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Optimal control of differential-algebraic equations from an ordinary differential equation perspective

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Summary

We study the optimal control problem (OCP) for regular linear differential-algebraic systems. To this end, we introduce the input index, which allows, on the one hand, to characterize the space of consistent initial values in terms of a Kalman-like matrix and, on the other hand, the necessary smoothness properties of the control. The latter is essential to make the problem accessible from a numerical point of view. Moreover, we derive an augmented system as the key to analyze the OCP with tools well known from optimal control of ordinary differential equations. The new concepts of the input index and the augmented system provide easily checkable sufficient conditions, which ensure that the stage costs are consistent with the differential-algebraic system.

KEYWORDS

consistent initial values, differential-algebraic equation, feedback control, optimal control

1 | INTRODUCTION

We consider time-invariant, single-input differential-algebraic systems (DAEs) described, for $[E, A, b] \in \mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n}\times\mathbb{R}^n$ and $x^0 \in \mathbb{R}^n$, by

$$\boxed{\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + bu(t), \quad (Ex)(0) = x^0,} \tag{DAE}$$

in *quasi-Weierstraß* form, ie,

$$E = \begin{bmatrix} I_{n_J} & 0 \\ 0 & N \end{bmatrix}, \quad A = \begin{bmatrix} J & 0 \\ 0 & I_{n_N} \end{bmatrix}, \quad b = \begin{pmatrix} b_J \\ b_N \end{pmatrix},$$

with $J \in \mathbb{R}^{n_J\times n_J}$, $b_J \in \mathbb{R}^{n_J}$, $b_N \in \mathbb{R}^{n_N}$, and nilpotent $N \in \mathbb{R}^{n_N\times n_N}$. It is well known (see theorem 2.6 in the work of Berger et al[1]) that every *regular* DAE system, ie, $\det(sE - A) \neq 0_{\mathbb{R}[s]}$, can be transformed into quasi-Weierstraß form and that the dimension $n_N \in \mathbb{N}$ of the algebraic part is unique, whereas the matrices J and N are unique up to similarity. We assume $n_N \geq 1$, otherwise (DAE) is an ordinary differential equation. For convenience, $x = (x_J, x_N)$ is partitioned according to the structure of the quasi-Weierstraß form.

The tuple $(x, u) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n \times \mathbb{R})$ is called a *solution* of (DAE) if and only if $Ex \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ and u satisfy (DAE) for almost all $t \geq 0$. An initial value $x^0 \in \mathbb{R}^n$ is called *consistent* if and only if a solution (x, u) exists with $(Ex)(0) = x^0$; the space of *consistent initial values* is denoted by $\mathcal{X}$.

Essential for our analysis will be the *input index* ω of (DAE) defined by

$$\omega := \begin{cases} \max\{i \in \mathbb{N} \,|\, N^i b_N \neq 0\} & \text{if } N b_N \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The input index is uniquely determined since N and b_N in the quasi-Weierstraß form are unique up to similarity; $\omega = 0$ holds if $b_N = 0$ or $Nb_N = 0$. As will be seen later, the input index is the number of times a function u must be differentiable almost everywhere to be a candidate for a solution (x, u).

The space $\mathcal{X}$ of consistent initial values will be characterized in terms of a Kalman-like matrix and it will be shown that $\dim \mathcal{X} = n_J + \omega$. This sets us in a position to define an *augmented system* of ordinary differential equations, which is equivalent to the (DAE).

Consider, for given weighting matrix $S = S^\top \in \mathbb{R}^{(n+1)\times(n+1)}$ and consistent initial value x^0, the *optimal control problem* (OCP) over time horizon $[0, T)$, $T \in (0, \infty]$

$$\boxed{\text{Minimize} \int_0^T \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^\top S \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t \quad \text{subject to} \quad (x, u) \text{ solves (DAE).}} \tag{OCP}$$

We determine the optimal value of the (OCP) and an optimal feedback law by using the augmented system. This allows to use *classical* ODE results to solve (OCP).

The last aspect, namely, to exploit *classical* ODE results in a DAE setting, is the key motivation of our approach because our long-term goal is the solution of (nonlinear) constrained OCPs on an infinite time horizon. However, constraints, eg, mixed control-state constraints of the form $Fx + gu \leq \mathbb{1}$ with matrix $F \in \mathbb{R}^{p\times n}$ and vector $g \in \mathbb{R}^p$, in combination with an infinite optimization horizon render the problem, in general, computationally intractable.[2] Hence, we want to apply model predictive control (MPC) to approximately solve this problem. In MPC, a sequence of *constrained* OCPs on a finite time horizon $T \in (0, \infty)$ is solved. While the theory is well developed for ODE systems, a rigorous stability analysis for the DAE case is still missing. In addition, existing approaches do not preserve the particular structure of the underlying DAE.[3,4]

Essentially, there are two approaches to guarantee asymptotic stability of the origin w.r.t. the MPC closed loop for ODE-constrained systems. The first is based on stabilizing terminal constraints and costs.[5] Here, using the representation (17) of the optimal control allows to write the constraint as $(F + g\hat{k}_\alpha G^\dagger)x \leq \mathbb{1}$ and to show existence of a weakly control invariant terminal set[6,7] $\{x^0 \in \mathcal{X} \,|\, V_\infty(x^0) \leq \rho\}$, $\rho > 0$. Since the terminal set is a sublevel set of the value function V_∞, V_∞ can be used to construct a suitable terminal cost. In conclusion, if Assumptions (A1) to (A5) hold, recursive feasibility and asymptotic stability can be ensured by combining our results[8] with well-established techniques.[9] A similar approach,[10] again based on the results of this paper, works for MPC without stabilizing terminal constraints and costs.[11,12] Furthermore, we are going to consider nonlinear differential-algebraic systems, whose linearization at the origin is stabilizable in a suitable sense.[1] Again, *standard* techniques for nonlinear ODEs should be applicable to exploit the presented results for a rigorous stability framework as it was done for ODEs.[13,14]

The OCP for DAEs has been studied by various authors. Basically, there exist three different approaches.[15-17] The first, which is also related to the work,[18-20] uses projectors. The second, which is further explicated in the work of Reis and Voigt,[21] uses the KYP inequality and Lur'e equations to characterize basic properties like feasibility and regularity of the (OCP) with a zero terminal constraint at $T = \infty$ by *storage functions* in the sense of the works of Willems and Trentelmann.[22,23] The third approach uses adjoint-based techniques and is applicable to nonlinear systems.[24] However, in none of them the optimal control u is explicitly specified, which is a basic requirement for the numerical treatment of OCPs governed by DAEs and, thus, for the usage of the "*first discretize, then optimize*" approach.[25] Foremost, the main motivation to solve the OCP via the quasi-Weierstraß form is that we consider this approach as very accessible from an ODE point of view. Hence, we can use the developed techniques to transfer well-known methods for ODE systems to DAE-constrained systems as indicated earlier w.r.t. MPC. Similar approaches to ours were considered,[26-28] but they did not explicitly exploit the necessary smoothness of the control and therefore impose too restrictive conditions. We plan to use this methodology as a blueprint to derive similar results based on the more sophisticated optimal control techniques[15-17] in a second step.

Overall, we propose a relatively simple approach to rewrite, under the condition that the DAE is already in quasi-Weierstraß form, (OCP) as an OCP constrained by *ordinary* differential equations. The key ingredients are the *input index* and the *augmented system*.

The structure of this paper is as follows. In Section 2, we exploit the quasi-Weierstraß form and show how much "freedom" in the input function there is and how the DAE is equivalent to an ODE. In Section 3, the OCP for (DAE) is investigated. The optimal value function of the DAE is shown to coincide with an optimal value function of an associated ODE OCP. Finally, in Section 4, it is proved how the optimal control can be expressed as a state feedback for the DAE.

The Appendix consists of two parts. In Part A, classical results on optimal control of ODEs necessary for our results are recalled. In Part B, proofs of the results are given.

