Zero dynamics and stabilization for analytic linear systems

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Abstract

The feedback stabilization problem is studied for time-varying real analytic systems. We investigate structural properties of the zero dynamics in terms of a system operator over a skew polynomial ring. The concept of (A, B)-invariant time-varying subspaces included in the kernel of C is used to obtain a condition for stabilizability. This condition is equivalent to autonomy of the zero dynamics in case of time-invariant systems. We derive a zero dynamics form for systems which satisfy an assumption close to autonomous zero dynamics; this in some sense resembles the Byrnes-Isidori form for systems with strict relative degree. Some aspects of the latter are also proved. Finally, we show for square systems with autonomous zero dynamics that there exists a linear state feedback such that the Lyapunov exponent of the closed-loop system equals the Lyapunov exponent of the zero dynamics; some boundedness conditions are required, too. If the zero dynamics are exponentially stable this implies that the system can be exponentially stabilized. These results are to some extent also new for time-invariant systems.

Keywords: Time-varying linear systems, feedback stabilization, zero dynamics, strict relative degree, Byrnes-Isidori form, geometric control theory, algebraic systems theory

1 Introduction

We study the class of linear time-varying systems with real analytic coefficients and m-inputs and p-outputs of the form

$$\dot{x} = A(t)\,x + B(t)\,u(t) \tag{1.1a}$$

$$y(t) = C(t)\,x(t)\,, \tag{1.1b}$$

where $(A, B, C) \in \mathcal{A}^{n\times n} \times \mathcal{A}^{n\times m} \times \mathcal{A}^{p\times n}$; this class is denoted by $\Sigma_{n,m,p}$ and we write $(A, B, C) \in \Sigma_{n,m,p}$ for short. The functions $u : \mathbb{R} \to \mathbb{R}^m$ and $y : \mathbb{R} \to \mathbb{R}^p$ are called *input* and *output* of the system, resp. A trajectory $(x, u, y) : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ is said to be a *solution* of (1.1) if, and only if, it belongs to the *behaviour* of (1.1):

$$\mathfrak{B}_{(1.1)} := \left\{ (x, u, y) \in \mathcal{AC}^n \times \mathcal{PC}^m \times \mathcal{AC}^p \;\middle|\; \begin{array}{l} (x, u, y) \text{ solves } (1.1) \\ \text{for almost all } t \in \mathbb{R} \end{array} \right\}.$$

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Systems of the form (1.1) result from a linearization of nonlinear input/output systems

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad y(t) = h(t, x(t))$$

along a solution, where f and h are real analytic functions.
A fundamental problem for systems of the form (1.1) is to find a time-varying state feedback $u(t) = F(t)x(t)$ for some $F : \mathbb{R} \to \mathbb{R}^{m\times n}$ such that, if applied to (1.1), the closed-loop system $\dot{x} = (A(t) + B(t)F(t))x$ is asymptotically stable. This problem was solved for time-invariant systems $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ under the condition that the system has autonomous and asymptotically stable zero dynamics (for the latter see Definition 2.1 and 5.1): see Corollary 5.4 in the present paper and [17, pp. 298-300].
The guiding research idea of the present paper is to see how this stabilization result can be extended to time-varying systems (1.1). To this end, it was necessary to combine different concepts:
The *algebraic concept* of the skew polynomial ring $\mathcal{M}[D]$, introduced in [15], is crucial for studying time-varying linear systems in polynomial operator description. Hence, we frequently consider $\mathcal{M}[D]$ with indeterminate D, real meromorphic coefficients, or $\mathcal{A}[D]$ with real analytic coefficients, and *multiplication rule*

$$\forall\, f \in \mathcal{M} : \quad Df = fD + \dot{f}. \tag{1.2}$$

The algebraic properties of $\mathcal{M}[D]$ and matrices over this ring will prove useful in our analysis, and we have delegated the important properties of this ring needed in the present paper to Appendix A.
From the field of *dynamical systems* we require the concept of (asymptotically stable) zero dynamics. The latter is the crucial assumption to resolve the stabilization problem. Zero dynamics was introduced for time-invariant nonlinear systems in [7], see also the textbook [17, Sec. 4.3, 5.1, 6.1]. We think that the importance of the concept of zero dynamics has been underestimated and we treat them in detail for time-varying linear systems.
The *geometric concept* of (A, B)-invariant time-varying subspaces is important to understand the zero dynamics. Geometric control theory was introduced by [2, 24] and for time-varying linear systems by [12]. The geometric description of the zero dynamics allows for the derivation of the zero dynamics form which is interesting in its own right and also essential for proving the stabilization result.
Different *canonical forms* for time-varying systems, such as the Byrnes-Isidori form [14], the zero dynamics form (new), the Teichmüller-Nakayama form [8], and the Hermite form [10] are a recurrent theme in the present paper. On the one hand they are instrumental for making connections between algebraic, geometric, and dynamic objects; on the other hand we refine some of these forms where needed for the stabilization result.
The concepts of *stability* and *Lypunov exponents* of time-varying linear systems, see e.g. [11], provide the main technical tool to derive the stabilization result.
The concept of *behaviour* is the general framework in the present paper. See the textbook [20] for time-invariant systems, and see [13] for time-varying systems and the ring $\mathcal{M}[D]$. Note that if we set

$$R(D) \;=\; \begin{bmatrix} DI_n - A & -B & 0 \\ -C & 0 & I_p \end{bmatrix} \in \mathcal{A}[D]^{(n+p)\times(n+m+p)}, \qquad \text{for } (A, B, C) \in \Sigma_{n,m,p},$$

then the behaviour can be written as $\ker_{\mathcal{AC}^n \times \mathcal{PC}^m \times \mathcal{AC}^p} R(\frac{\mathrm{d}}{\mathrm{d}t}) \;=\; \mathfrak{B}_{(1.1)}$.
In Section 2, we investigate the zero dynamics; i.e., those dynamics which are not visible at the output. It is shown that they are a dynamical system or, in other words, a behaviour. Autonomy and also triviality of the zero dynamics are closely related to full column rank and left invertibility of an operator, resp. The relations are depicted in Fig. 1.

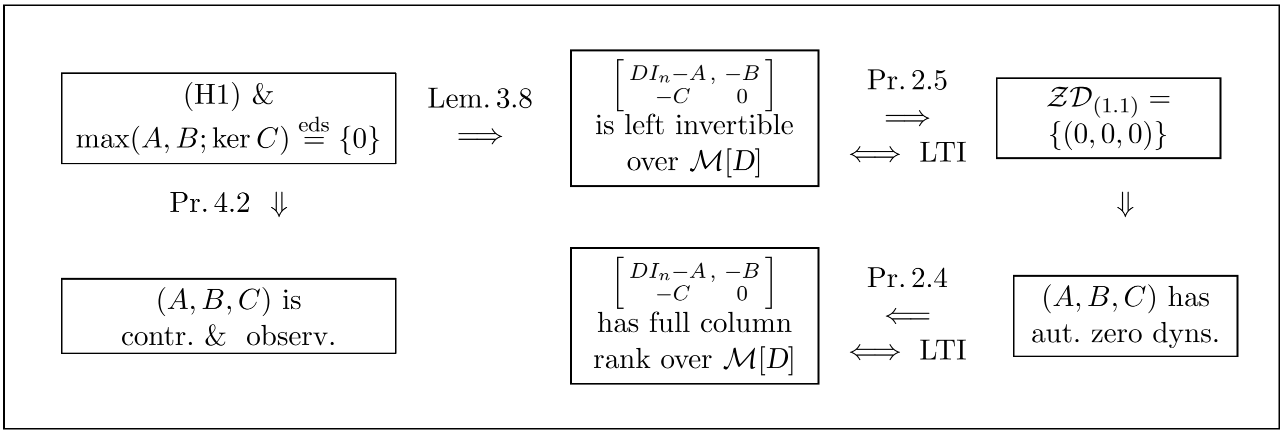

Fig. 1: Implications relating algebraic properties of (A, B, C) with properties of the zero dynamics $\mathcal{ZD}_{(1.1)}$. Arrows labeled with LTI point out stronger versions of the results in the linear time-invariant case. For each unidirectional arrow there is a counterexample showing that the converse implication is false.

In Section 3, we discuss (A, B)-invariant time-varying subspaces and their generators. Because of the close relation to the zero dynamics we focus on the maximal (A, B)-invariant subspace included in the kernel of C. A main step consists in the introduction of the assumptions (H1)-(H2), which require (i) that the input matrix B as well as the generator of the maximal (A, B)-invariant subspace included in the kernel of C to be of constant full rank and (ii) that these subspaces have trivial intersection. This assumption is used to derive a zero dynamics form (3.15) and it is shown by example that these assumptions cannot be weakened in a straightforward way. Furthermore, (H1)-(H2) guarantee the existence of a vector space isomorphism between the zero dynamics and the maximal (A, B)-invariant subspace included in the kernel of C. See Fig. 2.

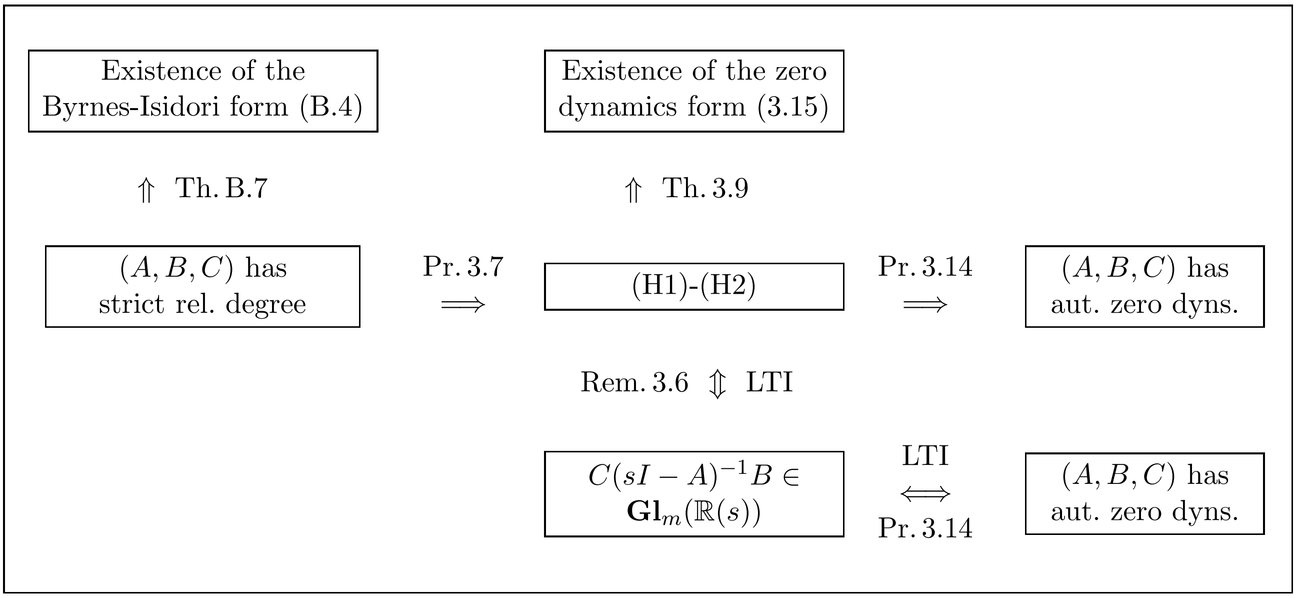

Fig. 2: Implications relating relative degree, the assumptions (H1)-(H2) and autonomy of the zero dynamics. Arrows labeled with LTI point out stronger versions of the results in the linear time-invariant case. For each unidirectional arrow there is a counterexample showing that the converse implication is false.

The assumption of a strict relative degree is stronger than Assumptions (H1)-(H2) and allows to show that the zero dynamics are a direct summand of the behaviour, and that the maximal (A, B)-invariant subspace included in the kernel of C, or equivalently the zero dynamics, can be viewed as the kernel of a certain matrix.
A further important aspect is that (H1)-(H2) imply the autonomy of the zero dynamics. For time-invariant systems, these assumptions are even equivalent to the autonomy. This fact was not realized in [17, pp. 298-300] although there (H1)-(H2) are used to derive the zero dynamics form in the time-invariant case.
In Section 4, we exploit an operator characterization of controllability and observability of time-varying systems [15] to show the relationship between triviality of the maximal (A, B)-invariant subspace included in $\ker C$ and controllability and observability. See Fig. 1.
In Section 5 the stabilization problem is solved using the tools presented so far. For square systems, i.e. $p = m$, it is shown that the Assumptions (H1)-H(2) – which are for time-invariant systems equivalent to autonomous zero dynamics – together with exponential stability of the zero dynamics are sufficient for the existence of an exponentially stabilizing feedback, provided we are dealing with bounded data. The proof relies on the transformation to the zero dynamics form and the analysis of the system in this simpler form. The result is also true for the problem of uniform exponential stabilization if uniform stability properties are required for the zero dynamics form. It also gives a deeper and sharper understanding in case of time-invariant systems.
Some basic algebraic facts about the ring $\mathcal{M}[D]$ are presented in Appendix A and in Appendix B we study some aspects of the (strict) relative degree of time-varying systems and recall the definition of the Byrnes-Isidori form and show some new aspects thereof.

We close the introduction with a list of nomenclature used in the present paper.

$\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$		the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and real numbers, resp.
$\mathcal{R}^{n\times m}$		the set of $n \times m$ matrices with entries in a ring $\mathcal{R}$
$\mathbf{Gl}_n(\mathcal{R})$		the general linear group of invertible $n \times n$ matrices over $\mathcal{R}$
$\\|x\\|$	=	$\sqrt{x^\top x}$, the Euclidean norm of $x \in \mathbb{R}^n$
$\\|A\\|$	=	$\max \{ \\|A\,x\\| \mid x \in \mathbb{R}^m,\ \\|x\\| = 1 \}$, induced matrix norm of $A \in \mathbb{R}^{n\times m}$
$\mathcal{A}$		the ring of real analytic functions $f : \mathbb{R} \to \mathbb{R}$
$\mathcal{M}$		the field of real meromorphic functions, i.e. the quotient field of $\mathcal{A}$
$\mathcal{A}_{\text{pw}}$	=	$\{ f : \mathbb{R} \setminus \mathcal{I} \to \mathbb{R} \mid f$ is real analytic and $\mathcal{I} \subset \mathbb{R}$ a discrete set $\}$, the set of piecewise analytic functions; a set $\mathcal{I} \subseteq \mathbb{R}$ is called *discrete* if, and only if, for any compact $K \subseteq \mathbb{R}$, we have that $K \cap \mathcal{I}$ is finite
$\mathcal{AC}$		the set of absolutely continuous functions $f : \mathbb{R} \to \mathbb{R}$, see [11, Def. A.3.12]
$\mathcal{PC}$		the set of piecewise continuous functions $f : \mathbb{R} \to \mathbb{R}$, i.e., f is left continuous everywhere, has only finitely many discontinuities on any compact subset of $\mathbb{R}$, and the right limits exist at the discontinuities
$\mathcal{C}^\ell$		the set of ℓ-times continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ for $\ell \in \mathbb{N}_0 \cup \{\infty\}$
$\mathcal{L}^\infty$		the set of measurable, essentially bounded functions $f : \mathbb{R} \to \mathbb{R}$
$\operatorname{dom} f$		the domain of definition of the function f
$f\|_{\mathcal{I}}$		the restriction of the function f to a set $\mathcal{I} \subseteq \operatorname{dom} f$
eds		means "for all $t \in \mathbb{R}$ with the exception of a discrete set", i.e., the respective statement is valid for all $t \in \mathbb{R} \setminus \mathcal{I}$, where $\mathcal{I} \subseteq \mathbb{R}$ is a discrete set
$\Phi_A(\cdot, \cdot)$		the transition matrix of $\dot{x} = A(t)x$ for $A \in \mathcal{A}^{n\times n}$.

2 Zero dynamics

In this section we introduce the crucial concept of zero dynamics for system (1.1) as well as the notion of autonomous zero dynamics. The concept of zero dynamics has been introduced by Byrnes and Isidori [7] and is well investigated for nonlinear systems [17, Sec. 4.3, 5.1, 6.1] and time-invariant linear (differential-algebraic) systems [3, 5, 6]; for time-varying systems they have not yet been studied. The zero dynamics are, loosely speaking, those dynamics which are not visible at the output. The concept of autonomy stems from the behavioural approach, see [20, Def. 3.2.1]. Several algebraic criteria for the autonomy of the zero dynamics are derived. Examples show the limitations of these criteria.

Definition 2.1 (Zero dynamics)**.**
The *zero dynamics* of system (1.1) are defined as the set of trajectories

$$\mathcal{ZD}_{(1.1)} := \left\{ \ (x,u,y) \in \mathfrak{B}_{(1.1)} \ \middle| \ y = 0 \ \right\} .$$

The zero dynamics $\mathcal{Z}D_{(1.1)}$ are called *autonomous* if, and only if,

$$\forall\, w_1, w_2 \in \mathcal{Z}D_{(1.1)} \ \forall\, I \subseteq \mathbb{R} \text{ an open interval}: \quad w_1|_I = w_2|_I \quad \Longrightarrow \quad w_1 = w_2 \,. \tag{2.1}$$

◇

It will be advantageous to rewrite the zero dynamics as follows

$$\forall\,(x,u,y) \in \mathcal{AC}^n \times \mathcal{PC}^m \times \mathcal{AC}^p: \quad (x,u,y) \in \mathcal{ZD}_{(1.1)} \quad \Longleftrightarrow \quad \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} I_n - A & -B \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0 \,. \tag{2.2}$$

Remark 2.2 (Autonomous zero dynamics)**.**
By linearity of (1.1), the set $\mathcal{ZD}_{(1.1)}$ is a real vector space. Therefore, the zero dynamics $\mathcal{Z}D_{(1.1)}$ are autonomous if, and only if, for any $w \in \mathcal{Z}D_{(1.1)}$ which satisfies $w|_I = 0$ on some open interval $I \subseteq \mathbb{R}$, it follows that $w = 0$. ◇

Next we show that the zero dynamics carries in a certain sense the structure of a dynamical system.

