r-1}b,c)} = \frac{(A^{*^r}c,x)}{(b,A^{*^{r-1}}c)} = Kx$$

for all $x \in H_{c,A}^{\perp}$. ∎

Formula (3.2) can be also obtained from the results in [17, 22]; we have derived it in our setup only for completeness.

The stability concepts for zero dynamics and semigroups are defined as follows.

DEFINITION 3.3. *A system* $(A,b,c) \in \Sigma_r$ *is said to have* exponentially stable zero dynamics *if and only if*

$$\exists\, M,\mu > 0\ \forall\,(x,u,0) \in \mathcal{ZD}_{(A,b,c)}\ \forall\, t \geq 0\ :\ \|(x(t),u(t))\| \leq M\|x(0)\|\mathrm{e}^{-\mu t}. \tag{3.3}$$

It is said to have strongly stable zero dynamics if and only if

$$\forall\,(x,u,0) \in \mathcal{ZD}_{(A,b,c)}\ :\ \lim_{t\to\infty}\|(x(t),u(t))\| = 0.$$

A semigroup $(T(t))_{t\geq 0}$ *on a Hilbert space* H *is called exponentially stable if and only if*

$$\exists M,\mu > 0\ \forall\, t \geq 0\ :\ \|T(t)\| \leq M\mathrm{e}^{-\mu t}, \tag{3.4}$$

and strongly stable if and only if

$$\forall x \in H\ :\ \lim_{t\to\infty}\|T(t)x\| = 0.$$ ∎

Now Proposition 3.2 allows for a characterization of the stability of the zero dynamics in terms of the Byrnes–Isidori form.

PROPOSITION 3.4. *A system* $(A,b,c) \in \Sigma_r$ *has exponentially stable zero dynamics if and only if the operator* Q *in Theorem* 2.6 *generates an exponentially stable semigroup. It has strongly stable zero dynamics if and only if* Q *generates a strongly stable semigroup.*

Proof. First note that Theorem 2.6 implies that Q generates a strongly continuous semigroup T_Q. If this semigroup is exponentially stable, then the assertion is an immediate consequence of Proposition 3.2. Assume on the other hand that (A,b,c) has exponentially stable zero dynamics and let $x^0 \in H_{A,b}$ be arbitrary. Then $x^0 = P^\perp x^0$ and (3.1) shows that

$$\left(U^*\begin{pmatrix} 0_{\mathbb{R}^r} \\ T_Q(\cdot)x^0\end{pmatrix}, -\frac{ST_Q(\cdot)x^0}{(A^{r-1}b,c)}, 0\right)\ \in\ \mathcal{ZD}_{(A,b,c)}.$$

Thus, the stability assumption (3.3) implies that

$$\forall\, t \geq 0\ :\ \left\|U^*\begin{pmatrix} 0_{\mathbb{R}^r} \\ T_Q(t)x^0\end{pmatrix}\right\| \leq M\left\|U^*\begin{pmatrix} 0_{\mathbb{R}^r} \\ x^0\end{pmatrix}\right\|\mathrm{e}^{-\mu t}.$$

Since U^* is boundedly invertible, we conclude $\|T_Q(t)x^0\| \leq \|U^*\|\|U^{-*}\|Me^{-\mu t}\|x^0\|$. This shows the exponential stability of the semigroup, because M and μ are, by assumption, independent of x^0.

The part about strong stability follows in the same manner from Proposition 3.2. ∎

COROLLARY 3.5. *Assume that $(A,b,c) \in \Sigma_r$ has exponentially stable zero dynamics. Then its transfer function G satisfies*

$$G(\lambda) \neq 0 \qquad \forall\ \lambda \in \rho(A) \cap \mathbb{C}_{\geq 0}$$

and

$$\rho(A) \cap \mathbb{C}_{\geq 0} = \left\{ \lambda \in \mathbb{C}_{\geq 0} \;\middle|\; \lambda^r - \sum_{k=0}^{r-1} P_k \lambda^k - S(\lambda - Q)^{-1} R \neq 0 \right\}.$$

Proof. By Proposition 3.4 and [7, Theorem 5.15], the exponential stability of the zero dynamics is equivalent to the conditions $\mathbb{C}_{\geq 0} \subset \rho(Q)$ and $\sup_{\lambda \in \mathbb{C}_{\geq 0}} \|(\lambda - Q)^{-1}\| < \infty$. Therefore, the denominator of the transfer function in (2.22) is finite at every point $\lambda \in \rho(A) \cap \mathbb{C}_{\geq 0}$ and the claim follows. ∎

In view of the internal loop form in Figure 1 we may observe that if $(A,b,c) \in \Sigma_r$ has exponentially stable zero dynamics, then Proposition 3.4 and (3.4) show that $\mathbf{T}_{\eta^0}$ maps bounded functions to bounded functions. This property is crucial for the high-gain stabilizability results that we derive in the next two sections.

4. High-gain stabilizability.

In this section we concentrate on systems with relative degree $r = 1$ and show *high-gain stabilizability*: if $(A,b,c) \in \Sigma_1$ has exponentially stable zero dynamics, then it is stabilizable by proportional output feedback $u(t) = -k \operatorname{sgn}(b,c)\, y(t)$ provided the gain $k > 0$ is sufficiently large. This feedback applied to (1.1) yields a closed-loop system

$$\dot{x}(t) = Ax(t) - \operatorname{sgn}(b,c)\, k\, b(x(t),c),$$

which is by the following proposition exponentially stable.

PROPOSITION 4.1 (high-gain stabilizability). *Let $(A,b,c) \in \Sigma_1$ have relative degree $r = 1$ and exponentially stable zero dynamics. Then there exists a $k^* > 0$ such that for all $k \geq k^*$ the operator*

$$A_k : \operatorname{dom} A \subset H \to H, \quad x \mapsto Ax - \operatorname{sgn}(b,c)\, k\, b(x,c),$$

generates an exponentially stable semigroup.

Proof. Note that according to [8, section III.1.3], A_k generates a semigroup since $A - A_k$ is a bounded operator. Furthermore, by Theorem 2.6 we have for $x \in H$

$$b(x,c) = U^* \widehat{b}(U^{-*}x, \widehat{c}) = U^* \begin{pmatrix} (b,c) \\ 0 \end{pmatrix} \left(U^{-*}x, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = U^* \begin{bmatrix} -(b,c) & 0 \\ 0 & 0 \end{bmatrix} U^{-*}x$$

and hence

$$A_k = U^* \left(\widehat{A} - \begin{bmatrix} k\,|(b,c)| & 0 \\ 0 & 0 \end{bmatrix} \right) U^{-*} = U^* \begin{bmatrix} P_0 - k\,|(b,c)| & S \\ R & Q \end{bmatrix} U^{-*}$$

with the boundedly invertible transformation U^*. So it suffices to show that the semigroup generated by

$$\widehat{A}_k : \operatorname{dom} \widehat{A} \subset \mathbb{R}^r \times H_{A,b} \to \mathbb{R}^r \times H_{A,b}, \qquad \widehat{A}_k := \begin{bmatrix} P_0 - k\,|(b,c)| & S \\ R & Q \end{bmatrix},$$

is exponentially stable for large k. By [7, Theorem 5.1.5] this is the case if and only if $\sigma(\widehat{A}_k) \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0\}$ and

$$\sup_{\operatorname{Re} \lambda > 0} \|(\lambda - \widehat{A}_k)^{-1}\| < \infty. \tag{4.1}$$

