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ROBUST STABILIZATION BY LINEAR OUTPUT DELAY FEEDBACK*

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Abstract. The main result of this paper establishes that if a controller C (comprising of a linear feedback of the output and its *derivatives*) globally stabilizes a (nonlinear) plant P, then global stabilization of P can also be achieved by an output feedback controller $C[h]$, where the output derivatives in C are replaced by an Euler approximation with sufficiently small delay $h > 0$. This is proved within the conceptual framework of the nonlinear gap metric approach to robust stability. The main result is then applied to finite dimensional linear minimum phase systems with unknown coefficients but known relative degree and known sign of the high frequency gain. Results are also given for systems with nonzero initial conditions.

Key words. robust stabilization, gap metric, high-gain output feedback, delay feedback

AMS subject classifications. 93D15, 93D21, 93B52, 34D23, 34K20

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Nomenclature.

$\mathbb{C}_+$, $\mathbb{C}_-$	$= \{s \in \mathbb{C} \,\vert\, \operatorname{Re} s > 0\}$, $\{s \in \mathbb{C} \,\vert\, \operatorname{Re} s < 0\}$, respectively.
$\vert x\vert$	$= \sqrt{x^T x}$, the Euclidean norm of $x \in \mathbb{R}^n$.
$\vert A\vert$	$= \max\{\vert A\,x\vert \,\vert\, x \in \mathbb{R}^m,\ \vert x\vert = 1\}$, the induced matrix norm for $A \in \mathbb{R}^{n\times m}$.
$[a_1/a_2/\ldots/a_m]$	$= [a_1^T, a_2^T, \ldots, a_m^T]^T \in \mathbb{R}^{m\times n}$ for $a_1, a_2, \ldots, a_m \in \mathbb{R}^{1\times n}$.
$e_k^{(n)}$	$= [0, \ldots, 0, 1, 0, \ldots, 0]^T$, the kth unit vector in $\mathbb{R}^n$, for $k, n \in \mathbb{N}$, and $k \le n$.
$\operatorname{spec}(A)$	the spectrum of $A \in \mathbb{R}^{n\times n}$.
$\operatorname{im} A$, $\ker A$	the image and kernel of $A \in \mathbb{R}^{n\times m}$.
$\Vert v\Vert_{\mathcal{V}}$	the norm of $v \in \mathcal{V}$, for a normed vector space $\mathcal{V}$.
$\operatorname{map}(E \to F)$	the set of all maps from the set E to the set F.
$\mathcal{C}^r(I \to \mathbb{R}^\ell)$	the set of r-times continuous differentiable functions $y\colon I \to \mathbb{R}^\ell$, where $r \in \mathbb{N} \cup \{\infty\}$ and $I \subset \mathbb{R}$ is an interval.
$\mathcal{C}_{\mathrm{pw}}(I \to \mathbb{R}^\ell)$	the set of piecewise continuous functions $y\colon I \to \mathbb{R}^\ell$, $I \subset \mathbb{R}$ an interval.
$L^p(I \to \mathbb{R}^\ell)$	the space of p-integrable functions $y\colon I \to \mathbb{R}^\ell$, $1 \le p < \infty$, $I \subset \mathbb{R}$ an interval, with norm $\Vert y\Vert_{L^p(I\to\mathbb{R}^\ell)} = \left(\int_I \vert y(t)\vert^p\,\mathrm{d}t\right)^{\frac{1}{p}}$.
$L^\infty(I \to \mathbb{R}^\ell)$	the space of essentially bounded functions $y\colon I \to \mathbb{R}^\ell$, $I \subset \mathbb{R}$ an interval, with norm $\Vert y\Vert_{L^\infty(I\to\mathbb{R}^\ell)} = \operatorname{ess\,sup}_{t\in I} \vert y(t)\vert$.
$CL^p(I \to \mathbb{R}^\ell)$	$= \mathcal{C}(I \to \mathbb{R}^\ell) \cap L^p(I \to \mathbb{R}^\ell)$, $1 \le p \le \infty$, $I \subset \mathbb{R}$ an interval, with norm $\Vert y\Vert_{CL^p(I\to\mathbb{R}^\ell)} = \Vert y\Vert_{L^p(I\to\mathbb{R}^\ell)}$.

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$CW^{r,p}(I \to \mathbb{R}^\ell) = \{y \in \mathcal{C}^r(I \to \mathbb{R}^\ell) \mid \forall\, i \in \{0,\dots,r\} : y^{(i)} \in L^p(I \to \mathbb{R}^\ell)\}$, $r \in \mathbb{N}$, $1 \le p \le \infty$, $I \subset \mathbb{R}$ an interval, with norm $\|y\|_{CW^{r,p}(I\to\mathbb{R}^\ell)} = \sum_{i=0}^r \|y^{(i)}\|_{L^p(I\to\mathbb{R}^\ell)}$.

$$CW_0^{r,p}(I \to \mathbb{R}^\ell) = \left\{ y \in \mathcal{C}^r(I \to \mathbb{R}^\ell) \;\middle|\; \begin{array}{l} \forall\, i \in \{0,\dots,r\} : y^{(i)} \in L^p(I \to \mathbb{R}^\ell)\,, \\ \text{if } 0 \in I, \text{ then} \\ \forall\, i \in \{0,\dots,r-1\} : y^{(i)}(0) = 0 \end{array} \right\},$$

$r \in \mathbb{N}$, $1 \le p \le \infty$, $I \subset \mathbb{R}$ an interval, with norm $\|y\|_{CW^{r,p}(I\to\mathbb{R}^\ell)}$.

$$CW_0^{\infty,p}(I \to \mathbb{R}^\ell) = \left\{ y \in \mathcal{C}^\infty(I \to \mathbb{R}^\ell) \;\middle|\; \begin{array}{l} \forall\, i \in \mathbb{N}_0 : y^{(i)} \in L^p(I \to \mathbb{R}^\ell)\,, \\ \sum_{i=0}^\infty \|y^{(i)}\|_{L^p(I\to\mathbb{R}^\ell)} < \infty\,, \\ \text{if } 0 \in I\,, \text{ then } \forall\, i \in \mathbb{N}_0 : y^{(i)}(0) = 0 \end{array} \right\},$$

$1 \le p \le \infty$, $I \subset \mathbb{R}$ an interval, with norm $\|y\|_{CW^{\infty,p}(I\to\mathbb{R}^\ell)}$.

1. Introduction. We present conditions under which a feedback controller based on the measured output and its derivatives can be replaced by a feedback controller based on the measured output and *numerical* derivatives. Derivative feedback occurs frequently in control; for example, proportional-derivative (PD) controllers are of this type, as are state feedback of systems of full relative degree, and as are suitable partial state feedbacks for systems of nonzero relative degree.

The problem is studied in the setup of the classical feedback configuration shown in Figure 1. We are concerned with the concept of gain stability, that is, with the existence and size of a finite gain from the external disturbances (u_0, y_0) to the internal signals (u_1, y_1), which is the quantity

$$\gamma := \sup_{(u_0,y_0)\in\mathcal{U}\times\mathcal{Y}\setminus\{0\}} \frac{\|(u_1,y_1)\|_{\mathcal{U}\times\mathcal{Y}}}{\|(u_0,y_0)\|_{\mathcal{U}\times\mathcal{Y}}} < \infty,$$

for some appropriate choices of signal spaces $\mathcal{U}, \mathcal{Y}$. We show that if P is stabilizable (P may be nonlinear) by some derivative feedback controller

$$C_k : y_2 \mapsto u_2 = -\sum_{i=0}^{r-1} k_{i+1}\, y_2^{(i)}\,, \qquad k = (k_1,\dots,k_r) \in \mathbb{R}^{1\times r}\,, \tag{1.1}$$

then stabilization can also be achieved by replacing C_k by the delay feedback controller $C_k^{\text{Euler}}[h]$ for sufficiently small $h > 0$, given by

$$C_k^{\text{Euler}}[h] : y_2 \mapsto u_2 = -\sum_{i=0}^{r-1} k_{i+1}\, \Delta_h^i y_2; \tag{1.2}$$

here $\Delta_h^0 y_2 = y_2$ and $\Delta_h^i y_2$, for $i \ge 1$, denotes the Euler approximation of the ith

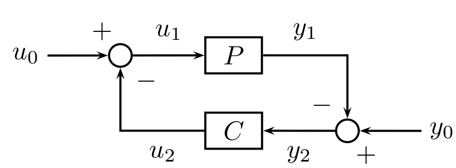

FIG. 1. *The closed-loop system* $[P, C]$.

derivative of y_2 defined by

$$\Delta_h^i y_2 = \underbrace{\Delta_h \circ \cdots \circ \Delta_h}_{i \text{ times}} y_2\,, \qquad \text{where} \quad (\Delta_h y_2)\,(t) = \frac{y_2(t) - y_2(t-h)}{h}\,.$$

The signal spaces for which these results hold depend on structural properties of the plant P. For concreteness we consider the case of single input–single output linear plants, which are minimum phase and of relative degree $\rho \geq 1$, and we show that the choices $\mathcal{U} = CL^p(\mathbb{R}_{\geq 0} \to \mathbb{R})$, $\mathcal{Y} = CW^{k,p}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ are valid, where $k = \rho$ and either $r = \rho - 1$ if $p < \infty$ or $r = \rho$ if $p = \infty$. The key motivation for this linear study is to establish the appropriate signal space settings, whereby the degree of regularity required in $\mathcal{Y}$ is determined by the relative degree. In the case of $r = \rho$, a stabilizing high-gain feedback is constructed and an explicit upper bound on the permitted delays is given. The case of $k = \infty$ is also considered. In the linear setting, the results are also extended to incorporate systems with nonzero initial conditions.

The results are established by computing the gap distance between C_k and $C_k^{\text{Euler}}[h]$ and using variants on nonlinear robust stability theory [9] to deduce the stability of the closed-loop containing the Euler controller from the stability of the derivative feedback controlled closed-loop.

In practice, a PD controller or (partial) state feedback is often implemented by such approximations. In the context of nonlinear plants, there are often limited options for the implementation of a (partial) state feedback: nonlinear observers are available only for limited classes of plants. Of course, in practice, a direct implementation of $C_k^{\text{Euler}}[h]$ is as problematic as the direct implementation of C_k from measurement of y_2 only: we are simply replacing the problem of *calculating* the derivative of the measurement with the problem of *storing* a finite interval of past measurements (so that the delays can be evaluated). However, sampled versions of $C_k^{\text{Euler}}[h]$ can also be analyzed utilizing the techniques of this paper, and such realizations, which coincide with common engineering practice, give analogous results. A variety of sampled versions of these results will be given in the companion paper [6], which also extends the results for fully nonlinear controllers and to the important case of semiglobal stabilization.

It is perhaps surprising that there are relatively few theoretical results available on closed-loop stability for such delay-based controllers. For linear time-invariant systems with relative degree 2 controlled by the delay feedback (1.2), exponential stability of the resulting closed-loop delay differential system was established in [11]. An analogous result for higher relative degree has not been previously established. Stabilization of (nonlinear) systems via delays has been considered by some authors: in [14] the authors give a control strategy with multiple delays that stabilizes a simple system of the form $y^{(n)} = u$. In [13] necessary conditions for multiple delay controllers that stabilize linear systems are shown, but no explicit control strategy is given. In [12] the author considers nonlinear systems with several constraints and gives a control strategy that achieves a bounded output. As observed by a reviewer of this paper, it is anticipated that it is also possible to study such delay feedbacks using the concept of w-stability [7], and we will discuss this approach further in what follows.

The paper is organized as follows. In section 2 we introduce the background theory and establish a key robust stability result. Section 3 contains the main theorem of the paper which shows that a stabilizing derivative feedback controller may be substituted by a delay feedback controller if the delay is sufficiently small. In section 4 we consider applications to linear systems to demonstrate structural features of the conditions, giving results establishing both external (gain) stability and internal stability of the closed-loop system. An illustrative example is developed in section 5, and conclusions are given in section 6.

2. Background. The material in this section is based on [9, section II], [4, section 2], and [5, section 2] and contains the gap metric results necessary for proving robustness in section 3.

2.1. Terminology. Let $\mathcal{X}$ be a nonempty set. For $0 < \omega \leq \infty$ let $\mathcal{S}_\omega$ denote the set of all locally integrable maps in $\mathrm{map}([0,\omega) \to \mathcal{X})$. For ease of notation define $\mathcal{S} := \mathcal{S}_\infty$. For $0 < \tau < \omega \leq \infty$ define a truncation operator T_τ and the restriction of maps as follows:

$$
\begin{aligned}
T_\tau \,:\, \mathcal{S}_\omega \to \mathcal{S}\,, \qquad & v \mapsto T_\tau v := \left(t \mapsto \begin{cases} v(t), & t \in [0,\tau) \\ 0, & t \in [\tau,\infty) \end{cases} \right), \\
(\cdot)\big|_{[0,\tau)} \,:\, \mathcal{S}_\omega \to \mathcal{S}_\tau\,, \qquad & v \mapsto v|_{[0,\tau)} := (t \mapsto v(t), \quad t \in [0,\tau))\;.
\end{aligned}
$$

With $\mathcal{V} \subset \mathcal{S}$ we associate spaces as follows:

$$
\begin{aligned}
\mathcal{V}[0,\tau) &= \left\{ v \in \mathcal{S}_\tau \,\middle|\, \exists\, w \in \mathcal{V} \text{ with } \|T_\tau w\|_{\mathcal{V}} < \infty \;:\; v = w|_{[0,\tau)} \right\} \text{ for } \tau > 0\,; \\
\mathcal{V}_e &= \left\{ v \in \mathcal{S} \,\middle|\, \forall\, \tau > 0 \,:\, v|_{[0,\tau)} \in \mathcal{V}[0,\tau) \right\}, \text{ the extended space}\,; \\
\mathcal{V}_\omega &= \left\{ v \in \mathcal{S}_\omega \,\middle|\, \forall\, \tau \in (0,\omega) \,:\, v|_{[0,\tau)} \in \mathcal{V}[0,\tau) \right\} \text{ for } 0 < \omega \leq \infty\,; \\
\mathcal{V}_a &= \textstyle\bigcup_{\omega \in (0,\infty]} \mathcal{V}_\omega\,, \text{ the ambient space}\,.
\end{aligned}
$$

For L^p spaces these definitions coincide with the definitions of ambient and extended spaces given in [4, 5, 9]; however, note that the definitions in [4, 5, 9] are not applicable for subspaces of continuously differentiable functions as considered in the present paper. This is due to the fact that $CW^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ is not closed under the action of T_τ, $\tau > 0$.