Remark 2.3 (Zero dynamics as a dynamical system)**.**
We now show that the zero dynamics of $(A, B, C) \in \Sigma_{n,m,p}$ carries the structure of an $\mathbb{R}$-linear dynamical system as defined in [11, Defs. 2.1.1, 2.1.26]. For any $(t_0, x^0, u(\cdot)) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{PC}^m$ there exists a unique maximal solution of the initial value problem (1.1), $x(t_0) = x^0$, defined on $\mathbb{R}$. Denote this solution by $\varphi(\cdot\,; t_0, x^0, u(\cdot)) : \mathbb{R} \to \mathbb{R}^n$. Then the *state transition map* of (1.1) is the map defined on its domain of definition

$$\mathcal{D}_\varphi := \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{PC}^m \quad \text{by} \quad \varphi : (t, t_0, x^0, u) \mapsto \varphi(t, t_0, x^0, u) \in \mathbb{R}^n \,.$$

The *output map* of (1.1) is defined by

$$\eta : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p, \ (t, x, u) \mapsto C(t)x \,.$$

We now restrict φ to the set

$$\mathcal{D}_\varphi^0 := \left\{ \ (t, t_0, x^0, u(\cdot)) \in \mathcal{D}_\varphi \ \middle| \ \forall\, \tau \in \mathbb{R}: \ C(\tau)\varphi(\tau; t_0, x^0, u(\cdot)) = 0 \ \right\} ,$$

and by abuse of notation we write the same symbol for the restriction. It is readily verified that the structure $(\mathbb{R}, \mathbb{R}^m, \mathcal{PC}^m, \mathbb{R}^n, \mathbb{R}^p, \varphi, \eta)$, with the restricted state transition map $\varphi : \mathcal{D}_\varphi^0 \to \mathbb{R}^n$ satisfies all the requirements of a linear dynamical system.

The set $\mathcal{D}^0_\varphi$ determines the zero dynamics in an equivalent manner. More precisely, for $(x,u,y) \in \mathcal{ZD}_{(1.1)}$ we have for all $t_0, t \in \mathbb{R}$ that $(t, t_0, x(t^0), u) \in \mathcal{D}^0_\varphi$. Conversely, if we introduce an equivalence relation on $\mathcal{D}^0_\varphi$ by

$$(t, t_0, x^0, u) \sim (t', t'_0, x^1, u') \;:\Leftrightarrow\; \varphi(t'_0; t_0, x_0, u) = x^1 \text{ and } u = u',$$

then the equivalence classes $[(t, t_0, x^0, u)]$ correspond to maximal trajectories that generate an output that is vanishing identically. So the equivalence classes are in one-to-one correspondence to the elements of $\mathcal{ZD}_{(1.1)}$. In this sense $\mathcal{ZD}_{(1.1)}$ describes a dynamical system, as it is the space of trajectories of a dynamical system. ◇

The analysis of the zero dynamics via the differential operator in (2.2) is closely related to the algebraic operator $\begin{bmatrix} DI_n - A, & -B \\ -C & 0 \end{bmatrix} \in \mathcal{A}[D]^{(n+p)\times(n+m)}$. As a first step we show that the latter has full column rank if the zero dynamics are autonomous.

Proposition 2.4 (Autonomous zero dynamics implies full column rank)**.**
Let $(A,B,C) \in \Sigma_{n,m,p}$. Then

$$\mathcal{ZD}_{(1.1)} \text{ are autonomous} \quad \begin{matrix} \Longrightarrow \\ \not\Longleftarrow \end{matrix} \quad \mathrm{rk}_{\mathcal{M}[D]} \begin{bmatrix} DI_n - A & -B \\ -C & 0 \end{bmatrix} = n + m\,,$$

and equivalence holds for time-invariant systems $(A,B,C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p\times n}$.

Proof: Let $q := n+m$, $g := n+p$, $R(D) := \begin{bmatrix} DI_n - A, & -B \\ -C & 0 \end{bmatrix} \in \mathcal{A}[D]^{g\times q}$, and $\ell := \mathrm{rk}_{\mathcal{M}[D]} R(D)$.
$\Longrightarrow$: Suppose that $R(D)$ is in Teichmüller-Nakayama form (A.1). In view of (1.2), we may rewrite $V(D)$ in (A.1) as

$$V(D) = \left(\sum_{k=0}^{N} D^k \frac{n^k_{ij}}{d^k_{ij}} \right)_{i,j} \qquad \text{for appropriate coprime } n^k_{ij},\, d^k_{ij} \in \mathcal{A}\,.$$

Let

$$\gamma \in \mathcal{A} \setminus \{0\} \quad \text{be the product of all } d^k_{ij},\; i,j \in \{1,\dots,n\},\; k \in \{0,\dots,N\}.$$

Seeking a contradiction, assume that $R(D)$ does not have full column rank, i.e., $q - \ell > 0$. Now choose $z \in \mathcal{C}^\infty$ and an open interval $I \subseteq \mathbb{R}$ such that $z|_I = 0$ and

$$w := V(\tfrac{\mathrm{d}}{\mathrm{d}t}) \left(\gamma \begin{bmatrix} 0_{q-1} \\ z \end{bmatrix} \right) \neq 0.$$

Note that $w \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^q)$ as the singularities in V are canceled by γ. Now we have for all $t \in \mathbb{R}$ that

$$R(\tfrac{\mathrm{d}}{\mathrm{d}t}) w(t) = U^{-1}(\tfrac{\mathrm{d}}{\mathrm{d}t}) \operatorname{diag}\left\{ I_{\ell-1},\, r(\tfrac{\mathrm{d}}{\mathrm{d}t}),\, 0_{(g-\ell)\times(q-\ell)} \right\} \left(\gamma(t) \begin{bmatrix} 0_{q-1} \\ z(t) \end{bmatrix} \right) = 0\,,$$

and hence $\big([I_n, 0]w, [0, I_m]w, [C, 0]w\big) \in \mathcal{ZD}_{(1.1)}$ and $w|_I = 0$. But $w \neq 0$, which contradicts autonomy of the zero dynamics.
$\not\Longleftarrow$: Consider the system (1.1) with

$$A(t) = \begin{bmatrix} 0 & 6t^2 + 2 \\ 0 & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ 3t^2 + 2 \end{bmatrix}, \quad C(t) = [1, -t^3], \quad t \in \mathbb{R},$$

and define

$$u(t) := \begin{cases} 0, & t \le 0 \\ \mathrm{e}^{-1/t^2}, & t > 0, \end{cases} \qquad x_1(t) := t^6 u(t), \quad x_2(t) := t^3 u(t), \quad t \in \mathbb{R}.$$

It can be verified that $u, x_1, x_2 \in \mathcal{C}^\infty$ and $t^3\dot{u}(t) = 2u(t)$ for all $t \in \mathbb{R}$. Furthermore, it is a simple calculation that $x := (x_1, x_2)^\top$, u, and $y := 0$ solve (1.1) for all $t \in \mathbb{R}$, thus $(x, u, y) \in \mathcal{ZD}_{(1.1)}$. However, $(x, u)|_{(-\infty,0)} = 0$ and $(x, u) \neq 0$, hence $\mathcal{ZD}_{(1.1)}$ is not autonomous. On the other hand,

$$R(D) = \begin{bmatrix} DI_2 - A & -B \\ -C & 0 \end{bmatrix} = \begin{bmatrix} D & -6t^2 - 2 & 0 \\ 0 & D & -3t^2 - 2 \\ -1 & t^3 & 0 \end{bmatrix}$$

has full column rank over $\mathcal{M}[D]$. This may be seen by the factorization using

$$X(D) := \begin{bmatrix} D & t^3D - 3t^2 - 2 & t^3 \\ 0 & D & 1 \\ -1 & 0 & 0 \end{bmatrix} \in \mathbf{Gl}_3(\mathcal{M}[D]), \quad \text{and} \quad R(D) = X(D) \begin{bmatrix} 1 & -t^3 & 0 \\ 0 & 1 & -t^3 \\ 0 & 0 & t^3D - 2 \end{bmatrix}.$$

$\Longleftarrow$ for $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n}$: If $R(D) \in \mathbb{R}[D]^{g\times q}$ has column rank q over $\mathcal{M}[D]$, then by Corollary A.2, $R(D)$ has full column rank over $\mathbb{R}[D]$. Autonomous zero dynamics can then be inferred from [3, Prop. 3.6]. This completes the proof of the proposition. □

We now strengthen the condition that $\left[\begin{smallmatrix} DI_n-A, & -B \\ -C & 0 \end{smallmatrix}\right]$ has full column rank to that of left invertibility over $\mathcal{M}[D]$; then we can show that the zero dynamics are trivial.

Proposition 2.5 (Left invertibility implies trivial zero dynamics).
Let $(A, B, C) \in \Sigma_{n,m,p}$. Then

$$\begin{bmatrix} DI_n - A & -B \\ -C & 0 \end{bmatrix} \text{ is left invertible over } \mathcal{M}[D] \quad \begin{array}{c}\Longrightarrow \\ \not\Longleftarrow\end{array} \quad \mathcal{ZD}_{(1.1)} = \{(0, 0, 0)\},$$

and equivalence holds for time-invariant systems $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n}$.

Proof: Let $q := n + m$, $g := n + p$ and $R(D) := \left[\begin{smallmatrix} DI_n-A, & -B \\ -C & 0 \end{smallmatrix}\right] \in \mathcal{A}[D]^{g\times q}$.
$\Longrightarrow$: Let $(x, u, y) \in \mathcal{ZD}_{(1.1)}$ and $w := (x^\top, u^\top)^\top$. Then $R(\frac{\mathrm{d}}{\mathrm{d}t})w(t) = 0$ for almost all $t \in \mathbb{R}$. Since $R(D)$ is left invertible, there exists $T(D) \in \mathcal{M}[D]^{q\times g}$ such that $T(D)R(D) = I_q$, and hence $T(\frac{\mathrm{d}}{\mathrm{d}t})R(\frac{\mathrm{d}}{\mathrm{d}t})w(t) = w(t) = 0$ for almost all $t \in \mathbb{R}$ with the exception of the discrete set. The latter finding together with $(x, u) \in \mathcal{AC}^n \times \mathcal{PC}^m$ yields $(x, u) = 0$ and proves that the zero dynamics are trivial.
$\not\Longleftarrow$: Consider the system (1.1) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C(t) = [1, t], \quad t \in \mathbb{R},$$

and observe that any $(x, u, y) \in \mathcal{ZD}_{(1.1)}$ satisfies

$$x_1(t) = \frac{t^2}{2}u(t), \quad x_2(t) = -\frac{t}{2}u(t), \quad t\dot{u}(t) = -3u(t), \quad t \in \mathbb{R}.$$

The latter differential equation implies that $u(t) = ct^{-3}$ for some $c \in \mathbb{R}$ on both $(-\infty, 0)$ and $(0, \infty)$. Since u is piecewise continuous this yields $u = 0$. Therefore, $\mathcal{ZD}_{(1.1)} = \{(0, 0, 0)\}$. However, the Teichmüller-Nakayama form (A.1) is given by

$$R(D) = \begin{bmatrix} DI_2 - A & -B \\ -C & 0 \end{bmatrix} = \begin{bmatrix} D & -1 & 0 \\ 0 & D & -1 \\ 1 & t & 0 \end{bmatrix} = \begin{bmatrix} D & -tD - 2 & t/2 \\ 0 & D & -1/2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & tD + 3 \end{bmatrix} \begin{bmatrix} 1 & -t & t^2/2 \\ 0 & 1 & -t/2 \\ 0 & 0 & 1 \end{bmatrix},$$

and hence we see that $R(D)$ is not left invertible.
$\Longleftarrow$ for time-invariant systems: This is a consequence of [3, Prop. 3.10] and the observation that any left inverse of $R(D)$ over $\mathbb{R}[D]$ is also a left inverse of $R(D)$ over $\mathcal{M}[D]$. This completes the proof of the proposition. $\square$

Next we exploit the Byrnes-Isidori form, which is introduced in Appendix B, to show that the zero dynamics are a direct summand of the behaviour $\mathfrak{B}_{(1.1)}$.

Proposition 2.6.
Suppose $(A, B, C) \in \Sigma_{n,m,m}$ has strict relative degree $\rho \in \mathbb{N}$. Then we have, for any $t_0 \in \mathbb{R}$ and $U(\cdot)$ from Theorem B.7, that

$$\mathfrak{B}_{(1.1)} \;=\; \mathcal{ZD}_{(1.1)} \oplus \left\{\, (x,u,y) \in \mathfrak{B}_{(1.1)} \;\middle|\; [0,...,0,I_{n-\rho m}]\,U(t_0)x(t_0) = 0 \,\right\}.$$

Proof: By Theorem B.7 (i), the parameter $\Gamma(t)$ is invertible for all $t \in \mathbb{R}$. It is then a direct consequence of the Byrnes-Isidori form (B.4) that the zero dynamics are given by

$$\mathcal{ZD}_{(1.1)} = \left\{ (x,u,y) \in \mathcal{AC}^n \times \mathcal{PC}^m \times \mathcal{AC}^p \;\middle|\; \begin{array}{l} (x,u,y) = \left(U^{-1}[0,\ldots,0,\eta^\top]^\top, -\Gamma^{-1}S\eta, 0\right) \\ \text{solves (1.1) for a.a. } t \in \mathbb{R}, \text{ where } \dot\eta = Q(t)\eta \end{array} \right\}.$$

Now for any $(x,u,y) \in \mathfrak{B}_{(1.1)}$ we have

$$[0,...,0,I_{n-\rho m}]U(t_0)\left(x(t_0) - U(t_0)^{-1}[0,\ldots,0,\eta(t_0)^\top]^\top\right) = 0.$$

Thus, for $\eta(\cdot)$ defined by $\dot\eta = Q(t)\eta$, $\eta(t_0) = [0,...,0,I_{n-\rho m}]U(t_0)x(t_0)$, it follows that

$$\begin{aligned} &\big(x(\cdot), u(\cdot), y(\cdot)\big) \\ &\quad = \left(U(\cdot)^{-1}\begin{bmatrix} 0_{\rho m} \\ \eta(\cdot)\end{bmatrix}, -\Gamma(\cdot)^{-1}S(\cdot)\eta(\cdot), 0\right) + \left(x(\cdot) - U(\cdot)^{-1}\begin{bmatrix} 0_{\rho m} \\ \eta(\cdot)\end{bmatrix}, u(\cdot) + \Gamma(\cdot)^{-1}S(\cdot)\eta(\cdot), y(\cdot)\right). \end{aligned}$$

Finally, the claim follows since by linearity $\mathfrak{B}_{(1.1)}$ is a vector space over $\mathbb{R}$. $\square$

The next proposition is an immediate consequence of Proposition B.5.

Proposition 2.7 (Characterization of zero dynamics)**.**
Suppose $(A, B, C) \in \Sigma_{n,m,m}$ has strict relative degree $\rho \in \mathbb{N}$. Then

$$(x,u,y) \in \mathcal{ZD}_{(1.1)} \iff \begin{array}{ll} (i) & y = 0\,, \\ (ii) & u = -\left[\left(\tfrac{\mathrm{d}}{\mathrm{d}t}I + A_r\right)^{\rho-1}(C)B\right]^{-1}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}I + A_r\right)^{\rho}(C)\,x\,, \\ (iii) & x \text{ solves } \dot x = \left[A - B\left[\left(\tfrac{\mathrm{d}}{\mathrm{d}t}I + A_r\right)^{\rho-1}(C)B\right]^{-1}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}I + A_r\right)^{\rho}(C)\right]x\,. \end{array}$$

3 $(\mathbf{A}, \mathbf{B})$-invariant subspaces

The zero dynamics are the linear space of system trajectories that have zero output. In this section we show that, given the assumptions (H1)-(H2) described below, this space is isomorphic to the supremal (in fact maximal) (A, B)-invariant time-varying subspace which is included in $\ker C$ for almost all times. As the main result of this section we derive the so-called zero dynamics form in Theorem 3.9.

A basic tool in the analysis are time-varying subspaces $\mathcal{V}$ generated by a piecewise analytic matrix-valued function V as introduced similarly in [12]. Given a time-dependent subspace of $\mathbb{R}^n$ denoted by $\mathcal{V} = \big(\mathcal{V}(t)\big)_{t\in\mathbb{R}}$ and a matrix-valued function $V : \mathbb{R}\setminus\mathcal{I} \to \mathbb{R}^{n\times k}$, where $\mathcal{I} \subseteq \mathbb{R}$ is a discrete set, we write

$$\mathcal{V} \overset{\text{eds}}{=} \operatorname{im} V \quad :\Longleftrightarrow \quad \mathcal{V}(t) = \operatorname{im} V(t) \ \text{ for all } t \in \mathbb{R} \text{ with the exception of a discrete set,}$$

define

$$\mathcal{W}_n := \left\{ \mathcal{V} = \big(\mathcal{V}(t)\big)_{t\in\mathbb{R}} \;\middle|\; \begin{array}{l} \mathcal{V}(t) \text{ is a subspace of } \mathbb{R}^n \text{ for all } t \in \mathbb{R} \text{ and} \\ \exists\, k \in \mathbb{N}\ \exists\, V \in \mathcal{A}_{\text{pw}}^{n\times k} :\ \mathcal{V} \overset{\text{eds}}{=} \operatorname{im} V \end{array} \right\},$$

and endow this set with the partial order

$$\mathcal{V}_1 \overset{\text{eds}}{\subset} \mathcal{V}_2 \quad :\Longleftrightarrow \quad \mathcal{V}_1(t) \subseteq \mathcal{V}_2(t) \ \text{ for all } t \in \mathbb{R} \text{ with the exception of a discrete set.}$$

The matrix $V \in \mathcal{A}_{\text{pw}}^{n\times k}$ is called a *generator of* $\mathcal{V} \in \mathcal{W}_n$ if, and only if, $\mathcal{V} \overset{\text{eds}}{=} \operatorname{im} V$. Note that the partial order $\overset{\text{eds}}{\subset}$ and the equality $\overset{\text{eds}}{=}$ allow for the definition of equivalence classes $[\mathcal{V}]$ for $\mathcal{V} \in \mathcal{W}_n$ as follows:

$$[\mathcal{V}] := \left\{ \mathcal{W} \in \mathcal{W}_n \;\middle|\; \mathcal{V} \overset{\text{eds}}{=} \mathcal{W} \right\}.$$

Note also that $\mathcal{V} \in \mathcal{W}_n$ does not have a unique generator, and different generators may have rank drops and singularities at different points. However, among the set of all generators there is one with piecewise constant rank. This is the content of the following straightforward modification of [12, Prop. 2.6].

Lemma 3.1 (Piecewise constant rank generators)**.**
For any $k \in \mathbb{N}$ and $V \in \mathcal{A}_{\text{pw}}^{n\times k}$ there exist $\hat{V} \in \mathcal{A}_{\text{pw}}^{n\times k}$ and a piecewise constant function $r : \operatorname{dom} V \to \mathbb{N}_0$ such that

$$\operatorname{dom} V = \operatorname{dom} \hat{V} \quad \wedge \quad \operatorname{im} V \overset{\text{eds}}{=} \operatorname{im} \hat{V} \quad \wedge \quad \operatorname{rk} \hat{V} \overset{\text{eds}}{=} r \quad \wedge \quad \Big[\forall\, t \in \operatorname{dom} V :\ \operatorname{im} V(t) \subseteq \operatorname{im} \hat{V}(t)\Big].$$

Now we introduce a concept of (A, B)-invariance for time-varying systems; it stems from [12, Sec. 4] but is slightly different.