By Proposition 3.4, Q generates an exponentially stable semigroup and thus there exists a $k_Q > 0$ with

$$\sup_{\operatorname{Re}\lambda>0} \|(\lambda - Q)^{-1}\| \le k_Q. \tag{4.2}$$

Now choose k^* such that

$$k^*|(b,c)| - P_0 > k_Q\|R\|\|S\|.$$

Then for all $k \ge k^*$ and all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > 0$, the number $h_\lambda := \lambda - P_0 + k|(b,c)| - S(\lambda - Q)^{-1}R \ne 0$ and

$$\left|\frac{1}{h_\lambda}\right| \le \left|\frac{1}{|\lambda - P_0 + k|(b,c)|\,| - \|S\|\|R\|k_Q}\right| < \frac{1}{k^*|(b,c)| - P_0 - \|S\|\|R\|k_Q}. \tag{4.3}$$

It is well known (see, e.g., [19, Lemma A.4.2(iii)]) that

$$\lambda - \widehat{A}_k = \begin{bmatrix} \lambda - P_0 + k\,|(b,c)| & -S \\ -R & \lambda - Q \end{bmatrix}$$

is invertible if the operators h_λ and $\lambda - Q$ are invertible and that in this case

$$(\lambda - \widehat{A}_k)^{-1} = \begin{bmatrix} \frac{1}{h_\lambda} & \frac{1}{h_\lambda}S(\lambda - Q)^{-1} \\ -(\lambda - Q)^{-1}R\frac{1}{h_\lambda} & (\lambda - Q)^{-1} + (\lambda - Q)^{-1}R\frac{1}{h_\lambda}S(\lambda - Q)^{-1} \end{bmatrix}.$$

The uniform bounds (4.2) and (4.3) imply that (4.1) holds and hence, the claim follows from [7, Theorem 5.1.5]. ◻

The high-gain result in Proposition 4.1 has some drawbacks: First, the size of the gain depends on the system data, and these data may not be given explicitly. In fact, the assumptions that (A, b, c) is in Σ_1 and has exponential stability are only structural. So if we did not have to determine the size of k in Proposition 4.1 a priori, we would need very little information to stabilize the system. Second, if a feasible size of the gain is chosen, it may be too large so that corruption of the output is amplified. A different feedback to resolve these drawbacks is introduced in the next section.

5. Funnel control.

In this section, we assume that the system $(A, b, c) \in \Sigma_r$ has exponentially stable zero dynamics and relative degree $r = 1$. Note that these assumptions on the system are only structural, that means no system data are required. We are going to design a special time-varying proportional feedback gain such that the closed-loop system has a global solution that is bounded with respect to the state space norm. Besides this stability, we desire to achieve two further control objectives: The first one is approximate tracking, by the output y, of reference signals y_{ref} of class $W^{1,\infty}(\mathbb{R}_{\ge 0}, \mathbb{R})$. In particular, for arbitrary $\lambda > 0$, we seek an output feedback strategy which ensures that, for every $y_{\text{ref}} \in W^{1,\infty}(\mathbb{R}_{\ge 0}, \mathbb{R})$, the closed-loop system has bounded solution and the tracking error $e(t) = y(t) - y_{\text{ref}}(t)$ is ultimately bounded by λ (that is, $\|e(t)\| \le \lambda$ for all t sufficiently large). The second control objective is prescribed transient behavior of the tracking error signal. We capture both objectives in the concept of a performance funnel

$$\mathcal{F}_\varphi := \Big\{(t, e) \in \mathbb{R}_{\ge 0} \times \mathbb{R} \;\Big|\; e \in \big(-\tfrac{1}{\varphi(t)}, \tfrac{1}{\varphi(t)}\big)\Big\},$$

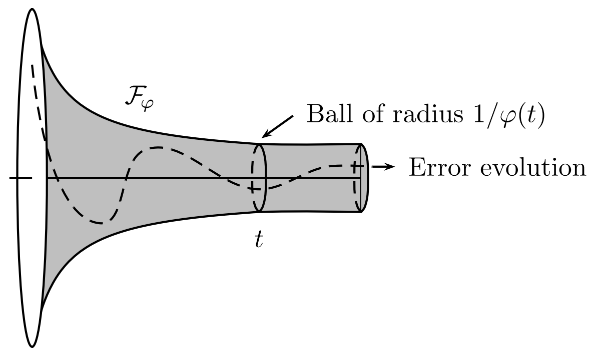


FIG. 2. *Prescribed performance funnel* $\mathcal{F}_\varphi$.

the boundary of which is determined by the reciprocal of a function φ belonging to

$$\Phi := \left\{ \varphi \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}) \;\middle|\; \begin{array}{c} \varphi(0) = 0,\, \varphi(s) > 0 \;\forall\, s > 0,\; \liminf_{s\to\infty} \varphi(s) > 0 \\ \forall\, \delta > 0 \;\exists \text{ global Lipschitz bound of } \varphi^{-1} \text{ on } [\delta, \infty) \end{array} \right\}.$$

The aim is an output feedback strategy ensuring that, for every reference signal $y_{\mathrm{ref}} \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R})$, the tracking error $e = y - y_{\mathrm{ref}}$ evolves within the funnel $\mathcal{F}_\varphi$; see Figure 2. For example, if $\liminf_{t\to\infty} \varphi(t) > 1/\lambda$, then evolution within the funnel ensures that the first control objective is achieved. If φ is chosen as the function $t \mapsto \min\{t/T, 1\}/\lambda$, then evolution within the funnel ensures that the prescribed tracking accuracy $\lambda > 0$ is achieved within the prescribed time $T > 0$.

Loosely speaking, funnel control exploits an inherent benign high-gain property of the system by designing—with appropriate choice of $\varphi \in \Phi$—a proportional error feedback $u(t) = -k(t)\, e(t)$ in such a way that $k(t)$ becomes large if $|e(t)|$ approaches the performance funnel boundary (equivalently, if $\varphi(t)|e(t)|$ approaches the value 1), thereby precluding contact with the funnel boundary. We emphasize that the gain is nonmonotone and decreases as the error recedes from the funnel boundary. The essence of the proof of the main result lies in showing that the closed-loop system is well-posed in the sense that u and k are bounded functions and the error evolves strictly within the performance funnel.

For $\varphi \in \Phi$, the *funnel controller* can be expressed in its simplest form as

$$u(t) = -k(t)\,\mathrm{sgn}\,(b, c)\, e(t), \qquad k(t) = \frac{1}{1 - \varphi(t)|e(t)|}, \qquad e(t) = y(t) - y_{\mathrm{ref}}(t)\ . \tag{5.1}$$

If (5.1) is applied to any system $(A, b, c) \in \Sigma_1$ with exponentially stable zero dynamics, then the following theorem shows that the tracking error $e(t)$ evolves within the performance funnel $\mathcal{F}_\varphi$ and moreover, the error evolution is strictly bounded away from the funnel boundary, thereby ensuring that the gain function $k(\cdot)$ and the control function $u(\cdot)$ are bounded.

The following proposition is from [11, Theorem 7]. We present it here in a version tailored to our setup.