Notation. The $(n \times n)$-identity matrix is denoted by I_n. A matrix $S \in \mathbb{R}^{n\times n}$ is called *positive semidefinite*, denoted by $S \geq 0$, if and only if $x^\top Sx \geq 0$ holds for all $x \in \mathbb{R}^n$; $N \in \mathbb{R}^{n\times n}$ is called *nilpotent* if and only if there exists $i \in \mathbb{N}$ such that $N^i = 0$. The sets $\mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ and $\mathcal{W}^{k,1}_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ are the space of locally Lebesgue-integrable functions and the Sobolev space of functions whose derivatives up to the order $k \in \mathbb{N}_0$ exist almost everywhere and are in $\mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$, respectively. The set $\mathcal{AC}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ denotes the space of functions that are absolutely continuous on each compact interval $K \subseteq \mathbb{R}_{\geq 0}$.

2 | DIFFERENTIAL-ALGEBRAIC SYSTEMS

As convenient technical tools, we define, for (DAE) and the input index ω as in (1), the *Kalman-like matrix*

$$K := N\left[\, b_N \;\dots\; N^{\omega-1}b_N \,\right] = \left[\, Nb_N \;\dots\; N^{\omega}b_N \,\right] \in \mathbb{R}^{n_N\times\omega} \quad \text{and} \quad F := \begin{bmatrix} I_{n_J} & 0 \\ 0 & -K \end{bmatrix} \in \mathbb{R}^{n\times(n_J+\omega)} \tag{2}$$

with the convention that K is void if $\omega = 0$.

The Kalman-like matrix helps us to characterize solutions of (DAE). The following proposition (see Appendix B for a proof) is a slight generalization of the solution formulas presented, eg, in lemma 2.8 in the work of Kunkel and Mehrmann[29] or equation (2.3.6) in the work of Brenan et al.[30] Whereas: these classical results a priori assume that input u is smooth, we can explicitly characterize the required differentiability conditions (cf (3a)). Some considerations on the required smoothness have been carried out in remark 3.4(ii) in the work of Berger and Reis[31] for the multi-input case, albeit without interpreting u as an input and thus without specifying the degree of freedom for u.

Proposition 1. *Consider (DAE). Then, (x, u) is a solution to (DAE) if and only if*

$$u(\cdot) \in \mathcal{W}^{\omega,1}_{loc}(\mathbb{R}_{\geq 0}, \mathbb{R}), \tag{3a}$$

$$x_J(t) = \mathrm{e}^{Jt}x_J(0) + \int_0^t \mathrm{e}^{J(t-s)}b_J u(s)\,\mathrm{d}s \qquad \text{for all } t \geq 0, \tag{3b}$$

$$x_N(t) = -\sum_{i=0}^{\omega} N^i b_N u^{(i)}(t) = -[b_N, K]\big(u(t), \dots, u^{(\omega)}(t)\big)^\top \qquad \text{for almost all } t \geq 0. \tag{3c}$$

Proposition 1 implies in particular that not every initial value $x^0 \in \mathbb{R}^n$ is consistent. In the following Proposition 2, a reformulation of the classical results presented in theorem 2.12 in the work of Kunkel and Mehrmann,[29] some properties and a characterization of the space of consistent initial data $\mathcal{X} \subseteq \mathbb{R}^n$ are presented. We give a short proof of i in Appendix B, from which the other assertions follow easily.

Proposition 2. *For (DAE) and the notation as in (2) we have:*

i. The matrix K defined by (2) has full rank, ie, $\operatorname{rank} K = \omega \leq n_N$.
ii. $\mathcal{X} = \operatorname{im} F$.
iii. F has a left inverse $F^\dagger \in \mathbb{R}^{(n_J+\omega)\times n}$.
iv. $\dim \mathcal{X} = n_J + \omega$.

As an immediate consequence of Proposition 1 and Proposition 2, we collect the following corollary.

Corollary 1. *If (x, u) is a solution to (DAE), then*

$$F\begin{pmatrix} x_J(0) \\ u(0) \\ \vdots \\ u^{(\omega-1)}(0) \end{pmatrix} = (Ex)(0) = x^0 \in \mathcal{X}.$$

For ODEs, ie, $n_N = 0$ in (DAE), it follows from Proposition 1, (3b) that, for any initial value $x^0 \in \mathbb{R}^n$ and any input function $u \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R})$, there exists a unique solution (x, u) of (DAE). In other words, the mapping

$$\mathbb{R}^n \times \mathcal{L}^1_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}) \to \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n), \quad t \mapsto x(t) = \mathrm{e}^{Jt}x_J(0) + \int_0^t \mathrm{e}^{J(t-s)}b_J u(s)\,\mathrm{d}s$$

is well defined. Thus, (x^0, u) can be "freely" chosen and determines a unique x so that (x, u) solves the initial value problem (DAE). In case of (DAE) with $n_N \geq 1$, formula (3c) shows that x and u must satisfy an algebraic equation. Hence, x^0 and u cannot be chosen freely. To be precise, we define, for K as in (2), the set

$$\mathcal{F} := \left\{ (x^0, u) \in \mathcal{X} \times \mathcal{W}^{\omega,1}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}) \;\middle|\; F\begin{pmatrix} x_J(0) \\ u(0) \\ \vdots \\ u^{(\omega-1)}(0) \end{pmatrix} = x^0 \right\}$$

and the mapping

$$\varphi : \mathcal{F} \to \mathcal{L}^1_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n), \quad (x^0, u) \mapsto x = \begin{pmatrix} x_J \\ x_N \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{J\cdot}x_J(0) + \int_0^{\cdot} \mathrm{e}^{J(\cdot-s)}b_J u(s)\,\mathrm{d}s \\ -[b_N, K](u(\cdot), \dots, u^{(\omega)}(\cdot))^\top \end{pmatrix}.$$

Then, for any $(x^0, u) \in \mathcal{F}$, the tuple $(\varphi(x^0, u), u)$ solves the initial value problem (DAE).

However, there is some freedom in u left as the following remark shows.

Remark 1. Consider (DAE) and define

$$\mathcal{U}^0 := \left\{ u \in \mathcal{W}^{\omega,1}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}) \,\middle|\, u(0) = \dots = u^{(\omega-1)}(0) = 0 \right\}.$$

Then, for any $(x^0, u) \in \mathcal{X} \times \mathcal{W}^{\omega,1}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R})$, we have

$$(x^0, u) \in \mathcal{F} \iff \forall \widetilde{u} \in \mathcal{U}^0 : (x^0, u + \widetilde{u}) \in \mathcal{F}.$$

Remark 1 shows that, whenever (x, u) is a solution to (DAE), then, for any $\widetilde{u} \in \mathcal{U}^0$, there exists a solution for $(x(0), u + \widetilde{u})$.

Remark 2. If $\omega = 0$, then K is void, F is given by $\begin{bmatrix} I_{n_J} & 0_{n_J \times n_N} \end{bmatrix}^\top$, and Proposition 2 states that $x^0 \in \mathcal{X}$ holds if and only if $x^0_i = 0$ for all $i \in \{n_J + 1, \dots, n\}$. Hence, Proposition 1 shows that (DAE) reduces to an initial value problem for the ordinary differential equation

$$\dot{x}_J(t) = Jx_J(t) + b_J u(t), \tag{4}$$

with initial value $x_J(0) = F^\dagger x^0$ and "output" $x_N(t) = -b_N u(t)$.