Definition 3.2 ((A, B)-invariance)**.**
Let $(A, B) \in \mathcal{A}^{n\times n} \times \mathcal{A}^{n\times m}$ and $\mathcal{V} \in \mathcal{W}_n$ with generator $V \in \mathcal{A}_{\text{pw}}^{n\times k}$ for some $k \in \mathbb{N}$. Then $\mathcal{V}$ is called (A, B)*-invariant* if, and only if,

$$\exists\, N \in \mathcal{A}_{\text{pw}}^{k\times k}\ \exists\, M \in \mathcal{A}_{\text{pw}}^{m\times k} :\ \big(\tfrac{\mathrm{d}}{\mathrm{d}t} I - A\big) V \overset{\text{eds}}{=} VN + BM\,. \tag{3.1}$$

◇

Note that the set $\mathcal{A}_{\text{pw}}$ of piecewise analytic functions includes in particular the set $\mathcal{M}$ of meromorphic functions. The use of piecewise analytic N and M in Definition 3.2 is necessary as the following example shows. Consider

$$V(t) = \begin{bmatrix} 0 \\ t \end{bmatrix}, \qquad A = 0_{2\times 2}, \qquad B(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad t \in \mathbb{R}\,.$$

Then

$$\forall\, t \in \mathbb{R}\setminus\{0\} \ :\ \big(\tfrac{\mathrm{d}}{\mathrm{d}t} I - A(t)\big)(V(t)) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} \cdot t^{-1} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot 0\,.$$

Note that the rank drop in $V(0)$ leads to a coefficient matrix $N(\cdot)$ with a pole at zero.

Now we consider, for $(A,B,C) \in \Sigma_{n,m,p}$, the set of all (A,B)-invariant subspaces in $\mathcal{W}_n$ which are included in the kernel of C; more precisely

$$\mathcal{L}(A,B;\ker C) := \left\{ \mathcal{V} \in \mathcal{W}_n \;\middle|\; \mathcal{V} \text{ is } (A,B)\text{-invariant and } \mathcal{V} \overset{\text{eds}}{\subset} \ker C \right\}.$$

It is easy to see that this set is nonempty and closed under (pointwise) subspace addition. Hence it is an upper semi lattice with partial order $\overset{\text{eds}}{\subset}$. Using the existence of generators with piecewise constant rank (see Lemma 3.1) it follows that there is a $\mathcal{V} \in \mathcal{L}(A,B;\ker C)$ with maximal, piecewise constant dimension with the exception of a discrete set. That is, its piecewise analytic and piecewise constant rank generator $V(\cdot)$ satisfies

$$\operatorname{rk} V(t) \geq \operatorname{rk} W(t) \quad \text{for all } t \in \mathbb{R}, \text{ eds},$$

and any $\mathcal{W} \in \mathcal{L}(A,B;\ker C)$ with piecewise constant rank generator W. Similarly to the time-invariant case, see [24, Lemma 4.4], we thus have the existence of a maximal element

$$\max(A,B;\ker C) \;:=\; \sup \mathcal{L}(A,B;\ker C) \;=\; \max \mathcal{L}(A,B;\ker C) \;\in \mathcal{W}_n;$$

this maximal element is unique relative to $\overset{\text{eds}}{=}$. . In the following we will identify $\max(A,B;\ker C)$ with its equivalence class $[\max(A,B;\ker C)]$.

The following proposition shows that $\max(A,B;\ker C)$ has a simple representation if (A,B,C) has a strict relative degree.

Proposition 3.3 (Representation of $\max(A,B;\ker\, C)$)**.**
Let $(A,B,C) \in \Sigma_{n,m,m}$ and use $\mathcal{C}$ as defined in Theorem B.7. Then we have

$$(A,B,C) \textit{ has strict relative degree } \rho \in \mathbb{N} \qquad \Longrightarrow \qquad \max(A,B;\ker\, C) \overset{\text{eds}}{=} \ker \mathcal{C}.$$

Proof: Let $U \in \mathbf{Gl}_n(\mathcal{A})$ be as in Theorem B.7 and $(\hat{A},\hat{B},\hat{C})$ as in (B.8). On several occasions, we will make use of the fact that

$$\forall\, t \in \mathbb{R}: \; U(t)\, \ker \mathcal{C}(t) = \ker \left(\mathcal{C}(t) U(t)^{-1} \right). \tag{3.2}$$

Note that this is a consequence of the invertibility of $U(t)$ and does not depend on the special structure of U or $\mathcal{C}$.
Step 1: We first show

$$\max(A,B;\ker\, C) \overset{\text{eds}}{=} U^{-1}\; \max(\hat{A},\hat{B};\ker \hat{C})$$

which is equivalent to

$$U\; \max(A,B;\ker\, C) \overset{\text{eds}}{=} \max(\hat{A},\hat{B};\ker \hat{C})\,. \tag{3.3}$$

Let $\mathcal{V} \in \mathcal{W}_n$ be any (A,B)-invariant time-varying subspace included in $\ker C$ and generated by $V \in \mathcal{A}_{\text{pw}}^{n\times k}$, $k \in \mathbb{N}$. Then (3.1) holds for some $N \in \mathcal{A}_{\text{pw}}^{k\times k}$, $M \in \mathcal{A}_{\text{pw}}^{m\times k}$ and hence we have

$$\begin{aligned}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}I - \hat{A}\right)(UV) &\overset{\text{eds}}{=} \dot{U}V + U\dot{V} - \hat{A}UV \underset{\text{(B.8)}}{\overset{\text{eds}}{=}} \dot{U}V + U\dot{V} - (UA + \dot{U})V \\ &\overset{\text{eds}}{=} U\left(\tfrac{\mathrm{d}}{\mathrm{d}t}I - A\right)(V) \underset{\text{(B.8), (3.1)}}{\overset{\text{eds}}{=}} (UV)N + \hat{B}M.\end{aligned}$$

Since $UV \in \mathcal{A}_{\rm pw}^{n\times k}$, $U\mathcal{V} = \big(U(t)\mathcal{V}(t)\big)_{t\in\mathbb{R}}$ is $(\hat{A},\hat{B})$-invariant. Furthermore,

$$\operatorname{im}(U(t)V(t)) = U(t)\operatorname{im}V(t) \subseteq U(t)\ker C(t) \overset{(3.2)}{=} \ker\big(C(t)U(t)^{-1}\big) \overset{(B.8)}{=} \ker \hat{C}(t)$$

for all $t\in\mathbb{R}$, eds, and so "$\overset{\rm eds}{\subset}$" in (3.3) follows. The proof of "$\overset{\rm eds}{\supset}$" in (3.3) is analogous and omitted.

Step 2: We show that

$$U^{-1}\ \max(\hat{A},\hat{B};\ker\hat{C}) \overset{\rm eds}{=} \ker\mathcal{C}$$

which, in view of (3.2) and (B.18), is equivalent to

$$\max(\hat{A},\hat{B};\ker\hat{C}) \overset{\rm eds}{=} \mathcal{X} := \operatorname{im}\,[0,...,0,I_{n-\rho m}]^\top \in \mathcal{W}_n\,. \tag{3.4}$$

Step 2a: We show "$\overset{\rm eds}{\supset}$" in (3.4). The family $\mathcal{X}$ is $(\hat{A},\hat{B})$-invariant, since

$$\big(\tfrac{\rm d}{{\rm d}t}I-\hat{A}\big)\begin{bmatrix}0\\ \vdots\\ 0\\ I_{n-\rho m}\end{bmatrix} \overset{(B.9)}{=} -\begin{bmatrix}0\\ \vdots\\ 0\\ S\\ Q\end{bmatrix} = \begin{bmatrix}0\\ \vdots\\ 0\\ I_{n-\rho m}\end{bmatrix}(-Q) + \begin{bmatrix}0\\ \vdots\\ 0\\ \Gamma\\ 0\end{bmatrix}(-\Gamma^{-1}S) = \begin{bmatrix}0\\ \vdots\\ 0\\ I_{n-\rho m}\end{bmatrix}N + \hat{B}M,$$

where $N := -Q \in \mathcal{A}^{(n-\rho m)\times(n-\rho m)}$ and $M := -\Gamma^{-1}S \in \mathcal{A}^{m\times(n-\rho m)}$. Furthermore,

$$\forall\, t\in\mathbb{R}\ :\ \operatorname{im}\begin{bmatrix}0\\ \vdots\\ 0\\ I_{n-\rho m}\end{bmatrix} \overset{(B.18)}{=} \ker\big(\mathcal{C}(t)U(t)^{-1}\big) \overset{(B.8)}{=} \ker\begin{bmatrix}\hat{C}(t)\\ \big(\tfrac{\rm d}{{\rm d}t}I + A(t)_r\big)\big(C(t)\big)U(t)^{-1}\\ \vdots\\ \big(\tfrac{\rm d}{{\rm d}t}I + A(t)_r\big)^{\rho-1}\big(C(t)\big)U(t)^{-1}\end{bmatrix} \subseteq \ker\hat{C}(t)$$

and therefore $\mathcal{X} \overset{\rm eds}{\subset} \max(\hat{A},\hat{B};\ker\hat{C})$.

Step 2b: We show $\overset{\rm eds}{\subset}$ in (3.4), i.e., that any $(\hat{A},\hat{B})$-invariant time-varying subspace $\hat{\mathcal{V}}\in\mathcal{W}_n$ included in $\ker\hat{C}$ satisfies $\hat{\mathcal{V}} \overset{\rm eds}{\subset} \mathcal{X}$. Let $\hat{V}\in\mathcal{A}_{\rm pw}^{n\times k}$, $k\in\mathbb{N}$, and $N\in\mathcal{A}_{\rm pw}^{k\times k}$, $M\in\mathcal{A}_{\rm pw}^{m\times k}$ be such that

$$\operatorname{im}\hat{V} \overset{\rm eds}{\subset} \ker\hat{C} \quad\text{and}\quad \big(\tfrac{\rm d}{{\rm d}t}I-\hat{A}\big)\hat{V} \overset{\rm eds}{=} \hat{V}N + \hat{B}M\,. \tag{3.5}$$

It suffices to show that

$$\forall\, j=1,...,\rho\ :\ S_j\hat{V} \overset{\rm eds}{=} 0\,, \tag{3.6}$$

where

$$S_j := \operatorname{diag}\{\underbrace{I_m,\ldots,I_m}_{j\text{-times}},0,\ldots,0\} \in \mathbb{R}^{n\times n}, \quad j=1,...,\rho.$$

We show (3.6) by induction. If $j=1$, then

$$S_1\hat{V} \underset{(B.9)}{\overset{\rm eds}{=}} \hat{C}^\top\hat{C}\hat{V} \underset{(3.5)}{\overset{\rm eds}{=}} 0.$$

Suppose $S_j\hat{V} \overset{\rm eds}{=} 0$ holds for some $j\in\{1,\ldots,\rho-1\}$, whence $\tfrac{\rm d}{{\rm d}t}S_j\hat{V} \overset{\rm eds}{=} S_j\tfrac{\rm d}{{\rm d}t}\hat{V} \overset{\rm eds}{=} 0$. Define

$$\hat{V}_i := \operatorname{diag}\{\underbrace{0_{m\times m},\ldots,0_{m\times m}}_{(i-1)\text{-times}},I_m,0,\ldots,0\}\hat{V}\,, \quad i=2,\ldots,j+1\,.$$

We obtain, using $j \le \rho - 1$ in the first and the last equality,

$$\begin{bmatrix} \hat V_2 \\ \vdots \\ \hat V_{j+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \underset{(B.9)}{\overset{\text{eds}}{=}} S_j \hat A \hat V \overset{\text{eds}}{=} -S_j\big(\tfrac{\mathrm{d}}{\mathrm{d}t} I - \hat A\big)(\hat V) \underset{(3.5)}{\overset{\text{eds}}{=}} -S_j \hat V N - S_j \hat B M \underset{(B.9)}{\overset{\text{eds}}{=}} -S_j \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \Gamma \\ 0 \end{bmatrix} M \overset{\text{eds}}{=} 0\,.$$

By continuity, we find $S_{j+1}\hat V \overset{\text{eds}}{=} 0$. The proof of Step 2 is complete, and the proof of the proposition follows from Step 1 and Step 2. □

For the remainder of this section we introduce the following assumptions for $(A, B, C) \in \Sigma_{n,m,p}$.

(H1)	$\forall\, t \in \mathbb{R}:\ \operatorname{rk} B(t) = m\,,$
(H2)	$\exists\, k \in \mathbb{N}_0\ \exists\, V \in \mathcal{A}^{n\times k}$ with constant rank k such that
	$\operatorname{im} V \overset{\text{eds}}{=} \max(A, B; \ker C)$ and $\forall\, t \in \mathbb{R}:\ \operatorname{im} B(t) \cap \operatorname{im} V(t) = \{0\}\,.$

For time-invariant systems $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ with the same number of inputs and outputs, we will see that the Assumptions (H1)-(H2) are equivalent to invertibility of the transfer function (see Remark 3.6) and also to the autonomy of the zero dynamics (see Proposition 3.14).
The following lemma is crucial. If Assumptions (H1)-(H2) hold and the largest (A, B)-invariant subspace included in the kernel of C is considered, then it is possible to require analyticity in the definition of (A, B)-invariant subspaces.

Lemma 3.4.
Let $(A, B) \in \mathcal{A}^{n\times n} \times \mathcal{A}^{n\times m}$ such that (H1)-(H2) are satisfied and let $V \in \mathcal{A}^{n\times k}$ be as in (H2). Then there exist $N \in \mathcal{A}^{k\times k}$ and $M \in \mathcal{A}^{m\times k}$ such that

$$\big(\tfrac{\mathrm{d}}{\mathrm{d}t} I - A\big)V = VN + BM\,. \tag{3.7}$$

Proof: By (H2) we have that $(\operatorname{im} V(t))_{t\in\mathbb{R}} \in \mathcal{W}_n$ is (A, B)-invariant, i.e., there exist $N \in \mathcal{A}_{\text{pw}}^{k\times k}$ and $M \in \mathcal{A}_{\text{pw}}^{m\times k}$ such that (3.1) is satisfied. As the left hand side of that equality is analytic, it follows from the identity theorem, that $VN + BM$ can be extended to an analytic function, so that the equality holds for all $t \in \mathbb{R}$. By (H2) it then follows that each of the summands VN and BM is analytic, as singular points in one summand cannot be canceled by the other one. For the proof that then N, resp. M are analytic we will use the full rank condition of V, resp. B. Clearly it is sufficient to do this once. So assume that M in (3.7) is not analytic. By (H1) and [22, Thm. 1] there exists $S \in \mathbf{Gl}_n(\mathcal{A})$ such that $B^\top S = [F, 0]$, where $F \in \mathbf{Gl}_m(\mathcal{A})$. Therefore, we find that

$$\begin{bmatrix} F^{-\top} & 0 \\ 0 & I_{n-m} \end{bmatrix} S^\top B M = \begin{bmatrix} M \\ 0 \end{bmatrix}.$$

As the left hand side is analytic, this implies that M is analytic. □

In the following we show that if (A, B, C) is time-invariant, then $\max(A, B; \ker C)$ has a time-invariant generator.

Proposition 3.5 (Time-invariant systems and generators)**.**
Let $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n}$ *and let* $\mathcal{V}^* \subseteq \mathbb{R}^n$ *be the largest subspace of* $\mathbb{R}^n$ *such that*

$$A\mathcal{V}^* \subseteq \mathcal{V}^* + \operatorname{im} B \;\wedge\; \mathcal{V}^* \subseteq \ker C. \tag{3.8}$$

Then the sequence $(\mathcal{V}_i)_{i\in\mathbb{N}_0}$ *defined by* $\mathcal{V}_0 := \ker C$ *and*

$$\forall\, i \in \mathbb{N}:\; \mathcal{V}_i := A^{-1}(\mathcal{V}_{i-1} + \operatorname{im} B) \cap \ker C$$

is nested, terminates and satisfies

$$\exists\, k^* \in \mathbb{N}\; \forall\, j \in \mathbb{N}:\; \mathcal{V}_0 \supsetneq \mathcal{V}_1 \supsetneq \cdots \supsetneq \mathcal{V}_{k^*} = \mathcal{V}_{k^*+j} = A^{-1}(\mathcal{V}_{k^*} + \operatorname{im} B) \cap \ker C. \tag{3.9}$$

Furthermore, we have that

$$\mathcal{V}^* = \mathcal{V}_{k^*} \overset{\text{eds}}{=} \max(A, B; \ker C).$$

Proof: It follows from [3, Lem. 3.4] that (3.9) holds and that $\mathcal{V}^* = \mathcal{V}_{k^*}$. In order to show

$$\mathcal{V}_{k^*} \overset{\text{eds}}{=} \max(A, B; \ker C), \tag{3.10}$$

let $V \in \mathbb{R}^{n\times k}$ be such that $\operatorname{im} V = \mathcal{V}_{k^*}$ and observe that by (3.8) there exist $N \in \mathbb{R}^{k\times k}$, $M \in \mathbb{R}^{m\times k}$ such that $AV = VN + BM$ and $CV = 0$. This implies that

$$(\tfrac{\mathrm{d}}{\mathrm{d}t} I - A)V = VN + BM$$

and hence $\operatorname{im} V$ is (A, B)-invariant and by $CV = 0$, $\operatorname{im} V$ is included in $\ker C$. Therefore, we find that

$$\mathcal{V}_{k^*} \overset{\text{eds}}{\subset} \max(A, B; \ker C).$$

In order to show maximality of $\mathcal{V}_{k^*}$ let $\tilde{V} \in \mathcal{A}_{\text{pw}}^{n\times q}$, $\tilde{N} \in \mathcal{A}_{\text{pw}}^{q\times q}$ and $\tilde{M} \in \mathcal{A}_{\text{pw}}^{m\times q}$ be such that

$$(\tfrac{\mathrm{d}}{\mathrm{d}t} I - A)\tilde{V} \overset{\text{eds}}{=} \tilde{V}\tilde{N} + B\tilde{M} \;\wedge\; C\tilde{V} \overset{\text{eds}}{=} 0. \tag{3.11}$$

For future reference recall that for any $f \in \left(\mathcal{C}^1\right)^q$, an open set $\mathcal{T} \subseteq \mathbb{R}$ and a subspace $\mathcal{S} \subseteq \mathbb{R}^q$ we have, as a simple consequence of the definition of $\dot{f}$ via limits of difference quotients, that

$$(\forall\, t \in \mathcal{T}:\; f(t) \in \mathcal{S}) \quad\Longrightarrow\quad \left(\forall\, t \in \mathcal{T}:\; \dot{f}(t) \in \mathcal{S}\right). \tag{3.12}$$

Now let $x \in \mathbb{R}^q$. Define $y(\cdot) := \tilde{V}(\cdot)x \in \mathcal{A}_{\text{pw}}^q$ and observe that $y(t) \in \ker C$ for all $t \in \operatorname{dom}\tilde{V}$, thus, by (3.12), $\dot{y}(t) \in \ker C$ for all $t \in \operatorname{dom}\tilde{V}$. We may then infer from (3.11) that

$$y(t) \in A^{-1}\left(\dot{y}(t) - \tilde{V}(t)\tilde{N}(t)x - B\tilde{M}(t)x\right) \cap \ker C \subseteq A^{-1}(\ker C + \operatorname{im} B) \cap \ker C = \mathcal{V}_1$$

for all $t \in \operatorname{dom}\tilde{V} \cap \operatorname{dom}\tilde{N} \cap \operatorname{dom}\tilde{M}$. As $x \in \mathbb{R}^q$ was arbitrary, this implies $\operatorname{im}\tilde{V} \overset{\text{eds}}{\subset} \mathcal{V}_1$. Also a further application of (3.12) yields $\dot{y}(t) \in \mathcal{V}_1$ for all $t \in \mathbb{R}$, eds. We may combine these properties using a similar argument as above to obtain $y(t) \in \mathcal{V}_2$ for all $t \in \mathbb{R}$, eds, hence $\operatorname{im}\tilde{V} \overset{\text{eds}}{\subset} \mathcal{V}_2$ and $\dot{y}(t) \in \mathcal{V}_2$. Inductively, we obtain $\operatorname{im}\tilde{V} \overset{\text{eds}}{\subset} \mathcal{V}_{k^*}$ and this shows (3.10). □

Remark 3.6 (Time-invariant systems).
If (A, B, C) is time-invariant, then $\max(A, B; \ker C)$ has a generator which is a fixed subspace, independent of time, by Proposition 3.5. Now [24, Excercise 4.4] and also [17, Rem. 6.1.3] yield that for time-invariant systems with the same number of inputs and outputs we have

$$\text{(H1)-(H2) hold} \qquad \Longleftrightarrow \qquad C(sI - A)^{-1}B \ \text{ is invertible over } \mathbb{R}(s).$$

◇

In the next proposition we show that the existence of a strict relative degree implies that Assumptions (H1)-(H2) hold but not vice versa.