PROPOSITION 5.1. *Let* $f \in \mathcal{C}(\mathbb{R}^n \times \mathbb{R}; \mathbb{R})$ *and* $T : \mathcal{C}(R_{\geq 0}; \mathbb{R}) \to \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R})$ *satisfy the following conditions:*

1. *For every nonempty compact set $C \subset \mathbb{R}^n$ and every sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with $|u_n| \to \infty$ and $u_n \neq 0$ it holds*

$$\min_{v\in\mathcal{C}} (\mathrm{sgn} u_n) f(v, u_n) \to \infty \quad (n \to \infty);$$

2. *for every $\delta > 0$ there exists $\Delta > 0$ such that for all $x \in \mathcal{C}(R_{\geq 0}; \mathbb{R})$ with $\sup_{t\in\mathbb{R}_{\geq 0}} |x(t)| \leq \delta$ we have*

$$\sup_{t\in\mathbb{R}_{\geq 0}} |T(x)(t)| \leq \Delta;$$

3. *for all $x, \xi \in \mathcal{C}(R_{\geq 0}; \mathbb{R})$ with $x|_{[0,t]} = \xi|_{[0,t]}$ we have*

$$T(x)|_{[0,t]} = T(\xi)|_{[0,t]};$$

4. *for all $t > 0$ and $x \in \mathcal{C}([0;t]; \mathbb{R})$ there are $\tau, \delta, c > 0$ such that all $x_1, x_2 \in \mathcal{C}(R_{\geq 0}; \mathbb{R})$ with $x_i|_{[0,t]} = x$ and $\sup_{s\in[t,t+\tau]} |x_i(s) - x(t)| < \delta$ for $i = 1, 2$ satisfy*

$$\sup_{s\in[t,t+\tau]} |T(x_1)(s) - T(x_2)(s)| \leq c \sup_{s\in[t,t+\tau]} |x_1(s) - x_2(s)|.$$

Let $\alpha : [0, 1) \to \mathbb{R}_{\geq 0}$ be continuous, strictly increasing, and unbounded. Furthermore, let $\varphi \in \Phi$, $y_{\mathrm{ref}} \in W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R})$, and $y^0 \in \mathbb{R}$. Then the equation

$$\dot{y}(t) = f\big(T(y)(t), -\alpha\left(\varphi(t)|y(t) - y_{\mathrm{ref}}(t)|\right)(y(t) - y_{\mathrm{ref}}(t))\big), \qquad y(0) = y^0 \tag{5.2}$$

has a solution. Every solution can be extended to a maximal solution, and every maximal solution $y : [0, \omega) \to \mathbb{R}$ satisfies

(a) *$\omega = \infty$;*
(b) *$\exists \varepsilon \in (0, 1)\ \forall t > 0\ :\ |y(t) - y_{\mathrm{ref}}(t)| \ \leq\ (1 - \varepsilon)\,\varphi(t)^{-1}$;*
(c) *the functions u and k defined by (5.1) are bounded on $\mathbb{R}_{\geq 0}$.*

We are now in a position to show the main result of this section: Funnel control is feasible for the class Σ_1.

THEOREM 5.2. *Consider a system $(A, b, c) \in \Sigma_1$ with relative degree $r = 1$ and exponentially stable zero dynamics. Let $\varphi \in \Phi$ specify the performance funnel $\mathcal{F}_\varphi$. Then for an arbitrary reference signal $y_{\mathrm{ref}} \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R})$ the control (5.1) applied to (1.1) yields the closed-loop system*

$$\begin{aligned} &\dot{x}(t) = Ax(t) - b\frac{1}{1 - \varphi(t)|y(t) - y_{\mathrm{ref}}(t)|}\mathrm{sgn}\,(b, c)(y(t) - y_{\mathrm{ref}}(t)), \qquad x(0) = x^0, \\ &y(t) = (x(t), c). \end{aligned} \tag{5.3}$$

The system (5.3) has a solution $x \in \mathcal{C}(\mathbb{R}_{\geq 0}, H)$ in the sense that the following equations hold for all $t \geq 0$:

$$\begin{aligned} x(t) &= T(t)x^0 + \int_0^t T(t - s)\, bu(s)\, \mathrm{d}s \qquad \textit{with} \\ u(s) &= \frac{-\mathrm{sgn}\,(b, c)}{1 - \varphi(s)|(x(s), c) - y_{\mathrm{ref}}(s)|}((x(s), c) - y_{\mathrm{ref}}(s)). \end{aligned} \tag{5.4}$$

Every function that satisfies (5.4) on an interval can be extended to a solution on $\mathbb{R}_{\geq 0}$ in the sense of (5.4) and satisfies

(a) *the functions* u*,* k*, and* e *defined in* (5.1) *are bounded;*
(b) $\exists\, \varepsilon \in (0,1)\ \forall\, t > 0\ :\ |e(t)| \leq (1-\varepsilon)\,\varphi(t)^{-1}$.

Proof. We use the equivalence of (i) and (iii) in Proposition 2.12. Since $r = 1$, Proposition 2.12 states that (5.4) is, for any fixed initial value $x^0 \in H$, equivalent to the functional differential equation

$$
\begin{aligned}
\dot y(t) &= P_0 y(t) + (\mathbf{T}_{\eta_0} y)(t) + (b,c)u(t) \qquad a.e.,\\
u(t) &= \frac{-\operatorname{sgn}(b,c)}{1-\varphi(t)|y(t)-y_{\mathrm{ref}}(t)|}\,[y(t)-y_{\mathrm{ref}}(t)],\\
y(0) &= \pi_{\mathbb{R}} U^{-*} x^0 = (x^0, c)
\end{aligned} \tag{5.5}
$$

with the operator $\mathbf{T}_{\eta^0}$ parametrized by $\eta^0 := P^{\perp} x^0$. More precisely, for any solution x of (5.4), the function $y(t) := (x(t), c)$ satisfies (5.5), and, conversely, if (5.5) has a solution $y \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R})$, then the function $x(\cdot) := U^*(y(\cdot), \eta(\cdot))^\top$ with $\eta(t) := T_Q(t)\eta^0 + \int_0^t T_Q(t-s)\, Ry(s)\,\mathrm{d}s$, fulfils $(x,u,y) \in \mathfrak{B}_{(A,b,c)}$, which by definition means that (5.4) holds.

In order to apply Proposition 5.1, we write (5.5) in the equivalent form

$$
\dot y(t) = f((Ty)(t), -\alpha(\varphi(t)|y(t)-y_{\mathrm{ref}}(t)|)(y(t)-y_{\mathrm{ref}}(t))), \tag{5.6}
$$

where

$$
\begin{aligned}
&f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}, && (x_1,x_2) \mapsto f(x_1,x_2) := x_1 + \operatorname{sgn}(b,c)\,(b,c)x_2,\\
&T : \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}) \to C(\mathbb{R}_{\geq 0}, \mathbb{R}), && y \mapsto P_0 y(\cdot) + \mathbf{T}_{\eta^0}(y),\\
&\alpha : [0,1) \to \mathbb{R}, && \alpha(e) = \frac{1}{1-e}.
\end{aligned}
$$

Then it is easy to see that f satisfies the prerequisites of Proposition 5.1. Furthermore, as described in [11, section 4.2], T has the properties 2–4 needed in Proposition 5.1 because the semigroup T_Q is exponentially stable by Proposition 3.4, So all the prerequisites of Proposition 5.1 are fulfilled and therefore there is a solution $y \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R})$ of (5.6), all solutions of (5.6) can be extended to solutions on $\mathbb{R}_{\geq 0}$, and all solutions on $\mathbb{R}_{\geq 0}$ satisfy (a) and (b). Since (5.6) is equivalent to (5.5) and (5.4) is equivalent to (5.5), it follows that all claims hold for solutions of (5.4). ∎

6. Example heat equation. We consider a metal bar of length one that can be heated on every point simultaneously according to

$$
\begin{aligned}
\partial_t x(\xi,t) &= \partial_\xi^2 x(\xi,t) + u(t), && \xi \in [0,1],\ t \geq 0,\\
x(\xi,0) &= x^0(\xi), && \xi \in [0,1],\\
\partial_\xi x(0,t) &= \partial_\xi x(1,t) = 0, && t \geq 0,\\
y(t) &= \textstyle\int_0^1 \cos^2(\pi\xi) x(\xi,t)\ \mathrm{d}\xi.
\end{aligned} \tag{6.1}
$$

The evaluation of the function $x(\xi,t)$ represents the temperature at position ξ and time t; the initial temperature profile is $x^0(\xi)$, and $u(\xi,t)$ denotes the input function for the heat. It is well known (see [7, Example 2.1.1]) that the partial differential equation (6.1) can be modeled in the form (1.1) by choosing $H = L^2(0,1)$ and

$$
\begin{aligned}
A : \operatorname{dom} A \subset H \to H, \qquad Af := f'',\\
\operatorname{dom} A = \{f \in W^{1,2}(0,1) \mid f'(0) = f'(1) = 0\},
\end{aligned}
$$