If $v, w \in \mathcal{V}_a$ with $v|_I = w|_I$ on $I = \mathrm{dom}(v) \cap \mathrm{dom}(w)$, then we write $v = w$. For $(u,y) \in \mathcal{V}_a \times \mathcal{V}_a$, the domains of u and y may be different; we adopt the convention

$$
\mathrm{dom}(u,y) \;:=\; \mathrm{dom}(u) \cap \mathrm{dom}(y)\,.
$$

We say $\mathcal{V} \subset \mathcal{S}$ is a *signal space* if and only if it is a normed vector space. For our main results we will consider different types of signal spaces, which are specified in (3.2).

2.2. Well posedness. A mapping $Q\colon \mathcal{U}_a \to \mathcal{Y}_a$ is said to be *causal* if and only if

$$
\forall\, x, y \in \mathcal{U}_a,\; \forall\, \tau \in \mathrm{dom}(x) \cap \mathrm{dom}(Qx) \,:\; \left[x|_{[0,\tau)} = y|_{[0,\tau)} \;\Rightarrow\; (Qx)\big|_{[0,\tau)} = (Qy)\big|_{[0,\tau)} \right].
$$

Consider $P\colon \mathcal{U}_a \to \mathcal{Y}_a$, $u_1 \mapsto y_1$, and $C\colon \mathcal{Y}_a \to \mathcal{U}_a$, $y_2 \mapsto u_2$ to be causal mappings representing the plant and the controller, respectively, and satisfying the closed-loop equations

$$
[P,C]\colon y_1 = Pu_1, \quad u_2 = Cy_2, \quad u_0 = u_1 + u_2, \quad y_0 = y_1 + y_2\,, \tag{2.1}
$$

corresponding to the closed-loop shown in Figure 1.

For $w_0 = (u_0, y_0) \in \mathcal{W} := \mathcal{U} \times \mathcal{Y}$ a pair $(w_1, w_2) = ((u_1, y_1), (u_2, y_2)) \in \mathcal{W}_a \times \mathcal{W}_a$, $\mathcal{W}_a := \mathcal{U}_a \times \mathcal{Y}_a$, is a *solution* if and only if (2.1) holds on $\mathrm{dom}(w_1, w_2)$. The (possibly empty) set of solutions is denoted by

$$\mathcal{X}_{w_0} := \{(w_1, w_2) \in \mathcal{W}_a \times \mathcal{W}_a \mid (w_1, w_2) \text{ solves } (2.1)\}.$$

The closed-loop system $[P, C]$, given by (2.1), is said to have

- the *existence property* if and only if $\mathcal{X}_{w_0} \neq \emptyset$;
- the *uniqueness property* if and only if

$$\forall\, w_0 \in \mathcal{W}\colon \Big[(\hat{w}_1, \hat{w}_2), (\tilde{w}_1, \tilde{w}_2) \in \mathcal{X}_{w_0} \\ \Longrightarrow (\hat{w}_1, \hat{w}_2) = (\tilde{w}_1, \tilde{w}_2) \quad \text{on} \quad \mathrm{dom}(\hat{w}_1, \hat{w}_2) \cap \mathrm{dom}\,(\tilde{w}_1, \tilde{w}_2)\Big].$$

Assume that $[P, C]$ has the existence and uniqueness properties. For each $w_0 \in \mathcal{W}$, define $\omega_{w_0} \in (0, \infty]$ by the property

$$[0, \omega_{w_0}) := \cup_{(\hat{w}_1, \hat{w}_2) \in \mathcal{X}_{w_0}} \mathrm{dom}(\hat{w}_1, \hat{w}_2),$$

and define $(w_1, w_2) \in \mathcal{W}_a \times \mathcal{W}_a$, with $\mathrm{dom}(w_1, w_2) = [0, \omega_{w_0})$, by the property $(w_1, w_2)|_{[0,t)} \in \mathcal{X}_{w_0}$ for all $t \in [0, \omega_{w_0})$. This construction induces the operator

$$H_{P,C} : \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a, \quad w_0 \mapsto (w_1, w_2).$$

For $\Omega \subset \mathcal{W}$, the closed-loop system $[P, C]$, given by (2.1), is said to be

- *locally well posed on* Ω if and only if it has the existence and uniqueness properties and the operator $H_{P,C}\big|_{\Omega}\colon \Omega \to \mathcal{W}_a \times \mathcal{W}_a,\ w_0 \mapsto (w_1, w_2)$, is causal;
- *globally well posed on* Ω if and only if it is locally well posed on Ω and $H_{P,C}(\Omega) \subset \mathcal{W}_e \times \mathcal{W}_e$;
- *regularly well posed* if and only if it is locally well posed and

$$\text{(2.2)} \quad \forall\, w_0 \in \mathcal{W} \ \Big[\omega_{w_0} < \infty \ \Rightarrow\ \big\|(H_{P,C} w_0)|_{[0,\tau)}\big\|_{\mathcal{W}_\tau \times \mathcal{W}_\tau} \to \infty \text{ as } \tau \to \omega_{w_0}\Big].$$

2.3. Graphs, the nonlinear gap metric, and gain stability. For the plant operator $P\colon \mathcal{U}_a \to \mathcal{Y}_a$ and the controller operator $C\colon \mathcal{Y}_a \to \mathcal{U}_a$ define the *graph* $\mathcal{G}_P$ of the plant and the *graph* $\mathcal{G}_C$ of the controller, respectively, as follows:

$$\mathcal{G}_P := \left\{ \begin{pmatrix} u \\ Pu \end{pmatrix} \,\middle|\, u \in \mathcal{U},\ Pu \in \mathcal{Y} \right\} \subset \mathcal{W}, \quad \mathcal{G}_C := \left\{ \begin{pmatrix} Cy \\ y \end{pmatrix} \,\middle|\, Cy \in \mathcal{U},\ y \in \mathcal{Y} \right\} \subset \mathcal{W}.$$

Note that $\mathcal{G}_P$ and $\mathcal{G}_C$ are, strictly speaking, not subsets of $\mathcal{W}$; however, by abusing the notation we identify $\mathcal{G}_P \ni \binom{u}{Pu} = (u, Pu) \in \mathcal{W}$ and $\mathcal{G}_C \ni \binom{Cy}{y} = (Cy, y) \in \mathcal{W}$. An operator $P\colon \mathcal{U}_a \to \mathcal{Y}_a$ is said to be *causally extendible* [8] (or stabilizable in [4]) if and only if

$$\forall\, \tau > 0\ \forall\, w_1 = (u_1, y_1) \in \mathcal{W}_a \text{ with } T_\tau y_1 = T_\tau P u_1\ \exists\, w_1^* \in \mathcal{G}_P\colon T_\tau w_1 = T_\tau w_1^*.$$

Given normed signal spaces $\mathcal{X}$ and $\mathcal{V}$ and $\Omega \subset \mathcal{X}$, a causal operator $Q\colon \mathcal{X} \to \mathcal{V}_a$ is said to be *gain stable on* Ω if and only if $Q(\Omega) \subset \mathcal{V}$, $Q(0) = 0$ and

$$\big\|Q|_\Omega\big\|_{\mathcal{X},\mathcal{V}} := \sup \left\{ \frac{\|(Qx)|_{[0,\tau)}\|_{\mathcal{V}_\tau}}{\|x|_{[0,\tau)}\|_{\mathcal{X}_\tau}} \,\middle|\, x \in \Omega,\ \tau > 0,\ x|_{[0,\tau)} \neq 0 \right\} < \infty.$$

Given normed signal spaces $\mathcal{U}$, $\mathcal{Y}$, and $\mathcal{W} := \mathcal{U} \times \mathcal{Y}$ and causal operators $P : \mathcal{U}_a \to \mathcal{Y}_a$, $C : \mathcal{Y}_a \to \mathcal{U}_a$ we make the following definitions. The closed-loop system $[P, C]$ given by (2.1) with the associated operator $H_{P,C} : \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a$ is said to be $\mathcal{W}$-*stable* if and only if it is globally well posed and $H_{P,C}(\mathcal{W}) \subset \mathcal{W} \times \mathcal{W}$. It is said to be $\mathcal{W}$-*gain stable* if and only if it is $\mathcal{W}$-stable and $H_{P,C}$ is gain stable on $\mathcal{W}$.

Next, associate with the closed-loop system $[P, C]$ given by (2.1) the following two parallel projection operators:

$$\Pi_{P//C} : \mathcal{W} \to \mathcal{W}_a\,,\ w_0 \mapsto w_1 \qquad \text{and} \qquad \Pi_{C//P} : \mathcal{W} \to \mathcal{W}_a\,,\ w_0 \mapsto w_2\,.$$

Note that gain stability of either $\Pi_{P//C}$ or $\Pi_{C//P}$ implies gain stability of the closed-loop system $[P, C]$, and that $\|\Pi_{P//C}\|_{\mathcal{W},\mathcal{W}}, \|\Pi_{C//P}\|_{\mathcal{W},\mathcal{W}} \geq 1$ since $\Pi_{P//C} = \Pi^2_{P//C}$, $\Pi_{C//P} = \Pi^2_{C//P}$.

For $P_1, P_2 \in \Gamma(\mathcal{U}, \mathcal{Y}) := \{P : \mathcal{U}_a \to \mathcal{Y}_a \mid P \text{ is causal}\}$, define the *directed gap* by

$$\vec{\delta} : \Gamma(\mathcal{U}, \mathcal{Y}) \times \Gamma(\mathcal{U}, \mathcal{Y}) \to [0, \infty]\,,$$
$$(P_1, P_2) \mapsto \inf_{\Phi \in \mathcal{O}_{P_1,P_2}} \sup_{x \in \mathcal{G}_{P_1} \setminus \{0\}} \left(\frac{\|(\Phi - I)|_{\mathcal{G}_{P_1}}(x)\|_{\mathcal{W}}}{\|x\|_{\mathcal{W}}} \right),$$

where $\mathcal{O}_{P_1,P_2}$ is the (possibly empty) set

$$\mathcal{O}_{P_1,P_2} := \{\Phi : \mathcal{G}_{P_1} \to \mathcal{G}_{P_2} \mid \Phi \text{ is causal, surjective and } \Phi(0) = 0\}\,.$$

Here we adopt the convention that $\vec{\delta}(P_1, P_2) := \infty$ if $\mathcal{O}_{P_1,P_2} = \emptyset$. The *nonlinear gap* is defined as

$$\delta : \Gamma(\mathcal{U}, \mathcal{Y}) \times \Gamma(\mathcal{U}, \mathcal{Y}) \to [0, \infty]\,, \quad (P_1, P_2) \mapsto \delta(P_1, P_2) := \max\{\vec{\delta}(P_1, P_2), \vec{\delta}(P_2, P_1)\}.$$

2.4. Robust stability. We now prove the robust stability theorem on which the main result in this paper is based. This result is based on [9, Thm. 1], but extends the scope of that result in several directions. First, the result is established in the language of ambient signal spaces to handle finite escape times (cf. [9, Thm. 8]). More important, the implicit requirement in [9] of well posedness of $[P_1, C]$ is extended to include the often weaker requirement of regular well posedness. This eases the application of the result in general, as global well posedness is nontrivial to verify a priori, and regular well posedness is often easier to establish (for $p = \infty$, regular well posedness follows from standard results on the finite escape time properties of differential equations).

Note that we state this theorem in a form where stability of $[P_1, C]$ is inferred from $[P, C]$; however, in what follows we will apply this theorem in the setting whereby stability of $[P, C_1]$ is to be inferred from $[P, C]$. Such applications of the theorem follow from a trivial interchange of P and C and $\mathcal{U}$, $\mathcal{Y}$; we elect to present the theorem in the context of P, P_1 to follow the convention of the literature and since, in contrast to this paper, most applications of such robust stability results concern uncertainty in the plant P.