For $\omega \geq 1$, the aforementioned findings allow to rewrite (DAE) equivalently as an ordinary differential equation, where lower-order derivatives are introduced as new system states. This concept was first introduced by Müller.[26]

Proposition 3. *Consider the initial value problem (DAE) with $Nb_N \neq 0$, let the input index ω be given by (1) and define*

$$\hat{x} = \left(x_J^\top, u, \dots, u^{(\omega-1)}\right)^\top, \qquad \hat{u} = u^{(\omega)}, \quad \text{and} \quad \hat{n} = n_J + \omega.$$

Then, (x, u) with $(Ex)(0) = x^0 \in \mathcal{X}$ is a solution to (DAE) if and only if $(\hat{x}, \hat{u}, x_N)$ with $\hat{x}(0) = F^\dagger x^0$ is a solution to the augmented system

$$\dot{\hat{x}}(t) = \underbrace{\begin{bmatrix} J & b_J & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix}}_{=:\hat{A} \in \mathbb{R}^{\hat{n} \times \hat{n}}} \hat{x}(t) + \underbrace{\begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}}_{=:\hat{b} \in \mathbb{R}^{\hat{n}}} \hat{u}(t) \tag{5a}$$

$$x_N(t) = -\begin{bmatrix} 0_{n_N\times n_J} & b_N & K \end{bmatrix}\begin{pmatrix} \hat{x}(t) \\ \hat{u}(t) \end{pmatrix}. \tag{5b}$$

Proof. Using Proposition 1 and reading (DAE) and system (5) row-wise, it follows that (x, u) solves (DAE) if and only if $(\hat{x}, \hat{u}, x_N)$ solves (5). The correspondence of the initial values follows from

$$F\hat{x}(0) = \begin{bmatrix} I_{n_J} & 0 \\ 0 & -\begin{bmatrix} Nb_N & \dots & N^{\omega}b_N \end{bmatrix} \end{bmatrix}\begin{pmatrix} x_J(0) \\ u(0) \\ \vdots \\ u^{\omega-1}(0) \end{pmatrix} = \begin{pmatrix} x_J(0) \\ -\sum_{i=1}^{\omega} N^i b_N u^{(i-1)}(0) \end{pmatrix} = \begin{pmatrix} x_J(0) \\ Nx_N(0) \end{pmatrix} = (Ex)(0).$$

□

The advantage of the augmented system (5) is that its input $\hat{u}$, unlike the input of (DAE), can be chosen freely from $\mathcal{L}^1_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R})$. Furthermore, it inherits stabilizability properties of (DAE), which will be useful for the OCP considered in the next section.

Definition 1. (DAE) is called *behaviorally stabilizable* if and only if

$$\forall (x, u) \text{ solution of (DAE)} \, \exists (\tilde{x}, \tilde{u}) \text{ solution of (DAE)} : \; (x, u)|_{(-\infty,0)} \overset{\text{ae}}{=} (\tilde{x}, \tilde{u})|_{(-\infty,0)} \wedge \lim_{t\to\infty} \operatorname{ess\,sup}_{[t,\infty)} \|(\tilde{x}, \tilde{u})\| = 0.$$

The following algebraic characterization of behavioral stability is given in corollary 4.3 in the work of Berger and Reis.[31]

Proposition 4. *(DAE) is behaviorally stabilizable if and only if*

$$\forall \lambda \in \overline{\mathbb{C}}_+ : \operatorname{rank}_{\mathbb{C}}[\lambda E - A, b] = \operatorname{rank}_{\mathbb{R}(s)}[sE - A, b].$$

Remark 3. An alternative, equivalent characterization of behavioral stabilizability is

$$\forall (x, u) \text{ solution of (DAE)} \, \exists \tilde{u} \in \mathcal{U}^0 : \lim_{t\to\infty} \operatorname{ess\,sup}_{[t,\infty)} \|(\varphi((Ex)(0), u + \tilde{u}), u + \tilde{u})\| = 0.$$

Lemma 1. *Consider (DAE) with $\omega > 0$. Then, (DAE) is behaviorally stabilizable if and only if the ODE system $(\hat{A}, \hat{b})$ given by the augmented system (5) is stabilizable.*

Proof. Note that (DAE) is in quasi-Weierstraß form and $\operatorname{rank}(\lambda N - I) = n_N$ for all $\lambda \in \mathbb{C}$. By Proposition 4 and the Hautus criterion, we therefore obtain the following:

$$\begin{aligned}
\text{(DAE) is behaviorally stabilizable} \quad &\iff \forall \lambda \in \overline{\mathbb{C}}_+ : \operatorname{rank}_{\mathbb{C}}[\lambda E - A, b] = \operatorname{rank}_{\mathbb{R}(s)}[sE - A, b] \\
&\iff \forall \lambda \in \overline{\mathbb{C}}_+ : \operatorname{rank}_{\mathbb{C}}[\lambda I_{n_J} - J, b_J] = n_J \\
&\iff \forall \lambda \in \overline{\mathbb{C}}_+ : \operatorname{rank}_{\mathbb{C}}[\lambda I_n - \hat{A}, \hat{b}] = \operatorname{rank}_{\mathbb{C}} \begin{bmatrix} \lambda I_{n_J} - J & b_J & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & 1 \end{bmatrix} = \hat{n} \\
&\iff (\hat{A}, \hat{b}) \text{ is stabilizable.}
\end{aligned}$$

□

Example 1. To illustrate the previous results, consider the DAE

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(t) = x(t) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u(t), \quad \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x\right)(0) = x^0. \tag{6}$$

It has the input index $\omega = 1$ and the consistent initial values are, due to Proposition 2, given by

$$\mathcal{X} = \operatorname{im} F = \operatorname{im}(-K) = \operatorname{im}(-1, 0, 0)^{\top}.$$

Proposition 3 yields the augmented system

$$\dot{\hat{x}}(t) = \hat{u}(t), \qquad x(t) = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} \hat{x}(t) \\ \hat{u}(t) \end{pmatrix} = \begin{pmatrix} -\hat{u}(t) \\ -\hat{x}(t) \\ 0 \end{pmatrix},$$

where $\hat{x} = u$, $\hat{u} = \dot{u}$, and with an initial value of $\hat{x}(0) = F^{\dagger}x^0 = (-1, 0, 0)x^0 = -x^0_1$.

3 | OPTIMAL CONTROL

For a given weighting matrix $S = S^\top \in \mathbb{R}^{(n+1)\times(n+1)}$ and time horizon $T \in (0, \infty]$, the *cost functional* J_T assigns to each solution (x, u) of (DAE) the value

$$J_T(x,u) = \int_0^T \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^\top S \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \mathrm{d}t \in \mathbb{R} \cup \{\pm\infty\}. \tag{7}$$

Our goal is to find the infimum of the cost functional for given consistent initial value $x^0 \in \mathcal{X}$, denoted by the *optimal value function*

$$\begin{aligned} V_T :\ & \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\} \\ & x^0 \mapsto \inf_{(x,u)} J(x,u) \quad \text{subject to } (x,u) \text{ solves (DAE)}. \end{aligned} \tag{8}$$

We define the optimal value function associated to the augmented system (5) in a similar fashion. The aforementioned weighting matrix S is partitioned according to the structure of the quasi-Weierstraß form (DAE) as

$$S =: \begin{bmatrix} Q_J & Q_{JN} & h_J \\ Q_{JN}^\top & Q_N & h_N \\ h_J^\top & h_N^\top & r \end{bmatrix}, \qquad \text{where } Q_J \in \mathbb{R}^{n_J \times n_J},\ Q_N \in \mathbb{R}^{n_N \times n_N},\ r \in \mathbb{R},$$

and a new symmetric matrix is defined as

$$\hat{S} = \begin{bmatrix} Q_J & h_J - Q_{JN} b_N & -Q_{JN} K \\ h_J^\top - b_N^\top Q_{JN}^\top & r + b_N^\top Q_N b_N - 2h_N^\top b_N & \left(b_N^\top Q_N - h_n^\top\right) K \\ -K^\top Q_{JN}^\top & K^\top (Q_N b_N - h_n) & K^\top Q_N K \end{bmatrix} \in \mathbb{R}^{(\hat{n}+1)\times(\hat{n}+1)}, \tag{9}$$

where $\hat{n} = n_J + \omega$. If $\omega = 0$, then K is void and the matrix $\hat{S}$ reduces to a (2×2)-block matrix. The *cost functional* $\hat{J}_T$ assigns to each solution $(\hat{x}, \hat{u})$ of (5) the value

$$\hat{J}_T(\hat{x},\hat{u}) = \int_0^T \begin{pmatrix} \hat{x}(t) \\ \hat{u}(t) \end{pmatrix}^\top \hat{S} \begin{pmatrix} \hat{x}(t) \\ \hat{u}(t) \end{pmatrix} \mathrm{d}t \in \mathbb{R} \cup \{\pm\infty\}. \tag{10}$$

Finally, the *optimal value function* is

$$\hat{V}_T : \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\}, \quad x^0 \mapsto \inf_{(\hat{x},\hat{u})} \hat{J}(\hat{x},\hat{u}) \quad \text{subject to } (\hat{x},\hat{u}) \text{ solves (5) and } \hat{x}(0) = F^\dagger x^0.$$

We will show in the next theorem that the DAE OCP is equivalent to the ODE OCP

$$\boxed{\begin{array}{c} \text{Minimize } \hat{J}_T(\hat{x},\hat{u}) \quad \text{subject to } \hat{u} \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}),\ \hat{x}(0) = F^\dagger x^0 \\ \text{and the ODE } \begin{cases} (5a) & \text{for } \omega \geq 1, \\ (4) & \text{for } \omega = 0. \end{cases} \end{array}} \tag{ODE-OCP}$$

Equivalent means that the optimal values are the same for a given consistent initial value x^0.