Proposition 3.7 (Strict relative degree implies (H1)-(H2)).
For $(A, B, C) \in \Sigma_{n,m,m}$ we have:

$$(A, B, C) \textit{ has strict relative degree } \rho \in \mathbb{N} \qquad \Longrightarrow \qquad \textit{(H1)-(H2) hold},$$

and the converse is false in general, even for time-invariant systems.

Proof: Let (A, B, C) have strict relative degree $\rho \in \mathbb{N}$. We may assume, without restriction of generality due to Theorem B.7, that system (A, B, C) is in Byrnes-Isidori form (B.4). Then Assumption (H1) follows since $\Gamma(t)$ is invertible for all $t \in \mathbb{R}$. Invoking Proposition 3.3, a (full rank) generator $V \in \mathcal{A}^{n\times(n-\rho m)}$ for $\max(A, B; \ker C) \overset{\text{eds}}{=} \ker \mathcal{C}$ is given in Theorem B.7. Finally, Assumption (H2) is a consequence of $\operatorname{im} V = \ker \mathcal{C}$ and the relation $\mathcal{C}B = [0, ..., 0, \Gamma^\top]^\top$.
To show that the converse is false in general, consider $(A, B, C) = \left(\begin{bmatrix} 1&0&0\\0&0&1\\0&1&0 \end{bmatrix}, \begin{bmatrix} 1&0\\0&1\\0&0 \end{bmatrix}, \begin{bmatrix} 1&0&0\\0&0&1 \end{bmatrix}\right)$. Then $CB = \begin{bmatrix} 1&0\\0&0 \end{bmatrix}$, and so (A, B, C) does not have strict relative degree. Furthermore, (H1) is satisfied and, since $\ker C = \operatorname{im}\left([0, 1, 0]^\top\right)$, we find that any generator $V \in \mathcal{A}_{\text{pw}}^3$ of $\max(A, B; \ker C) \overset{\text{eds}}{\subset} \ker C$ takes the form $V(\cdot) = [0, v(\cdot), 0]^\top$ for some $v \in \mathcal{A}_{\text{pw}}$. Now, by (A, B)-invariance, we have

$$\begin{bmatrix} 0\\ \dot v(t)\\ 0 \end{bmatrix} - \begin{bmatrix} 0\\ 0\\ v(t) \end{bmatrix} = V(t)N(t) + B(t)M(t) = \begin{bmatrix} *\\ *\\ 0 \end{bmatrix} \qquad \text{for all } t \in \mathbb{R}, \text{ eds}.$$

Therefore, $v \overset{\text{eds}}{=} 0$ and thus $\max(A, B; \ker C) \overset{\text{eds}}{=} \{0\}$. So (H2) is satisfied and the system does not have a strict relative degree. □

Before we derive the main result of this section, we prove that if the maximal (A, B)-invariant subspace included in $\ker C$ is trivial, then $\begin{bmatrix} DI_n - A, & -B\\ -C & 0 \end{bmatrix}$ is left invertible, and hence by Proposition 2.5 the zero dynamics are trivial.

Lemma 3.8.
For any $(A, B, C) \in \Sigma_{n,m,p}$ we have :

$$\textit{(H1)} \ \wedge \ \max(A, B; \ker C) \overset{\text{eds}}{=} \{0\} \qquad \Longrightarrow \qquad \begin{bmatrix} DI_n - A, & -B\\ -C & 0 \end{bmatrix} \textit{ is left invertible over } \mathcal{M}[D].$$

Proof: The assumption $\max(A, B; \ker C) \overset{\text{eds}}{=} \{0\}$ is equivalent to:

$$\forall\, k \in \mathbb{N}\ \forall\, Z \in \mathcal{A}_{\text{pw}}^{n\times k}, N \in \mathcal{A}_{\text{pw}}^{k\times k}, M \in \mathcal{A}_{\text{pw}}^{m\times k} : \quad \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} I - A & -B\\ -C & 0 \end{bmatrix} \begin{bmatrix} Z\\ M \end{bmatrix} \overset{\text{eds}}{=} \begin{bmatrix} ZN\\ 0 \end{bmatrix} \quad \Longrightarrow \quad Z \overset{\text{eds}}{=} 0. \quad (3.13)$$

We use the factorization (A.1) and accompanying notation from the proof of Proposition 2.4. Set

$$1_{q-\ell} := (1, \ldots, 1)^\top \in \mathbb{R}^{q-\ell} \quad \text{ and let}$$

$$\gamma \in \mathcal{A} \setminus \{0\} \quad \text{ be the product of all } d_{ij}^k,\ i, j \in \{1, \ldots, n\},\ k \in \{0, \ldots, N\}.$$

Then by construction
$$w := V(\tfrac{\mathrm{d}}{\mathrm{d}t})\left(\gamma \begin{bmatrix} 0_\ell \\ 1_{q-\ell} \end{bmatrix}\right) \in \mathcal{A}^q .$$

Step 1: We show that $\ell = q$. Seeking a contradiction, assume that $\ell < q$. Then we have

$$R(\tfrac{\mathrm{d}}{\mathrm{d}t})w(t) = R(\tfrac{\mathrm{d}}{\mathrm{d}t})\left(V(\tfrac{\mathrm{d}}{\mathrm{d}t})\left(\gamma(t)\begin{bmatrix} 0_\ell \\ 1_{q-\ell} \end{bmatrix}\right)\right)$$

$$\stackrel{(A.1)}{=} U(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1}\mathrm{diag}\left\{I_{\ell-1},\, r(\tfrac{\mathrm{d}}{\mathrm{d}t}),\, 0_{(g-\ell)\times(q-\ell)}\right\}\left(\gamma(t)\begin{bmatrix} 0_\ell \\ 1_{q-\ell} \end{bmatrix}\right) = 0$$

for all $t \in \mathbb{R}$ and hence $k := 1$, $Z := [I_n, 0_{n\times m}]w$, $N := 0$, $M := [0_{m\times n}, I_m]w$ satisfy the left hand side of (3.13), thus $Z = 0$. This implies $BM = 0$ and by (H1) it follows $M = 0$. Therefore, $w = \left[Z^\top, M^\top\right]^\top = 0$ and as V is unimodular it follows that $\gamma = 0$, a contradiction.
Step 2: We show that $r(D) = r \in \mathcal{M} \setminus \{0\}$. Seeking a contradiction assume that $\deg r(D) \geq 1$. Then there exists $z \in \mathcal{A}_{\mathrm{pw}} \setminus \{0\}$ such that $r(\tfrac{\mathrm{d}}{\mathrm{d}t})z(t) = 0$ for all $t \in \mathbb{R}$, eds. It follows that for $w := V(\tfrac{\mathrm{d}}{\mathrm{d}t})(0_{q-1}, z)^\top \in \mathcal{A}_{\mathrm{pw}}^q$ we have $R(\tfrac{\mathrm{d}}{\mathrm{d}t})w(t) = 0$ for all $t \in \mathbb{R}$, eds. Now set $k := 1$, $Z := [I_n, 0]w \in \mathcal{A}_{\mathrm{pw}}^{n\times 1}$ and $M := [0, I_m]w \in \mathcal{A}_{\mathrm{pw}}^{m\times 1}$ and observe that, similar to Step 1, (3.13) implies that $Z \stackrel{\mathrm{eds}}{=} 0$ and $M \stackrel{\mathrm{eds}}{=} 0$. Thus we arrive at $w \stackrel{\mathrm{eds}}{=} 0$, whence the contradiction $z \stackrel{\mathrm{eds}}{=} 0$. This completes the proof of the lemma. ☐

The Byrnes-Isidori form (B.4) is derived for systems with strict relative degree and is fundamental for the analysis of the stabilization problem. The zero dynamics form (3.15) derived in the following for systems satisfying Assumptions (H1)-(H2) is of equal importance. Note that (H1)-(H2) are weaker than strict relative degree and in case of time-invariant systems they are equivalent to the autonomy of the zero dynamics; see Fig. 2.
The zero dynamics form has been derived in [17, Rem. 6.1.3] for time-invariant ODE systems; however it was not mentioned that it is based on the assumption of the autonomy of the zero dynamics, although technically (H1)-(H2) are assumed. In the present paper the zero dynamics form is used (i) to construct a vector space isomorphism between the zero dynamics of system (1.1) and the maximal (A, B)-invariant time-varying subspace included in $\ker C$, and (ii) for proving stabilization results in Section 5.

Theorem 3.9 (Zero dynamics form)**.**
Consider $(A, B, C) \in \Sigma_{n,m,p}$ and suppose Assumptions (H1)-(H2) are satisfied. Let $V \in \mathcal{A}^{n\times k}$ be given by (H2), where $k \in \{0, \ldots, n\}$. Then there exists $W \in \mathcal{A}^{n\times(n-k)}$ such that $[V, W] \in \mathbf{Gl}_n(\mathcal{A})$ and the coordinate transformation
$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} := [V(t),\, W(t)]^{-1}\, x(t) \tag{3.14}$$
converts (1.1) *into the form*
$$\boxed{\begin{array}{rcl} \dot z_1 &=& A_1(t)\, z_1 + A_2(t) z_2 \\ \dot z_2 &=& B_2(t)K(t)\, z_1 + A_4(t) z_2 + B_2(t)\, u(t) \\ y(t) &=& C_2(t)\, z_2(t) \end{array}} \tag{3.15}$$
where $A_1 \in \mathcal{A}^{k\times k}$, $A_2 \in \mathcal{A}^{k\times(n-k)}$, $A_4 \in \mathcal{A}^{(n-k)\times(n-k)}$, $B_2 \in \mathcal{A}^{(n-k)\times m}$, $C_2 \in \mathcal{A}^{p\times(n-k)}$, $K \in \mathcal{A}^{m\times k}$ and
$$\max(A_4, B_2; \ker C_2) \stackrel{\mathrm{eds}}{=} \{0\} \quad \wedge \quad \forall\, t \in \mathbb{R}:\ \mathrm{rk}\, B_2(t) = m.$$

Proof: *Step 1*: We show the form (3.15). First note that by (H2) we have $k+m \le n$. Then, by Assumptions (H1)-(H2),

$$\forall\, t \in \mathbb{R}:\ \operatorname{rk}[V(t),\, B(t)] = k+m.$$

If $k+m = n$, define $W := B$. If $k+m < n$, then we find that by [22, Thm. 1] that there exist $X \in \mathcal{A}^{n\times(m+k)}$ and $Y \in \mathcal{A}^{n\times(n-m-k)}$ such that $[X,Y] \in \mathbf{Gl}_n(\mathcal{A})$ and

$$\begin{bmatrix} V^\top \\ B^\top \end{bmatrix} \begin{bmatrix} X & Y \end{bmatrix} = \begin{bmatrix} F & 0 \end{bmatrix},$$

where $F \in \mathbf{Gl}_{m+k}(\mathcal{A})$. In particular, the invertibility of F follows from the constant full column rank of $[V,B]$. Let $Z := [X,Y]^{-\top}[0_{(n-m-k)\times(m+k)}, I_{n-m-k}]^\top \in \mathcal{A}^{(n-m-k)\times n}$ and observe that

$$\begin{bmatrix} V & B & Z \end{bmatrix} = \begin{bmatrix} X^\top \\ Y^\top \end{bmatrix}^{-1} \begin{bmatrix} F^\top & 0 \\ 0 & I_{n-m-k} \end{bmatrix} \in \mathbf{Gl}_n(\mathcal{A})\,.$$

We may thus define $W := [B,Z]$. Now in both cases

$$\forall\, t \in \mathbb{R}:\ \operatorname{im} B(t) \subseteq \operatorname{im} W(t)$$

and the coordinate transformation (3.14) converts (1.1) into the form

$$\begin{aligned} \dot z_1 &= A_1(t)\, z_1 + A_2(t)\, z_2 + B_1(t)\, u(t) \\ \dot z_2 &= A_3(t)\, z_1 + A_4(t)\, z_2 + B_2(t)\, u(t) \\ y(t) &= C_1(t)\, z_1(t) + C_2(t)\, z_2(t), \end{aligned}$$

where

$$\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, [C_1, C_2] \right) = \left([V,W]^{-1} A [V,W] - [V,W]^{-1} [\dot V, \dot W], [V,W]^{-1} B, C[V,W] \right) \tag{3.16}$$

are partitioned appropriately. Therefore all entries of A_i, B_i ,and C_i are real analytic.
The equality $C_1 = CV = 0$ is a consequence of $\operatorname{im} V(t) \overset{\text{eds}}{\subseteq} \ker C(t)$. The claim $B_1 = 0$ may be seen as follows: Let $t \in \mathbb{R}$ and $x \in \mathbb{R}^m$. Then $B(t) = V(t)B_1(t) + W(t)B_2(t)$ together with $\operatorname{im} B(t) \subseteq \operatorname{im} W(t)$ gives $V(t)B_1(t)x = B(t)x - W(t)B_2(t)x \in \operatorname{im} W(t) \cap \operatorname{im} V(t) = \{0\}$. Since $V(t)$ has full rank, we obtain $B_1(t)x = 0$, and so $B_1(t) = 0$ as x was arbitrary. By (H1) it is then clear that $\operatorname{rk} B_2(t) = m$ for all $t \in \mathbb{R}$.
Step 2: We show that there exists $K \in \mathcal{A}^{m\times k}$ such that $A_3 = B_2 K$.
The (A,B)-invariance of $(\operatorname{im} V(t))_{t\in\mathbb{R}} \in \mathcal{W}_n$ together with Lemma 3.4 yield the existence of $N \in \mathcal{A}^{k\times k}, M \in \mathcal{A}^{m\times k}$ such that

$$(\tfrac{\mathrm{d}}{\mathrm{d}t} I - A)V = VN + BM.$$

From (3.16) we see $\dot V - AV = -[V,W] \left[\begin{smallmatrix} A_1 \\ A_3 \end{smallmatrix}\right]$ and $B = WB_2$. Inserting these relations in the previous equation we obtain

$$-VA_1 - WA_3 = VN + WB_2M$$

as an equality of analytic functions. Invoking $\operatorname{im} V(t) \cap \operatorname{im} W(t) = \{0\}$ for all $t \in \mathbb{R}$ and using the full rank of V, resp. W, we arrive at

$$N(t) = -A_1(t) \quad \wedge \quad B_2(t)M(t) = -A_3(t) \quad \forall\, t \in \mathbb{R}. \tag{3.17}$$

The choice $K := -M$ yields the assertion.
Step 3: We show that $\max(A_4, B_2; \ker C_2) = \{0\}$.
To this end consider an (A_4, B_2)-invariant subspace contained in $\ker C_2$. So let $p \in \mathbb{N}$, $Z \in \mathcal{A}_{\rm pw}^{(n-k)\times p}$, $X \in \mathcal{A}_{\rm pw}^{p\times p}$, $Y \in \mathcal{A}_{\rm pw}^{m\times p}$ be such that

$$\begin{bmatrix} \frac{\rm d}{{\rm d}t} I_{n-k} - A_4 & -B_2 \\ -C_2 & 0 \end{bmatrix} \begin{bmatrix} Z \\ Y \end{bmatrix} = \begin{bmatrix} ZX \\ 0 \end{bmatrix}.$$

Let $\mathcal{V}(t) := \operatorname{im}[V(t), W(t)Z(t)]$ for $t \in \operatorname{dom} Z$ and $\mathcal{V}(t) := \{0\}$ for $t \in \mathbb{R} \setminus \operatorname{dom} Z$. Then $\mathcal{V} \in \mathcal{W}_n$ with generator $[V, WZ] \in \mathcal{A}_{\rm pw}^{n\times(k+p)}$ and we will show that $\mathcal{V}$ is (A, B)-invariant and included in $\ker C$. For M as in (3.17) we have

$$\begin{aligned}
(\tfrac{\rm d}{{\rm d}t} I - A)[V, WZ] &\overset{\rm eds}{=} [\dot V, \tfrac{\rm d}{{\rm d}t}(WZ)] - [V, W] \begin{bmatrix} A_1 & A_2 Z \\ A_3 & A_4 Z \end{bmatrix} - [\dot V, \dot W Z] \\
&\overset{\rm eds}{=} [0, W\dot Z] - [V, W] \begin{bmatrix} A_1 & A_2 Z \\ A_3 & A_4 Z \end{bmatrix} \\
&\overset{\rm eds}{=} [-VA_1 - WA_3, W(A_4 Z + ZX + B_2 Y) - VA_2 Z - WA_4 Z] \\
&\underset{A_3 = -B_2 M}{\overset{\rm eds}{=}} [-VA_1, -VA_2 Z + WZX] + [WB_2 M, WB_2 Y] \\
&\overset{\rm eds}{=} [V, WZ] \begin{bmatrix} -A_1 & -A_2 Z \\ 0 & X \end{bmatrix} + \underbrace{[V, W] \begin{bmatrix} 0 \\ B_2 \end{bmatrix}}_{=B} [M, Y],
\end{aligned}$$

and $\left[\begin{smallmatrix} -A_1 & -A_2 Z \\ 0 & X \end{smallmatrix}\right] \in \mathcal{A}_{\rm pw}^{(k+p)\times(k+p)}$, $[M, Y] \in \mathcal{A}_{\rm pw}^{m\times(k+p)}$. This shows the desired (A, B)-invariance.
It follows directly from

$$C[V, WZ] \overset{\rm eds}{=} [0, CWZ] \overset{\rm eds}{=} [0, C_2 Z] \overset{\rm eds}{=} [0, 0],$$

that $\mathcal{V}$ is included in $\ker C$ eds.
Now, since $(\operatorname{im} V(t))_{t\in\mathbb{R}}$ is (up to $\overset{\rm eds}{=}$) the largest (A, B)-invariant time-varying subspace included in $\ker C$, it follows that $W(t)Z(t) \in \operatorname{im} V(t) \cap \operatorname{im} W(t) = \{0\}$ for all $t \in \mathbb{R}$, eds, and therefore, since $W(t)$ has full column rank for all $t \in \mathbb{R}$, $Z \overset{\rm eds}{=} 0$. This implies the assertion and concludes the proof of the theorem. ☐

Note that for the counterexamples in Proposition 2.4 and 2.5, the Assumptions (H1)-(H2) are not satisfied since $\operatorname{im} V(t) \cap \operatorname{im} B(t) \neq \{0\}$ for $t = 0$. The necessity of the latter for all $t \in \mathbb{R}$ is stressed by the following example.