THEOREM 2.1. *Let $\mathcal{U}$, $\mathcal{Y}$ be signal spaces and $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$. Consider $P : \mathcal{U}_a \to \mathcal{Y}_a$, $P_1 : \mathcal{U}_a \to \mathcal{Y}_a$, and $C : \mathcal{Y}_a \to \mathcal{U}_a$ with $P(0) = 0$, $C(0) = 0$. Suppose $[P, C]$ is gain stable on $\mathcal{W}$, P_1 is causally extendible, and $[P_1, C]$ is either* (a) *globally or* (b) *regularly well posed. If*

$$\vec{\delta}(P, P_1) < \|\Pi_{P//C}\|^{-1}_{\mathcal{W},\mathcal{W}}, \tag{2.3}$$

then the closed-loop system $[P_1, C]$ *is gain stable on* $\mathcal{W}$ *with*

$$\left\|\Pi_{P_1//C}\right\|_{\mathcal{W},\mathcal{W}} \le \left\|\Pi_{P//C}\right\|_{\mathcal{W},\mathcal{W}} \frac{1+\vec{\delta}(P,P_1)}{1-\left\|\Pi_{P//C}\right\|_{\mathcal{W},\mathcal{W}}\vec{\delta}(P,P_1)}. \tag{2.4}$$

Proof. Since $\left\|\Pi_{P//C}\right\|_{\mathcal{W},\mathcal{W}} \ge 1$, it follows that $\vec{\delta}(P,P_1) < \infty$, and hence there exists a causal surjective mapping $\Phi\colon \mathcal{G}_P \to \mathcal{G}_{P_1}$ such that

$$\gamma := \|(\Phi - I)\Pi_{P//C}\| \le \|(\Phi - I)\| \cdot \|\Pi_{P//C}\| < 1. \tag{2.5}$$

Let $w \in \mathcal{W}$, and let $[0, \omega_w)$ be the maximal interval of existence for $H_{P_1,C}(w)$. Let $0 < \tau < \omega_w$. Consider the equation

$$w|_{[0,\tau)} = ((I + (\Phi - I)\Pi_{P//C})(x))|_{[0,\tau)} \tag{2.6}$$

$$= ((\Pi_{C//P} + \Phi\Pi_{P//C})(x))|_{[0,\tau)}. \tag{2.7}$$

By either well posedness assumption (a) or (b), we know that $[P_1, C]$ is locally well posed and hence satisfies the existence and uniqueness properties on $[0, \tau)$. Hence there exist $w_1 = (u_1, y_1), w_2 = (u_2, y_2) \in \mathcal{W}_{\omega_w}$ such that $y_1 = P_1 u_1$, $u_2 = C y_2$, and $w|_{[0,\tau)} = w_1|_{[0,\tau)} + w_2|_{[0,\tau)}$. Since P_1 is stabilizable, there exists $w_1'' \in \mathcal{G}_{P_1}$ such that $w_1''|_{[0,\tau)} = w_1|_{[0,\tau)}$. By the definition of $\mathcal{W}_{\omega_w}$ we have $w_2|_{[0,\tau)} \in \mathcal{W}[0,\tau)$, and hence there exists $w_2' \in \mathcal{W}$ such that $w_2'|_{[0,\tau)} = w_2|_{[0,\tau)}$. Since Φ is surjective it follows that there exists $w_1' \in \mathcal{G}_P$ such that $\Phi(w_1') = w_1''$, and hence $(\Phi(w_1'))|_{[0,\tau)} = w_1''|_{[0,\tau)} = w_1|_{[0,\tau)}$. It can now be seen that $x = w_1' + w_2' \in \mathcal{W}$ satisfies $x|_{[0,\tau)} = (w_1' + w_2)|_{[0,\tau)}$, and x is a solution of (2.7).

Since Φ, $\Pi_{P_1//C}$, $\Pi_{P//C}$, $\Pi_{C//P}$ are causal, it follows from (2.7) that

$$(\Pi_{P_1//C}(w))|_{[0,\tau)} = \left(\Pi_{P_1//C}\left(\Pi_{C//P}x + \Phi\Pi_{P//C}(x)\right)\right)|_{[0,\tau)} = (\Phi\Pi_{P//C}(x))|_{[0,\tau)}. \tag{2.8}$$

It follows from (2.6) that $\left\|x|_{[0,\tau)}\right\| \le \frac{1}{1-\gamma}\left\|w|_{[0,\tau)}\right\|$, and hence, in view of (2.3), (2.5), and (2.8),

$$\begin{aligned}\left\|\Pi_{P_1//C}(w)|_{[0,\tau)}\right\| &= \left\|\Phi\Pi_{P//C}(x)|_{[0,\tau)}\right\| \\ &\le \left\|\Pi_{P//C}(x)|_{[0,\tau)}\right\| + \left\|(\Phi - I)\Pi_{P//C}(x)|_{[0,\tau)}\right\| \\ &\le \left\|\Pi_{P//C}\right\|_{\mathcal{W},\mathcal{W}} \frac{1+\vec{\delta}(P,P_1)}{1-\left\|\Pi_{P//C}\right\|_{\mathcal{W},\mathcal{W}}\vec{\delta}(P,P_1)}\left\|w|_{[0,\tau)}\right\|. \end{aligned} \tag{2.9}$$

If $[P_1, C]$ is globally well posed, $\omega_w = \infty$, so inequality (2.9) holds for all $\tau > 0$, and the proof is complete.

Suppose that $[P_1, C]$ is regularly well posed. Since we have shown that $(\Pi_{P_1//C}(w))|_{[0,\tau)} \in \mathcal{W}[0,\tau)$ is uniformly bounded for all $\tau \in (0, \omega_w)$ and since $[P_1, C]$ is regularly well posed, it follows that $\omega_w = \infty$, so inequality (2.9) holds for all $\tau > 0$. This completes the proof. ∎

3. Robust stabilization by delay feedback. Our main result will establish conditions under which a derivative feedback controller (1.1) may be replaced by Euler approximation (1.2). We first formally define, for $h > 0$, the Euler approximation

$$\begin{aligned}&\Delta_h\colon \mathrm{map}(\mathbb{R}_{\ge 0} \to \mathbb{R}) \to \mathrm{map}(\mathbb{R}_{\ge 0} \to \mathbb{R}), \\ &\qquad (t \mapsto y(t)) \mapsto \left(t \mapsto \frac{y(t) - y(t-h)}{h}\right), \quad \text{where } y(s) = 0 \text{ if } s < 0,\end{aligned}$$

of the derivative of y, and, for higher derivatives $y^{(i)}$, $i \in \mathbb{N}$,

$$
\begin{aligned}
\Delta_h^i \colon \operatorname{map}(\mathbb{R}_{\geq 0} \to \mathbb{R}) \quad &\to \quad \operatorname{map}(\mathbb{R}_{\geq 0} \to \mathbb{R})\,, \\
y \quad &\mapsto \quad \Delta_h^i(y) \;:=\; \begin{cases} \Delta_h^{i-1}(\Delta_h(y)) & \text{if } i \geq 2, \\ \Delta_h(y) & \text{if } i = 1, \\ y & \text{if } i = 0\,. \end{cases}
\end{aligned}
\tag{3.1}
$$

Our results will hold in the following signal space settings (A)–(C):

$$
\begin{aligned}
&\text{(A) } \mathcal{W} = \mathcal{U} \times \mathcal{Y} = CL^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) \times CW^{r,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})\,, \\
&\quad\;\; \mathcal{W}_0 = \mathcal{U}_0 \times \mathcal{Y}_0 = CL^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) \times CW_0^{r,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})\,, \; r \in \mathbb{N}\,, \; p = \infty; \\
&\text{(B) } \mathcal{W} = \mathcal{U} \times \mathcal{Y} = CL^p(\mathbb{R}_{\geq 0} \to \mathbb{R}) \times CW^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R})\,, \\
&\quad\;\; \mathcal{W}_0 = \mathcal{U}_0 \times \mathcal{Y}_0 = CL^p(\mathbb{R}_{\geq 0} \to \mathbb{R}) \times CW_0^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R})\,, \; r \in \mathbb{N}\,, \; p \in [1, \infty); \\
&\text{(C) } \mathcal{W} = \mathcal{U} \times \mathcal{Y} = CW^{\infty,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}) \times CW^{\infty,p}(\mathbb{R}_{\geq 0} \to \mathbb{R})\,, \\
&\quad\;\; \mathcal{W}_0 = \mathcal{U}_0 \times \mathcal{Y}_0 = CW_0^{\infty,p}(\mathbb{R}_{\geq 0} \to \mathbb{R}) \times CW_0^{\infty,p}(\mathbb{R}_{\geq 0} \to \mathbb{R})\,, \; p \in [1, \infty)\,.
\end{aligned}
\tag{3.2}
$$

The spaces $\mathcal{W}_0$ will be utilized for results whereby the initial conditions of the system are zero, while the spaces $\mathcal{W}$ are utilized in the general setting with nonzero initial conditions. The spaces of types (A) and (B) are standard; the need for spaces with constrained derivatives arises from the setting whereby derivative-based controllers are being considered. In section 4 we will motivate the spaces of type (C), which allows for more general controllers (we will not require $k_r = 0$ for controller C_k given by (1.1) as for signal spaces of type (B); see below) at the price of greater regularity constraints on the disturbances.

We are now in a position to state the main result of this section, namely that if C_k gain stabilizes P, it follows that $C_k^{\text{Euler}}[h]$ is also a gain stabilizing controller of P for sufficiently small $h > 0$. The idea behind the proof is to show that the gap $\vec{\delta}(C_k, C_k^{\text{Euler}}[h])$ is small if $h > 0$ is small and hence deduce the result from Theorem 2.1.

THEOREM 3.1. *Let $1 \leq p \leq \infty$, $r \in \mathbb{N}$ and consider signal spaces $\mathcal{U}_0$, $\mathcal{Y}_0$, and $\mathcal{W}_0$ of type* (A), (B), *or* (C) *in* (3.2). *Suppose that there exists $k = (k_1, \ldots, k_r) \in \mathbb{R}^{1\times r} \setminus \{0\}$, with $k_r = 0$ for case* (B), *such that controller $C_k : \mathcal{Y}_{0a} \to \mathcal{U}_{0a}$ given by* (1.1) *applied to a causal plant $P : \mathcal{U}_{0a} \to \mathcal{Y}_{0a}$ with $P(0) = 0$ yields a closed-loop system $[P, C_k]$ which is gain stable on $\mathcal{W}_0 := \mathcal{U}_0 \times \mathcal{Y}_0$ with*

$$1 \;\leq \gamma := \left\| \Pi_{C_k//P} \right\|_{\mathcal{W}_0, \mathcal{W}_0} \;<\; \infty\,.$$

Suppose $h^* > 0$ satisfies

$$h^* \leq \left(\gamma \sum_{i=1}^{r-1} |k_{i+1}| \cdot i \eta_p(h^*, i) \right)^{-1}, \tag{3.3}$$

$$\text{where, for } h > 0,\ \eta_p(h, i) := \begin{cases} 1 & \text{in case (A)}, \\ 2^{\frac{1}{p}} (1 + ihp)^{\frac{1}{p}} & \text{in case (B)}, \\ 2^{\frac{1}{p}} (1 + ihp) & \text{in case (C)}. \end{cases}$$

Let $h \in (0, h^*)$ and suppose that $[P, C_k^{\text{Euler}}[h]]$ is either globally or regularly well posed, where the controller $C_k^{\text{Euler}}[h] \colon \mathcal{Y}_{0a} \to \mathcal{U}_{0a}$ is given by (1.2). *Then the closed-loop* *system* $[P, C_k^{\text{Euler}}[h]]$ *is gain stable on* $\mathcal{W}_0$ *with*

$$\|\Pi_{C_k^{\text{Euler}}[h]//P}\|_{\mathcal{W}_0,\mathcal{W}_0} \le \|\Pi_{C_k//P}\|_{\mathcal{W}_0,\mathcal{W}_0} \frac{1 + h\sum_{i=1}^{r-1}|k_{i+1}|\, i\eta_p(h,i)}{1 - \|\Pi_{C_k//P}\|_{\mathcal{W}_0,\mathcal{W}_0} h\sum_{i=1}^{r-1}|k_{i+1}|\, i\eta_p(h,i)}\,. \tag{3.4}$$

In all three signal space settings (A), (B), and (C) condition (3.3) on h^* can always be met for sufficiently small $h^* > 0$, e.g., by taking $h^* = \big(\gamma\sum_{i=1}^{r-1}|k_{i+1}|\cdot i\big)^{-1}$ in case (A) and by taking $h^* = \min\big\{\frac{1}{rp}, \big(2^{\frac{1+p}{p}}\gamma\sum_{i=1}^{r-1}|k_{i+1}|\cdot i\big)^{-1}\big\}$ in cases (B) and (C).

The condition that the nominal closed-loop gain is bounded enforces key attenuation properties, which vary with the different choices of signal space $\mathcal{W}_0$. Within the context of linear systems, a key purpose of section 4 is to explore how these properties can be enforced by structural requirements on the relative degree and order of the controller. For example, Theorem 4.2 shows that in the case of the signal space setting of (A) stabilizing controllers whose order is less than or equal to the relative degree of the plant can be replaced by suitable Euler controllers, and that in the signal space setting of (B), the stabilizing controllers are required to have an order strictly less than the order of the plant. Theorem 4.2 also shows that the final signal space setting (C) overcomes the structural limitation of the choice (B) by again allowing stabilizing controllers whose order is less than or equal to the relative degree of the plant, but with considerable extra signal regularity requirements.

The extra requirement that $k_r = 0$ in the signal space setting (B) arises from the application of the Mean Value Theorem in the proof of Theorem 4.2. In the case of $p = \infty$, i.e., signal space setting (A), it follows from the Mean Value Theorem that $\|y(\cdot) - y(\cdot - h)\|_{L^\infty} \le h\|\dot y\|_{L^\infty} \le h\|y\|_{W^{1,\infty}}$, whereas in the case of $p < \infty$, i.e., signal space setting (B), again by the Mean Value Theorem, $\|y(\cdot) - y(\cdot - h)\|_{L^p} \le h\|M_h[\dot y](\cdot)\|_{L^p}$, which in turn by Proposition 3.2 below yields a bound of the form $\|y(\cdot)-y(\cdot-h)\|_{L^p} \le 2h\|M_h[y](\cdot)\|_{W^{2,p}}$. The requirement of bounding an extra derivative then leads to the additional requirement that $k_r = 0$ (signal space setting (A)), or alternatively, that derivatives of all orders are bounded (signal space setting (C)).