Theorem 1. *The optimal values of the OCPs and (ODE-OCP) coincide, ie,*

$$V_T(x^0) = \hat{V}_T(x^0) \qquad \text{for any consistent initial value } x^0 \in \mathcal{X}.$$

The theorem is proved in Appendix B.

Theorem 1 allows to apply well-known results of optimal control for ODEs and, for example, to derive a representation of the optimal value function V_T in terms of a quadratic form and to give a sufficient condition for the feasibility of (OCP), ie, V_T is finite on $\mathcal{X}$. To define the differential Riccati equation, we partition the symmetric weighting matrix $\hat{S}$ given by (9) as

$$\hat{S} = \begin{bmatrix} \hat{Q} & \hat{h} \\ \hat{h}^\top & \hat{r} \end{bmatrix} \quad \text{with} \quad \hat{Q} \in \mathbb{R}^{\hat{n}\times\hat{n}} \quad \text{and} \quad \hat{r} \in \mathbb{R}$$

and the relevant *differential Riccati equation* becomes

$$\dot{P}(t) = \hat{A}^\top P(t) + P(t)\hat{A} + \hat{Q} - (P(t)\hat{b} + \hat{h})\hat{r}^{-1}(P(t)\hat{b} + \hat{h})^\top, \quad P(0) = 0, \tag{11}$$

where $\hat{A}$ and $\hat{b}$ are given by (5b) (or $\hat{A} = J$, $\hat{b} = b_J$ if $\omega = 0$). The limit $\lim_{t\to\infty} P(t)$ exists, cf Appendix A, and is denoted by $P(\infty)$.

The following assumptions are formulated in terms of $\hat{S}$ defined by (9) and $\hat{A}$ given as in (5a) for $\omega \geq 1$ or J for $\omega = 0$:

(A1) $\hat{S} \geq 0$;
(A2) (DAE) is stabilizable, ie, the pair (J, b_J) is stabilizable;
(A3) $\hat{r} = (\,0\ \dots\ 0\ 1\,)\hat{S}(\,0\ \dots\ 0\ 1\,)^\top > 0$;
(A4) $(\hat{A}, \hat{Q}) = (\hat{A}, [\,I_{\hat{n}}\ 0_{\hat{n}\times 1}\,]\hat{S}[\,I_{\hat{n}}\ 0_{\hat{n}\times 1}\,]^\top)$ is observable;
(A5) $\operatorname{rank} \hat{S} = \operatorname{rank} \hat{Q} + 1 = \operatorname{rank}([\,I_{\hat{n}}\ 0_{\hat{n}\times 1}\,]\hat{S}[\,I_{\hat{n}}\ 0_{\hat{n}\times 1}\,]^\top) + 1$.

The first three assumptions imply that the value function V_T can be represented by a quadratic function, thus, exhibits finite values on $\mathcal{X}$. If, in addition, also the fourth and fifth assumption hold, the optimal value function is positive definite.

Theorem 2. *Suppose (A1) to (A3) hold and let $P(T)$ and V_T be defined by (11) and (8), respectively. Then, the following holds for $T \in (0, \infty]$:*

i. $\forall x^0 \in \mathcal{X} : V_T(x^0) = (x^0)^\top (F^\dagger)^\top P(T) F^\dagger x^0$ *and* $P(T) \geq 0$.
ii. If, in addition, (A4) and (A5) hold, then $P(T) > 0$, *ie,* $\forall x^0 \in \mathcal{X} \setminus \{0\} : V_T(x^0) > 0$.

Proof. By Lemma 1, Assumption (A2) is equivalent to the stabilizability of the augmented system (5a). Therefore, the ODE (5a) (or (4) for $\omega = 0$) fulfills the assumptions of Proposition 7. By Theorem 1, the ODE OCP in Theorem 1 is equivalent to (OCP), thus showing the assertion. □

4 | THE OPTIMAL CONTROL AS A FEEDBACK

It is well known that, in case of ODEs, the optimal control can be expressed as a state feedback. We will show that a similar result holds for (DAE). For $\omega = 0$, this is immediately clear from Proposition 5 as $x^* = \hat{x}^*$ holds. For $\omega > 0$, we need some further algebraic manipulations to obtain an optimal feedback. We stress that, in case of DAEs, the closed-loop system is not necessarily regular; this is shown in the following example.

Example 2. Consider the scalar DAE $0 = x(t) + u(t)$, which has the form (DAE), where $n_J = 0$, $N = 0$, $b_N = 1$. Choosing the feedback $u(t) = -x(t)$, we obtain the closed-loop system $0 = x(t) - x(t) = 0$. Hence, any function $x \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R})$ is a solution of the closed-loop system, which is not regular any more. This happens because the feedback only provides the superfluous information $u = -x$, which is already containted in the algebraic constraints of the DAE.

The following Lemma 2 characterizes when a feedback leads to a regular closed-loop system.

Lemma 2. *Consider (DAE) and $k = (k_J, k_N) \in \mathbb{R}^{1\times n_J} \times \mathbb{R}^{1\times n_N}$. Then,*

$$\det(sE - (A - bk)) = 0_{\mathbb{R}[s]} \quad \Longleftrightarrow \quad k_J(sI_{n_J} - J)^{-1} b_J = 0 \wedge k_N b_N = 1 \wedge k_N K = 0_{1\times\omega}. \tag{12}$$

The proof is given in Appendix B.

Consider the (OCP) for (DAE) with solution (x^*, u^*) and consistent initial value x^0. Then, x^* and u^* are called *optimal state trajectory* and *optimal control* for time horizon $T \in (0, \infty]$ if and only if $J_T(x^*, u^*) = V_T(x^0)$.

We show that the optimal control of the (OCP) equals the $(n_J + 1)$-th component of the augmented state $\hat{x}$, where the latter solves an ODE.

Proposition 5. *Consider the OCP with optimization horizon $T \in (0, \infty]$ and suppose that Assumptions (A1) to (A3) hold. Then, the unique optimal control is given by $u^* = \hat{x}^*_{n_J+1}(t)$, where $\hat{x}^*$ solves the ODE*

$$\dot{\hat{x}}^*(t) = \left[\hat{A} - \hat{b}\hat{r}^{-1}(\hat{b}^\top P(T-t) + \hat{h}^\top)\right]\hat{x}^*(t), \quad \hat{x}^*(0) = F^\dagger x^0, \tag{13}$$

and P is the solution of the differential Riccati equation (11). If $\omega = 0$, then $u^*(t) = -\hat{r}^{-1}(\hat{b}^\top P(T-t) + \hat{h}^\top)\hat{x}^*(t)$.

Proof. Let $x^0 \in \mathcal{X}$ be arbitrary, and consider the ODE OCP for the augmented system (5a) stated in Theorem 1. In view of Proposition 7, the optimal trajectory $\hat{x}^*$ and optimal control $\hat{u}^*$ for the augmented system are given by the solution of (13) and $\hat{u}^*(t) = -\hat{r}^{-1}(\hat{b}^\top P(T-t) + \hat{h}^\top)\hat{x}^*(t)$, respectively. Therefore, Theorem 1 yields

$$V_T(x^0) = \int_0^T \begin{pmatrix} \hat{x}^*(t) \\ \hat{u}^*(t) \end{pmatrix}^\top \hat{S} \begin{pmatrix} \hat{x}^*(t) \\ \hat{u}^*(t) \end{pmatrix} \mathrm{d}t.$$

By Proposition 3 and Theorem 1,

$$\begin{pmatrix} x_J^* \\ x_N^* \\ u^* \end{pmatrix} = \begin{bmatrix} I_{n_J} & 0 & 0 \\ 0 & -b_N & -K \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} \hat{x}^* \\ \hat{u}^* \end{pmatrix}$$

is a solution of (DAE) that fulfills

$$J_T(x^*, u^*) = \int_0^T \begin{pmatrix} \hat{x}^*(t) \\ \hat{u}^*(t) \end{pmatrix}^\top \hat{S} \begin{pmatrix} \hat{x}^*(t) \\ \hat{u}^*(t) \end{pmatrix} \mathrm{d}t = V_T(x^0).$$

Therefore, u^* is an optimal control.