Example 3.10 (Necessity of (H2))**.**
Assumption (H2) states that the intersection of $\operatorname{im} V(t)$ and $\operatorname{im} B(t)$ must be trivial for *all* $t \in \mathbb{R}$. In fact, this assumption cannot, in general, be weakened to "for all $t \in \mathbb{R}$, eds". The following example illustrates that the assumption

$$\max(A, B; \ker C) \cap \operatorname{im} B \overset{\rm eds}{=} \{0\} \tag{3.18}$$

is not sufficient for the existence of a zero dynamics form. Consider (1.1) with

$$A(t) = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}, \quad B(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}, \quad C(t) = [t, -1], \quad t \in \mathbb{R},$$

and note that B satisfies Assumption (H1). Then any $\tilde{V} \in \mathcal{A}_{\rm pw}^{2\times 2}$ with $C\tilde{V} = 0$ has the form $\left[\begin{smallmatrix} v_1(t) & v_2(t) \\ tv_1(t) & tv_2(t) \end{smallmatrix}\right]$, where $v_1, v_2 \in \mathcal{A}_{\rm pw}$, and clearly $\operatorname{im} \tilde{V}(t) \subseteq \operatorname{im}[1,t]^\top =: \operatorname{im} V(t)$ for all $t \in \mathbb{R}$, eds. Furthermore, $\operatorname{im} V$ is (A,B)-invariant since

$$(\tfrac{\rm d}{{\rm d}t} I - A)V(t) = \begin{bmatrix} 1+t \\ 1+t \end{bmatrix} = \begin{bmatrix} 1 \\ t \end{bmatrix} \cdot 1 + \begin{bmatrix} t \\ 1 \end{bmatrix} \cdot 1 = V(t)N + B(t)M.$$

Therefore, V is a generator of $\max(A,B;\ker\, C)$ with constant rank. Now,

$$[V(t), B(t)] = \begin{bmatrix} 1 & t \\ t & 1 \end{bmatrix},$$

which is invertible for all $t \in \mathbb{R}$ with $|t| \neq 1$, hence (3.18) is satisfied, but (H2) is not. Clearly, $[V,B]$ does not constitute a basis transformation. Furthermore, (A,B,C) cannot be put into the form (3.15), which can be seen as follows: Let $T \in \mathbf{Gl}_2(\mathcal{A})$ be such that

$$\text{(a) } TB = \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \quad \text{and} \quad \text{(b) } CT^{-1} = [0, c_2].$$

Then, by (a), $T(t) = \begin{bmatrix} \alpha(t) & -t\alpha(t) \\ \beta(t) & \gamma(t) \end{bmatrix}$ for all $t \in \mathbb{R}$ and some $\alpha, \beta, \gamma \in \mathcal{A}$. Therefore,

$$T(t)^{-1} = \big(\det T(t)\big)^{-1} \begin{bmatrix} \gamma(t) & t\alpha(t) \\ -\beta(t) & \alpha(t) \end{bmatrix}, \qquad t \in \mathbb{R},$$

and by (b) if follows $\beta(t) = t\gamma(t)$. This implies that $\det T(t) = (1-t^2)\alpha(t)\gamma(t) = 0$ for $|t| = 1$, a contradiction. ◇

We are now in a position to characterize the zero dynamics of a system (1.1) in terms of the maximal time-varying (A,B)-invariant subspace included in $\ker C$.

Corollary 3.11 (Characterization of zero dynamics)**.**
Let $(A,B,C) \in \Sigma_{n,m,p}$ and suppose that Assumptions (H1)-(H2) hold. If $(x,u,y) \in \mathfrak{B}_{(1.1)}$, then

$$(x,u,y) \in \mathcal{ZD}_{(1.1)} \quad \Longleftrightarrow \quad \Big[\ x(t) \in \max(A,B;\ker\, C)(t) \quad \textit{for all } t \in \mathbb{R},\ \textit{eds}\ \Big].$$

Proof: $\Longrightarrow$: Let $(x,u,y) \in \mathcal{ZD}_{(1.1)}$. Applying the coordinate transformation $(z_1^\top, z_2^\top)^\top = [V,W]^{-1}x$ from Theorem 3.9 we find that

$$\begin{aligned} \dot{z}_2 &= A_4 z_2 + B_2(u + Kz_1), \\ 0 &= C_2 z_2. \end{aligned}$$

Therefore $(z_2, u + Kz_1, 0) \in \mathcal{ZD}_{(A_4,B_2,C_2)}$, and since $\max(A_4, B_2; \ker C_2) \overset{\rm eds}{=} \{0\}$ it follows from Lemma 3.8 that $\mathcal{ZD}_{(A_4,B_2,C_2)} = \{(0,0,0)\}$. This yields $z_2 = 0$ and $u = -Kz_1$, thus $x(t) = V(t)z_1(t) \in \operatorname{im} V(t) \overset{\rm eds}{=} \max(A,B;\ker C)(t)$ for all $t \in \mathbb{R}$, eds.
$\Longleftarrow$: By assumption,

$$x(t) \in \max(A,B;\ker\, C)(t) \overset{\rm eds}{\subset} \ker C(t) \qquad \text{for all } t \in \mathbb{R}, \text{eds}.$$

Hence $y(t) = C(t)x(t) = 0$ for all $t \in \mathbb{R}$ by continuity, which gives $(x,u,y) \in \mathcal{ZD}_{(1.1)}$. □

In the following corollary we show that $\max(A, B; \ker C)$, i.e. the maximal (A, B)-invariant subspace included in the kernel of C, is isomorphic to the zero dynamics $\mathcal{ZD}_{(1.1)}$. However, some care is required in the formulation of the isomorphism since $\max(A, B; \ker C)$ is an equivalence class, see page 10. Therefore we choose an appropriate generator of the equivalence class, namely V in Assumption (H2), to formulate the isomorphism.

Corollary 3.12 (Vector space isomorphism).
Let $(A, B, C) \in \Sigma_{n,m,p}$ satisfy Assumptions (H1)-(H2) and let V be as in (H2). Then, for all $t_0 \in \mathbb{R}$, the linear map

$$\begin{array}{rcl} L_{t_0}\colon \operatorname{im} V(t_0) & \to & \mathcal{ZD}_{(1.1)} \\ x^0 & \mapsto & \left(x(\cdot),\ F(\cdot)x(\cdot),\ C(\cdot)x(\cdot)\right), \\ & & \text{where } F := -[K, 0]\,[V, W]^{-1} \text{ for } K, V, W \text{ as in Theorem 3.9} \\ & & \text{and } x(\cdot) \text{ solves } \dot{x} = (A + BF)\,x,\ x(t_0) = x^0, \end{array}$$

is a vector space isomorphism.

Proof: *Step 1*: We show that L_{t_0} is well-defined. Thus we have to show that for arbitrary $x^0 \in \operatorname{im} V(t_0)$, the solution of

$$\dot{x} = (A + BF)\,x, \qquad x(t_0) = x^0 \tag{3.19}$$

satisfies

$$(x, u, y) := (x, Fx, Cx) \in \mathcal{ZD}_{(1.1)}. \tag{3.20}$$

First note that $A + BF \in \mathcal{A}^{n\times n}$ and hence $x \in \mathcal{A}^n$. It is then immediate that $(x, u, y) \in \mathfrak{B}_{(1.1)}$, hence it remains to show that $y = 0$.
Applying the coordinate transformation $(z_1^\top, z_2^\top)^\top = [V, W]^{-1}x$ from Theorem 3.9 and invoking

$$BF = [V, W] \begin{bmatrix} 0 & 0 \\ -B_2K & 0 \end{bmatrix} [V, W]^{-1},$$

we find that

$$\begin{array}{rcrcl} \dot{z}_1 & = & A_1 z_1 & + & A_2 z_2, \\ \dot{z}_2 & = & & & A_4 z_2\,. \end{array}$$

Also the initial value satisfies

$$V(t_0)z_1(t_0) + W(t_0)z_2(t_0) = x(t_0) \in \operatorname{im} V(t_0)\,.$$

Thus $W(t_0)z_2(t_0) = x(t_0) - V(t_0)z_1(t_0) \in \operatorname{im} W(t_0) \cap \operatorname{im} V(t_0) = \{0\}$. Then $W(t_0)z_2(t_0) = 0$ and the full column rank of $W(t_0)$ gives $z_2(t_0) = 0$ which yields $z_2 = 0$. Therefore, $x(t) = V(t)z_1(t) \in \operatorname{im} V(t) \subseteq \ker C(t)$ for all $t \in \mathbb{R}$ with the exception of a discrete set and hence, by continuity, $y = Cx = 0$.
Step 2: We show that L_{t_0} is injective. Let $x^1, x^2 \in \operatorname{im} V(t_0)$ so that $L_{t_0}(x^1)(\cdot) = L_{t_0}(x^2)(\cdot)$. Then $x^1 = x^2$ because

$$(x^1, *, *) = L_{t_0}(x^1)(\cdot)\big|_{t=t_0} = L_{t_0}(x^2)(\cdot)\big|_{t=t_0} = (x^2, *, *).$$

Step 3: We show that L_{t_0} is surjective. Let $(x, u, y) \in \mathcal{ZD}_{(1.1)}$. Then Corollary 3.11 yields that $x(t) \in \max(A, B; \ker C)(t)$ for all $t \in \mathbb{R}$, eds. Hence, applying the coordinate transformation $(z_1^\top, z_2^\top)^\top = [V, W]^{-1}x$ from Theorem 3.9 to (1.1) gives $V(t)z_1(t) + W(t)z_2(t) = x(t) \in \operatorname{im} V(t)$ for all $t \in \mathbb{R}$, eds, and, similarly to Step 1, we may conclude $z_2 \overset{\text{eds}}{=} 0$ and, by continuity, $z_2 = 0$. Therefore,

$$\begin{array}{rcl} \dot{z}_1(t) & = & A_1(t)z_1(t), \\ 0 & = & B_2(t)K(t)z_1(t) + B_2(t)u(t) \end{array} \tag{3.21}$$

for all $t \in \mathbb{R}$. Due to $\operatorname{rk} B_2(t) = m$ for all $t \in \mathbb{R}$ the second equation in (3.21) now gives

$$u = -Kz_1 = -K[I, 0][V, W]^{-1}x = Fx.$$

Finally, simple calculations show that $x = Vz_1$ satisfies $\dot{x} = (A + BF)x$ and, clearly, $x(t_0) = V(t_0)z_1(t_0) \in \operatorname{im} V(t_0)$. $\square$

We record that Proposition 3.7 yields, in view of Corollary 3.12 and Proposition 3.3, that the zero dynamics are isomorphic to a certain kernel.

Corollary 3.13 (Characterization of zero dynamics)**.**
For any $(A, B, C) \in \Sigma_{n,m,m}$ with strict relative degree $\rho \in \mathbb{N}$ we have, for $\mathcal{C}$ defined in Theorem B.7,

$$\mathcal{ZD}_{(1.1)} \;\cong\; \ker \mathcal{C}(t_0) \qquad \text{for almost all } t_0 \in \mathbb{R}\,.$$

We may also show that the Assumptions (H1)-(H2) imply that the zero dynamics of the system (1.1) are autonomous.

Proposition 3.14 (Assumptions (H1)-(H2) imply autonomy)**.**
For $(A, B, C) \in \Sigma_{n,m,p}$ we have:

$$\text{(H1)-(H2) hold} \quad \overset{\Longrightarrow}{\not\Longleftarrow} \quad \mathcal{ZD}_{(1.1)} \text{ are autonomous},$$

and equivalence holds for time-invariant systems $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n}$.

Proof: $\Longrightarrow$: Let $(x, u, y) \in \mathcal{ZD}_{(1.1)}$ and $I \subseteq \mathbb{R}$ an open interval such that $(x, u)|_I = 0$. Applying the coordinate transformation $(z_1^\top, z_2^\top)^\top = [V, W]^{-1}x$ from Theorem 3.9 we may conclude, as in Step 3 of the proof of Corollary 3.12, that $z_2 = 0$ and that z_1 and u solve (3.21) for all $t \in \mathbb{R}$. Then $z_1|_I = Vx|_I = 0$ gives $z_1 = 0$ and hence $B_2u = 0$. The full rank of B_2 finally yields $u = 0$.
$\not\Longleftarrow$: Consider $\dot{x}(t) = x(t) + tu(t)$, $y(t) = x(t)$, which has trivial and hence autonomous zero dynamics, but for which (H1) is not satisfied.
$\Longleftarrow$ for $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n}$: (H1) follows from Proposition 2.4, hence $\ker B = \{0\}$. Now assume that (H2) does not hold. By Proposition 3.5 there exists $V \in \mathbb{R}^{n\times k}$ with $\operatorname{im} V \overset{\text{eds}}{=} \max(A, B; \ker C)$. As $\operatorname{im} V$ is (A, B)-invariant, it is well known that there exists an $F \in \mathbb{R}^{m\times n}$ such that $(A + BF)\operatorname{im} V \subseteq \operatorname{im} V$, [24]. Now suppose that $\operatorname{im} B \cap \operatorname{im} V \neq \{0\}$. Fix

$$F \in \mathbb{R}^{m\times n} : \; (A + BF)\operatorname{im} V \subseteq \operatorname{im} V \quad \text{and} \quad T = [T_1, T_2] \in \mathbf{Gl}_n(\mathbb{R}) \text{ with } \operatorname{im} T_2 = \operatorname{im} V\,.$$

Now $\operatorname{im} V \subseteq \ker C$ yields

$$\begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A + BF & -B \\ -C & 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} = \left[\begin{array}{cc|c} A_{11} & 0 & B_1 \\ A_{21} & Q & B_2 \\ \hline C_1 & 0 & 0 \end{array}\right]. \tag{3.22}$$

Since $\operatorname{im} B \cap \operatorname{im} V \neq \{0\}$, we may choose $v \in \mathbb{R}^m$ such that $0 \neq Bv \in \operatorname{im} V$ and so there exists $v \in \mathbb{R}^m \setminus \{0\}$ and $w \in \mathbb{R}^k \setminus \{0\}$ such that $0 \neq Bv = T\left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right] v = T_2 w = T\left[\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right]$. This implies $B_1 v = 0$ and $B_2 v = w$. Proposition 2.4 together with Corollary A.2 yield that $\left[\begin{smallmatrix} sI-A, & -B \\ -C & 0 \end{smallmatrix}\right]$ has full column rank over $\mathbb{R}[s]$. Hence there exists $\lambda \in \mathbb{C}$ such that $\left[\begin{smallmatrix} \lambda I-A, & -B \\ -C & 0 \end{smallmatrix}\right]$ has full column rank and $\lambda I - Q$ is invertible by (3.22). Finally,

$$\begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \lambda I_n - A & -B \\ -C & 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I \end{bmatrix} \begin{bmatrix} 0 \\ -(\lambda I - Q)^{-1} w \\ v \end{bmatrix}$$

$$= \begin{bmatrix} \lambda I - A_{11} & 0 & B_1 \\ -A_{21} & \lambda I - Q & B_2 \\ C_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -(\lambda I - Q)^{-1} w \\ v \end{bmatrix} = 0,$$

a contradiction. This completes the proof of the proposition. □

If we relax Assumption (H1) in Proposition 3.14 to the requirement that $\operatorname{rk}_{\mathcal{M}} B = m$, then we would also need to assume (3.18) instead of (H2) as B does only have full rank almost everywhere. In this case, however, we cannot obtain Proposition 3.14 with the presented proof, because the argument relies crucially on the zero dynamics form (3.15) from Theorem 3.9. Assumption (3.18) is not sufficient to derive this form, as shown in Example 3.10.

4 Controllability and observability

In this section we show that the triviality of the maximal (A, B)-invariant subspace included in $\ker C$ is sufficient for controllability and observability. The converse is false. We first recall the definitions for controllability and observability.

Definition 4.1 (Controllability and observability)**.**
A system $(A, B, C) \in \Sigma_{n,m,p}$ is called *completely controllable* if, and only if,

$$\forall\, t_0 \in \mathbb{R}\ \forall\, x^0, x^1 \in \mathbb{R}^n\ \exists\, t_1 > t_0\ \exists\, (x, u, y) \in \mathfrak{B}_{(1.1)} :\ x(t_0) = x^0 \ \wedge\ \ x(t_1) = x^1. \tag{4.1}$$

(A, B, C) is called *completely observable* if, and only if,

$$\forall\, t_0 \in \mathbb{R}\ \exists\, t_1 > t_0\ \forall\, (x, u, y) \in \mathfrak{B}_{(1.1)} :\ \Big(u|_{[t_0,t_1]} = 0\ \wedge\ y|_{[t_0,t_1]} = 0 \Big) \implies x|_{[t_0,t_1]} = 0.$$

◇

In the sequel we will not use the qualifying "completely" as we do not consider other concepts of controllability or observability. Note that by linearity, controllability is equivalent to (4.1) with $x_1 = 0$ at every instance, see [23, Lem. 3.1.7].
For system with identical input and output dimensions we are now in a position to show the relationship between trivial maximal (A, B)-invariant subspace included in $\ker C$ and controllability and observability.

Proposition 4.2 (Trivial maximal (A, B)-invariant subspace included in $\ker C$ yields controllability and observability)**.**
If $(A, B, C) \in \Sigma_{n,m,p}$ and (H1) holds, then

$$m = p\ \wedge\ \max(A, B; \ker C) \overset{\text{eds}}{=} \{0\} \quad \Longrightarrow \quad (A, B, C) \textit{ is controllable and observable.}$$

The converse implication is false in general, even for time-invariant systems, and the implication is not true in general for $m \neq p$.