Before giving the proof of Theorem 3.1, we first establish the key bound which will be required in the proof of Theorem 3.1 for the signal space choices (B) and (C) as discussed above.

PROPOSITION 3.2. *For* $y \in \mathcal{C}(\mathbb{R}_{\ge 0} \to \mathbb{R})$ *and* $\varrho > 0$*, define the function*

$$M_\varrho[y]\colon \mathbb{R}_{\ge 0} \to \mathbb{R}\,, \qquad t \mapsto \max_{\tau\in[t-\varrho,t]} |y(\tau)|\,, \quad \text{where } y(s) = 0 \text{ if } s < 0. \tag{3.5}$$

Then, for every $y \in CW_0^{1,p}(\mathbb{R}_{\ge 0} \to \mathbb{R})$ *and* $1 \le p < \infty$,

$$\forall T > 0 \;:\; \|M_\varrho[y]\|^p_{L^p([0,T)\to\mathbb{R})} \le 2\|y\|^{p-1}_{L^p([0,T)\to\mathbb{R})}\left(\|y\|_{L^p([0,T)\to\mathbb{R})} + \varrho p\|\dot y\|_{L^p([0,T)\to\mathbb{R})}\right)\,. \tag{3.6}$$

Proof. Let $y \in CW_0^{1,p}(\mathbb{R}_{\ge 0} \to \mathbb{R})$, $1 \le p < \infty$, $T > 0$, and $\varepsilon > 0$. By the density of $C_0^2([0,T] \to \mathbb{R})$ in $C_0([0,T] \to \mathbb{R})$ it follows from [16, Thm. 4.12] applied on the interval $[0,T]$ that there exists a (piecewise cubic) function $G_0\colon [0,T] \to \mathbb{R}$ such that G_0 is nowhere locally constant, and

$$|G_0(t) - y(t)| \le \varepsilon\,, \quad |\dot G_0(t) - \dot y(t)| \le \varepsilon\,, \quad t \in [0,T]\,.$$

We now define $G \in L^p(\mathbb{R}_{\geq 0} \to \mathbb{R})$ by $G = T_T G_0$. Suppose, for the time being,

$$\|M_\varrho[G]\|^p_{L^p([0,T)\to\mathbb{R})} \leq 2\,\|G\|^{p-1}_{L^p([0,T)\to\mathbb{R})}\big(\|G\|_{L^p([0,T)\to\mathbb{R})} + \varrho p\|\dot G\|_{L^p([0,T)\to\mathbb{R})}\big). \tag{3.7}$$

Then in view of

$$M_\varrho[y](t) = M_\varrho[y+G-G](t) \leq M_\varrho[G](t) + M_\varrho[y-G](t) \leq M_\varrho[G](t) + M_\varrho[\varepsilon](t)\,,$$

and since $M_\varrho[\varepsilon](t) = \varepsilon$ for $t \geq 0$, it follows that

$$\|M_\varrho[y]\|^p_{L^p([0,T)\to\mathbb{R})} \leq 2\|G\|^{p-1}_{L^p([0,T)\to\mathbb{R})}\Big(\|G\|_{L^p([0,T)\to\mathbb{R})} + \varrho p\|\dot G\|_{L^p([0,T)\to\mathbb{R})}\Big) + T\varepsilon^p\,.$$

Since

$$\begin{aligned}\|G\|_{L^p([0,T)\to\mathbb{R})} &\leq \|G-y\|_{L^p([0,T)\to\mathbb{R})} + \|y\|_{L^p([0,T)\to\mathbb{R})} \leq T^{\frac{1}{p}}\,\varepsilon + \|y\|_{L^p([0,T)\to\mathbb{R})}\,,\\ \|\dot G\|_{L^p([0,T)\to\mathbb{R})} &\leq \|\dot G-\dot y\|_{L^p([0,T)\to\mathbb{R})} + \|\dot y\|_{L^p([0,T)\to\mathbb{R})} \leq T^{\frac{1}{p}}\,\varepsilon + \|\dot y\|_{L^p([0,T)\to\mathbb{R})}\,,\end{aligned}$$

it follows that

$$\begin{aligned}\|M_\varrho[y]\|^p_{L^p([0,T)\to\mathbb{R})} &\leq 2\big(T^{\frac{1}{p}}\,\varepsilon + \|y\|_{L^p([0,T)\to\mathbb{R})}\big)^{p-1}\\ &\quad\cdot\Big(T^{\frac{1}{p}}\,\varepsilon + \|y\|_{L^p([0,T)\to\mathbb{R})} + \varrho p\big(T^{\frac{1}{p}}\,\varepsilon + \|\dot y\|_{L^p([0,T)\to\mathbb{R})}\big)\Big) + T\varepsilon^p.\end{aligned}$$

As this holds for all $\varepsilon > 0$, inequality (3.6) follows as required.

It remains to show (3.7). Let

$$\mathcal{R}(G) := \{t \in [0,T]\,|\,|G(t)| \text{ is a local maximum of } |G|\}\,.$$

Since $|G|$ is piecewise polynomial, $G \not\equiv 0$, $\mathcal{R}(G)$ is nonempty and has a finite or countable number of elements. For every point $t \in \mathcal{R}(G)$ we define

$$\begin{aligned}t^M &:= \inf\left(\{T\} \cup \{\tau \in [t,T]\ |\ |G(\tau)| \text{ is a local minimum of } |G|\}\right)\,,\\ t^R &:= \min\left\{t+\varrho\,, T\,,\ \inf\left\{\tau \in \mathcal{R}(G)\,\middle|\,\tau > t\right\}\right\}\,.\end{aligned}$$

We estimate the L^p-norm of $M_\varrho[G]$ by the L^p-norm of G and the sum of parts of the areas of the hatched boxes; see Figure 2.

By the definition of $M_\varrho[G]$ we have

$$\begin{aligned}\|M_\varrho[G]\|^p_{L^p([0,T)\to\mathbb{R})} &= \int_0^T \left(\max_{\tau\in[t-\varrho,t]}|G(\tau)|\right)^p\ \mathrm{d}t\\ \leq \int_0^T |G(t)|^p\ \mathrm{d}t &+ \sum_{t\in\mathcal{R}(G)}\Big(\left[t^R - t\right]|G(t)|^p\\ &\qquad + \min\{t+\varrho-t^R, T-t^R\}\cdot\max\left\{0, |G(t)|^p - |G(t^R)|^p\right\}\Big),\end{aligned}$$

where $\left(\left[t^R - t\right]|G(t)|^p\right)$ is the area of the hatched box of height $|G(t)|^p$ between the local maximum t and either the following local maximum t^R on the right or the minimum of the points T or $t+\varrho$. Furthermore, $\min\{t+\varrho-t^R, T-t^R\}\cdot\max\left\{0, |G(t)|^p - |G(t^R)|^p\right\}$ is the area of the box which remains by subtracting a box with the height $|G(t^R)|^p$ of the following maximum value t^R from a box with height $|G(t)|^p$ and length

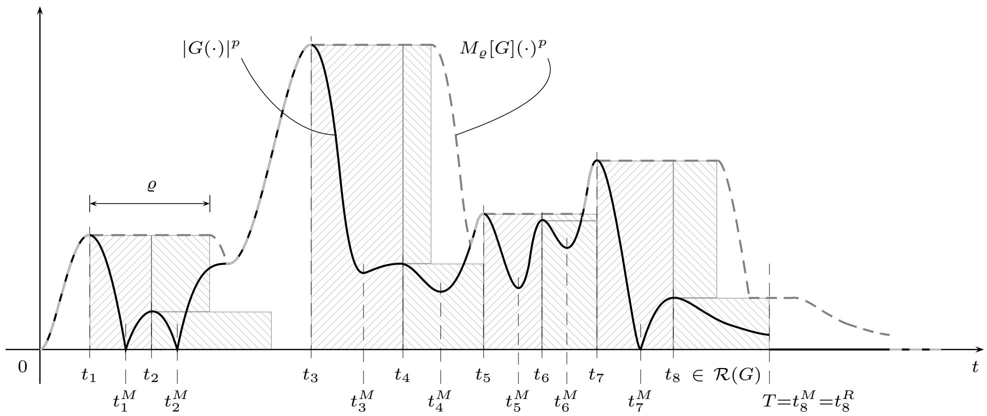

FIG. 2. *Example function* $|G|^p$ *and* $M_\varrho[G]^p$*, here:* $t_1^R = t_2$, $t_2^R = t_2 + \varrho$, $t_3^R = t_4$, $t_4^R = t_5$, $t_5^R = t_6$, $t_6^R = t_7$, $t_7^R = t_8$, *and* $t_8^R = T = t_8^M$.

$\min\{t+\varrho-t^R, T-t^R\}$. Since $|G(t^R)| \geq |G(t^M)|$, $(t,t^R)\cap(s,s^R) = \emptyset$ for all $t,s \in \mathcal{R}(G)$ and $t^R \leq T$, we have

$$\sum_{t\in\mathcal{R}(G)} \left[t^R - t\right]|G(t^M)|^p \leq \sum_{t\in\mathcal{R}(G)} \int_t^{t^R} |G(t)|^p \,\mathrm{d}t \leq \int_0^T |G(t)|^p \,\mathrm{d}t,$$

and hence

$$\begin{aligned}
&\|M_\varrho[G]\|^p_{L^p([0,T)\to\mathbb{R})}\\
&\leq \int_0^T |G(t)|^p \ \mathrm{d}t + \sum_{t\in\mathcal{R}(G)} \Big(\left[t^R - t\right] \left(|G(t)|^p - |G(t^M)|^p\right) + \left[t^R - t\right] |G(t^M)|^p \\
&\qquad + \min\{t+\varrho-t^R, T-t^R\} \cdot \max\left\{0, |G(t)|^p - |G(t^R)|^p\right\} \Big)\\
&\leq \int_0^T |G(t)|^p \ \mathrm{d}t + \sum_{t\in\mathcal{R}(G)} \Big(\left[t^R - t\right] \left(|G(t)|^p - |G(t^M)|^p\right) + \left[t^R - t\right] |G(t^M)|^p \\
&\qquad + \left[t+\varrho-t^R\right] \left(|G(t)|^p - |G(t^M)|^p\right) \Big)
\end{aligned}$$

$$\leq 2\int_0^T |G(t)|^p \ \mathrm{d}t + \varrho \sum_{t\in\mathcal{R}(G)} \left(|G(t)|^p - |G(t^M)|^p\right) . \tag{3.8}$$

Since $G|_{(t,t^M)}$ is either strictly positive or negative, $|G|$ is continuously differentiable on (t,t^M), and partial integration yields

$$\begin{aligned}\sum_{t\in\mathcal{R}(G)} \left(|G(t)|^p - |G(t^M)|^p\right) &\leq \sum_{t\in\mathcal{R}(G)} \int_t^{t^M} p \ |G(t)|^{p-1} \ |\dot{G}(t)| \,\mathrm{d}t \\
&\leq p\|G^{p-1}\dot{G}\|_{L^1([0,T)\to\mathbb{R})} ,\end{aligned} \tag{3.9}$$

where the second inequality above follows from $(t,t^M)\cap(s,s^M) = \emptyset$ for all $t,s \in \mathcal{R}(G)$ and since $t^M \leq T$. Let $1 < q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$; then by Hölder's inequality,

$$\begin{aligned}&\|G^{p-1}\dot{G}\|_{L^1([0,T)\to\mathbb{R})}\\
&\qquad \leq \|G^{p-1}\|_{L^q([0,T)\to\mathbb{R})}\|\dot{G}\|_{L^p([0,T)\to\mathbb{R})} = \|G\|^{p-1}_{L^p([0,T)\to\mathbb{R})}\|\dot{G}\|_{L^p([0,T)\to\mathbb{R})}.\end{aligned} \tag{3.10}$$

Finally, inequalities (3.8), (3.9), and (3.10) give the claimed inequality (3.7). ◻

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Let $1 \le p \le \infty$, $r \in \mathbb{N}$, let signal spaces $\mathcal{U}_0$, $\mathcal{Y}_0$, and $\mathcal{W}_0$ of type (A), (B), or (C) be given by (3.2), and let $k = (k_1, \ldots, k_r) \in \mathbb{R}^{1\times r}$, $k_r = 0$, in case (B). We claim that if $h \in (0, h^*)$, then

$$\vec{\delta}\left(C_k, C_k^{\text{Euler}}[h]\right) \le \; h \sum_{i=1}^{r-1} |k_{i+1}| \cdot i\eta_p(h,i)\,, \tag{3.11}$$

and hence,

$$\vec{\delta}(C_k, C_k^{\text{Euler}}[h]) \overset{(3.11)}{\le} h \sum_{i=0}^{r-1} |k_{i+1}| \cdot i\eta_p(h,i) \overset{(3.3)}{<} \gamma^{-1} = \left\|\Pi_{C_k//P}\right\|^{-1}_{\mathcal{W}_0,\mathcal{W}_0}.$$

By assumption, P, C are causal, $C(0) = P(0) = 0$, $[P, C_k]$ is gain stable on $\mathcal{W}_0$, and $[P, C_k^{\text{Euler}}[h]]$ is either globally or regularly well posed. Finally, since $C_k^{\text{Euler}}[h](\mathcal{Y}_0) \subset \mathcal{U}_0$ it follows that $C_k^{\text{Euler}}[h]$ is causally extendible. Applying Theorem 2.1 with the roles of P and C interchanged we see that (3.4) is a consequence of inequalities (2.4) and (3.11).