$b(\xi) := 1$, and $c(\xi) := \cos^2(\pi\xi)$. Clearly, A is a self-adjoint operator in H, the vectors b and c are in $\operatorname{dom} A = \operatorname{dom} A^*$, and since

$$(b, c) = \int_0^1 \cos^2(\pi\xi)\,\mathrm{d}\xi = \frac{1}{2},$$

it follows that $(A, b, c) \in \Sigma_1$. According to Theorem 2.6 we may transform (A, b, c) into Byrnes–Isidori form. We calculate the entries of this form:

$$P^0 x = P_1^0 x = \frac{(x, b)}{(b, c)} = 2\int_0^1 x(\xi)\,\mathrm{d}\xi,$$
$$P_{A,b} x = x - cP^0 x = x - 2\cos^2(\pi\,\cdot\,)\int_0^1 x(\xi)\,\mathrm{d}\xi \qquad \forall x \in H,$$

and by (2.2) we have $H_{A,b} = \{b\}^\perp$. Since

$$(A^* c)(\xi) = \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\cos^2(\pi\xi) = 2\pi^2\left(\sin^2(\pi\xi) - \cos^2(\pi\xi)\right)$$
$$\text{and} \qquad Ab = 0 \qquad \text{for all } \xi \in [0, 1],$$

it follows that

$$\begin{aligned}
P_0 &= P^0 A^* c = 2\textstyle\int_0^1 2\pi^2\left(\sin^2(\pi\xi) - \cos^2(\pi\xi)\right)\,\mathrm{d}\xi = 0,\\
R &= \tfrac{1}{(b,c)} Ab = 0,\\
S\eta &= (P_{A,b} A^* c, \eta) = 2\pi^2 \textstyle\int_0^1 \left(\sin^2(\pi\xi) - \cos^2(\pi\xi), \eta(\xi)\right)\,\mathrm{d}\xi \qquad && \forall\, \eta \in \{b\}^\perp,\\
Q\eta &= P^\perp A\eta - R(c, \eta) = P^\perp A\eta, && \forall\, \eta \in \{b\}^\perp \cap \operatorname{dom} A.
\end{aligned}$$

It is well known that the eigenvalues of A are $\{-n^2\pi^2 \,|\, n = 0, 1, 2, \dots\}$ with corresponding eigenvectors $\phi_n(\xi) := \sqrt{2}\cos(n\pi\xi)$; see, for example, [7, Example 2.3.7]. Since $\phi_0 = b$ and $\{\phi_n(\xi) \,|\, n = 0, 1, \dots\}$ is an orthonormal basis of $L^2(0, 1)$, we have

$$H_{A,b} = \overline{\operatorname{ls}\{\phi_n \,|\, n = 1, 2, \dots\}},$$

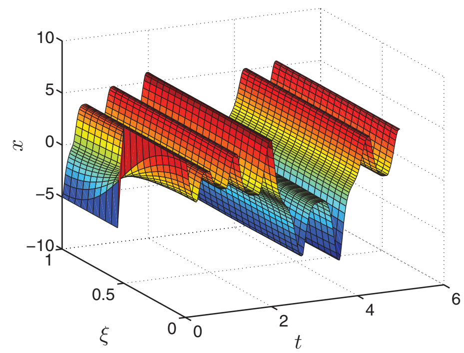
which shows that $H_{A,b}$ is invariant under A and hence

$$Q = P^\perp A|_{H_{A,b} \cap \operatorname{dom} A} = A|_{H_{A,b} \cap \operatorname{dom} A}, \qquad \operatorname{dom} Q = \{b\}^\perp \cap \operatorname{dom} A.$$

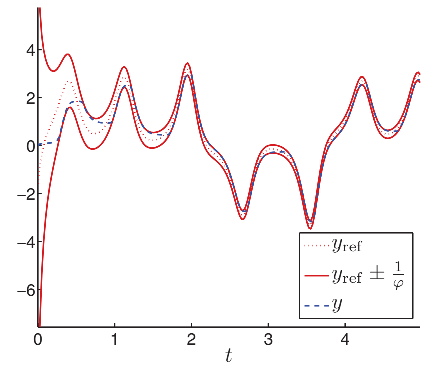
So the Byrnes–Isidori form reads

$$\widehat{A} = \begin{bmatrix} P_0 & S \\ R & Q \end{bmatrix} = \begin{bmatrix} 0 & S \\ 0 & A|_{\{b\}^\perp \cap \operatorname{dom} A} \end{bmatrix}, \qquad \widehat{b} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \quad \widehat{c} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

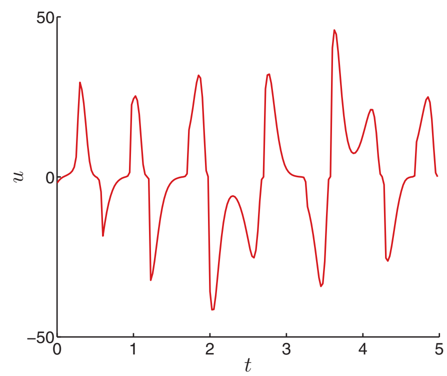
By Theorem 2.6, the original system (A, b, c) is similar to $(\widehat{A}, \widehat{b}, \widehat{c})$. The latter has a more revealing structure. For example, the fact that R is zero shows that the second component of its infinite-dimensional state space $\mathbb{R} \times H_{A,b}$ is completely unaffected by inputs. Due to Proposition 3.4, we also obtain information about the zero dynamics of (A, b, c) from the Byrnes–Isidori form: By Proposition 3.4, the system (A, b, c) has exponentially stable zero dynamics if and only if Q generates an exponentially stable semigroup in $H_{A,b}$. The latter is true because the operator $A|_{H_{A,b} \cap \operatorname{dom} A}$ has the eigenvalues $\{-n^2\pi^2 \,|\, n = 1, 2, \dots\}$ with corresponding eigenvectors $\{\phi_n \,|\, n = 1, 2, \dots\}$ and hence, by [19, Examples 3.3.3 and 3.3.5], it generates a semigroup with growth bound $-\pi^2$.



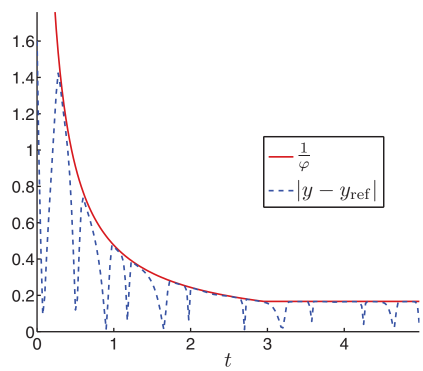
(a) Solution of (6.1) with feedback (5.1).



(b) The output trajectory (dashed) and the funnel (red) around y_{ref} (dotted).



(c) The input signal u produced by the controller (5.1).



(d) The error $e = |y - y_{\mathrm{ref}}|$.

FIG. 3. *Funnel controller* (5.1) *applied to the heat equation* (6.1) *to track a chaotic reference signal* $y_{\mathrm{ref}}(\cdot) := \zeta_1$ *from the Lorenz equation* (6.2) *within the funnel specified by* $\varphi(t) = \min\{2t, 6\}$.

Therefore, the system (6.1) fulfills the prerequisites of Theorem 5.2 which implies the following: The original system (A, b, c) controlled by the funnel controller (5.1) has a global solution whose state trajectory is bounded and whose output trajectory evolves strictly within the prescribed funnel.