Proof: *Step 1*: Recall [15, Thm. 6.4 and Prop. 6.5] that (A, B, C) is controllable and observable if $[DI_n - A, -B]$ is right invertible over $\mathcal{M}[D]$ and $\left[\begin{smallmatrix} DI_n - A \\ -C \end{smallmatrix}\right]$ is left invertible over $\mathcal{M}[D]$.
Step 2: Now suppose the presupposition holds. Then Lemma 3.8 yields that $\left[\begin{smallmatrix} DI_n - A, & -B \\ -C & 0 \end{smallmatrix}\right]$ is left invertible over $\mathcal{M}[D]$, and hence invertible, and the implication follows from Step 1.
Step 3: We show that the converse implication is not true in general. The time-invariant system $(A, B, C) = \left(\left[\begin{smallmatrix} 0 & 1 \\ 0 & 2 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]\right)$ is controllable and observable by the well-known Kalman test, and for $F = \begin{bmatrix} 0 & -1 \end{bmatrix}$ we have $A + BF = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, which has a nontrivial invariant subspace contained in $\ker C$.
Step 4: We show that the implication is in general not true for $m \neq p$. Consider $(A, B, C) = \left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]\right)$. Since $\ker C = \{0\}$, also $\max(A, B; \ker C) \stackrel{\text{eds}}{=} \{0\}$. The system is observable but not controllable. ☐

5 Stabilization by state feedback

In this section we introduce the notion of (asymptotically and exponentially) stable zero dynamics and show that any system (A, B, C) with analytic coefficients, satisfying (H1)-(H2) and with exponentially stable zero dynamics, is stabilizable via state feedback. For time-invariant systems, this has been mentioned as a short note in [17, Rem. 6.1.3] and for time-invariant, differential-algebraic systems this is shown in [4], but apart from that this result is new. First we define the notions of stability we use in this paper for behaviours, which can then be applied to both linear systems (A, B, C) and the zero dynamics $\mathcal{ZD}_{(1.1)}$.

Definition 5.1 (Stable behaviour)**.**
Let $\mathfrak{B} \subseteq \mathcal{PC}^p$ be a linear behaviour; i.e., for any $w_1, w_2 \in \mathfrak{B}$ and $\alpha \in \mathbb{R}$ it holds that $\alpha w_1 + w_2 \in \mathfrak{B}$. Then $\mathfrak{B}$ is called

$$\begin{aligned}
\textit{stable} \quad &:\Longleftrightarrow \quad \forall\, \varepsilon > 0\ \forall\, t_0 \in \mathbb{R}\ \exists\, \delta > 0\ \forall\, w \in \mathfrak{B} \text{ s.t. } w(t_0) \in \mathcal{B}_\delta(0) : \\
&\qquad\quad \forall\, t \geq t_0 : \ w(t) \in \mathcal{B}_\varepsilon(0). \\
\textit{attractive} \quad &:\Longleftrightarrow \quad \forall\, w \in \mathfrak{B} : \lim_{t\to\infty} w(t) = 0. \\
\textit{asymptotically stable} \quad &:\Longleftrightarrow \quad \mathfrak{B} \text{ is stable and attractive.} \\
\textit{exponentially stable} \quad &:\Longleftrightarrow \quad \exists\, \lambda > 0\ \forall\ t_0 \in \mathbb{R}\ \exists\, M > 0\ \ \forall\, w \in \mathfrak{B}\ \forall\, t \geq t_0 : \\
&\qquad\quad \|w(t)\| \leq M \mathrm{e}^{-\lambda(t-t_0)} \|w(t_0)\|. \\
\textit{uniformly exponentially stable} \quad &:\Longleftrightarrow \quad \exists\, M, \lambda > 0\ \forall\, t \geq t_0 \in \mathbb{R}\ \forall\, w \in \mathfrak{B} : \\
&\qquad\quad \|w(t)\| \leq M \mathrm{e}^{-\lambda(t-t_0)} \|w(t_0)\|.
\end{aligned}$$

The *Lyapunov exponent* of a behaviour is defined as

$$k_{\mathrm{L}}(\mathfrak{B}) \ := \ \inf \left\{ \lambda \in \mathbb{R} \ \middle|\ \exists\, M_\lambda > 0\ \forall\, w \in \mathfrak{B}\ \forall\, t \geq 0 : \ \|w(t)\| \leq M_\lambda \ \mathrm{e}^{\lambda\, t} \|w(0)\| \ \right\} \in \mathbb{R} \cup \{-\infty, \infty\}.$$

◇

The above concept sets us in a position to speak about stability of the zero dynamics, and to relate this to linear systems of the form

$$\dot{x} = A(t)x \tag{5.1}$$

where $A \in \mathcal{A}^{n\times n}$ is bounded. The linear equation (5.1) (or, more precisely, the zero solution) is said to be stable, resp. attractive, asymptotically stable, uniformly exponentially stable if, and only if, the behaviour

$$\{\ x \in (\mathcal{C}^\infty)^n \ \mid \forall\, t \in \mathbb{R} : \ \dot{x}(t) = A(t)x(t) \ \}$$

has the respective property. Note that for linear systems (behaviours) attractivity is equivalent to asymptotic stability. The Lyapunov exponent becomes

$$k_{\mathrm{L}}(A) \;:=\; \inf\left\{\lambda\in\mathbb{R}\;\middle|\;\exists\, M_\lambda>0\;\forall\, t\geq 0:\;\|\Phi_A(t,0)\|\leq M_\lambda\,\mathrm{e}^{\lambda\, t}\right\}\in\mathbb{R}\cup\{-\infty,\infty\}\,,$$

and it is well-known that (5.1) is exponentially stable if, and only if, $k_{\mathrm{L}}(A)\in[-\infty,0)$; see [11, Sect. 3.3] for more details.
We require the following estimate for the constants λ, M_λ in the definition of the Lyapunov exponent.

Lemma 5.2.
Let $A\in\mathcal{A}^{n\times n}$ be bounded. If $k_{\mathrm{L}}(A)<\lambda$ for some $\lambda\in\mathbb{R}$, then there exists $M>0$ such that, with $c:=\|A\|_\infty+|\lambda|$,

$$\forall\; t_1\geq t_0\in\mathbb{R}:\quad \|\Phi_A(t_1,t_0)\|\leq M\mathrm{e}^{c|t_0|}\mathrm{e}^{\lambda(t_1-t_0)}\,. \tag{5.2}$$

Proof: By [11, Lemma 3.3.4] we have with $a:=\|A\|_\infty$ that $\|\Phi_A(t_1,t_0)\|\leq \mathrm{e}^{a|t_1-t_0|}$ for all $t_1,t_0\in\mathbb{R}$. Since $k_{\mathrm{L}}(A)\leq\lambda$ there exists $M>0$ such that $\|\Phi_A(t,0)\|\leq M\mathrm{e}^{\lambda t}$ for all $t\geq 0$. Then using the cocycle property of the evolution operator we obtain

$$\forall\; t_1\geq t_0\in\mathbb{R}:\quad \|\Phi_A(t_1,t_0)\|\leq\|\Phi_A(t_1,0)\|\cdot\|\Phi_A(0,t_0)\|\leq M\mathrm{e}^{(a+|\lambda|)|t_0|}\mathrm{e}^{\lambda(t_1-t_0)}\,.$$

This shows the assertion. □

We are now ready to prove the main result of the present paper which concerns stabilizability of a system with stable zero dynamics. The zero dynamics form will be a main tool in the proof. Here an additional complication arises as the state transformations which leave the property of uniform exponential stability invariant are the so-called *Bohl transformations*, see [11, Chapter 3]. In order that we can infer from the stability properties of the transformed system those of the original system we have to restrict ourselves to these transformations. In the following result we will use the slightly more restrictive notion of a Lyapunov transformation [11, Chapter 3]. A time-varying transformation $S\in\mathbf{Gl}_n(\mathcal{A})$ is called a *Lyapunov transformation* if, and only if, S, S^{-1} and $\dot S$ are bounded.
The following result that if a square system (1.1) satisfies Assumptions (H1)-(H2) (which is closely related to the autonomy of the zero dynamics) and various boundedness conditions hold, then we may choose a state feedback $u(t)=F(t)x(t)$ for some $F\in\mathcal{A}^{m\times n}$ such that if applied to (1.1) the Lyapunov exponent of the closed-loop system $\dot x=[A(t)+B(t)F(t)]x$ is equal to the Lyapunov exponent of the zero dynamics of (1.1).

Theorem 5.3 (Lyapunov exponents and state feedback)**.**
Consider a square system $(A,B,C)\in\Sigma_{n,m,m}$ and suppose

(α) A,B,C are bounded and Assumptions (H1)-(H2) hold,

(β) $[V,W]$ from Theorem 3.9 is a Lyapunov transformation and K from Theorem 3.9 is bounded.

If $k_{\mathrm{L}}\left(\mathcal{ZD}_{(1.1)}\right)=-\infty$ (equivalently, the zero dynamics are trivial), then for all $\mu\in\mathbb{R}$ there exists $F\in\mathcal{A}^{m\times n}$ such that

$$k_{\mathrm{L}}(A+BF)\;\leq\;\mu. \tag{5.3}$$

If $k_{\mathrm{L}}\left(\mathcal{ZD}_{(1.1)}\right)\neq-\infty$, then there exists $F\in\mathcal{A}^{m\times n}$ such that the state feedback $u(t)=F(t)x(t)$ applied to (1.1) *yields*

$$k_{\mathrm{L}}(A+BF)\;=\;k_{\mathrm{L}}\left(\mathcal{ZD}_{(1.1)}\right). \tag{5.4}$$

Proof: Consider the transformation (3.14) and the decomposition (3.15). Note that by presupposition (β) every matrix in (3.15) is bounded. Furthermore, the Lyapunov exponent is invariant under the transformation since $[V, W]$ is a Lyapunov transformation.

Step 1: By Theorem 3.9, $\max(A_4, B_2; \ker C_2) \overset{\text{eds}}{=} \{0\}$ and hence, in view of Proposition 4.2, (A_4, B_2, C_2) is controllable. We may thus apply [1, Theorem 3.6] to conclude that (A_4, B_2) is exponentially stabilizable with arbitrary decay, i.e., for arbitrary $\lambda > 0$ there exists $G \in \mathcal{A}^{m\times(n-k)}$ such that

$$k_{\mathrm{L}}(A_4 + B_2 G) < -\lambda. \tag{5.5}$$

To be precise, in [1] the general case of system matrices over $\mathcal{L}^\infty$ are considered; however, inspection of the proof yields that for real analytic system matrices, the feedback matrix G may also be chosen to be real analytic.

Step 2: We show that $k_{\mathrm{L}}(A_1) = k_{\mathrm{L}}\left(\mathcal{ZD}_{(1.1)}\right) \in \mathbb{R} \cup \{-\infty\}$.

Note that $k = 0$ if, and only if, the zero dynamics $\mathcal{ZD}_{(1.1)}$ is trivial; and if this holds, then $k_{\mathrm{L}}(A_1) = k_{\mathrm{L}}\left(\mathcal{ZD}_{(1.1)}\right) = -\infty$. If $k > 0$, then $k_{\mathrm{L}}(A_1) \in \mathbb{R}$ since A_1 is bounded.

Step 2a: We show "$\leq$". Let $z^0 \in \mathbb{R}^k$ and define $z(\cdot) := \Phi_{A_1}(\cdot, 0) z^0 \in (\mathcal{C}^\infty)^k$ and $u := -Kz \in (\mathcal{C}^\infty)^m$. Then (3.16) yields, for K as in Theorem 3.9,

$$(\tfrac{\mathrm{d}}{\mathrm{d}t} I - A)V = -VA_1 - WB_2K = -VA_1 - BK,$$

and therefore

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(Vz) = \dot{V}z + V\dot{z} = (\dot{V} + VA_1)z = (AV - BK)z = A(Vz) + Bu.$$

Furthermore, $C(Vz) = 0$, thus $(x, u, 0) := (Vz, -Kz, 0) \in \mathcal{ZD}_{(1.1)}$. Since $[V, W]$, $[V, W]^{-1}$ and K are bounded, the estimate

$$\|z(t)\| = \left\| [I, 0, 0] \begin{bmatrix} I \\ 0 \\ -K(t) \end{bmatrix} z(t) \right\| \leq \left\| \begin{bmatrix} I \\ 0 \\ -K(t) \end{bmatrix} z(t) \right\| \leq \left\| \begin{bmatrix} [V(t), W(t)]^{-1} & 0 \\ 0 & I \end{bmatrix} \right\| \cdot \left\| \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \right\|$$

shows the claim.

Step 2b: We show "$\geq$". Let $(x, u, 0) \in \mathcal{ZD}_{(1.1)}$ and observe that, as in Step 3 of the proof of Corollary 3.12, $x = Vz_1$ for some $z_1 \in \mathcal{AC}^k$ and (z_1, u) solve (3.21) for all $t \in \mathbb{R}$. Since $B_2(t)$ has full column rank for all $t \in \mathbb{R}$ we have $B_2^\top B_2 \in \mathbf{Gl}_m(\mathcal{A})$. Multiplying the second equation in (3.21) from the left by $(B_2^\top B_2)^{-1} B_2^\top$ yields that $u = -Kz_1$. Now $\dot{z}_1 = A_1(t) z_1$ and the estimate

$$\left\| \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \right\| \leq \left\| \begin{bmatrix} [V(t), W(t)] & 0 \\ 0 & I \end{bmatrix} \right\| \cdot \left\| \begin{bmatrix} I \\ 0 \\ -K(t) \end{bmatrix} \right\| \cdot \|z_1(t)\|$$

proves the assertion.

Step 3: We show that (5.4) is satisfied if $k_{\mathrm{L}}\left(\mathcal{ZD}_{(1.1)}\right) \neq -\infty$. In this case, we have that $k > 0$ and we choose $F := [-K, G][V, W]^{-1} \in \mathcal{A}^{m\times n}$, where G satisfies (5.5) with

$$\lambda > 2|k_{\mathrm{L}}(A_1)| + \|A_1\|_\infty + 1.$$

First observe that

$$\begin{aligned} [V, W]^{-1}(A + BF)[V, W] - [V, W]^{-1}[\dot{V}, \dot{W}] &= \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix} + [V, W]^{-1} B [0, G][V, W]^{-1}[V, W] \\ &= \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 + B_2 G \end{bmatrix}, \end{aligned}$$

and the closed-loop system takes the form

$$\dot{z} = \begin{bmatrix} A_1(t) & A_2(t) \\ 0 & A_4(t) + B_2(t)G(t) \end{bmatrix} z \,. \tag{5.6}$$

Step 3a: We show "$\geq$" in (5.4). Since for any solution $z_1 \in (\mathcal{C}^\infty)^k$ of $\dot{z}_1 = A_1(t)z_1$ the function $z = (z_1^\top, 0)^\top$ solves (5.6), the claim follows from Step 3.
Step 3b: We show "$\leq$" in (5.4). Consider $k_{\mathrm{L}}(A_1) = k_{\mathrm{L}}\left(\mathcal{ZD}_{(1.1)}\right) \in \mathbb{R}$ and let $\mu \in \left(k_{\mathrm{L}}(A_1), k_{\mathrm{L}}(A_1) + 1\right)$ be arbitrary. We may apply Lemma 5.2 and choose some $M_1 > 0$ such that

$$\forall\, t \geq t_0 \geq 0 \;:\; \|\Phi_{A_1}(t, t_0)\| \leq M_1 \mathrm{e}^{ct_0} \mathrm{e}^{\mu(t-t_0)},$$

where $c = \|A_1\|_\infty + |\mu|$. It is a simple calculation that $\lambda + \mu - c > 0$. Then, by (5.5),

$$\exists\, M_2 > 0 \;\forall\, t \geq 0 : \;\|\Phi_{A_4 + B_2 G}(t, 0)\| \leq M_2 \mathrm{e}^{-\lambda t}.$$

Let (z_1, z_2) be any solution of (5.6). Then

$$\forall\, t \geq 0 : \|z_2(t)\| \leq M_2 \mathrm{e}^{-\lambda t} \|z_2(0)\|$$

and variation of constants yields, for all $t \geq 0$, and in view of boundedness of A_2,

$$\begin{aligned} \|z_1(t)\| &= \left\| \Phi_{A_1}(t, 0)\, z_1(0) + \int_0^t \Phi_{A_1}(t, \tau)\, A_2(\tau)\, z_2(\tau)\, \mathrm{d}\tau \right\| \\ &\leq M_1 \mathrm{e}^{\mu t} \|z_1(0)\| + M_1 M_2 \|A_2\|_\infty \int_0^t \mathrm{e}^{c\tau} \mathrm{e}^{\mu(t-\tau)}\, \mathrm{e}^{-\lambda\tau} \|z_2(0)\|\, \mathrm{d}\tau \end{aligned}$$

and we continue, with $M := \max\{M_1, M_1 M_2 \|A_2\|_\infty\}$,

$$\begin{aligned} &\leq M \mathrm{e}^{\mu t} \|z_1(0)\| + M e^{\mu t} \|z_2(0)\| \int_0^t \mathrm{e}^{(c-\mu-\lambda)\tau}\, \mathrm{d}\tau \\ &\leq M \mathrm{e}^{\mu t} \|z_1(0)\| + \frac{M}{\mu + \lambda - c}\, e^{\mu t} \|z_2(0)\|. \end{aligned}$$

Since $[V, W]$ is a Lyapunov transformation by (β) and $-\lambda < \mu$, the above inequalities imply that $k_{\mathrm{L}}(A + BF) \leq \mu$. As $\mu > k_{\mathrm{L}}(A_1)$ is arbitrary the claim is shown.
Step 4: We show that (5.3) is satisfied if $k_{\mathrm{L}}\left(\mathcal{ZD}_{(1.1)}\right) = -\infty$. In this case, in Step 3 (except for Step 3a) the constant λ and the feedback F can be chosen in dependence of $\mu \in \mathbb{R}$ and the argumentation remains the same. This finishes the proof of the theorem. ☐

Note that the proof of Theorem 5.3 is constructive. The matrix G can be obtained as described in [1]. Theorem 5.3 has been proved for time-invariant systems by Isidori [17, pp. 298-300]; however, it was not realized that the Assumptions (H1)-(H2) are equivalent to the autonomy of the zero dynamics and the explicit decay estimate was not given. Now, in view of Theorem 5.3, Proposition 3.14, and the observation that in the time-invariant case a constant feedback F can be chosen, we are able to refine [17, pp. 298-300] as follows.