It remains to show (3.11).

Step 1. The graphs of C_k and $C_k^{\text{Euler}}[h]$ are given by

$$\mathcal{G}_{C_k} = \left\{ \begin{pmatrix} -\sum_{i=0}^{r-1} k_{i+1}\, y^{(i)} \\ y \end{pmatrix} \middle| \; -\sum_{i=0}^{r-1} k_{i+1}\, y^{(i)} \in \mathcal{U}\,,\; y \in \mathcal{Y} \right\} \;\subset\; \mathcal{U} \times \mathcal{Y}\,,$$

$$\mathcal{G}_{C_k^{\text{Euler}}[h]} = \left\{ \begin{pmatrix} -\sum_{i=0}^{r-1} k_{i+1}\, \Delta_h^i(y) \\ y \end{pmatrix} \middle| \; -\sum_{i=0}^{r-1} k_{i+1}\, \Delta_h^i(y) \in \mathcal{U}\,,\; y \in \mathcal{Y} \right\} \;\subset\; \mathcal{U} \times \mathcal{Y}\,.$$

Consider the surjective map

$$\Phi_h : \mathcal{G}_{C_k} \;\to\; \mathcal{G}_{C_k^{\text{Euler}}[h]}\,, \qquad \begin{pmatrix} -\sum_{i=0}^{r-1} k_{i+1}\, y^{(i)} \\ y \end{pmatrix} \;\mapsto\; \begin{pmatrix} -\sum_{i=0}^{r-1} k_{i+1}\, \Delta_h^i(y) \\ y \end{pmatrix}. \tag{3.12}$$

Since $\left\|\left(-\sum_{i=0}^{r-1} k_{i+1}\, y^{(i)}, y\right)^T\right\|_{\mathcal{U}_0\times\mathcal{Y}_0} \ge \|y\|_{\mathcal{Y}_0}$ and

$$\begin{aligned} \left\| \left(\sum_{i=0}^{r-1} k_{i+1}\, \Delta_h^i(y), y\right)^T - \left(\sum_{i=0}^{r-1} k_{i+1}\, y^{(i)}, y\right)^T \right\|_{\mathcal{W}_0} &= \left\| \sum_{i=1}^{r-1} k_{i+1}\, \left(\Delta_h^i(y) - y^{(i)}\right) \right\|_{\mathcal{U}_0} \\ &\le \sum_{i=1}^{r-1} |k_{i+1}| \left\|\Delta_h^i(y) - y^{(i)}\right\|_{\mathcal{U}_0}\,, \end{aligned}$$

it follows that

$$\vec{\delta}(C_k, C_k^{\text{Euler}}[h]) \le \|\Phi_h - I\|_{\mathcal{W}_0,\mathcal{W}_0} \le \sup_{y \in \mathcal{Y}_0 \setminus \{0\}} \frac{\sum_{i=1}^{r-1} |k_{i+1}| \left\|\Delta_h^i(y) - y^{(i)}\right\|_{\mathcal{U}_0}}{\|y\|_{\mathcal{Y}_0}}\,. \tag{3.13}$$

Note that (3.13) holds for all signal spaces $\mathcal{U}_0$ and $\mathcal{Y}_0$ considered in (A), (B), and (C).

Step 2. Recall that, for $y \in \mathcal{Y}_0$, the definition of $\mathcal{Y}_0$ gives $y^{(i)}(0) = 0$ for all $i \in \{0, \ldots, r-1\}$, in cases (A) and (B), and that $y^{(i)}(0) = 0$ for all $i \in \mathbb{N}_0$ in case (C). Also recall that by definition of Δ_h^i we have $\Delta_h^i(y)(t) = 0$ for $t < ih$. To simplify notation, without loss of generality, define $y(t) = 0$ for $t < 0$.

Let $y \in \mathcal{Y}_0$ and fix $i \in \{1, \ldots, r-1\}$. By $i+1$ applications of the Mean Value Theorem, there exist, for $j \in \{1, \ldots, i\}$, functions $\xi_j^i : [0, \infty) \to \mathbb{R}$ with $\xi_j^i(t) \in (0, jh]$ and $\xi_{i+1}^{i,0} : [0, \infty) \to \mathbb{R}$ with $\xi_{i+1}^{i,0}(t) \in (0, ih]$ such that, for all $t \geq 0$,

$$
\begin{aligned}
\left|\Delta_h^i(y)(t) - y^{(i)}(t)\right| &= \left|\Delta_h^{i-1}\left(\frac{1}{h}\big(y(\cdot) - y(\cdot - h)\big)\right)(t) - y^{(i)}(t)\right| \\
&= \left|\Delta_h^{i-1} y^{(1)}(t - \xi_1^i) - y^{(i)}(t)\right| \\
&\;\vdots \\
&= \left|\frac{1}{h}\left(y^{(i-1)}(t - \xi_{i-1}^i) - \big(y^{(i-1)}\big)(t - \xi_{i-1}^i - h)\right) - y^{(i)}(t)\right| \\
&= \left|y^{(i)}\left(t - \xi_i^i\right) - y^{(i)}(t)\right| \\
&\leq ih \left|y^{(i+1)}(t - \xi_{i+1}^{i,0}(t))\right| .
\end{aligned}
$$

Furthermore, in case (C) there exist, for all $\mu \in \mathbb{N}$, functions $\xi_{i+1}^{i,\mu} : [0, \infty) \to \mathbb{R}$ with $\xi_{i+1}^{i,\mu}(t) \in (0, ih]$ such that, for all $t \geq 0$,

$$
\left|\Delta_h^i(y^{(\mu)})(t) - y^{(\mu+i)}(t)\right| \leq ih \left|y^{(\mu+i+1)}(t - \xi_{i+1}^{i,\mu}(t))\right| .
$$

Hence, in case (A) for $p = \infty$, $\mu = 0$; in case (B) for $p \in [1, \infty)$, $\mu = 0$; and in case (C) for $p \in [1, \infty)$, $\mu \in \mathbb{N}_0$; the following inequality holds:

$$
\begin{aligned}
\left\|\Delta_h^i(y^{(\mu)}) - y^{(\mu+i)}\right\|_{L^p(\mathbb{R}_{\geq 0} \to \mathbb{R})} &\leq ih \left\|y^{(\mu+i+1)}(\cdot - \xi_{i+1}^{i,\mu}(\cdot))\right\|_{L^p(\mathbb{R}_{\geq 0} \to \mathbb{R})} \\
&\leq ih \left\|M_{ih}[y^{(\mu+i+1)}](\cdot)\right\|_{L^p(\mathbb{R}_{\geq 0} \to \mathbb{R})} . \qquad (3.14)
\end{aligned}
$$

Step 3. We show inequality (3.11) in case (A), i.e., for $\mathcal{U}_0 = CL^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$ and $\mathcal{Y}_0 = CW_0^{r,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$. Let $y \in \mathcal{Y}_0$. Observe that $\|y^{(i+1)}\|_{\mathcal{U}_0} \leq \|y\|_{\mathcal{Y}_0}$ for all $i \in \{1, \ldots, r-1\}$. Thus it follows from inequalities (3.13) and (3.14) that

$$
\vec{\delta}(C_k, C_k^{\text{Euler}}[h]) \leq \sup_{y \in \mathcal{Y}_0 \setminus \{0\}} \frac{\sum_{i=1}^{r-1} |k_{i+1}|\, ih \left\|y^{(i+1)}\right\|_{\mathcal{U}_0}}{\|y\|_{\mathcal{Y}_0}} \leq h \sum_{i=1}^{r-1} |k_{i+1}|\, i \, .
$$

This completes the proof in case (A).

Step 4. We show (3.11) in case (B) with $k_r = 0$; that is, for $p \in [1, \infty)$ we let $\mathcal{U}_0 = CL^p(\mathbb{R}_{\geq 0} \to \mathbb{R})$ and $\mathcal{Y}_0 = CW_0^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R})$. Let $y \in \mathcal{Y}_0$, $i \in \{1, \ldots, r-2\}$. Since $k = (k_1, \ldots, k_{r-1}, 0) \in \mathbb{R}^{1 \times r}$ and $y^{(i+1)} \in CW_0^{1,p}(\mathbb{R}_{\geq 0} \to \mathbb{R})$, it follows from (3.14) and Proposition 3.2 that

$$
\begin{aligned}
\left\|\Delta_h^i(y) - y^{(i)}\right\|_{\mathcal{U}_0} &\leq ih \left\|M_{ih}[y^{(i+1)}](\cdot)\right\|_{\mathcal{U}_0} \\
&\leq ih \left(2 \left\|y^{(i+1)}\right\|_{\mathcal{U}_0}^{p-1} \left(\|y^{(i+1)}\|_{\mathcal{U}_0} + ihp\|y^{(i+2)}\|_{\mathcal{U}_0}\right)\right)^{\frac{1}{p}} \\
&\leq 2^{\frac{1}{p}} ih(1 + ihp)^{\frac{1}{p}} \|y\|_{\mathcal{Y}_0}. \qquad (3.15)
\end{aligned}
$$

Then by (3.13) and (3.15),

$$\vec{\delta}(C_k, C_k^{\mathrm{Euler}}[h]) \le 2^{\frac{1}{p}}\, h \sum_{i=0}^{r-2} |k_{i+1}| \cdot i(1+ihp)^{\frac{1}{p}} \overset{k_r=0}{=} 2^{\frac{1}{p}}\, h \sum_{i=0}^{r-1} |k_{i+1}| \cdot i(1+ihp)^{\frac{1}{p}} .$$

This completes the proof in case (B) with $k_r = 0$.

Step 5. We show (3.11) in case (C); i.e., for $p \in [1,\infty)$, let $\mathcal{U}_0 = \mathcal{Y}_0 = CW_0^{\infty,p}(\mathbb{R}_{\ge 0} \to \mathbb{R})$. For brevity write $\|\cdot\|_{L^p} := \|\cdot\|_{L^p(\mathbb{R}_{\ge 0}\to\mathbb{R})}$. Since $y^{(i+1)} \in CW_0^{1,p}(\mathbb{R}_{\ge 0} \to \mathbb{R})$, it follows from Proposition 3.2 and inequality (3.14) that for all $y \in \mathcal{Y}_0$ and $i \in \{1,\dots,r-1\}$,

$$\begin{aligned}
&\left\|\Delta_h^i(y) - y^{(i)}\right\|_{\mathcal{U}_0} \le ih \left\| M_{ih}\left[y^{(i+1)}\right](\cdot)\right\|_{\mathcal{U}_0}\\
&\le ih \sum_{\mu=0}^{\infty} \left[2\, \|y^{(\mu+i+1)}\|_{L^p}^{p-1} \left(\|y^{(\mu+i+1)}\|_{L^p} + ihp \|y^{(\mu+i+2)}\|_{L^p}\right)\right]^{\frac{1}{p}}\\
&\le 2^{\frac{1}{p}} ih \left(\sum_{\mu=0}^{\infty} \|y^{(\mu+i+1)}\|_{L^p} + ihp \sum_{\mu=0}^{\infty} \|y^{(\mu+i+1)}\|_{L^p}^{\frac{p-1}{p}} \|y^{(\mu+i+2)}\|_{L^p}^{\frac{1}{p}} \right)\\
&\le 2^{\frac{1}{p}} ih \left(\sum_{\mu=0}^{\infty} \|y^{(\mu+i+1)}\|_{L^p} + ihp \left(\sum_{\mu=0}^{\infty} \|y^{(\mu+i+1)}\|_{L^p}^{\frac{p-1}{p}} \right) \left(\sum_{\mu=0}^{\infty} \|y^{(\mu+i+2)}\|_{L^p}^{\frac{1}{p}} \right) \right)\\
&\le 2^{\frac{1}{p}} ih \left(\sum_{\mu=0}^{\infty} \|y^{(\mu)}\|_{L^p} + ihp \left(\sum_{\mu=0}^{\infty} \|y^{(\mu)}\|_{L^p}^{\frac{p-1}{p}} \right) \left(\sum_{\mu=0}^{\infty} \|y^{(\mu)}\|_{L^p}^{\frac{1}{p}} \right) \right)\\
&\le 2^{\frac{1}{p}} ih(1+ihp) \sum_{\mu=0}^{\infty} \|y^{(\mu)}\|_{L^p} = 2^{\frac{1}{p}} ih(1+ihp)\|y\|_{\mathcal{Y}_0},
\end{aligned}$$

and so (3.13) yields

$$\vec{\delta}(C_k, C_k^{\mathrm{Euler}}[h]) \le 2^{\frac{1}{p}}\, h \sum_{i=0}^{r-1} |k_{i+1}| \cdot i(1+ihp),$$

which completes the proof in case (C) and concludes the proof. ◻

4. Applications to linear minimum phase systems. The main result, Theorem 3.1, is stated for various signal spaces (3.2). We now consider linear systems in detail to illustrate how the choice of signal space is determined by relative degree assumptions on the linear system and the stabilizability requirements in the various signal spaces. In particular we consider the class $\mathcal{P}_{n,r}$ of all state space triples (A,b,c) corresponding to n-dimensional, minimum phase, single input–single output systems with relative degree $r \in \{1,\dots,n\}$ and positive high frequency gain $cA^{r-1}b$. Let $(A,b,c) \in \mathcal{P}_{n,r}$, $x^0 \in \mathbb{R}^n$, and $P(A,b,c;x^0)\colon \mathcal{U}_e \to \mathcal{Y}_e$ be the associated plant operator $u_1 \mapsto y_1$ given by

$$\left\{ \begin{array}{rcl} \dot{x} &=& A\,x + b\,u_1\,, \qquad x(0) = x^0,\\ y_1 &=& c\,x\,, \end{array}\right. \tag{4.1}$$

where $\mathcal{U}$ and $\mathcal{Y}$ are any of the input/output signal spaces pairs given in (3.2). We establish stability properties for both the nominal closed-loop system $[P(A,b,c;x^0), C_k]$ and the closed-loop system with the delay-based controller $[P(A,b,c;x^0), C_k^{\mathrm{Euler}}[h]]$.