We illustrate this by a numerical simulation. The class $W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ of reference signals consists of bounded signals with essentially bounded derivative, or in other words, bounded functions that are uniformly Lipschitz. For our simulation, we have chosen a rather vivid signal: the first component of the chaotic Lorenz system

$$
\begin{aligned}
\dot{\zeta}_1 &= 10\,(\zeta_2 - \zeta_1), && \zeta_1(0) = -8\,,\\
\dot{\zeta}_2 &= 28\,\zeta_1 - \zeta_2 - \zeta_1\zeta_3, && \zeta_2(0) = 8\,,\\
\dot{\zeta}_3 &= \zeta_1\zeta_2 - \tfrac{8}{3}\,\zeta_3\,, && \zeta_3(0) = 27\,;
\end{aligned}
\tag{6.2}
$$

see [10, Ex. 3.1.27]. The signal $y_{\mathrm{ref}}(\cdot) := \zeta_1$ is indeed of class $W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ since the unique solution of (6.2) is bounded and has a bounded derivative as by [10, Ex. 3.2.33] solutions are attracted to a compact set. The function determining the funnel boundary was chosen to be $\varphi(t) = \min\{2t, 6\}$. In Figure 3 the numerical solution of the partial differential equation (6.1) controlled by the funnel controller (5.1) and initial value

$$x^0(\xi) = \begin{cases} 5, & \xi \in [0, 1/2], \\ -5, & \xi \in (1/2, 1], \end{cases}$$

is depicted. It can be seen in Figure 3 that the output trajectory y evolves strictly within the funnel, indicated by solid red lines, around the reference trajectory y_{ref}, and that the input is large only if the output is close to the reference signal.

Remark 6.1. If we switch from Neumann to Dirichlet boundary conditions in this example, i.e., we demand $x(0,t) = x(1,t) = 0$ in (6.1) and $f(0) = f(1) = 0$ in $\operatorname{dom} A$, then the vectors b and c no longer belong to $\operatorname{dom} A$ and so assumption (A2) is violated. Moreover, assumptions (A1)–(A3) are necessary to obtain the Byrnes–Isidori form from Definition 2.1.

A. Proofs of Theorem 2.6 and Proposition 2.14. We first collect three technical lemmas which are essential for the proof of Theorem 2.6.

LEMMA A.1. *Let $(A, b, c) \in \Sigma_r$. Then*

$$A^*\eta = (P^0 A^*\eta)c + P_{A,b}A^*\eta \in \operatorname{ls}\{c\} \dotplus H_{A,b} \qquad \forall\, \eta \in H_{A,b} \cap \operatorname{dom} A^*. \tag{A.1}$$

Proof. If $r = 1$, then (A.1) follows immediately from (2.2) and (2.7). Assume $r > 1$ and let $\eta \in H_{A,b} \cap \operatorname{dom} A^*$. Then (2.2) and (2.7) yield

$$\begin{aligned} &A^*\eta = \alpha_0 c + \alpha_1 A^* c + \cdots + \alpha_{r-1} {A^*}^{r-1} c + P_{A,b}A^*\eta \quad \text{with} \\ &\alpha_i = P^i A^*\eta \in \mathbb{R} \text{ and } P_{A,b}A^*\eta \in H_{A,b}, \end{aligned}$$

and thus

$$0 \overset{(2.2)}{=} (\eta, Ab) = (A^*\eta, b) \overset{(A3)}{=} \alpha_{r-1}({A^*}^{r-1}c, b)$$

and (A3) yields $\alpha_{r-1} = 0$. Next,

$$0 \overset{(2.2)}{=} (\eta, A^2 b) = (A^*\eta, Ab) \overset{(A3)}{=} \alpha_{r-2}({A^*}^{r-2}c, Ab) = \alpha_{r-2}({A^*}^{r-1}c, b)$$

and (A3) yields $\alpha_{r-2} = 0$. Proceeding in this way, we conclude

$$0 \overset{(2.2)}{=} (\eta, A^{r-1}b) = (A^*\eta, A^{r-2}b) \overset{(A3)}{=} \alpha_1(A^*c, A^{r-2}b) = \alpha_1({A^*}^{r-1}c, b)$$

and arrive at $0 = \alpha_{r-1} = \cdots = \alpha_1$. This proves the lemma. ∎

LEMMA A.2. *Let $(A, b, c) \in \Sigma_r$. Then, for any $m = 0, \ldots, r-1$, the operator $P^m A^*$ is closable and densely defined, and its closure is the bounded operator*

(A.2)

$$\begin{aligned} \overline{P^m A^*} : H \to \mathbb{R}, \quad x \mapsto\ & \frac{(x, A^{r-m}b)}{(c, A^{r-1}b)} \\ & - \sum_{j=m+2}^{r} \left(P^m_{m+1} {A^*}^{j-1} c - \sum_{k=m+2}^{j-1} P^m_k {A^*}^{j-1} c \right) \frac{(x, A^{r+1-j}b)}{(c, A^{r-1}b)}. \end{aligned}$$

The operator $P_{A,b}A^*|_{H_{A,b}}$ with domain $H_{A,b} \cap \operatorname{dom} A^*$ is a closed and densely defined operator in $H_{A,b}$ and satisfies

(A.3)

$$P_{A,b}A^*\eta = A^*\eta - \left(\overline{P^0 A^*}\eta\right)c = A^*\eta - \left(P^0 A^*\eta\right)c \qquad \forall\, \eta \in H_{A,b} \cap \operatorname{dom} A^*.$$

Proof. Obviously, for $x \in \operatorname{dom} A^*$ the mapping defined in (A.2) coincides with $P^m A^* x$; see Lemma 2.3. Since b belongs to $\operatorname{dom} A^r$, the right-hand side of (A.2) is also defined for arbitrary $x \in H$, hence $P^m A^*$ is closable and its closure $\overline{P^m A^*}$ is given by (A.2).

Statement (A.3) follows from (A.1) and the fact that $P^0 A^* \eta = \overline{P^0 A^*}\eta$ for $\eta \in \operatorname{dom} A^*$. Since A^* is closed and densely defined in H and $\overline{P^0 A^*}$ is a bounded operator by (A.2), $P_{A,b} A^*|_{H_{A,b}}$ is by (A.3) a closed and densely defined operator in $H_{A,b}$. This completes the proof of the lemma. ∎

LEMMA A.3. *Let* $(A, b, c) \in \Sigma_r$ *and let* P^m *be as in Lemma* 2.3. *Assume* P_i, R, *and* S *are given by* (2.14)–(2.16). *Then*

$$R = \left(P^0 A^*\right)^* = (\overline{P^0 A^*})^*, \quad R^* = \overline{P^0 A^*}, \tag{A.4}$$

$$S^* \alpha = \alpha P_{A,b} {A^*}^r c \quad \forall \alpha \in \mathbb{R}\,. \tag{A.5}$$

Let Q *be defined by* (2.17), *then* Q *is a densely defined closed operator in* $H_{A,b}$ *and satisfies*

(A.6)

$$Q = \left(P_{A,b} A^*|_{H_{A,b}}\right)^*, \qquad Q^* = P_{A,b} A^*|_{H_{A,b}} \quad \textit{with} \quad \operatorname{dom} Q^* = H_{A,b} \cap \operatorname{dom} A^*.$$

Proof. The operator $\overline{P^0 A^*}$ is bounded; see Lemma A.2. Hence $(P^0 A^*)^* = (\overline{P^0 A^*})^*$. If $P^\perp$ denotes the orthogonal projector onto $H_{A,b}$ in H, we have, for all $\eta \in H_{A,b}$ and all $\alpha \in \mathbb{R}$,

$$\left(\overline{P^0 A^*}\eta, \alpha\right)_{\mathbb{R}} \overset{(A.2),(2.2)}{=} \left(\frac{(\eta, A^r b)_H}{(c, A^{r-1} b)}, \alpha\right)_{\mathbb{R}} = \left(\eta, \alpha \frac{P^\perp A^r b}{(c, A^{r-1} b)}\right)_{H_{A,b}} \overset{(2.15)}{=} (\eta, R\alpha)_{H_{A,b}}\ .$$

This proves (A.4). Equation (A.5) follows immediately from the definition (2.16).