To prove the uniqueness, assume that we have a solution (x', u') of (DAE) with $J_T(x', u') = J_T(x^*, u^*)$. Proposition 3 and Theorem 1 yield that

$$\hat{x}' = \left(x_J'^\top,\ u',\ \dots,\ u'^{(\omega-1)}\right)^\top, \quad \hat{u}' = u'^{(\omega)}$$

is an optimal solution of the ODE OCP for the augmented system (5a). Thus, uniqueness of the ODE optimal control (cf Proposition 7ii) yields $\hat{u}' = \hat{u}^*$ and hence $\hat{x}' = \hat{x}^*$. For $\omega = 0$, this immediately shows the uniqueness of u^* as $u' = \hat{u}' = \hat{u}^* = u^*$. For $\omega > 0$, it follows from Proposition 3 that

$$u' = \hat{x}'_{n_J+1} = \hat{x}^*_{n_J+1} = u^*,$$

which concludes the proof. □

In the infinite optimization horizon case $T = \infty$, note that, by convention, $P(\infty - t) = P(\infty)$ for all $t \geq 0$, so that (13) is a time-invariant ODE.

Example 3. Revisit Example 1 and define for the DAE (6) the cost functional

$$J_T(x, u) = \int_0^T \|x(t)\|^2 + u(t)^2 \,\mathrm{d}t.$$

To solve the OCP, rewrite the OCP as an ODE OCP for the augmented system as described in Theorem 1. The cost functional for the augmented system is

$$\hat{J}_T(\hat{x}, \hat{u}) = \int_0^T \begin{pmatrix} \hat{x}(t) \\ \hat{u}(t) \end{pmatrix}^\top \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x}(t) \\ \hat{u}(t) \end{pmatrix} \mathrm{d}t,$$

so the Riccati Equation (11) takes the form

$$\dot{P}(t) = 2 - P(t)^2, \quad P(0) = 0,$$

and hence

$$\forall T \geq 0 : P(T) = \frac{\sqrt{2}\left(\mathrm{e}^{2\sqrt{2}T} - 1\right)}{\mathrm{e}^{2\sqrt{2}T} + 1}, \quad P(\infty) = \lim_{t\to\infty} P(t) = \sqrt{2}.$$

Theorem 1 yields the optimal value

$$V_T(x^0) = \hat{V}_T(x^0) = (x^0)^\top (F^\dagger)^\top P(T) F^\dagger x^0 = P(T) \cdot \left(x_1^0\right)^2.$$

In view of Proposition 5, the optimal control for the DAE (6) satisfies the ODE initial value problem

$$\dot{u}^*(t) = -P(T-t)u^*(t), \quad u^*(0) = -x_1^0. \tag{14}$$

In case of an infinite time horizon $T = \infty$, this leads to the optimal trajectory and control given for almost all $t \geq 0$ by

$$x^*(t) = \begin{pmatrix} -\sqrt{2}e^{-\sqrt{2}t}x_1^0 \\ e^{-\sqrt{2}t}x_1^0 \\ 0 \end{pmatrix}, \quad u^*(t) = -e^{-\sqrt{2}t}x_1^0. \tag{15}$$

We are now in a position to state the main result of this section, that is, how the optimal control can be expressed as a feedback that yields a regular closed-loop system.

Theorem 3. *Consider (OCP) with optimization horizon $T = \infty$ and $Nb_N \neq 0$ and suppose that Assumptions (A1) to (A3) hold. Define*

$$\hat{k}_\alpha := \alpha \left(\hat{r}^{-1}(\hat{b}^\top P(\infty) + \hat{h}^\top) \;\; 1 \right) + \left(0_{1\times n_J} \;\; 1 \;\; 0_{1\times\omega} \right) \in \mathbb{R}^{1\times(\hat{n}+1)}, \quad \alpha \in \mathbb{R},$$

where $P(\infty) = \lim_{t\to\infty} P(t) \in \mathbb{R}^{\hat{n}\times\hat{n}}$ and P is the solution of the Riccati Equation (11). Furthermore, let $G^\dagger \in \mathbb{R}^{(\hat{n}+1)\times n}$ be a left inverse of

$$G = \begin{bmatrix} I_{n_J} & 0 & 0 \\ 0 & -b_N & -K \end{bmatrix}.$$

Then, for any $\alpha \neq 0$, the closed-loop system

$$\frac{\mathrm{d}}{\mathrm{d}t} Ex(t) = (A + b\hat{k}_\alpha G^\dagger)x(t), \quad (Ex)(0) = x^0 \tag{16}$$

obtained by the feedback

$$u(t) = \hat{k}_\alpha G^\dagger x(t) \tag{17}$$

is regular and its solution is an optimal state trajectory for (OCP).

The proof is carried out in Appendix B.

Remark 4. While the optimal *control* of (DAE) is unique, the optimal *feedback* is generally not unique; any $\alpha \neq 0$ and any left inverse of G can be chosen for the feedback, giving rise to a possible multitude of optimal feedbacks. Roughly speaking, this is due to the fact that the states $u, \dots, u^{(\omega-1)}$ of the augmented system, which are necessary for the optimal control, can be derived from (DAE) in multiple equivalent ways. Nevertheless, all feedbacks will lead to the same optimal control input.

Example 4. We continue with Example 3. For $T = \infty$, the optimal control can be expressed as a state feedback. Using the family of left inverses

$$G^\dagger = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & \beta \end{bmatrix}, \quad \beta \in \mathbb{R},$$

we arrive by Theorem 3 at the family of distinct optimal feedbacks

$$u = \left(-\alpha \;\; -\sqrt{2}\alpha - 1 \;\; \beta \right) x, \quad \alpha \in \mathbb{R}\setminus\{0\}, \; \beta \in \mathbb{R},$$

leading to the closed-loop system

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(t) = \begin{bmatrix} 1 & 0 & 0 \\ -\alpha & -\sqrt{2}\alpha & \beta \\ 0 & 0 & 1 \end{bmatrix} x(t), \quad \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x \right)(0) = x^0.$$

The solution of the closed-loop system fulfills for almost all $t \geq 0$

$$x(t) = \begin{pmatrix} -\sqrt{2}e^{-\sqrt{2}t}x_1^0 \\ e^{-\sqrt{2}t}x_1^0 \\ 0 \end{pmatrix},$$

coinciding with the optimal trajectory stated in (15).

5 | CONCLUSIONS AND OUTLOOK

Most of the presented results should be directly generalizable to multi-input DAEs by defining the input index as a vector, whose entries correspond to the columns of matrix B. Furthermore, we conjecture that similar results can also be established for nonregular DAEs.

The presented results on optimal control of regular DAEs may be worth knowing in its own right. However, our guided research interest stems from MPC with finite time horizon $T \in (0, \infty)$ to approximately solve (OCP) with mixed control-state constraints

$$Fx + gu \leq \mathbb{1}, \qquad (F, g) \in \mathbb{R}^{p\times n} \times \mathbb{R}^{p} \tag{18}$$

on an infinite time horizon. As outlined in the introduction, this hope is justified (see the works of Ilchmann et al[8,10] for details). In particular, our approach helps to identify the control and also to characterize initial feasibility in terms of the consistency space. Moreover, also a suitable combination of terminal region and terminal costs was deduced in the work of Ilchmann et al[8] such that the origin is asymptotically stable w.r.t. the MPC closed loop. In conclusion, the proposed approach allows to directly transfer *standard* techniques used in MPC to systems governed by DAEs.