In particular, we show exponential stability of the initial value problems for closed-loop systems with zero disturbances $u_0 \equiv y_0 \equiv 0$ and gain stability of the closed-loop systems with arbitrary u_0, y_0 from signal spaces in cases (A), (B), and (C). For the following stabilization results, we consider the high-gain control design

$$C_{k,\kappa,\nu} \colon \mathcal{Y}_e \to \mathcal{U}_e\,, \quad y_2 \mapsto u_2 = -\nu \sum_{i=0}^{r-1} \kappa^{r-i} k_{i+1} y_2^{(i)}\,, \tag{4.2}$$

where $\kappa, \nu \geq 1$ are suitably large scalars which are to be determined and $k = (k_1, \dots k_r)$ is such that $k_r > 0$, and the polynomial $s \mapsto \sum_{i=0}^{r-1} k_{i+1}\, s^i$ is Hurwitz, i.e., has all roots in $\mathbb{C}_-$.

4.1. Exponential stability of $[P(A,b,c;x^0), C_{k,\kappa,\nu}]$ with $u_0 \equiv y_0 \equiv 0$. In Proposition 4.1 we present how the high-gain derivative feedback controller of form (4.2) stabilizes systems $(A,b,c) \in \mathcal{P}_{n,r}$, $r \leq n$, given by (4.1). We show that there exist $\kappa, \nu \geq 1$ such that an application of controller $C_{k,\kappa,\nu}$ to a linear system (A,b,c) yields an exponentially stable closed-loop system $[P(A,b,c;x^0), C_{k,\kappa,\nu}]$ with $u_0 \equiv y_0 \equiv 0$.

With controller $C_{k,\kappa,\nu}$ we use a static feedback of derivatives of the output signal to stabilize linear systems. Note that only structural conditions of the considered system (A,b,c) are known: the relative degree is known, and the system is minimum phase and has positive high-frequency gain $cA^{r-1}b$.

Proposition 4.1. *Let, for $r, n \in \mathbb{N}$ with $r \leq n$, $(A,b,c) \in \mathcal{P}_{n,r}$ and $x^0 \in \mathbb{R}^n$. Suppose $k = (k_1, \dots, k_r) \in \mathbb{R}^{1\times r}$ with $k_r > 0$ and $s \mapsto \sum_{i=0}^{r-1} k_{i+1}\, s^i$ Hurwitz. Then, for sufficiently large $\kappa, \nu \geq 1$, the closed-loop system $[P(A,b,c;x^0), C_{k,\kappa,\nu}]$ given by (4.1), (4.2), (2.1) with $u_0 \equiv y_0 \equiv 0$ is exponentially stable, in the sense that*

$$\exists\, \nu^* \geq 1\ \forall\, \nu \geq \nu^*\ \exists\, \kappa^* \geq 1\ \exists\, M > 0\ \exists\, \alpha > 0\ \ \forall\, \kappa \geq \kappa^*\ \forall\, t \geq 0\ \ \forall\, x^0 \in \mathbb{R}^n\ \ : \\ \big|x(t;x^0)\big| \leq M e^{-\alpha t} |x^0|,$$

where $x(\cdot;x^0)$ denotes the solution of (4.1), (4.2), (2.1) with $u_0 \equiv y_0 \equiv 0$. Therefore, for every system $(A,b,c) \in \mathcal{P}_{n,r}$ of form (4.1) with $r, n \in \mathbb{N}$, $r \leq n$, we may choose $\widetilde{k} \in \mathbb{R}^{1\times r}$ such that

$$\operatorname{spec}\big(A + b\widetilde{k}[c/\dots/cA^{r-1}]\big) \subset \mathbb{C}_-\,. \tag{4.3}$$

A proof for Proposition 4.1 can be found in the appendix.

Proposition 4.1 shows the existence of parameters $\kappa, \nu \geq 1$ with which controller (4.2) stabilizes system (4.1). Explicit bounds for κ and ν, which are not given here, depend only on the system matrices A, b, c and the vector $k = (k_1, \dots, k_r)$.

With the transformation matrix $V \in \mathbb{R}^{n\times n}$ given by (A.1) and $\widetilde{k} = (\widetilde{k}_1, \dots, \widetilde{k}_r)$ defined by $\widetilde{k}_i := \nu\kappa^{r+1-i} k_i$, $i \in \{1, \dots, r\}$, (4.3) follows from

$$\operatorname{spec}\big(A + b\widetilde{k}[c/\dots/cA^{r-1}]\big) = \operatorname{spec}\big(VAV^{-1} + Vb(\widetilde{k} \mid 0)\big) \subset \mathbb{C}_-.$$

In the remainder of this section we consider signal spaces of type (B) in (3.2) and assume that $\widetilde{k}_r = 0$ and $\operatorname{spec}\big(A + b\widetilde{k}[c/\dots/cA^{r-1}]\big) \subset \mathbb{C}_-$. In this case we cannot refer to Proposition 4.1.

4.2. Stability properties of the closed-loop system $[P(A,b,c;x^0),C_k]$. Now we show for $(A,b,c)\in\mathcal{P}_{n,r}$ with $k\in\mathbb{R}^{1\times r}$ such that $\operatorname{spec}(A+bk[c/\dots/cA^{r-1}])\subset\mathbb{C}_-$, and for appropriate input/output signal spaces of types (A), (B), and (C) in (3.2), that if $x^0=0$, then the closed-loop system $[P(A,b,c;x^0),C_k]$ is gain stable on $\mathcal{W}_0$. For the input/output signal spaces of type (A) or (B), we also show that the closed-loop system $[P(A,b,c;x^0),C_k]$ is $\mathcal{W}$-stable for any initial conditions $x^0\in\mathbb{R}^n$.

THEOREM 4.2. *Let, for $r,n\in\mathbb{N}$ with $r\le n$, $(A,b,c)\in\mathcal{P}_{n,r}$ given by* (4.1)*, and choose $k\in\mathbb{R}^{1\times r}$ such that* $\operatorname{spec}\left(A+bk[c/\dots/cA^{r-1}]\right)\subset\mathbb{C}_-$*. Let the signal spaces $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{W}$, $\mathcal{W}_0$ be of type* (A), (B), *or* (C) *in* (3.2)*; in case* (B) *suppose $ke_r^{(r)}=0$. Consider the controller operator $C_k\colon\mathcal{Y}_e\to\mathcal{U}_e$, as defined by* (1.1)*, and the associated plant operator $P(A,b,c;x^0)\colon\mathcal{U}_e\to\mathcal{Y}_e$ with initial value $x^0\in\mathbb{R}^n$ as defined by* (4.1)*. Then the closed-loop system $[P(A,b,c;0),C_k]$ is $\mathcal{W}_0$-gain stable. In the case of signal spaces given by* (A) *or* (B)*, the closed-loop system $[P(A,b,c;x^0),C_k]$ is also $\mathcal{W}$-stable.*

Proof. *Step* 1. Consider $\mathcal{W}$ of type (A), (B), or (C) given by (3.2) and let $(u_0,y_0)\in\mathcal{W}$. The closed-loop system $[P(A,b,c;x^0),C_k]$ given by (4.1), (1.1), (2.1) is, in view of coordinate transformation (A.2), equivalent to (A.3), (1.1), (2.1). Invoking Lemma A.1 and applying a variation of constants yields

(4.4)
$$\forall\, t\ge 0\,:\ \begin{pmatrix}\xi\\ \eta\end{pmatrix}(t)=e^{V\left(A+bk\begin{bmatrix}c\\ \vdots\\ cA^{r-1}\end{bmatrix}\right)V^{-1}t}\begin{pmatrix}\xi^0\\ \eta^0\end{pmatrix}+\int_0^t e^{V\left(A+bk\begin{bmatrix}c\\ \vdots\\ cA^{r-1}\end{bmatrix}\right)V^{-1}(t-s)}\varphi(s)\mathrm{d}s,$$

where, in view of $u_0\in\mathcal{U}$ and $y_0\in\mathcal{Y}$,

$$\varphi(\cdot):=\begin{pmatrix}0_{r-1}\\ cA^{r-1}b\\ 0_{n-r}\end{pmatrix}\left[u_0(\cdot)+\big(C_ky_0\big)(\cdot)\right]\in\mathcal{U}\,. \tag{4.5}$$

Step 2. Consider case (A) or (B), i.e., $\mathcal{U}\times\mathcal{Y}=CL^p(\mathbb{R}_{\ge0}\to\mathbb{R})\times CW^{r,p}(\mathbb{R}_{\ge0}\to\mathbb{R})$, $1\le p\le\infty$, $r\le n$. Taking norms in (4.4) and invoking the well-known inequality $\left\|\int_0^{\cdot} f(\cdot-s)\,g(s)\,\mathrm{d}s\right\|_{L^p}\le\|f\|_{L^1}\|g\|_{L^p}$, for $f\in L^1$ and $g\in L^p$, we obtain, for some $\beta_1,\beta_2>0$,

$$\begin{aligned}\left\|\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right\|_{L^p(\mathbb{R}_{\ge0}\to\mathbb{R}^n)}&\le\beta_1\left[\left|\begin{pmatrix}\xi^0\\ \eta^0\end{pmatrix}\right|+\|\varphi\|_{L^p(\mathbb{R}_{\ge0}\to\mathbb{R}^n)}\right]\\ &\le\beta_1\left|\begin{pmatrix}\xi^0\\ \eta^0\end{pmatrix}\right|+\beta_1\beta_2\left[\|u_0\|_{L^p(\mathbb{R}_{\ge0}\to\mathbb{R})}+\sum_{i=0}^{r-1}|k_{i+1}|\big\|y_0^{(i)}\big\|_{L^p(\mathbb{R}_{\ge0}\to\mathbb{R})}\right]\end{aligned}$$

and thus,

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix}\in L^p(\mathbb{R}_{\ge0}\to\mathbb{R}^n)\,.$$

Now, by (A.3),

$$\begin{aligned}y_1^{(i)}&=\xi_{i+1}\in L^p(\mathbb{R}_{\ge0}\to\mathbb{R})\quad\text{for } i=0,\dots,r-1,\\ y_1^{(r)}&=\dot\xi_r=\left(\sum_{i=1}^{r}\left(R_i-cA^{r-1}bk_i\right)\xi_i\right)+S\eta+\varphi\in L^p(\mathbb{R}_{\ge0}\to\mathbb{R}),\end{aligned}$$

and with (4.5) it follows that $y_1\in CW^{r,p}(\mathbb{R}_{\ge0}\to\mathbb{R})=\mathcal{Y}$. Finally,

$$u_1=u_0-C_k(y_2)=u_0-C_k(y_0)+C_k(y_1)\in CL^p(\mathbb{R}_{\ge0}\to\mathbb{R})=\mathcal{U}\,,$$

and we have shown that the closed-loop system $[P(A,b,c;x^0),C_k]$ is $\mathcal{W}$-stable in cases (A) and (B).

Step 3. Let $x^0=0$ and let $\mathcal{W}_0$ be as in (A) or (B), i.e., $\mathcal{U}_0\times\mathcal{Y}_0 = CL^p(\mathbb{R}_{\geq 0}\to\mathbb{R})\times CW_0^{r,p}(\mathbb{R}_{\geq 0}\to\mathbb{R})$, $1\leq p\leq\infty$, $r\leq n$. It is straightforward to see that $y^{(i)}(0)=0$ for $i=0,\dots,r$, and hence one can show as in Step 2 that, for some $\beta_1,\dots,\beta_5\geq 1$,

$$\begin{aligned}\|y_1\|_{CW_0^{r,p}(\mathbb{R}_{\geq 0}\to\mathbb{R})} &\leq \beta_1\beta_2\beta_3\left[\|u_0\|_{CL^p(\mathbb{R}_{\geq 0}\to\mathbb{R})} + \sum_{i=0}^{r-1}|k_{i+1}|\|y_0^{(i)}\|_{CL^p(\mathbb{R}_{\geq 0}\to\mathbb{R})}\right]\\ &\leq \beta_4\left[\|u_0\|_{CL^p(\mathbb{R}_{\geq 0}\to\mathbb{R})} + \|y_0\|_{CW_0^{r-1,p}(\mathbb{R}_{\geq 0}\to\mathbb{R})}\right]\end{aligned}$$

and

$$\begin{aligned}&\|u_1\|_{CL^p(\mathbb{R}_{\geq 0}\to\mathbb{R})}\\ &\leq \|u_0\|_{CL^p(\mathbb{R}_{\geq 0}\to\mathbb{R})} + \|C_k y_2\|_{CL^p(\mathbb{R}_{\geq 0}\to\mathbb{R})}\\ &\leq \|u_0\|_{CL^p(\mathbb{R}_{\geq 0}\to\mathbb{R})} + \beta_5\,\|y_2\|_{CW_0^{r-1,p}(\mathbb{R}_{\geq 0}\to\mathbb{R})}\\ &\leq \|u_0\|_{CL^p(\mathbb{R}_{\geq 0}\to\mathbb{R})} + \beta_5\left[\|y_1\|_{CW_0^{r-1,p}(\mathbb{R}_{\geq 0}\to\mathbb{R})} + \|y_0\|_{CW_0^{r-1,p}(\mathbb{R}_{\geq 0}\to\mathbb{R})}\right],\end{aligned}$$

and thus $\mathcal{W}_0$ gain stability in cases (A) and (B) follows.