Corollary 5.4 (Lyapunov exponents of time-invariant systems).
Let $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ have autonomous zero dynamics and let $\mathcal{V}^*$ be as in Proposition 3.5. Then

$$\begin{aligned} \dim \mathcal{V}^* > 0 :&\quad \exists\, F \in \mathbb{R}^{m\times n} :\; k_{\mathrm{L}}(A + BF) = k_{\mathrm{L}}\left(\mathcal{ZD}_{(1.1)}\right), \\ \dim \mathcal{V}^* = 0 :&\quad \forall\, \mu \in \mathbb{R}\; \exists\, F \in \mathbb{R}^{m\times n} :\; k_{\mathrm{L}}(A + BF) \leq \mu. \end{aligned}$$

Proof: It is clear that $k_{\mathrm{L}}\left(\mathcal{ZD}_{(1.1)}\right) = -\infty$ if, and only if, $\mathcal{ZD}_{(1.1)} = \{(0,0,0)\}$. By [3, Prop. 3.10], the latter is equivalent to $\mathcal{V}^* = \{0\}$ and $\operatorname{rk} B = m$, hence the corollary follows from Theorem 5.3 and the fact that a constant F can be chosen. $\square$

Theorem 5.3 in particular shows that exponentially stable zero dynamics imply existence of a feedback such that the closed-loop system is exponentially stable. Provided that the two diagonal systems in (5.6) are uniformly exponentially stable it is possible to show uniform exponential stabilizability.

Corollary 5.5 (Uniform exponential stabilizability).
Under the assumptions of Theorem 5.3: If, with the notation in (3.15)*, the system* (A_4, B_2) *is uniformly exponentially stabilizable (i.e., there exists* $G \in \mathcal{A}^{m\times(n-k)}$ *such that* $\dot z = (A_4 + B_2 G)z$ *is uniformly exponentially stable) and the zero dynamics* $\mathcal{ZD}_{(1.1)}$ *are uniformly exponentially stable, then* F *may be chosen so that* $\dot x = [A(t) + B(t)F(t)]x$ *is uniformly exponentially stable.*

Proof: We inspect the steps in the proof of Theorem 5.3. By assumption, in Step 1 G can be chosen so that $A_4 + B_2 G$ defines a uniformly exponentially stable system. In Step 2 we have in fact also shown that $\mathcal{ZD}_{(1.1)}$ is uniformly exponentially stable if, and only if, $\dot z = A_1(t)z$ is uniformly exponentially stable. If G in Step 1 can be chosen so that $A_4 + B_2 G$ defines a uniformly exponentially stable system and if the zero dynamics are uniformly exponentially stable, then the estimate in Step 3 can be performed uniformly for all $t_0 \geq 0$; where t_0 denotes the initial time, and $t_0 = 0$ in Step 3. Moreover, in this case the constant M is independent of t_0 and we see that the coupled system (5.6) is uniformly exponentially stable. $\square$

Remark 5.6 (Feedback and strict relative degree).
The following observation may also be worth knowing for time-invariant systems.
In view of Proposition 3.7, Theorem 5.3 is in particular applicable to systems $(A, B, C) \in \Sigma_{n,m,m}$ with strict relative degree ρ, and then the Byrnes-Isidori form allows to construct the stabilizing feedback F in Theorem 5.3 explicitly.
Let $U(\cdot)$, $U(\cdot)^{-1}$ be given as in Theorem B.7 and assume that U is a Lyapunov transformation. Consider the Byrnes-Isidori form (B.4) and let

$$p(s) = p_1 + p_2 s + \ldots + p_\rho s^{\rho-1} + s^\rho \in \mathbb{R}[s]$$

be a Hurwitz polynomial. Then the feedback

$$u(t) = \underbrace{-\Gamma(t)^{-1}\left[R_1(t) + p_1 I_m, \ldots, R_\rho(t) + p_\rho I_m, S\right]}_{=:\ \tilde G(t)} \begin{pmatrix} \underline{y}(t) \\ \eta(t) \end{pmatrix}, \qquad \text{where } \underline{y}(t) := \begin{pmatrix} y(t) \\ y^{(1)}(t) \\ \vdots \\ y^{(\rho-1)}(t) \end{pmatrix}$$

applied to (B.4) yields the closed-loop system

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\underline{y}(t) &= \underbrace{\begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & I_m & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & I_m \\ -p_1 I_m & -p_2 I_m & \cdots & -p_{\rho-1} I_m & -p_\rho I_m \end{bmatrix}}_{=:K} \underline{y}(t) \\ \dot\eta(t) &= P(t)\, y(t) \qquad\qquad + \; Q(t)\,\eta(t)\,. \end{aligned} \tag{5.7}$$

Since $\det(sI-K)=p(s)^m$, it follows that K is Hurwitz. From the representation of the zero dynamics in Proposition 2.6, the (uniform) exponential stability of the zero dynamics, and the boundedness of U and U^{-1} it follows that $\dot\eta = Q(t)\eta$ is (uniformly) exponentially stable with exponent $\lambda > 0$. We may choose $p(s)$ such that there exist $M, L \geq 0$, and $\mu > \lambda > 0$ satisfying

$$\|e^{K(t-t_0)}\| \leq M e^{-\mu(t-t_0)} \quad \text{and} \quad \|\Phi_Q(t,t_0)\| \leq L e^{-\lambda(t-t_0)} \qquad \text{for all} \quad t \geq t_0 \geq 0\,.$$

Now an application of variation of constants to (5.7) and invoking the boundedness of P it follows that

$$\begin{aligned} \|\eta(t)\| &= \left\| \Phi_Q(t,t_0)\eta(t_0) + \int_{t_0}^t \Phi_Q(t,s)P(s)y(s)\,\mathrm{d}\,s \right\| \\ &\leq L e^{-\lambda(t-t_0)}\|\eta(t_0)\| + \int_{t_0}^t L e^{-\lambda(t-s)}\|P(\cdot)\|_\infty M e^{-\mu(s-t_0)}\|\underline{y}(t_0)\|\mathrm{d}\,s \\ &\leq L e^{-\lambda(t-t_0)}\|\eta(t_0)\| + \tfrac{L\|P(\cdot)\|_\infty M}{\mu-\lambda}\; e^{-\lambda(t-t_0)}\; \|\underline{y}(t_0)\|\,. \end{aligned}$$

and a straightforward calculation shows (uniform) exponential stability of (5.7). Finally, invoking again the boundedness of U and U^{-1}, the claim follows for $F := \tilde G\,U$.

Appendix A Algebraic properties of the skew polynomial ring $\mathcal{M}[D]$

We have chosen the multiplication rule (1.2) for the skew polynomial ring $\mathcal{M}[D]$. This rule is a consequence of the associative rule $(Df)h = D(fh)$ for all differentiable functions f, h which yields $(Df)(h) = \frac{\mathrm{d}}{\mathrm{d}t}f \cdot h + f \cdot \frac{\mathrm{d}}{\mathrm{d}t}h = \left(\frac{\mathrm{d}}{\mathrm{d}t}f \,+\, fD\right)(h)$. In contrast to the commutative ring $\mathbb{R}[D]$ used in the time-invariant case, $\mathcal{M}[D]$ is non-commutative. It is obvious, that $\mathcal{M}[D]$ does not have any zero divisors, allows a right and left division algorithm, and hence is a right and left *Euclidean domain*, and even a *principle ideal domain*.
Matrices over this ring may be viewed as $R(D) = \sum_{i=0}^n R_i D^i \in \mathcal{M}[D]^{g\times q} \cong \mathcal{M}^{g\times q}[D]$. The *left row rank* (*right column rank*) of a matrix $R(D) \in \mathcal{M}[D]^{g\times q}$ is defined as the rank of the free left (right) $\mathcal{M}[D]$-module of the rows (columns) of $R(D)$, resp. As a consequence of Theorem A.1, the row and column rank coincide and hence we denote the *rank* of $R(D)$ by $\mathrm{rk}_{\mathcal{M}[D]} R(D)$.

Theorem A.1 (Teichmüller-Nakayama canonical form [8, Sect. 8])**.**
For any $R(D) \in \mathcal{M}[D]^{g\times q}$ with $\mathrm{rk}_{\mathcal{M}[D]} R(D) = \ell$, there exist $\mathcal{M}[D]$-unimodular matrices $U(D) \in \mathcal{M}[D]^{g\times g}$, $V(D) \in \mathcal{M}[D]^{q\times q}$ and nonzero $r(D) \in \mathcal{M}[D]$ such that

$$R(D) = U(D)^{-1}\,\mathrm{diag}\left\{I_{\ell-1},\, r(D),\, 0_{(g-\ell)\times(q-\ell)}\right\} V(D)^{-1}\,, \tag{A.1}$$

where the scalar $r(D)$ is unique modulo similarity, that means for any other $\tilde r(D) \in \mathcal{M}[D]$ such that $r(D)\,a(D) = \tilde a(D)\,\tilde r(D)$ for some $a(D), \tilde a(D) \in \mathcal{M}[D]$, the only common left (right) divisors of $r(D), \tilde a(D)$ $(a(D), \tilde r(D))$ are units.

An immediate consequence of Theorem A.1 is that the degree of $r(D)$ is unique; and the diagonal matrix in (A.1) is canonical if $r(D)$ is chosen to be monic. See Remark B.9 for the definition of a canonical form.
Another canonical form is the so called Hermite form, see e.g. [10, Thm. 2.4 and Thm. 6.1]. If instead of $\mathcal{M}[D]$, the commutative ring $\mathbb{R}[s]$ is considered, then the Hermite form over $\mathbb{R}[s]$ is well known, cf. [20, Thm. 2.5.14]. The following corollary is an immediate consequence of the two Hermite forms.

Corollary A.2.
Any $R(s) \in \mathbb{R}[s]^{g\times q}$ satisfies $\mathrm{rk}_{\mathcal{M}[D]} R(D) = \mathrm{rk}_{\mathbb{R}[s]} R(s)$.

Appendix B Relative degree and Byrnes-Isidori form

The Byrnes-Isidori form, exploited at several places of the present paper, is interesting in its own right. We study the Byrnes-Isidori form for time-varying systems with strict relative degree. It is well-known for time-invariant (nonlinear) systems [17, p. 137, 220], for time-invariant multi-input multi-output systems [16], and for time-varying systems [14]. However, to the best of our knowledge, it has not been investigated before in which sense the Byrnes-Isidori form is "close" to a canonical form.

Although we only consider real analytic systems in the preceding sections, the Byrnes-Isidori form is studied in the more general setup of sufficiently smooth matrices. To this end, we introduce the notation $\Sigma^{\ell}_{n,m}$ for the class of systems (1.1) with $(A,B,C) \in (\mathcal{C}^{\ell})^{n\times n} \times (\mathcal{C}^{\ell})^{n\times m} \times (\mathcal{C}^{\ell})^{m\times n}$ and $\ell \in \mathbb{N}_0 \cup \{\infty\}$; we write $(A,B,C) \in \Sigma^{\ell}_{n,m}$.

As a technically useful notation (see [9, 14, 21] for time-varying linear systems), we introduce the operator $\left(\frac{\mathrm{d}}{\mathrm{d}t} I + A(t)_r\right)$, where the subscript r in $A_r(C)$ indicates that A acts on C by multiplication from the right:

Notation B.1 (The operator $(\frac{\mathrm{d}}{\mathrm{d}t} I + A(t)_r)^k$)**.**
Let $\ell \in \mathbb{N}_0$, $A \in (\mathcal{C}^{\ell})^{n\times n}$, and $C \in (\mathcal{C}^{\ell})^{m\times n}$. Set

$$\begin{aligned}
\forall\, t \in \mathbb{R} \ :\ \left(\tfrac{\mathrm{d}}{\mathrm{d}t} I + A(t)_r\right)^0 \left(C(t)\right) &:= C(t)\,,\\
\forall\, t \in \mathbb{R} \ :\ \left(\tfrac{\mathrm{d}}{\mathrm{d}t} I + A(t)_r\right) \left(C(t)\right) &:= \dot C(t) + C(t)A(t)\,,\\
\forall\, t \in \mathbb{R}\ \forall\, k \in \{1,\ldots,\ell\} \ :\ \left(\tfrac{\mathrm{d}}{\mathrm{d}t} I + A(t)_r\right)^k \left(C(t)\right) &:= \left(\tfrac{\mathrm{d}}{\mathrm{d}t} I + A(t)_r\right)\left(\left(\tfrac{\mathrm{d}}{\mathrm{d}t} I + A(t)_r\right)^{k-1} \left(C(t)\right)\right).
\end{aligned}$$

◇

The concept of relative degree is defined as follows, see [14, Def. 2.2, Thm. 2.7].

Definition B.2 (Relative degree)**.**
Let $\rho, \ell \in \mathbb{N}$ with $\rho \le \ell$ and $(A,B,C) \in \Sigma^{\ell}_{n,m}$. Then (A,B,C) is said to have *strict relative degree* ρ if, and only if,

$$\left.\begin{aligned}
\forall\, t \in \mathbb{R}\quad \forall\, k = 0,\ldots,\rho-2 \ :\ \left(\tfrac{\mathrm{d}}{\mathrm{d}t} I + A(t)_r\right)^k \left(C(t)\right) B(t) &= 0_{m\times m}\\
\forall\, t \in \mathbb{R} \ :\ \left(\tfrac{\mathrm{d}}{\mathrm{d}t} I + A(t)_r\right)^{\rho-1} \left(C(t)\right) B(t) &\in \mathbf{Gl}_m(\mathbb{R})\,.
\end{aligned}\right\} \tag{B.1}$$

◇

The concept of relative degree is well-known for time-invariant nonlinear SISO systems [17, p. 137], time-invariant nonlinear MIMO systems [17, p. 220], [18], and for time-varying nonlinear MIMO systems [14, Def. 2.2]. It can even be generalized to differential-algebraic systems [3, App. B].

Remark B.3 (Relative degree for time-invariant systems)**.**
If system (1.1) is a time-invariant system, i.e. $(A,B,C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$, then it is straightforward to see that

$$\forall\, k \in \mathbb{N}_0 \ :\ \left(\tfrac{\mathrm{d}}{\mathrm{d}t} I + A(\cdot)_r\right)^k \left(C(\cdot)\right) B(\cdot) = CA^kB$$

and hence the conditions in (B.1) are equivalent to

$$CA^{\rho-1}B \in \mathbf{Gl}_m(\mathbb{R}) \qquad \text{and} \qquad \forall\, k = 0,\ldots,\rho-2 \ :\ CA^kB = 0\,.$$

◇

Remark B.4 (Vector relative degree).
The notion 'strict' is superfluous for single-input single-output systems. However, even for multivariable time-invariant systems, we may have $CA^kB = 0$ for all $k = 0, \dots, \rho - 2$ and $CA^{\rho-1}B \neq 0$ but $CA^{\rho-1}B \notin \mathbf{Gl}_m(\mathbb{R})$. In this case, one may introduce the concept of a vector relative degree: the vector $(\rho_1, \dots, \rho_m) \in \mathbb{N}^m$ collects the smallest number of times ρ_j one has to differentiate $y_j(\cdot)$ so that the input occurs explicitly in $y_j^{(\rho_j)}(\cdot)$. This is not considered in the present note, for further details see [17, Sec. 5.1] and [3, 19]. $\diamond$

The relative degree ρ is the least number of times one has to differentiate the output $y(\cdot)$ so that the input $u(\cdot)$ occurs explicitly in $y^{(\rho)}(\cdot)$; this is well-known for time-invariant systems. That this also holds for time-varying systems is made explicit in the following proposition.

Proposition B.5 (Relative degree and output representation).
Let $\ell \in \mathbb{N}$. Suppose $(A, B, C) \in \Sigma_{n,m}^\ell$ has strict relative degree $\rho \leq \ell$. Then every $(x, u, y) \in \mathfrak{B}_{(1.1)}$ satisfies the following:

$$\forall\, j = 0, \dots, \rho - 1: \ y^{(j)} = \left(\tfrac{\mathrm{d}}{\mathrm{d}t} I + A_r\right)^j (C)\, x\,, \tag{B.2}$$

$$y^{(\rho)} = \left(\tfrac{\mathrm{d}}{\mathrm{d}t} I + A_r\right)^\rho (C)\, x + \left[\left(\tfrac{\mathrm{d}}{\mathrm{d}t} I + A_r\right)^{\rho-1} (C) B\right] u\,. \tag{B.3}$$

Proof: We show (B.2) by induction over $j = 0, \dots, \rho - 1$. For $j = 0$ the statement is clear. Suppose it holds for some $j \in \{0, \dots, \rho - 2\}$. Then, invoking Definition B.2 we have, for all $t \in \mathbb{R}$,

$$\begin{aligned}
y^{(j+1)}(t) &= \tfrac{\mathrm{d}}{\mathrm{d}t} \left[\left(\tfrac{\mathrm{d}}{\mathrm{d}t} I + A(t)_r\right)^j \big(C(t)\big) x(t)\right] \\
&= \left[\tfrac{\mathrm{d}}{\mathrm{d}t} \left(\tfrac{\mathrm{d}}{\mathrm{d}t} I + A(t)_r\right)^j \big(C(t)\big)\right] x(t) + \left(\tfrac{\mathrm{d}}{\mathrm{d}t} I + A(t)_r\right)^j \big(C(t)\big) \big(A(t)x(t) + B(t)u(t)\big) \\
&= \left[\tfrac{\mathrm{d}}{\mathrm{d}t} \left(\tfrac{\mathrm{d}}{\mathrm{d}t} I + A(t)_r\right)^j \big(C(t)\big) + \left(\tfrac{\mathrm{d}}{\mathrm{d}t} I + A(t)_r\right)^j \big(C(t)\big) A(t)\right] x(t) + \left(\tfrac{\mathrm{d}}{\mathrm{d}t} I + A(t)_r\right)^j \big(C(t)\big) B(t) u(t) \\
&\overset{(B.1)}{=} \left(\tfrac{\mathrm{d}}{\mathrm{d}t} I + A(t)_r\right)^{j+1} \big(C(t)\big) x(t)\,.
\end{aligned}$$

Now we may derive that

$$\forall\, t \in \mathbb{R}: \ y^{(\rho)}(t) = \left(\tfrac{\mathrm{d}}{\mathrm{d}t} I + A(t)_r\right)^\rho \big(C(t)\big) x(t) + \left(\tfrac{\mathrm{d}}{\mathrm{d}t} I + A(t)_r\right)^{\rho-1} \big(C(t)\big)\, B(t) u(t). \qquad \square$$

We now define the Byrnes-Isidori form and show its existence and uniqueness modulo transformations of the zero dynamics under the assumption of a strict relative degree.