It remains to show (A.6). We have, for arbitrary $\eta \in H_{A,b} \cap \operatorname{dom} A^*$ and $\xi \in H_{A,b} \cap \operatorname{dom} A$,

$$\begin{aligned}(P_{A,b} A^* \eta, \xi)_{H_{A,b}} &\overset{(A.3)}{=} (A^*\eta, \xi)_H - \left((\overline{P^0 A^*}\eta) c, \xi\right)_H = (\eta, A\xi)_H - \left(\overline{P^0 A^*}\eta, 1\right)_{\mathbb{R}} (c, \xi)_H \\ &\overset{(A.4)}{=} \left(\eta, P^\perp A\xi - R(c,\xi)_H\right)_{H_{A,b}} \overset{(2.17)}{=} (\eta, Q\xi)_{H_{A,b}}\ .\end{aligned}$$

Hence, $\operatorname{dom} Q \subset \operatorname{dom}(P_{A,b} A^*|_{H_{A,b}})^*$ and $Qx = (P_{A,b} A^*|_{H_{A,b}})^* x$ for all $x \in \operatorname{dom} Q$. Thus it remains to show that $\operatorname{dom}(P_{A,b} A^*|_{H_{A,b}})^* \subset \operatorname{dom} Q$. Since $\operatorname{dom} Q = H_{A,b} \cap \operatorname{dom} A$ it is sufficient to show $\operatorname{dom}(P_{A,b} A^*|_{H_{A,b}})^* \subset \operatorname{dom} A$. Let $\xi \in \operatorname{dom}(P_{A,b} A^*|_{H_{A,b}})^*$ and $x \in \operatorname{dom} A^*$. We write x in the form (2.8) and observe that, by assumption (A2), $\eta := P_{A,b} x \in \operatorname{dom} A^*$. We compute

$$\begin{aligned}(A^* x, \xi) &\overset{(2.8)}{=} (A^*\eta, \xi) + \sum_{j=0}^{r-1} \left(P^j x\right) \left({A^*}^{j+1} c, \xi\right) \\ &\overset{(A.1)}{=} \left((P^0 A^* \eta) c, \xi\right) + (P_{A,b} A^* \eta, \xi) + \sum_{j=0}^{r-1} \left(P^j x\right) \left({A^*}^{j+1} c, \xi\right) \\ &\overset{(A.4)}{=} (\eta, R(c,\xi)_H)_{H_{A,b}} + \left(\eta, (P_{A,b} A^*|_{H_{A,b}})^* \xi\right) + \sum_{j=0}^{r-1} \left(P^j x\right) \left({A^*}^{j+1} c, \xi\right).\end{aligned} \tag{A.7}$$

The mapping which maps $x \in \operatorname{dom} A^*$ to the right-hand side of (A.7) is continuous with respect to the norm of H, and thus $\xi \in \operatorname{dom}(A^*)^* = \operatorname{dom} A$. So the first equation in (A.6) holds and it implies that Q is closed. The second equation in (A.6) follows immediately from the first one and therefore the proof of the lemma is complete. $\square$

Proof of Theorem 2.6. We claim that $(\widehat{A}, \widehat{b}, \widehat{c})$ fulfills Definition 2.1 with $V = H_{A,b}$. For the proof we proceed in several steps.

Step 1: We show that $\widehat{b}$ and $\widehat{c}$ satisfy Definition 2.1(ii)(b). Applying U^{-*} to $b \in H_{A,b}^{\perp}$ yields, with the representation (2.11) and (A3), the equality $\widehat{b} = U^{-*}b = (0, \ldots, 0, (A^{r-1}b, c), 0)^\top$. Equations (2.5), (2.7), and (2.9) show that $\widehat{c} = Uc = (1, 0, \ldots, 0)^\top$.

Step 2: For $\widehat{A}$ we show that if $\widehat{A}$ is defined by (2.13), (2.18) with entries (2.14)–(2.17), then it satisfies

$$\operatorname{dom} \widehat{A} = U^{-*} \operatorname{dom} A \qquad \text{and} \qquad \widehat{A} = U^{-*} A U^*. \tag{A.8}$$

Since the entries of $\widehat{A}$ defined by (2.14)–(2.16) are bounded, the domain of the adjoint of $\widehat{A}$ is given by $\mathbb{R}^r \times \operatorname{dom} Q^*$ and Lemma A.3 yields

$$\operatorname{dom} \widehat{A}^* = \mathbb{R}^r \times (H_{A,b} \cap \operatorname{dom} A^*)\,. \tag{A.9}$$

Moreover, by Lemma A.3, the operator $\widehat{A}^*$ is given, with respect to the decomposition $\mathbb{R}^r \times H_{A,b}$, by

$$\begin{aligned} &\widehat{A}^* \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{pmatrix} \\ &= \begin{bmatrix} 0 & 0 & \cdots & 0 & P^0 {A^*}^r c & P^0 A^* \\ 1 & 0 & \cdots & 0 & P^1 {A^*}^r c & 0 \\ 0 & 1 & \ddots & \vdots & \vdots & 0 \\ \vdots & & \ddots & 0 & P^{r-2} {A^*}^r c & 0 \\ 0 & 0 & & 1 & P^{r-1} {A^*}^r c & 0 \\ 0 & 0 & \cdots & 0 & P_{A,b} {A^*}^r c & P_{A,b} A^* \end{bmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{pmatrix} \qquad \begin{array}{l} \forall\, \alpha_0, \ldots, \alpha_{r-1} \in \mathbb{R}, \\ \forall\, \eta \in H_{A,b} \cap \operatorname{dom} A^*\,. \end{array} \end{aligned} \tag{A.10}$$

Next, we show

$$\operatorname{dom} A^* = U^{-1} \operatorname{dom} \widehat{A}^* \qquad \text{and} \qquad U A^* x = \widehat{A}^* U x \qquad \forall\, x \in \operatorname{dom} A^*\,. \tag{A.11}$$

Let $(\alpha_0, \ldots, \alpha_{r-1}, \eta)^\top \in \operatorname{dom} \widehat{A}^*$. Then $\eta \in H_{A,b} \cap \operatorname{dom} A^*$, which, together with (A2), shows

$$U^{-1}(\alpha_0, \ldots, \alpha_{r-1}, \eta)^\top = \sum_{i=0}^{r-1} \alpha_i {A^*}^i c + \eta \in \operatorname{dom} A^*.$$

Conversely, fix $x \in \operatorname{dom} A^*$. Then

$$x \overset{(2.8)}{=} (P^0 x) c + (P^1 x) A^* c + \cdots + (P^{r-1} x) {A^*}^{r-1} c + P_{A,b} x\,,$$

and (A2) yields

$$\eta := P_{A,b}x \overset{(2.8)}{=} x - \sum_{j=0}^{r-1}(P^j x){A^*}^j c \in H_{A,b} \cap \operatorname{dom} A^*,$$

which implies, in view of the definition of U and (A.9), that $Ux \in \operatorname{dom} \widehat{A}^*$. Hence the first equality in (A.11) follows.