The choice of the quasi-Weierstraß form in (DAE) is mainly motivated by the fact that it makes DAE optimal control very accessible from an ODE point of view. In the future, we want to investigate whether similar results are achievable based on the feedback equivalence form (FEF)[17] or representation concepts introduced in the work of Embree and Keeler.[32] The FEF is numerically more robust in comparison to the quasi-Weierstraß form but may also allow for the application of *classical* ODE results because the lower right block of the transformed matrix A is invertible. Hence, we think that the presented approach is the starting point to design MPC schemes based on more sophisticated optimal control techniques.[15-17]

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APPENDIX A

OPTIMAL CONTROL OF ORDINARY DIFFERENTIAL EQUATIONS

Consider for $[\bar{A}, \bar{b}] \in \mathbb{R}^{\bar{n}\times\bar{n}} \times \mathbb{R}^{\bar{n}}$ and $x^0 \in \mathbb{R}^{\bar{n}}$ the initial value problem of ordinary differential equations

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{b}\bar{u}(t), \quad \bar{x}(0) = \bar{x}^0 \tag{A1}$$

with associated cost functional

$$\bar{J}_T(\bar{x}, \bar{u}) = \int_0^T \begin{pmatrix} \bar{x}(t) \\ \bar{u}(t) \end{pmatrix}^\top \bar{S} \begin{pmatrix} \bar{x}(t) \\ \bar{u}(t) \end{pmatrix} \mathrm{d}t,$$

where $\bar{S} = \bar{S}^\top \in \mathbb{R}^{(\bar{n}+1)\times(\bar{n}+1)}$ and OCP

$$\bar{V}_T : \mathbb{R}^{\bar{n}} \to \mathbb{R} \cup \{\pm\infty\}, \qquad x^0 \mapsto \inf_{(\bar{x},\bar{u})} \bar{J}(\bar{x},\bar{u}) \quad \text{subject to } (\bar{x},\bar{u}) \text{ solves (A1) and } \bar{x}(0) = \bar{x}^0.$$

To guarantee the feasibility of the OCP, ie, to ensure that $\bar{V}_T(\bar{x}^0)$ is finite for all $T \in (0,\infty]$ and all $\bar{x}^0 \in \mathbb{R}^n$, we introduce some standard assumptions.[33] These are already formalized in (A1) to (A5) for the *DAE* OCP that are identical to the assumptions for the ODE OCP, except that in (A2), the stabilizability of (DAE) needs to be replaced by the stabilizability of the ODE (A1).

Proposition 6. *Let the ODE (A1) be stabilizable and Assumptions (A1) to (A3) be fulfilled. Then, the associated differential Riccati Equation (11) satisfies the following.*

i. *There exists a solution $P(\cdot) : [0,\omega) \to \mathbb{R}^{n\times n}$ for some maximal $\omega \in (0,\infty]$.*
ii. *The life span of this solution can be extended on the whole half axis, ie, $\omega = \infty$.*
iii. *The solution is unique, symmetric, and positive semidefinite, ie, $P(t) = P(t)^\top \geq 0$ for all $t > 0$.*
iv. *The solution is monotonically nondecreasing in time, ie,*

$$\forall \delta > 0 \ \ \forall t \geq 0 : P(t+\delta) \geq P(t).$$

v. $\exists P(\infty) = P(\infty)^\top \in \mathbb{R}^{n\times n} : \lim_{t\to\infty} P(t) = P(\infty) \geq 0.$

If additionally Assumptions (A4) and (A5) hold, then

vi. $\forall t > 0 : P(t) > 0, P(\infty) > 0.$

Proof. According to theorem 16.4.3 in the work of Lancaster and Rodman,[33] the differential Riccati equation (11) has a unique solution on $[0,\infty)$ and this solution is monotonically nondecreasing. Using proposition 16.2.5 and theorem 16.4.4 in the work of Lancaster and Rodman,[33] we obtain that $P(\infty) := \lim_{t\to\infty} P(t)$ exists.

If Assumptions (A4) and (A5) hold, $P(\infty)$ is positive definite according to theorem 16.3.3 in the work of Lancaster and Rodman.[33] It remains to be shown that $P(t) > 0$ for all $t > 0$. This follows analogously to proposition 16.2.8 in the work of Lancaster and Rodman.[33] □

Remark 5. According to theorem 16.4.4 and lemma 10.11 in the work of Lancaster and Rodman,[33] the limit $P(\infty)$ in Proposition 6v is the minimal solution of the *algebraic Riccati equation*

$$0 = \bar{A}^\top P + P\bar{A} + \bar{Q} - (P\bar{b} + \bar{h})\bar{r}^{-1}(P\bar{b} + \bar{h})^\top, \tag{ARE}$$

ie, for all $P \in \mathbb{R}^{\bar{n}\times\bar{n}}$ that fulfill (ARE), it holds that $P - P(\infty) \geq 0$.

We are now in a position to explain the consequences of the differential Riccati equation (11) when applied to the OCP.

Proposition 7. *Let the ODE (A1) be stabilizable and suppose the associated OCP fulfills Assumptions (A1) to (A3). Then, we have, for $T \in (0,\infty]$, the following.*

i. *There exists a unique $P_T = P_T^\top \geq 0$ such that*

$$\forall \bar{x}^0 \in \mathbb{R}^{\bar{n}} : \bar{V}_T(\bar{x}^0) = (\bar{x}^0)^\top P_T \bar{x}^0$$

and this P_T is given by $P_T = P(T)$ in Proposition 6.

ii. *For every $\bar{x}^0 \in \mathbb{R}^{\bar{n}}$, the unique optimal control $\bar{u}^* \in \mathcal{L}^1_{\text{loc}}([0,T),\mathbb{R})$ such that $V_T(\bar{x}^0) = \bar{J}_T(\bar{x}^0, \bar{u}^*)$ is given by*

$$\begin{aligned}\dot{\bar{x}}^*(t) &= \left[\bar{A} - \bar{b}\bar{r}^{-1}(\bar{b}^\top P(T-t) + \bar{h}^\top)\right]\bar{x}^*(t), \quad \bar{x}^*(0) = \bar{x}^0 \\ \bar{u}^*(t) &= -\bar{r}^{-1}(\bar{b}^\top P(T-t) + \bar{h}^\top)\bar{x}^*(t),\end{aligned}$$

where $P(\cdot)$ is the solution of (11) and by convention $P(\infty - t) = P(\infty)$ for all $t \geq 0$.

If additionally Assumptions (A4) and (A5) hold, then

iii. $\forall T \in (0,\infty] \forall \bar{x}^0 \in \mathbb{R}^{\bar{n}} : \bar{V}_T(\bar{x}^0) > 0.$

Proof. This follows according to lemma 16.4.2 and theorem 16.3.3 in the work of Lancaster and Rodman,[33] for $T \in (0,\infty)$ and $T = \infty$, respectively. □

APPENDIX B

PROOFS

Proof of Proposition 1. (3b) follows immediately from the variation of constants formula. For (3a) and (3c), consider the algebraic part of (DAE), ie,

$$\frac{\mathrm{d}}{\mathrm{d}t} N x_N(t) = x_N(t) + b_N u(t). \tag{B1}$$

In passing, we note that $\varphi \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$ if and only if $\varphi_1, \dots, \varphi_n \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R})$ and therefore $M\varphi \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^l)$ for any $M \in \mathbb{R}^{l \times n}$.

Let $\left(\begin{pmatrix} x_J \\ x_N \end{pmatrix}, u\right)$ be a solution of (DAE), then (x_N, u) is a solution of (B1). We proceed in several steps.

Step 1: We show that

$$\forall i > \omega : N^i x_N = 0. \tag{B2}$$

For $i \geq \min\{i \in \mathbb{N} \,|\, N^i = 0\} > \omega$, we have $N^i = 0$ and therefore (B2) follows.

Assume that (B2) holds for $i > \omega + 1$. Then, we have

$$N^{i-1} x_N = N^{i-1}(x_N + b_N u) \overset{\text{ae}}{\underset{(\text{B1})}{=}} N^{i-1} \frac{\mathrm{d}}{\mathrm{d}t} N x_N = \frac{\mathrm{d}}{\mathrm{d}t} N^i x_N \overset{(\text{B2})}{=} 0.$$

Therefore, (B2) is shown for $i - 1$.