Step 4. Let $x^0=0$ and let $\mathcal{W}_0$ be as in (C), i.e., $\mathcal{U}_0=\mathcal{Y}_0=CW_0^{\infty,p}(\mathbb{R}_{\geq 0}\to\mathbb{R})$, $1\leq p<\infty$. First note that $\varphi\in CW_0^{\infty,p}(\mathbb{R}_{\geq 0}\to\mathbb{R})$. By [19, Proposition VI.3.1] we have, for all $i\in\mathbb{N}$ and $t\geq 0$,

$$\frac{\mathrm{d}^i}{\mathrm{d}t^i}\int_0^t e^{V\left(A+bk\begin{bmatrix}c\\ \vdots\\ cA^{r-1}\end{bmatrix}\right)V^{-1}(t-s)}\varphi(s)\mathrm{d}s = \int_0^t e^{V\left(A+bk\begin{bmatrix}c\\ \vdots\\ cA^{r-1}\end{bmatrix}\right)V^{-1}(t-s)}\varphi^{(i)}(s)\mathrm{d}s.$$

Hence it follows from (4.4) that $\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left(\begin{smallmatrix}\xi\\ \eta\end{smallmatrix}\right)\big|_{t=0}=0$, and so $y^{(i)}(0)=0\ \forall\, i\in\mathbb{N}$. It follows also that $u_1^{(i)}(0)=u_0^{(i)}(0)-C_k(y_0^{(i)}(0))+C_k(y_1^{(i)}(0))=0\ \forall\, i\in\mathbb{N}$. One can then show as in Step 3 that, for some $\beta_1,\beta_2\geq 1$,

$$\begin{aligned}\|y_1\|_{CW^{\infty,p}(\mathbb{R}_{\geq 0}\to\mathbb{R})} &= \sum_{j\geq 0}\|y_1\|_{CL^p(\mathbb{R}_{\geq 0}\to\mathbb{R})}\\ &\leq \beta_1\sum_{j\geq 0}\left[\|u_0^{(j)}\|_{CL^p(\mathbb{R}_{\geq 0}\to\mathbb{R})} + \|\left(C_k y_0\right)^{(j)}\|_{CL^p(\mathbb{R}_{\geq 0}\to\mathbb{R})}\right]\\ &\leq \beta_1\,\|u_0\|_{CW^{\infty,p}(\mathbb{R}_{\geq 0}\to\mathbb{R})} + \beta_2\sum_{j\geq 0}\sum_{i=0}^{r-1}\|y_0^{(i+j)}\|_{CL^p(\mathbb{R}_{\geq 0}\to\mathbb{R})}\\ &\leq \beta_1\,\|u_0\|_{CW^{\infty,p}(\mathbb{R}_{\geq 0}\to\mathbb{R})} + r\beta_2\,\|y_0\|_{CW^{\infty,p}(\mathbb{R}_{\geq 0}\to\mathbb{R})}\,.\end{aligned}$$

An analogous inequality for $\|u_1\|_{CW^{\infty,p}(\mathbb{R}_{\geq 0}\to\mathbb{R})}$ gives $\mathcal{W}_0$-gain stability as required. This completes the proof of the theorem. ∎

Theorem 4.2 shows, in combination with Proposition 4.1 for signal spaces of type (A) or (C) in (3.2), that if $r\leq n$ and $\kappa,\nu\geq 1$ sufficiently large, then $[P(A,b,c;x^0),C_{k,\kappa,\nu}]$ is $\mathcal{W}_0$-gain stable, with a bound for the gain given by

$$\left\|\Pi_{C_{k,\kappa,\nu}//P(A,b,c;0)}\right\|_{\mathcal{W}_0,\mathcal{W}_0}\leq\beta(k,\kappa,\nu) \tag{4.6}$$

for some $\beta(k,\kappa,\nu) > 0$ determined by the proofs of Theorem 4.2 and Proposition 4.1.

In the signal space setting of type (B) in (3.2), i.e., $p, r < \infty$, these stability results are proved only for $k_r = 0$, thus precluding the application of Proposition 4.1. However, there are many plants stabilizable in $\mathcal{P}_{n,r}$, $r \leq n-1$, including, for example, the class of plants stabilizable by PD controllers ($r = 2$, $n \geq 3$).

Since Proposition 4.1 gives stabilizability of plants in $\mathcal{P}_{n,r}$, $r \leq n$, and since the signal space setting (A) is only applicable when $p = \infty$, the setting (C) has been introduced to allow stability results in the context of $p < \infty$, without the assumption that $k_r = 0$ as in (B). However, the setting (C) does introduce extra regularity requirements on the external disturbances u_0, y_0.

4.3. Gain stability of the closed-loop system $[P(A,b,c;0), C_k^{\text{Euler}}[h]]$. We are now in a position to show that linear systems $(A,b,c) \in \mathcal{P}_{n,r}$ are gain stabilizable on $\mathcal{U} \times \mathcal{Y}$ by the delay feedback $C_k^{\text{Euler}}[h]$ defined in (1.2) for suitable $k \in \mathbb{R}^{1\times r}$, for sufficiently small $h > 0$, and for signal spaces of type (A), (B), or (C) in (3.2).

THEOREM 4.3. *Let, for $r, n \in \mathbb{N}$ with $r \leq n$, $(A,b,c) \in \mathcal{P}_{n,r}$, and choose $k \in \mathbb{R}^{1\times r}$ such that* spec $\left(A + bk[c/\dots/cA^{r-1}]\right) \subset \mathbb{C}_-$. *Let the signal spaces $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{W}$, $\mathcal{W}_0$ be of type* (A), (B), *or* (C) *in* (3.2); *in case* (B) *suppose $ke_r^{(r)} = 0$. Then $\gamma := \left\|\Pi_{C_k//P(A,b,c;0)}\right\|_{\mathcal{W}_0,\mathcal{W}_0} < \infty$. Suppose $h \in (0, h^*)$, where $h^* > 0$ satisfies* (3.3). *Then the delay feedback controller $C_k^{\text{Euler}}[h] \colon \mathcal{Y}_{0e} \to \mathcal{U}_{0e}$, defined in* (1.2), *applied to the plant $P(A,b,c;0) \colon \mathcal{U}_{0e} \to \mathcal{Y}_{0e}$ given by* (4.1) *yields*

$$\left\|\Pi_{C_k^{\text{Euler}}[h]//P(A,b,c;0)}\right\|_{\mathcal{W}_0,\mathcal{W}_0} \leq \gamma \frac{1 + h\sum_{i=1}^{r-1}|k_{i+1}|\, i\eta_p(h,i)}{1 - h\gamma\sum_{i=1}^{r-1}|k_{i+1}|\, i\eta_p(h,i)}\,. \tag{4.7}$$

Proof. By Theorem 4.2 it follows that $\gamma := \left\|\Pi_{C_k//P(A,b,c;0)}\right\|_{\mathcal{W}_0,\mathcal{W}_0} < \infty$. The result now follows from Theorem 3.1 since $[P(A,b,c;0), C_k^{\text{Euler}}[h]]$ is globally well posed and $P(0) = 0$. ◻

In the following let $C_{k,\kappa,\nu}^{\text{Euler}}[h] \colon y_2 \mapsto u_2 = -\nu\sum_{i=0}^{r-1}\kappa^{r-i}k_{i+1}\Delta_h^i y_2$ be the delay feedback controller corresponding to controller $C_{k,\kappa,\nu}$ given in (4.2).

Together with Proposition 4.1 and Theorem 4.2, Theorem 4.3 shows for signal spaces of type (A) or (C) in (3.2), that for sufficiently large $\kappa, \nu \geq 1$ (determined by Proposition 4.1), $\beta(k,\kappa,\nu)$ given in (4.6), and sufficiently small $h > 0$ (determined by Theorem 4.3), the closed-loop system $[P(A,b,c;0), C_{k,\kappa,\nu}^{\text{Euler}}[h]]$ is $\mathcal{W}_0$-gain stable and

$$\left\|\Pi_{C_{k,\kappa,\nu}^{\text{Euler}}[h]//P(A,b,c;0)}\right\|_{\mathcal{W}_0,\mathcal{W}_0} \leq \beta(k,\kappa,\nu)\frac{1 + h\nu\kappa^{r-1}\sum_{i=1}^{r-1}|k_{i+1}|\, i\eta_p(h,i)}{1 - h\beta(k,\kappa,\nu)\nu\kappa^{r-1}\sum_{i=1}^{r-1}|k_{i+1}|\, i\eta_p(h,i)}\,.$$

4.4. Gain stability of $[P(A,b,c;x^0), C_k^{\text{Euler}}[h]]$ with nonzero initial condition. To generalize Theorem 4.3 by allowing for nonzero initial conditions, we give the following result which will be applied to signal spaces of type (A) or (B) in (3.2). The proof of Theorem 4.4 is based on an extension of [4, Thm. 5.3].

THEOREM 4.4. *Let $r, n \in \mathbb{N}$ with $r \leq n$, and consider signal spaces $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{W}$, $\mathcal{W}_0$ of type* (A) *or* (B) *in* (3.2). *Let $k \in \mathbb{R}^{1\times r}$, and additionally, in case* (B) *suppose $ke_r^{(r)} = 0$. Let $(A,b,c) \in \mathcal{P}_{n,r}$, $x^0 \in \mathbb{R}^n$, and consider the operator $P(A,b,c;x^0) \colon \mathcal{U}_e \to \mathcal{Y}_e$ as defined in*(4.1). *Suppose that for $h > 0$, applying the feedback controllers*

$$C_k \colon \mathcal{Y}_e \to \mathcal{U}_e \qquad \text{and} \qquad C_k^{\text{Euler}}[h] \colon \mathcal{Y}_{0e} \to \mathcal{U}_e$$

as defined in (1.1) *and* (1.2)*, respectively, to* $P(A,b,c;0)$ *yields*

$$\left\|\Pi_{C_k//P(A,b,c;0)}\right\|_{\mathcal{W}_0,\mathcal{W}_0}<\infty \qquad \textit{and} \qquad \left\|\Pi_{C_k^{\mathrm{Euler}}[h]//P(A,b,c;0)}\right\|_{\mathcal{W}_0,\mathcal{W}_0}=:\gamma<\infty\,.$$

Then

$$\exists\,\lambda>0\;\forall x^0\in\mathbb{R}^n\;\forall w_0\in\mathcal{W}_0\;:\;\left\|\Pi_{C_k^{\mathrm{Euler}}[h]//P(A,b,c;x^0)}w_0\right\|_{\mathcal{W}}\le\lambda|x^0|+\gamma\left\|w_0\right\|_{\mathcal{W}_0}. \tag{4.8}$$

Proof. Note that we may consider $P(A,b,c;0)$ as an operator from $\mathcal{U}_e$ to $\mathcal{Y}_e$ or from $\mathcal{U}_e$ to $\mathcal{Y}_{0e}$. Furthermore, note that C_k and $C_k^{\mathrm{Euler}}[h]$ may be considered as operators from $\mathcal{Y}_e$ to $\mathcal{U}$ or from $\mathcal{Y}_{0e}$ to $\mathcal{U}$. Thus we may consider the graphs of $P(A,b,c;0)$, C_k, and $C_k^{\mathrm{Euler}}[h]$ in $\mathcal{W}_0$ or in $\mathcal{W}$. To identify in which signal space a graph is considered we add a superscript $\mathcal{W}_0$ or $\mathcal{W}$ such as in $\mathcal{G}^{\mathcal{W}_0}_{P(A,b,c;0)}\subset\mathcal{W}_0$ or $\mathcal{G}^{\mathcal{W}}_{P(A,b,c;0)}\subset\mathcal{W}$. For $x^0\neq0$ we have to consider $P(A,b,c;x^0)$ as an operator from $\mathcal{U}_e$ to $\mathcal{Y}_e$ with $\mathcal{G}^{\mathcal{W}}_{P(A,b,c;x^0)}\subset\mathcal{W}$.

Step 1. Let $y_0\equiv0$ and consider the map defined by $u_0\overset{(A.3),\,(1.1),\,(2.1)}{\longmapsto}y_1$ with transfer function

$$\begin{aligned}s\mapsto G(s)&=(1,0,\dots,0)(sI-(VAV^{-1}+Vb(k\mid 0)))^{-1}Vb\\&=(1,0,\dots,0)(V(sI-(A+bk[c/\dots/cA^{r-1}]))V^{-1})^{-1}Vb\,,\end{aligned}$$

where the matrix V is given by (A.1). By boundedness of $\left\|H_{P(A,b,c;0),C_k}\right\|_{\mathcal{W}_0,\mathcal{W}_0\times\mathcal{W}_0}$ and [3, Thm. 2, section 2.4] it follows that $G(\cdot)$ is stable. Since (A,b,c) is minimum phase, setting $F:=k[c/\dots/cA^{r-1}]$, [1, Thm. 10] yields that $\operatorname{spec}(A+bF)=\operatorname{spec}(A+bk[c/\dots/cA^{r-1}])\subset\mathbb{C}_-$. Since $\left\|H_{P(A,b,c;0),C_k}\right\|_{\mathcal{W}_0,\mathcal{W}_0\times\mathcal{W}_0}<\infty$, we may define maps $\widetilde{N}:\mathcal{U}\to\mathcal{U},\;u_0\mapsto u_1$ and $M:\mathcal{U}\to\mathcal{Y}_0,\;u_0\mapsto y_1$ by

$$\widetilde{N}u_0=(1\;0)\,\Pi_{P(A,b,c;0)//C_k}\begin{pmatrix}u_0\\0\end{pmatrix},\qquad Mu_0=(0\;1)\,\Pi_{P(A,b,c;0)//C_k}\begin{pmatrix}u_0\\0\end{pmatrix}.$$

Proposition 4.1 yields that the tuples $(u_0,u_1)=(u_0,\widetilde{N}u_0)$ and $(u_0,y_1)=(u_0,Mu_0)$ satisfy

$$\left\{\begin{array}{rcll}\dot{x}&=&(A+bF)x+bu_0\,,&\qquad x(0)=0,\\u_1&=&Fx+u_0\,,&\\y_1&=&cx\,.&\end{array}\right. \tag{4.9}$$

Step 2. We show $\widetilde{N}(\mathcal{U})=\mathcal{V}:=\{u\in\mathcal{U}\mid P(A,b,c;0)u\in\mathcal{Y}\}$.