The decomposition (2.8) applied to the vector ${A^*}^r c$ gives

$$(A.12) \qquad {A^*}^r c = \left(\sum_{j=0}^{r-1}(P^j {A^*}^r c){A^*}^j c + P_{A,b}{A^*}^r c \right)$$

and we conclude

$$\begin{aligned}
UA^*x &\overset{(2.8)}{=} U\left(A^*\eta + \sum_{j=0}^{r-1}(P^j x){A^*}^{j+1} c \right) \\
&\overset{(A.1)}{=} U\left((P^0 A^*\eta)c + P_{A,b}A^*\eta + \sum_{j=0}^{r-2}(P^j x){A^*}^{j+1} c + (P^{r-1}x){A^*}^r c \right) \\
&\overset{(A.12)}{=} U\Bigg((P^0 A^*\eta)c + P_{A,b}A^*\eta + \sum_{j=0}^{r-2}(P^j x){A^*}^{j+1} c \\
&\qquad + P^{r-1}x \left(\sum_{j=0}^{r-1}(P^j {A^*}^r c){A^*}^j c + P_{A,b}{A^*}^r c \right) \Bigg) \\
&\overset{(2.4),(2.5)}{=} \begin{pmatrix} P^0 {A^*}^r c(P^{r-1}x) + P^0 A^*\eta \\ P^0 x + P^1 {A^*}^r c(P^{r-1}x) \\ P^1 x + P^2 {A^*}^r c(P^{r-1}x) \\ \vdots \\ P^{r-2}x + P^{r-1}{A^*}^r c(P^{r-1}x) \\ P_{A,b}{A^*}^r c(P^{r-1}x) + P_{A,b}A^*\eta \end{pmatrix} \\
&= \begin{bmatrix} 0 & 0 & \cdots & 0 & P^0 {A^*}^r c & P^0 A^* \\ 1 & 0 & \cdots & 0 & P^1 {A^*}^r c & 0 \\ 0 & 1 & \ddots & \vdots & \vdots & 0 \\ \vdots & & \ddots & 0 & P^{r-2}{A^*}^r c & 0 \\ 0 & 0 & & 1 & P^{r-1}{A^*}^r c & 0 \\ 0 & 0 & \cdots & 0 & P_{A,b}{A^*}^r c & P_{A,b}A^* \end{bmatrix} \begin{pmatrix} P^0 x \\ P^1 x \\ \vdots \\ P^{r-1}x \\ \eta \end{pmatrix} \\
&= \widehat{A}^* U x\,.
\end{aligned}$$

This proves (A.10) and the remaining part of (A.11).

Recall that $(XT)^* = T^*X^*$ for any densely defined operator T and bounded operator X and, if in addition X is boundedly invertible, we have $(TX)^* = X^*T^*$;

see, e.g., [21, section 4.4]. Hence, (A.8) follows from (A.11). This completes the proof of (A.8). Since U^* is a boundedly invertible transformation and $(A,b,c) \in \Sigma_r$ we have $(\widehat{A},\widehat{b},\widehat{c}) \in \Sigma_r$.

Step 3: It remains to show that Q generates a semigroup, i.e., it fulfills Definition 2.1 (i). It is clear that $\widehat{A}$ generates a semigroup on $\mathbb{R}^r \times H_{A,b}$. In the block operator notation (2.19) for $\widehat{A}$ the operators $\widehat{A}_{11}$, $\widehat{A}_{12}$, and $\widehat{A}_{21}$ are bounded. So

$$\operatorname{diag}(0,Q) := \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} = \widehat{A} - \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ \widehat{A}_{21} & 0 \end{bmatrix}$$

is a bounded perturbation of $\widehat{A}$ and, in view of [8, section III.1.3], it is a semigroup generator whose domain equals $\operatorname{dom}\widehat{A}$. Obviously $\{0\} \times H_{A,b}$ is a closed, $\operatorname{diag}(0,Q)$-invariant subspace of $\mathbb{R}^r \times H_{A,b}$. Since the spectrum of Q is equal to the spectrum of $\operatorname{diag}(0,Q)$ up to the value 0, the condition (iv) of [19, Theorem 3.14.4] is satisfied. This theorem implies that $\operatorname{diag}(0,Q)|_{\{0\}\times H_{A,b}}$ with domain

$$\operatorname{dom}\widehat{A} \cap (\{0\} \times H_{A,b}) = \{0\} \times (H_{A,b} \cap \operatorname{dom} A),$$

generates a strongly continuous semigroup on $\{0\} \times H_{A,b}$. Now the identification of $H_{A,b}$ with $\{0\} \times H_{A,b}$ and Q with $\operatorname{diag}(0,Q)|_{\{0\}\times H_{A,b}}$ completes the proof. ∎

Proof of Proposition 2.14. We mimic the proof for finite-dimensional time-varying linear systems in [1]. Consider the bounded bijective linear operator

$$Y : \mathbb{R}^r \times H_{A,b} \to \mathbb{R}^r \times \widetilde{V}, \qquad Y := W^{-*}U^* \tag{A.13}$$

which admits a representation with respect to $\mathbb{R}^r \times H_{A,b}$ and $\mathbb{R}^r \times \widetilde{V}$ of the form

$$Y = \begin{bmatrix} Y_{00} & Y_{01} & \cdots & Y_{0r} \\ Y_{10} & Y_{11} & \cdots & Y_{1r} \\ \vdots & \vdots & & \vdots \\ Y_{r0} & Y_{r1} & \cdots & Y_{rr} \end{bmatrix} \quad \text{with bounded} \quad \begin{array}{ll} Y_{ij} : \mathbb{R} \to \mathbb{R}, & i,j \in \{0,\dots,r-1\}, \\ Y_{ir} : H_{A,b} \to \mathbb{R}, & i \in \{0,\dots,r-1\}, \\ Y_{rj} : \mathbb{R} \to \widetilde{V}, & j \in \{0,\dots,r-1\}, \\ Y_{rr} : H_{A,b} \to \widetilde{V}, & \end{array}$$

and write

$$(\widetilde{A}\,, \widetilde{b}\,, \widetilde{c}) := \left(W^{-*}AW^*, W^{-*}b, Wc\right). \tag{A.14}$$

Then we have, with $\widehat{A}$ from Theorem 2.6,

$$Y\widehat{A}Y^{-1} = W^{-*}U^*\widehat{A}U^{-*}W^* \overset{\text{(A.11)}}{=} W^{-*}AW^* \overset{\text{(A.14)}}{=} \widetilde{A}. \tag{A.15}$$

Simply applying $\widetilde{A}$ from (2.39) $(r-1)$-times to $\widetilde{b}$ yields

$$(A^{r-1}b,c) = (W^{-*}A^{r-1}W^*W^{-*}b, Wc) = (\widetilde{A}^{r-1}\widetilde{b},\widetilde{c}) = b_r.$$

Hence, we have shown (iii) and

$$Y\begin{pmatrix} 0 \\ \vdots \\ 0 \\ (A^{r-1}b,c) \\ 0 \end{pmatrix} \overset{(2.13)}{=} W^{-*}b \overset{(2.39)}{=} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (A^{r-1}b,c) \\ 0 \end{pmatrix}. \tag{A.16}$$

We calculate

$$\begin{pmatrix} Y_{00}^* \\ \vdots \\ Y_{0,r-1}^* \\ Y_{0r}^* \end{pmatrix} = Y^* \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \overset{(2.39)}{=} Y^* W c \overset{(A.13)}{=} Uc \overset{(2.13)}{=} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

This, together with (A.16), gives

$$Y = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ Y_{10} & Y_{11} & \cdots & Y_{1,r-2} & 0 & Y_{1r} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ Y_{r-2,0} & Y_{r-2,1} & \cdots & Y_{r-2,r-2} & 0 & Y_{r-2,r} \\ Y_{r-1,0} & Y_{r-1,1} & \cdots & Y_{r-1,r-2} & 1 & Y_{r-1,r} \\ Y_{r0} & Y_{r1} & \cdots & Y_{r,r-2} & 0 & Y_{rr} \end{bmatrix} \tag{A.17}$$

and

$$\begin{aligned} [0,1,0,\dots,0] &\overset{(2.1)}{=} [1,0,\dots,0]\widehat{A} \overset{(A.17)}{=} [1,0,\dots,0]Y\widehat{A} \\ &\overset{(A.15)}{=} [1,0,\dots,0]\widetilde{A}Y \overset{(2.39)}{=} [0,1,0,\dots,0]Y \\ &\overset{(A.13)}{=} [Y_{10},\cdots,Y_{1,r-2},0,Y_{1,r}], \\ [0,0,1,0,\dots,0] &\overset{(2.1)}{=} [0,1,0,\dots,0]\widehat{A} \overset{(A.18)}{=} [0,1,0,\dots,0]Y\widehat{A} \\ &\overset{(A.15)}{=} [0,1,0,\dots,0]\widetilde{A}Y \overset{(2.39)}{=} [0,0,1,0,\dots,0]Y \\ &\overset{(A.13)}{=} [Y_{20},\dots,Y_{2,r-2},0,Y_{2r}]. \end{aligned} \tag{A.18}$$