Definition B.6 (Byrnes-Isidori form).
$(A, B, C) \in \Sigma_{n,m}^\ell$, $\ell \in \mathbb{N}$, is said to be in *Byrnes-Isidori form* if, and only if, the matrices (A, B, C) are of the form, for some $\rho \in \mathbb{N}$,

$$A(t) = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_m & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & I_m & 0 \\ R_1(t) & R_2(t) & \cdots & R_{\rho-1}(t) & R_\rho(t) & S(t) \\ P(t) & 0 & \cdots & 0 & 0 & Q(t) \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Gamma(t) \\ 0 \end{bmatrix}, \quad C(t) = \begin{bmatrix} I_m \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}^\top \tag{B.4}$$

and

$$R_1, \dots, R_\rho, \Gamma \in (\mathcal{C}^1)^{m \times m}, \quad S, P^\top \in (\mathcal{C}^1)^{m \times (n - \rho m)}, \quad Q \in (\mathcal{C}^1)^{(n-\rho m) \times (n - \rho m)}\,. \tag{B.5}$$

$\diamond$

One advantage of the form (B.4) is that it expresses the dynamical properties of the system by allowing u only to affect the ρth derivative (ρ the relative degree) of the output and separating another part of the dynamics which is only influenced by y. This decomposition of the system into a main part (containing the relative degree and the high-frequency gain matrix) and an internal loop for $y \mapsto \hat{y}$ is depicted in Figure 1.

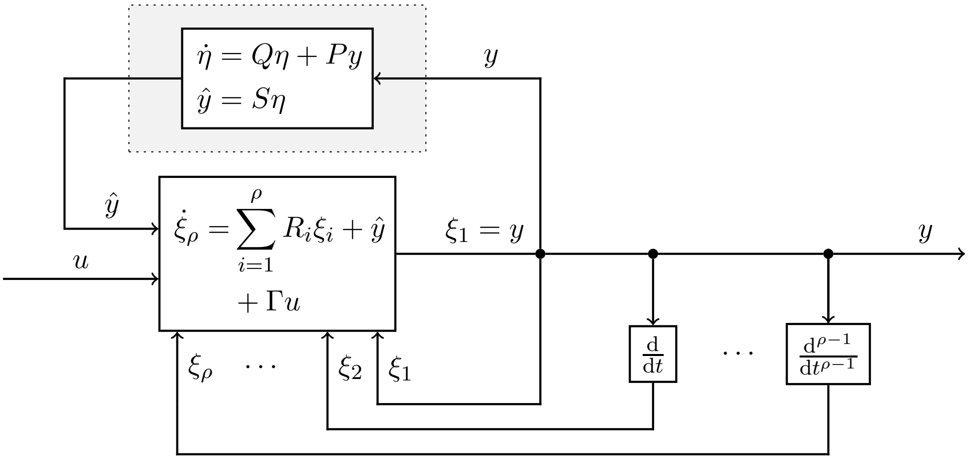

Figure 1: Byrnes-Isidori form

In the following theorem we show that for systems (1.1) with some strict relative degree a Byrnes-Isidori form always exists, we also clarify in which sense the entries are uniquely defined.

Theorem B.7 (Byrnes-Isidori form)**.**
Suppose $(A, B, C) \in \Sigma^{\ell}_{n,m}$, $\ell \in \mathbb{N}$, *has strict relative degree* $\rho \le \ell$. *Then there exists a coordinate transformation* $U \in \mathcal{C}^1(\mathbb{R}, \mathbf{Gl}_n(\mathbb{R}))$ *such that*

$$\begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} := \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \\ \vdots \\ \xi_\rho(t) \\ \eta(t) \end{pmatrix} := \begin{pmatrix} y(t) \\ y^{(1)}(t) \\ \vdots \\ y^{(\rho-1)}(t) \\ \eta(t) \end{pmatrix} = U(t)\, x(t), \tag{B.6}$$

transforms (1.1) *into Byrnes-Isidori form* (B.4) *with initial condition*

$$\begin{pmatrix} \xi(t_0) \\ \eta(t_0) \end{pmatrix} = \begin{pmatrix} \xi^0 \\ \eta^0 \end{pmatrix} = \begin{pmatrix} y(t_0) \\ \vdots \\ y^{(\rho-1)}(t_0) \\ \eta^0 \end{pmatrix} = U(t_0)\, x(t_0), \quad t_0 \in \mathbb{R}. \tag{B.7}$$

Set

$$\mathcal{B} := \left[B, \left(\tfrac{\mathrm{d}}{\mathrm{d}t} I - A\right)(B), \ldots, \left(\tfrac{\mathrm{d}}{\mathrm{d}t} I - A\right)^{\rho-1}(B) \right] \in (\mathcal{C}^1)^{n \times \rho m},$$

$$\mathcal{C} := \begin{bmatrix} C \\ \left(\tfrac{\mathrm{d}}{\mathrm{d}t} I + A_r\right)(C) \\ \vdots \\ \left(\tfrac{\mathrm{d}}{\mathrm{d}t} I + A_r\right)^{\rho-1}(C) \end{bmatrix} \in (\mathcal{C}^1)^{\rho m \times n},$$

then uniqueness of the entries of the Byrnes-Isidori form holds as follows:

(i) the entries

$$\begin{aligned}\Gamma &= \left(\tfrac{\mathrm{d}}{\mathrm{d}t}I + A_r\right)^{\rho-1}(C)B \in \mathcal{C}^1(\mathbb{R};\mathbf{Gl}_m(\mathbb{R})),\\ [R_1,\ldots,R_\rho] &= \left(\tfrac{\mathrm{d}}{\mathrm{d}t}I + A_r\right)^{\rho}(C)\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \in (\mathcal{C}^1)^{m\times n}\end{aligned}$$

are uniquely defined,

(ii) the subsystem $(Q,P,S) \in (\mathcal{C}^1)^{(n-\rho m)\times(n-\rho m)} \times (\mathcal{C}^1)^{(n-\rho m)\times m} \times (\mathcal{C}^1)^{m\times(n-\rho m)}$ *is unique up to* $(Z^{-1}QZ - Z^{-1}\dot{Z}, Z^{-1}P, SZ)$ *for any* $Z \in \mathcal{C}^1(\mathbb{R};\mathbf{Gl}_{n-\rho m}(\mathbb{R}))$*.*

A possible transformation (B.6) *of* (1.1) *into Byrnes-Isidori form* (B.4) *is feasible by*

$$U = \begin{bmatrix}\mathcal{C}\\ N\end{bmatrix}, \quad \textit{where } N := (V^\top V)^{-1}V^\top\left[I - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\right] \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^{(n-\rho m)\times n})$$

and $V \in \mathcal{L}^\infty(\mathbb{R};\mathbb{R}^{n\times(n-\rho m)}) \cap \mathcal{C}^1(\mathbb{R};\mathbb{R}^{n\times(n-\rho m)})$ *may be chosen such that*

$$(V^\top V)^{-1}V^\top \in \mathcal{L}^\infty(\mathbb{R};\mathbb{R}^{(n-\rho m)\times n}) \quad\wedge\quad \forall\, t\in\mathbb{R}\ :\ \operatorname{im} V(t) = \ker \mathcal{C}(t), \quad \operatorname{rk} V(t)^\top V(t) = n-\rho m.$$

If the coefficients of (A,B,C) *are in* $\mathcal{A}$ *(in* $\mathcal{C}^\infty$*), then the coefficients of* $\mathcal{C}$*,* $\mathcal{B}$*,* N*,* V*,* U *defined above and of all entries in* (B.4) *are in* $\mathcal{A}$ *(in* $\mathcal{C}^\infty$*).*

Proof: The proof can be found in [14, Thm. 3.5] except for the uniqueness properties and the case of coefficients in $\mathcal{A}$ and $\mathcal{C}^\infty$. The latter however is a simple calculation. (i) is also a consequence of [14, Thm. 3.5], so it remains to show (ii).
Let

$$(\hat{A},\hat{B},\hat{C}) := \left((UA+\dot{U})U^{-1}, UB, CU^{-1}\right) \tag{B.8}$$

for $U = \begin{bmatrix}\mathcal{C}\\ N\end{bmatrix}$. Then

$$\hat{A} = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 & 0\\ 0 & 0 & I_m & & & 0\\ \vdots & & \ddots & \ddots & & \vdots\\ 0 & 0 & \cdots & 0 & I_m & 0\\ R_1 & R_2 & \cdots & R_{\rho-1} & R_\rho & S\\ P & 0 & \cdots & 0 & 0 & Q\end{bmatrix}, \quad \hat{B} = \begin{bmatrix}0\\0\\ \vdots\\ 0\\ \Gamma\\ 0\end{bmatrix}, \quad \hat{C} = [I_m, 0, \ldots, 0] \tag{B.9}$$

holds (see [14, Thm. 3.5]) for Γ, R_i, S, P, Q given by [14, (3.6)-(3.12)].
Consider next

$$(\tilde{A}\,,\,\tilde{B}\,,\,\tilde{C}) = \left((WA+\dot{W})W^{-1},\, WB\,,\, CW^{-1}\right) \tag{B.10}$$

for any $W \in \mathcal{C}^1(\mathbb{R};\mathbf{Gl}_n(\mathbb{R}))$ such that

$$\tilde{A} = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 & 0\\ 0 & 0 & I_m & & & 0\\ \vdots & & \ddots & \ddots & & \vdots\\ 0 & 0 & \cdots & 0 & I_m & 0\\ \tilde{R}_1 & \tilde{R}_2 & \cdots & \tilde{R}_{\rho-1} & \tilde{R}_\rho & \tilde{S}\\ \tilde{P} & 0 & \cdots & 0 & 0 & \tilde{Q}\end{bmatrix}, \quad \tilde{B} = \begin{bmatrix}0\\0\\ \vdots\\ 0\\ \Gamma\\ 0\end{bmatrix}, \quad \tilde{C} = [I_m, 0, \ldots, 0]\,. \tag{B.11}$$

We show that

$$\left.\begin{array}{lll} \tilde R_i = R_i, & \tilde S = S\,Z^{-1}, & \forall\, i = 1,\dots,\rho,\\ \tilde P = ZP, & \tilde Q = Z\,Q\,Z^{-1} + \dot Z\,Z^{-1} & \text{for some } Z \in \mathcal{C}^1(\mathbb{R};\mathbf{Gl}_{n-\rho m}(\mathbb{R})). \end{array}\right\} \tag{B.12}$$

Set

$$WU^{-1} =: Y = \begin{bmatrix} Y^1\\ \vdots\\ Y^{\rho+1}\end{bmatrix} = [Y_1,\dots,Y_{\rho+1}] \tag{B.13}$$

for $Y^i, (Y_i)^\top \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^{m\times n})$, $i = 1,\dots,\rho$, and $Y^{\rho+1}, (Y_{\rho+1})^\top \in (\mathcal{C}^1)^{(n-\rho m)\times n}$. Then (B.8) and (B.10) together with $\frac{\mathrm{d}}{\mathrm{d}t}(U^{-1}) = -U^{-1}\dot U U^{-1}$ yield

$$\begin{aligned}(Y\hat A + \dot Y)Y^{-1} &= \big(WAU^{-1} + WU^{-1}\dot U U^{-1} + \dot W U^{-1} + W\tfrac{\mathrm{d}}{\mathrm{d}t}(U^{-1})\big)UW^{-1}\\ &= WAW^{-1} + \dot W W^{-1} = \tilde A\,.\end{aligned}$$

Thus

$$\begin{aligned}(Y\hat A + \dot Y)Y^{-1} &= \tilde A, & \text{(B.14)}\\ Y\hat B &= \tilde B, & \text{(B.15)}\\ \hat C &= \tilde C Y. & \text{(B.16)}\end{aligned}$$

This gives

$$Y^1 \overset{\text{(B.16)}}{=} [I_m, 0, \dots, 0] \qquad \text{and} \qquad Y_\rho \overset{\text{(B.15)}}{=} \begin{bmatrix} 0\\ \vdots\\ 0\\ I_m\\ 0\end{bmatrix}, \tag{B.17}$$

and we proceed

$$\begin{array}{rclclclcl} [0, I_m, 0, \dots, 0] & \overset{\text{(B.9)}}{=} & Y^1\hat A + \frac{\mathrm{d}}{\mathrm{d}t}Y^1 & \overset{\text{(B.17)}}{=} & Y^1(Y\hat A + \dot Y) & \overset{\text{(B.14)}}{=} & Y^1\tilde A Y & \underset{\text{(B.11)}}{\overset{\text{(B.13)}}{=}} & Y^2\\ [0, 0, I_m, 0, \dots, 0] & \overset{\text{(B.9)}}{=} & Y^2\hat A + \frac{\mathrm{d}}{\mathrm{d}t}Y^2 & = & Y^2(Y\hat A + \dot Y) & \overset{\text{(B.14)}}{=} & Y^2\tilde A Y & \underset{\text{(B.11)}}{\overset{\text{(B.13)}}{=}} & Y^3\\ & \vdots & & & & & & & \\ [0, \dots, 0, I_m, 0] & \overset{\text{(B.9)}}{=} & Y^{\rho-1}\hat A + \frac{\mathrm{d}}{\mathrm{d}t}Y^{\rho-1} & = & Y^{\rho-1}(Y\hat A + \dot Y) & \overset{\text{(B.14)}}{=} & Y^{\rho-1}\tilde A Y & \underset{\text{(B.11)}}{\overset{\text{(B.13)}}{=}} & Y^\rho. \end{array}$$

Therefore, Y is of the form

$$Y = \begin{bmatrix} I_m & 0 & \dots & 0 & 0\\ 0 & I_m & & & 0\\ \vdots & \ddots & \ddots & & \vdots\\ 0 & \dots & 0 & I_m & 0\\ Y_{\rho+1,1} & \dots & Y_{\rho+1,\rho-1} & 0 & Z\end{bmatrix} \qquad \text{for some } Z \in \mathcal{C}^1(\mathbb{R};\mathbf{Gl}_{n-\rho m}(\mathbb{R})).$$

Now consider the last $n - \rho m$ rows in $Y\hat A + \dot Y = \tilde A Y$, which read

$$\begin{aligned}&[ZP + \tfrac{\mathrm{d}}{\mathrm{d}t}Y_{\rho+1,1}, Y_{\rho+1,1} + \tfrac{\mathrm{d}}{\mathrm{d}t}Y_{\rho+1,2}, \dots, Y_{\rho+1,\rho-2} + \tfrac{\mathrm{d}}{\mathrm{d}t}Y_{\rho+1,\rho-1}, Y_{\rho+1,\rho-1}, ZQ + \dot Z]\\ &\qquad = [\tilde P + \tilde Q Y_{\rho+1,1}, \tilde Q Y_{\rho+1,2}, \dots, \tilde Q Y_{\rho+1,\rho-1}, 0, \tilde Q Z],\end{aligned}$$

and comparing successively the ρ^{th} block, ..., 1^{st} block yields $Y_{\rho+1,\rho-1} = 0, \dots, Y_{\rho+1,1} = 0$. Finally, $Y = \operatorname{diag}\{I_m, \dots, I_m, Z\}$ applied to (B.14)-(B.16) gives (B.12). $\square$

Remark B.8 (Byrnes-Isidori form)**.**

(i) In the time-invariant case $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$, all matrices in Theorem B.7 are constant matrices over $\mathbb{R}$.

(ii) The converse of Theorem B.7 is false in general even for time-invariant systems: any system (B.4) with non-invertible Γ does not have a strict relative degree.

(iii) A formula for (Q, P, S) in terms of the transformation $U = \begin{bmatrix} \mathcal{C} \\ N \end{bmatrix}$ is given in [14, (3.7)-(3.12)], but unfortunately with a typo in formula [14, (3.10)] for Q; the correct formula is

$$\begin{aligned} Q &= -(V^\top V)^{-1} V^\top \left[(\tfrac{\mathrm{d}}{\mathrm{d}t} I - A)V + B\Gamma^{-1} (\tfrac{\mathrm{d}}{\mathrm{d}t} I + A_r)^\rho (C)V \right], \\ P &= (-1)^\rho (V^\top V)^{-1} V^\top \left[I - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C} \right] (\tfrac{\mathrm{d}}{\mathrm{d}t} I - A)^\rho (B)\Gamma^{-1}, \\ S &= (\tfrac{\mathrm{d}}{\mathrm{d}t} I + A_r)^\rho (C)V. \end{aligned}$$

For its proof see the proof of [14, Thm. 3.5].

(iv) As a useful technicality we mention the fact that Theorem B.7 gives

$$\forall\, t \in \mathbb{R} : \ \mathcal{C}(t)U(t)^{-1} = \left[I_{\rho m}, 0_{\rho m\times(n-\rho m)} \right] \ \wedge \ \ker(\mathcal{C}(t)U(t)^{-1}) = \operatorname{im}\, [0, \dots, 0, I_{n-\rho m}]^\top. \quad \text{(B.18)}$$

$\diamond$

Remark B.9. (Canonical form) Recall the definition of a canonical form: given a group G, a set $\mathcal{S}$, a group action $\alpha : G \times \mathcal{S} \to \mathcal{S}$, we write

$$s \overset{\alpha}{\sim} s' \quad \Leftrightarrow \quad \exists\, U \in G : \alpha(U, s) = s'.$$

Then a map $\gamma : \mathcal{S} \to \mathcal{S}$ is called a *canonical form for* α if, and only if,

$$\forall\, s, s' \in S \ : \ \gamma(s) \overset{\alpha}{\sim} s \quad \wedge \quad \left[s \overset{\alpha}{\sim} s' \Leftrightarrow \gamma(s) = \gamma(s') \right].$$

In words: the set $\mathcal{S}$ is divided into disjoint orbits (i.e., equivalence classes) and the mapping γ picks a unique representative in each equivalence class. In the present setup, the group $\mathcal{C}^1(\mathbb{R}; \mathbf{Gl}_n(\mathbb{R}))$ yields an equivalence relation on the set $\Sigma^\ell_{n,m,p}$ of systems (1.1) by state space transformation:

$$(A, B, C) \sim (\hat{A}, \hat{B}, \hat{C}) \quad :\Longleftrightarrow \quad \exists\, U \in \mathcal{C}^1(\mathbb{R}, \mathbf{Gl}_n(\mathbb{R})) \ : \ \big(\hat{A}, \hat{B}, \hat{C}\big) = \big((UA + \dot{U})U^{-1}, UB, CU^{-1}\big).$$

Now it is clear that the Byrnes-Isidori form (B.4) is not a canonical form but "close" to a canonical form: the only non-unique entries are (Q, P, S), but they describe an internal loop (see Figure 1) and they are unique modulo a state space transformation. More precisely, the uniqueness of (Q, P, S) in Theorem B.7 holds modulo $(Z^{-1}QZ - Z^{-1}\dot{Z}, Z^{-1}P, SZ)$ for any $Z \in \mathcal{C}^1(\mathbb{R}; \mathbf{Gl}_{n-\rho m}(\mathbb{R}))$ corresponding to a coordinate transformation of the subsystem (Q, P, S). This may also be viewed as the freedom in choosing V such that the conditions in Theorem B.7 are satisfied. If V is replaced by VZ^{-1} for arbitrary $Z \in \mathcal{C}^1(\mathbb{R}; \mathbf{Gl}_{n-\rho m}(\mathbb{R}))$, then an easy calculation shows that N becomes ZN and therefore (Q, P, S) becomes $(Z^{-1}QZ - Z^{-1}\dot{Z}, Z^{-1}P, SZ)$. $\diamond$

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