Step 2: For $\omega = 0$, we have $x_N = \frac{\mathrm{d}}{\mathrm{d}t}(N x_N) - b_N u = -b_N u$ and (3c) is shown. For $\omega > 0$, we show the following statement by induction:

$$\forall i \in \{0, \dots, \omega - 1\} : u \in \mathcal{W}^{i+1,1}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}) \qquad \text{and} \tag{B3}$$

$$N^{\omega} b_N u^{(i)} \overset{\text{ae}}{=} -N^{\omega - i} x_N - \sum_{k=0}^{i-1} N^{\omega - i + k} b_N u^{(k)}. \tag{B4}$$

For $i = 0$, we have

$$0 \overset{(\text{B3})}{=} \frac{\mathrm{d}}{\mathrm{d}t} N^{\omega+1} x_N = N^{\omega} \frac{\mathrm{d}}{\mathrm{d}t} N x_N \overset{\text{ae}}{\underset{(B2)}{=}} N^{\omega} x_N + N^{\omega} b_N u \qquad \Rightarrow \qquad N^{\omega} b_N u \overset{\text{ae}}{=} -N^{\omega} x_N \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$$

and so (B4) follows. Furthermore, as $N^{\omega} b_N \neq 0$ and u is scalar, we have (B3).

Assume that (B3) and (B4) hold for $i \in \{0, 1, \dots, \omega - 2\}$. Then, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} N^{\omega} b_N u^{(i)} &\overset{\text{ae}}{\underset{(\text{B4})}{=}} \frac{\mathrm{d}}{\mathrm{d}t} \left(-N^{\omega - i} x_N - \sum_{k=0}^{i-1} N^{\omega - i + k} b_N u^{(k)} \right) \overset{\text{ae}}{\underset{(\text{B1})}{=}} -N^{\omega - (i+1)}(x_N + b_N u) - \sum_{k=0}^{i-1} N^{\omega - i + k} b_N u^{(k+1)} \\
&= -\underbrace{N^{\omega - (i+1)} x_N}_{\in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)} - \sum_{k=0}^{(i+1)-1} \underbrace{N^{\omega - (i+1) + k} b_N u^{(k)}}_{\overset{(\text{B3})}{\in} \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)}.
\end{aligned} \tag{B5}$$

This shows (B4) for $i + 1$.

Furthermore, as the right-hand side of (B5) is in $\mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$, $N^{\omega} b_N \neq 0$, and u is scalar, it follows that $u^{(i+1)} \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R})$. Therefore, (B3) holds for $i + 1$.

Step 3: For $\omega > 0$, reconsider (B4) for $i = \omega - 1$; by (B3), we know that $u \in \mathcal{W}^{i+1,1}(\mathbb{R}_{\geq 0}, \mathbb{R})$, so (B4) is differentiable. Carrying out the differentiation in the same way as in (B5), we arrive at

$$N^{\omega} b_N u^{(\omega)} \stackrel{\text{ae}}{=} -x_N - \sum_{k=0}^{\omega-1} N^k b_N u^{(k)}. \tag{B6}$$

Rearranging (B6) for x_N immediately gives (3c). □

Proof of Proposition 2. For i, we show by induction for $0 \leq i \leq \omega$ that

$$\operatorname{rank}\left[N^i b_N,\ N^{i+1} b_N\ \cdots\ N^{\omega} b_N\right] = \omega - i + 1. \tag{B7}$$

For $i = \omega$, (B7) follows from $N^{\omega} b_N \neq 0$.

Assume that (B7) holds for an $i \in \{1, \ldots, \omega\}$. Consider the linear combination

$$\alpha_{i-1} N^{i-1} b_N + \cdots + \alpha_{\omega} N^{\omega} b_N = 0 \tag{B8}$$

for $\alpha_{i-1}, \ldots, \alpha_{\omega} \in \mathbb{R}$. Multiplying (B8) from the left by N, we arrive at

$$0 = \alpha_{i-1} N^i b_N + \cdots + \alpha_{\omega-1} N^{\omega} b_N + \alpha_{\omega} N^{\omega+1} b_N = \alpha_{i-1} N^i b_N + \cdots + \alpha_{\omega-1} N^{\omega} b_N.$$

By (B7), we see that

$$\alpha_{i-1} = \cdots = \alpha_{\omega-1} = 0.$$

Therefore, by (B8) and $N^{\omega} b_N \neq 0$, we obtain $\alpha_{\omega} = 0$ and hence (B7) for $i - 1$. Now, (B7) for $i = 1$ shows the assertion.

For ii, note that, by Proposition 1, it follows that every solution (x, u) of (DAE) fulfills

$$(Ex)(0) = \begin{bmatrix} x_J(0) \\ N x_N(0) \end{bmatrix} = \begin{bmatrix} x_J(0) \\ -\sum_{i=1}^{\omega} N^i b_N u^{(i-1)}(0) \end{bmatrix} = \begin{bmatrix} I_{n_J} & 0 \\ 0 & -K \end{bmatrix} \begin{pmatrix} x_J(0) \\ u(0) \\ \vdots \\ u^{(\omega-1)}(0) \end{pmatrix},$$

where $x_J(0), u(0), \ldots, u^{\omega-1}(0)$ can be chosen freely.

Now, iii and iv immediately follow from i and ii. □

Proof of Theorem 1. As a first step, we show that, for any solution (x, u) of (DAE), the cost functional (7) is given by

$$J_T(x, u) = \hat{J}_T(\hat{x}, \hat{u}), \tag{B9}$$

where $(\hat{x}, \hat{u})$ is the solution of the augmented system (5a) (or of (4) for $\omega = 0$) with $\hat{x}(0) = F^{\dagger} x^0$; by Proposition 3 (or by $x_N = -b_N u = -b_N \hat{u}$ for $\omega = 0$), it follows that

$$\begin{pmatrix} x_J \\ x_N \\ u \end{pmatrix} \stackrel{\text{ae}}{=} \begin{bmatrix} I_{n_J} & 0 & 0 \\ 0 & -b_N & -K \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} \hat{x} \\ \hat{u} \end{pmatrix}.$$

Substituting this in (7) gives (B9) as

$$\begin{bmatrix} I_{n_J} & 0 & 0 \\ 0 & -b_N & -K \\ 0 & 1 & 0 \end{bmatrix}^{\top} \begin{bmatrix} Q_J & Q_{JN} & h_J \\ Q_{JN}^{\top} & Q_N & h_N \\ h_J^{\top} & h_N^{\top} & r \end{bmatrix} \begin{bmatrix} I_{n_J} & 0 & 0 \\ 0 & -b_N & -K \\ 0 & 1 & 0 \end{bmatrix} = \hat{S}.$$

Now, let $x^0 \in \mathcal{X}$ and $\varepsilon > 0$ be arbitrary. By $\hat{V}_T : \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\}$, we denote the optimal value function of the ODE OCP

$$\hat{V}_T(x^0) = \inf_{\hat{u} \in \mathcal{L}^1_{\text{loc}}} \hat{J}_T(\hat{x}, \hat{u}) \quad \text{subject to} \quad \text{(5a) and } \hat{x}(0) = F^{\dagger} x^0$$

(or (4) instead of (5a) for $\omega = 0$). Choose $\hat{u} \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}, \mathbb{R})$ such that $\hat{J}_T(\hat{x}, \hat{u}) \leq \hat{V}_T(x^0) + \varepsilon$, where $\hat{x}$ is a solution of the ordinary differential Equation (5a) (or (4) for $\omega = 0$) with $\hat{x}(0) = F^{\dagger} x^0$. Define

$$\begin{pmatrix} x_J \\ x_N \\ u \end{pmatrix} := \begin{bmatrix} I_{n_J} & 0 & 0 \\ 0 & -b_N & -K \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} \hat{x} \\ \hat{u} \end{pmatrix}.$$

Then, by Proposition 3, we have that $\left(\begin{pmatrix} x_J \\ x_N \end{pmatrix}, u\right)$ is a solution of (DAE) and

$$(Ex)(0) = \begin{pmatrix} x_J(0) \\ Nx_N(0) \end{pmatrix} = \begin{bmatrix} I_{n_J} & 0 \\ 0 & -K \end{bmatrix} \hat{x}(0) = F\hat{x}(0) = x^0.$$