We proceed by calculating the first r rows of Y in this way until

$$\begin{aligned} [0,\dots,0,1,0] &\overset{(2.1)}{=} [0,\dots,0,1,0,0]\widehat{A} = [0,\dots,0,1,0,0]Y\widehat{A} \\ &\overset{(A.15)}{=} [0,\dots,0,1,0,0]\widetilde{A}Y \overset{(2.39)}{=} [0,\dots,0,0,1,0]Y \\ &\overset{(A.13)}{=} [Y_{r-1,0},\dots,Y_{r-1,r-2},1,Y_{r-1,r}], \end{aligned}$$

and arrive at

$$Y = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ Y_{r,0} & \dots & Y_{r,r-2} & 0 & Y_{rr}. \end{bmatrix}. \tag{A.19}$$

The operator Y is bounded and bijective. Therefore,

$$\widetilde{Y} : H_{A,b} \to \widetilde{V}, \qquad \widetilde{Y} := Y_{rr}$$

is a bounded, bijective operator. This, together with (A.15), already shows that

$\widetilde{S} = S\widetilde{Y}^{-1}$. Now the special structure of $\widehat{A}$, $\widetilde{A}$, Y in (2.1), (2.39), (A.19), resp., yield

$$
\begin{aligned}
&[\widetilde{Y}R, Y_{r0}, \dots, Y_{r,r-2}, \widetilde{Y}Q] \\
&= \begin{bmatrix} 0, & \dots, & 0, & I \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ Y_{r0} & \dots & Y_{r,r-2} & 0 & \widetilde{Y} \end{bmatrix} \\
&\qquad \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & 1 & 0 & 0 \\ 0 & 0 & & 0 & 1 & 0 \\ P^0 A^{*^r} c & P^1 A^{*^r} c & \cdots & P^{r-2} A^{*^r} c & P^{r-1} A^{*^r} c & S \\ R & 0 & \cdots & 0 & 0 & Q \end{bmatrix} \\
&= \begin{bmatrix} 0, & \dots, & 0, & I \end{bmatrix} Y \widehat{A} \\
&\overset{(A.15)}{=} \begin{bmatrix} 0, & \dots, & 0, & I \end{bmatrix} \widetilde{A} Y \\
&= \begin{bmatrix} 0, & \dots, & 0, & I \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ \widetilde{P}_0 & \widetilde{P}_1 & \cdots & \widetilde{P}_{r-2} & \widetilde{P}_{r-1} & \widetilde{S} \\ \widetilde{R} & 0 & \cdots & 0 & 0 & \widetilde{Q} \end{bmatrix} \\
&\qquad \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ Y_{r0} & \dots & Y_{r,r-2} & 0 & \widetilde{Y} \end{bmatrix} \\
&= [\widetilde{R} + \widetilde{Q}Y_{r0}, \widetilde{Q}Y_{r1}, \dots, \widetilde{Q}Y_{r,r-2},\ 0\ , \widetilde{Q}\widetilde{Y}] \,.
\end{aligned}
$$

By successively comparing the blocks in order from $(r-2)$th to first and by finally considering the last entry, we see that

$$
Y_{r,r-2} = 0 = \cdots = Y_{r0} = 0 \qquad \text{and} \quad \widetilde{R} = \widetilde{Y}R, \quad \widetilde{Q} = \widetilde{Y}Q\,\widetilde{Y}^{-1}\,.
$$

Finally,

$$
Y = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & \widetilde{Y} \end{bmatrix}
$$

and (A.15) give (ii). This completes the proof. $\square$

Acknowledgments. It is our pleasure to thank Birgit Jacob (Wuppertal, Germany) and Kirsten Morris (Waterloo, Canada) for helpful discussions and the referees for their constructive comments.

REFERENCES

[1] T. Berger, A. Ilchmann, and F. Wirth, *Zero dynamics and stabilization for analytic linear systems*, Acta Appl. Math., 138 (2015), pp. 17–57.

[2] C. I. Byrnes, *Adaptive stabilization of infinite dimensional linear systems*, in Proceedings of the 27th IEEE Conference on Decision and Control, IEEE, Piscataway, NJ, 1987, pp. 1435–1440.

[3] C. I. Byrnes, I. G. Lauko, D. S. Gilliam, and V. I. Shubov, *Zero dynamics for relative degree one SISO distributed parameter systems*, in Proceedings of the 37th IEEE Conference on Decision and Control, IEEE, Piscataway, NJ, 1998, pp. 2390–2391.

[4] C. I. Byrnes and J. C. Willems, *Adaptive stabilization of multivariable linear systems*, in Proceedings of the 23rd IEEE Conference on Decision and Control, IEEE, Piscataway, NJ, 1984, pp. 1574–1577.

[5] R. F. Curtain, *(C, A, B)–pairs in infinite dimensions*, Systems Control Lett., 5 (1984), pp. 59–65.

[6] R. F. Curtain, *Invariance concepts in infinite dimensions*, SIAM J. Control Optim., 24 (1986), pp. 1009–1030.

[7] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer-Verlag, New York, 1995.

[8] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, New York, 2000.

[9] E. Hille and R. S. Phillips, *Functional Analysis and Semigroups*, AMS, Providence, RI, 1957.

[10] D. Hinrichsen and A. J. Pritchard, *Mathematical Systems Theory* I. *Modelling, State Space Analysis, Stability and Robustness*, Springer-Verlag, Berlin, 2005.

[11] A. Ilchmann, E. P. Ryan, and C. J. Sangwin, *Tracking with prescribed transient behaviour*, ESAIM: Control, Optim. Calc. Var., 7 (2002), pp. 471–493.

[12] A. Ilchmann, E. P. Ryan, and P. Townsend, *Tracking with prescribed transient behavior for nonlinear systems of known relative degree*, SIAM J. Control Optim., 46 (2007), pp. 210–230.

[13] A. Isidori, *Nonlinear Control Systems*, 3rd ed., Springer-Verlag, Berlin, 1995.

[14] H. Logemann and D. H. Owens, *Robust high-gain feedback control of infinite-dimensional minimum-phase systems*, IMA J. Math. Control & Information, 4 (1987), pp. 195–220.

[15] H. Logemann and S. B. Townley, *Adaptive control of infinite-dimensional systems without parameter estimation: An overview*, IMA J. Math. Control Inform., 14 (1997), pp. 175–206.

[16] H. Logemann and H. J. Zwart, *Some remarks on adaptive stabilization of infinite-dimensional systems*, Systems Control Lett., 16 (1991), pp. 199–207.

[17] K. A. Morris and R. Rebarber, *Feedback invariance of SISO infinite-dimensional systems*, Math. Control Signals Systems, 19 (2007), pp. 313–335.

[18] K. A. Morris and R. Rebarber, *Invariant zeros of SISO infinite-dimensional systems*, Internat. J. Control, 83 (2010), pp. 2573–2579.

[19] O. Staffans, *Well-Posed Linear Systems*, Cambridge University Press, Cambridge, 2005.

[20] C. Tretter, *Spectral Theory of Block Operator Matrices and Applications*, Imperial College Press, London, 2008.

[21] J. Weidmann, *Linear Operators in Hilbert Spaces*, Springer-Verlag, New York, 1985.

[22] H. Zwart, *Equivalence between open-loop and closed-loop invariance for infinite-dimensional systems: A frequency domain approach*, SIAM J. Control Optim., 26 (1988), pp. 1175–1199.