Therefore, by (B9), we have

$$V_T(x^0) \le J_T(x, u) = \hat{J}_T(\hat{x}, \hat{u}) \le \hat{V}_T(x^0) + \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we get

$$V_T(x^0) \le \hat{V}_T(x^0).$$

To prove the reverse inequality, let $x^0 \in \mathcal{X}$ and $\varepsilon > 0$ be arbitrary. Choose a solution (x, u) of (DAE) such that $(Ex)(0) = x^0$ and $J_T(x, u) \le V_T(x^0) + \varepsilon$. By Proposition 3, we have that

$$\hat{x} := \left(x_J^\top, \ u, \ \dots, \ u^{(\omega-1)} \right)^\top$$

solves the ordinary differential Equation (5a) (or (4) for $\omega = 0$). Furthermore,

$$(Nx_N)(0) = -N \begin{bmatrix} 0_{n_N \times n_J} & b_N & K \end{bmatrix} \begin{pmatrix} \hat{x}(0) \\ u^{(\omega)}(0) \end{pmatrix} = \begin{bmatrix} 0_{n_N \times n_J} & -K \end{bmatrix} \hat{x}(0),$$

therefore

$$F\hat{x}(0) = \begin{bmatrix} I_{n_J} & 0 \\ 0 & -K \end{bmatrix} \hat{x}(0) = \begin{pmatrix} x_J(0) \\ (Nx_N)(0) \end{pmatrix} = (Ex)(0).$$

Consequently,

$$\hat{V}_T(x^0) \le \hat{J}_T\left(\hat{x}, u^{(\omega-1)}\right) = J_T(x, u) \le V_T(x^0) + \varepsilon.$$

As for $\varepsilon \to 0$, we obtain

$$\hat{V}_T(x^0) \le V_T(x^0),$$

the assertion is proved. □

Proof of Lemma 2. As (DAE) is regular and therefore $(sE - A) \ne 0_{\mathbb{R}[s]}$, the Sherman-Morrison-Woodbury formula (cf fact 2.16.3 in the work of Bernstein[34])

$$\det((sE - A) + bk) = (1 + k(sE - A)^{-1}b) \det(sE - A)$$

yields

$$\det(sE - (A - bk)) = 0_{\mathbb{R}[s]} \quad \Longleftrightarrow \quad 1 + k(sE - A)^{-1}b = 0. \tag{B10}$$

Furthermore,

$$k(sE - A)^{-1}b = \begin{pmatrix} k_J & k_N \end{pmatrix} \begin{bmatrix} (sI_{n_J} - J)^{-1} & 0 \\ 0 & (sN - I_{n_N})^{-1} \end{bmatrix} \begin{pmatrix} b_J \\ b_N \end{pmatrix} = k_J(sI_{n_J} - J)^{-1}b_J + k_N(sN - I_{n_N})^{-1}b_N \in \mathbb{R}(s). \tag{B11}$$

By eq (4.4.23) in the work of Bernstein,[34] we have

$$(sI_{n_J} - J)^{-1} = \frac{\sum_{i=0}^{n_J-1} J_i s^i}{\det(sI_{n_J} - J)} \quad \text{for} \quad J_0, \dots, J_{n_J-2} \in \mathbb{R}^{n_J \times n_J} \quad \text{and} \quad J_{n_J-1} = I_{n_J} \tag{B12}$$

and the nilpotency of N gives

$$(sN - I_{n_N})^{-1} = -\sum_{i=0}^{n_N-1} N^i s^i. \tag{B13}$$

"⇒": Suppose that $\det(sE - (A - bk)) = 0_{\mathbb{R}[s]}$. Then, (B11) applied to (B10), using the two identities (B12) and (B13) yields

$$-\det(sI_{n_J} - J) = \sum_{i=0}^{n_J-1} k_J J_i b_J s^i - \sum_{i=0}^{n_N-1} k_N N^i b_N s^i \cdot \det(sI_{n_J} - J). \tag{B14}$$

Since

$$\det(sI_{n_J} - J) = s^{n_J} + \alpha_{n_J-1}s^{n_J-1} + \cdots + \alpha_0 s^0 \in \mathbb{R}[s],$$

we conclude from (B14) that

$$\sum_{i=0}^{n_N-1} k_N N^i b_N s^i = k_N b_N$$

and therefore

$$-1 = -k_N b_N, \quad k_N K = 0_{1\times\omega}. \tag{B15}$$

Substituting (B15) in (B14) now yields

$$\sum_{i=0}^{n_J-1} k_J J_i b_J s^i = 0.$$

This applied to (B12) shows the first of the necessary conditions in (12). The second necessary condition was already shown in (B15).

"⇐": Substituting the necessary conditions in (12) into (B11) together with (B13) directly shows

$$k(sE-A)^{-1}b = -1$$

and $\det(sE-(A-bk)) = 0_{\mathbb{R}[s]}$ follows from (B10). This concludes the proof. □

Proof of Theorem 3. Define $p := \hat{r}^{-1}(\hat{b}^\top P(\infty) + \hat{h}^\top)$ and note that

$$\hat{k}_\alpha \begin{bmatrix} I_{\hat{n}} \\ -p \end{bmatrix} = (\alpha(\,p\;\;1\,) + (\,0_{1\times n_J}\;\;1\;\;0_{1\times\omega}\,)) \begin{bmatrix} I_{\hat{n}} \\ -p \end{bmatrix} = (\,0_{1\times n_J}\;\;1\;\;0_{1\times\omega}\,).$$

Let (x^*, u^*) be an optimal trajectory for (OCP) according to Proposition 5. Furthermore, let

$$\hat{x}^* = \left(\left(x_J^*\right)^\top,\; u^*,\; \dots,\; u^{*(\omega-1)} \right)^\top, \quad \hat{u}^* = u^{*(\omega)}$$

be the corresponding solution of (5) according to Proposition 3. By Theorem 1 and Proposition 5, it follows that

$$u^* = \hat{x}^*_{n_J+1} = (\,0_{1\times n_J}\;\;1\;\;0_{1\times\omega}\,)\hat{x}^* = \hat{k}_\alpha \begin{bmatrix} I_{\hat{n}} \\ -p \end{bmatrix} \hat{x}^* = \hat{k}_\alpha \begin{pmatrix} \hat{x}^* \\ -p\hat{x}^* \end{pmatrix} = \hat{k}_\alpha \begin{pmatrix} \hat{x}^* \\ \hat{u}^* \end{pmatrix}. \tag{B16}$$

By (3c), we have

$$x = \begin{pmatrix} x_J^* \\ x_N^* \end{pmatrix} \overset{\text{ae}}{=} \begin{bmatrix} I_{n_J} & 0 \\ 0 & -[b_N,\, \dots, N^\omega b_N] \end{bmatrix} \begin{pmatrix} x_J^* \\ u^* \\ \vdots \\ u^{*(\omega)} \end{pmatrix} = G \begin{pmatrix} \hat{x}^* \\ \hat{u}^* \end{pmatrix}.$$

As in Proposition 2i, it follows that $[b_N, \dots, N^\omega b_N]$ has full column rank, so G is left invertible and

$$\begin{pmatrix} \hat{x}^* \\ \hat{u}^* \end{pmatrix} \overset{\text{ae}}{=} G^\dagger x^*(\cdot).$$

Substituting this into (B16) yields that the optimal trajectory of (OCP) fulfills

$$\frac{\mathrm{d}}{\mathrm{d}t}(Ex^*)(t) = Ax^*(t) + bu^*(t) = Ax^*(t) + b\hat{k}_\alpha G^\dagger x^*(t) = (A + b\hat{k}_\alpha G^\dagger)x^*(t),$$

so x^* is a solution of (16).

To ensure that x^* is the only solution of (16), it remains to show that the closed-loop system (16) is regular, ie, $\det(sE-(A+b\hat{k}_\alpha G^\dagger)) \neq 0_{\mathbb{R}[s]}$. Partition $-\hat{k}_\alpha G^\dagger =: (k_J, k_N) \in \mathbb{R}^{1\times n_J} \times \mathbb{R}^{1\times n_N}$ and note that

$$k_N K = (k_J, k_N) \begin{pmatrix} 0_{n_J\times 1} \\ b_N \end{pmatrix} = -\hat{k}_\alpha G^\dagger \begin{pmatrix} 0_{n_J\times\omega} \\ K \end{pmatrix} = \hat{k}_\alpha G^\dagger G \begin{pmatrix} 0_{(n_J+1)\times\omega} \\ I_\omega \end{pmatrix} = (\,p\;\;1\,) \begin{pmatrix} 0_{(n_J+1)\times\omega} \\ I_\omega \end{pmatrix} = (\,p_{n_J+2}\;\cdots\;p_{\hat{n}}\;\;1\,) \neq 0,$$

and regularity follows by Lemma 2. □