Suppose $u\in\mathcal{V}$, i.e., $u\in\mathcal{U}$ with $P(A,b,c;0)u\in\mathcal{Y}$. Then $\mathcal{Y}\ni P(A,b,c;0)u=cx=:y$ for x a solution of $\dot{x}=Ax+bu$, $x(0)=0$. Since (A,b,c) is minimum phase, and thus (A,c) is detectable, there exists $L\in\mathbb{R}^n$ such that $\operatorname{spec}(A+Lc)\subset\mathbb{C}_-$. Since $y\in\mathcal{Y}$ and $u\in\mathcal{U}$, writing

$$\dot{x}=(A+Lc)x-Lcx+bu=(A+Lc)x-Ly+bu$$

yields that $x\in CL^p(\mathbb{R}_{\ge0}\to\mathbb{R}^n)$. Thus $u_0:=u-Fx\in\mathcal{U}$, and (4.9) then yields that $u=u_1=\widetilde{N}(u_0)\in\widetilde{N}(\mathcal{U})$, which gives $\mathcal{V}\subset\widetilde{N}(\mathcal{U})$.

Conversely, suppose $u\in\widetilde{N}(\mathcal{U})$. Then there exists $u_0\in\mathcal{U}$ such that $u_0=u-Fx\in\mathcal{U}$. Since $\operatorname{spec}(A+bF)\subset\mathbb{C}_-$ it follows by (4.9) that $P(A,b,c;0)u=y=cx\in\mathcal{Y}$.

Hence $\widetilde{N}(\mathcal{U}) \subset \mathcal{V}$. Now $N\colon \mathcal{U} \to \mathcal{V}$, $u_0 \mapsto (1\ 0)\,\Pi_{P(A,b,c;0)//C_k}\left(\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right)$ is well defined, and writing

$$\begin{aligned} \dot{x} &= Ax + bu_1\,, \qquad x(0) = 0, \\ u_0 &= Fx - u_1\,, \end{aligned}$$

directly gives that N is invertible and $P(A,b,c;0) = MN^{-1}$.

Step 3. Set $\bar{A} := A + bF = (A + bk[c/\ldots/cA^{r-1}])$. We show

$$\mathcal{G}^{\mathcal{W}}_{P(A,b,c;x^0)} = Q := \left\{ \begin{pmatrix} N \\ M \end{pmatrix} v + \begin{pmatrix} F\exp(\bar{A}\cdot)x^0 \\ c\exp(\bar{A}\cdot)x^0 \end{pmatrix} \in \mathcal{W} \;\middle|\; \begin{array}{l} v \in \mathcal{U},\ N, M, F, \text{ and} \\ \bar{A} \text{ as in Step 1} \end{array} \right\}.$$

We show $Q \subset \mathcal{G}^{\mathcal{W}}_{P(A,b,c;x^0)}$.

Consider, for any $v \in \mathcal{U}$,

$$q_v = \begin{pmatrix} N \\ M \end{pmatrix} v + \begin{pmatrix} F\exp(\bar{A}\cdot)x^0 \\ c\exp(\bar{A}\cdot)x^0 \end{pmatrix} \in Q.$$

Let $u = Nv + F\exp(\bar{A}\cdot)x^0$. Since $Nv \in \mathcal{U}$ and $\exp(\bar{A}\cdot) \in CW^{r,p}(\mathbb{R}_{\geq 0} \to \mathbb{R})^{n\times n} = \mathcal{Y}^{n\times n}$, we have $u \in \mathcal{U}$. Observe that

$$\dot{x} = Ax + b(F\exp(\bar{A}\cdot)x^0)\,, \qquad x(0) = x^0 \in \mathbb{R}^n$$

has the solution $x(\cdot) = \exp(\bar{A}\cdot)x^0$. Thus it follows that

$$P(A,b,c;x^0)F\exp(\bar{A}\cdot)x^0 = c\exp(\bar{A}\cdot)x^0\,.$$

Hence

$$\begin{aligned} P(A,b,c;x^0)u &= P(A,b,c;x^0)Nv + P(A,b,c;x^0)\left(F\exp(\bar{A}\cdot)x^0\right) - P(A,b,c;x^0)0 \\ &= P(A,b,c;0)Nv + P(A,b,c;x^0)\left(F\exp(\bar{A}\cdot)x^0\right) \\ &= M(N)^{-1}Nv + c\exp(\bar{A}\cdot)x^0 \\ &= Mv + c\exp(\bar{A}\cdot)x^0 \in \mathcal{Y}\,. \end{aligned} \tag{4.10}$$

Thus $q_v = \left(\begin{smallmatrix} u \\ P(A,b,c;x^0)u \end{smallmatrix}\right) \in \mathcal{U}\times\mathcal{Y}$, so $q_v \in \mathcal{G}^{\mathcal{W}}_{P(A,b,c;x^0)}$ and $Q \subset \mathcal{G}^{\mathcal{W}}_{P(A,b,c;x^0)}$.

We show $\mathcal{G}^{\mathcal{W}}_{P(A,b,c;x^0)} \subset Q$.

Consider $\left(\begin{smallmatrix} u \\ P(A,b,c;x^0)u \end{smallmatrix}\right) \in \mathcal{G}^{\mathcal{W}}_{P(A,b,c;x^0)}$. Then

$$P(A,b,c;0)\left(u - F\exp(\bar{A}\cdot)x^0\right) = P(A,b,c;x^0)u - P(A,b,c;x^0)\left(F\exp(\bar{A}\cdot)x^0\right),$$

and since the right-hand side lies in $\mathcal{Y}$, it follows that $P(A,b,c;0)\left(u - F\exp(\bar{A}\cdot)x^0\right) \in \mathcal{Y}$. Therefore $u - F\exp(\bar{A}\cdot)x^0 \in \mathcal{V} = \mathrm{Im}(N)$, and so there exists $v \in \mathcal{U}$ such that $Nv = u - F\exp(\bar{A}\cdot)x^0$. Therefore (4.10) holds, and hence

$$\begin{pmatrix} u \\ P(A,b,c;x^0)u \end{pmatrix} = \begin{pmatrix} N \\ M \end{pmatrix} v + \begin{pmatrix} F\exp(\bar{A}\cdot)x^0 \\ c\exp(\bar{A}\cdot)x^0 \end{pmatrix} \in Q,$$

and so $\mathcal{G}^{\mathcal{W}}_{P(A,b,c;x^0)} \subset Q$. Therefore we have shown $\mathcal{G}^{\mathcal{W}}_{P(A,b,c;x^0)} = Q$ as claimed.

Step 4. Finally we show (4.8). For $w_0 \in \mathcal{W}_0$ and $x^0 \in \mathbb{R}^n$ let

$$w_0' := w_0 - v_1 - v_2, \quad v_1 := \begin{pmatrix} F\exp(\bar{A}\cdot)x^0 \\ c\exp(\bar{A}\cdot)x^0 \end{pmatrix} \quad v_2 := \begin{pmatrix} -C_k^{\mathrm{Euler}}[h](c\exp(\bar{A}\cdot)x^0) \\ -c\exp(\bar{A}\cdot)x^0 \end{pmatrix}.$$

Since $C_k^{\mathrm{Euler}}[h](\mathcal{Y}) \subset \mathcal{U}$, we have $w_0' \in \mathcal{W}_0$, and hence,

$$H_{P(A,b,c;0),C_k^{\mathrm{Euler}}[h]}(w_0') = (w_1, w_2) \in \mathcal{G}^{\mathcal{W}_0}_{P(A,b,c;0)} \times \mathcal{G}^{\mathcal{W}_0}_{C_k^{\mathrm{Euler}}[h]}.$$

In particular, $w_0' = w_1 + w_2$, and by rearranging we have $w_0 = (w_1 + v_1) + (w_2 + v_2)$. Since $w_1 \in \mathcal{G}^{\mathcal{W}_0}_{P(A,b,c;0)} \subset \mathcal{G}^{\mathcal{W}}_{P(A,b,c;0)}$, there exists $v \in \mathcal{U}$ such that $w_1 = \left(\begin{smallmatrix} N \\ M \end{smallmatrix}\right) v$, and hence $w_1 + v_1 \in Q = \mathcal{G}^{\mathcal{W}}_{P(A,b,c;x^0)}$. Since $w_2 \in \mathcal{G}^{\mathcal{W}_0}_{C_k^{\mathrm{Euler}}[h]} \subset \mathcal{G}^{\mathcal{W}}_{C_k^{\mathrm{Euler}}[h]}$ and $v_2 \in \mathcal{G}^{\mathcal{W}}_{C_k^{\mathrm{Euler}}[h]}$, it follows by linearity of $C_k^{\mathrm{Euler}}[h]$ that $w_2 + v_2 \in \mathcal{G}^{\mathcal{W}}_{C_k^{\mathrm{Euler}}[h]}$. Therefore, since $[P(A,b,c;x^0), C_k^{\mathrm{Euler}}[h]]$ has the uniqueness property, $H_{P(A,b,c;x^0),C_k^{\mathrm{Euler}}[h]}\colon \mathcal{W} \to \mathcal{W} \times \mathcal{W}$ is defined, and

$$H_{P(A,b,c;x^0),C_k^{\mathrm{Euler}}[h]} w_0 = (w_1 + v_1, w_2 + v_2) \in \mathcal{G}^{\mathcal{W}}_{P(A,b,c;x^0)} \times \mathcal{G}^{\mathcal{W}}_{C_k^{\mathrm{Euler}}[h]} \subset \mathcal{W} \times \mathcal{W}.$$

Now for

$$\lambda := \sup_{x_0 \in \mathbb{R}^n \setminus \{0\}} \frac{\|v_2\|_{\mathcal{W}}}{|x^0|} = \left\| \begin{pmatrix} -C_k^{\mathrm{Euler}}[h](c \exp(\bar{A}\cdot)) \\ -c \exp(\bar{A}\cdot) \end{pmatrix} \right\|_{\mathcal{W}^n} \tag{4.11}$$

it follows that

$$\left\| \Pi_{C_k^{\mathrm{Euler}}[h]//P(A,b,c;x^0)} w_0 \right\|_{\mathcal{W}\times\mathcal{W}} \le \|v_2\|_{\mathcal{W}} + \|w_2\|_{\mathcal{W}} \le \lambda |x^0| + \gamma \|w_0\|_{\mathcal{W}_0},$$

thus concluding the proof. □

We can now state the result for the delay feedback controller in the presence of both input/ouput disturbances and initial conditions.

THEOREM 4.5. *Let, for $r, n \in \mathbb{N}$ with $r \le n$, $(A,b,c) \in \mathcal{P}_{n,r}$, and choose $k \in \mathbb{R}^{1\times r}$ such that* $\operatorname{spec}\left(A + bk[c/\dots/cA^{r-1}]\right) \subset \mathbb{C}_-$. *Let the signal spaces $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{W}$, $\mathcal{W}_0$ be of type* (A) *or* (B) *in* (3.2)*; in case* (B) *suppose $k e_r^{(r)} = 0$. Then $\gamma := \left\| \Pi_{C_k//P(A,b,c;0)} \right\|_{\mathcal{W}_0,\mathcal{W}_0} < \infty$. Suppose $h \in (0, h^*)$, where $h^* > 0$ satisfies* (3.3). *Consider for (A,b,c) the plant operator $P(A,b,c;x^0)\colon \mathcal{U}_e \to \mathcal{Y}_e$ given by* (4.1) *and the delay feedback controller $C_k^{\mathrm{Euler}}[h]\colon \mathcal{Y}_{0e} \to \mathcal{U}_e$ defined in* (1.2)*. Then there exists $\lambda > 0$ such that, for all $w_0 \in \mathcal{W}_0$,*

$$\left\| \Pi_{C_k^{\mathrm{Euler}}[h]//P(A,b,c;x^0)} w_0 \right\|_{\mathcal{W},\mathcal{W}} \le \lambda |x^0| + \gamma \frac{1 + h \sum_{i=1}^{r-1} |k_{i+1}|\, i \eta_p(h,i)}{1 - h\gamma \sum_{i=1}^{r-1} |k_{i+1}|\, i \eta_p(h,i)} \|w_0\|_{\mathcal{W}_0}.$$

Proof. The result follows directly from Theorems 4.3 and 4.4. □

Together with Proposition 4.1 and Theorem 4.2, Theorem 4.5 shows for signal spaces of type (A) in (3.2) that, for sufficiently large $\kappa, \nu \ge 1$ (determined by Proposition 4.1), $\beta(k,\kappa,\nu)$ given in (4.6), and for sufficiently small $h > 0$ (determined by Theorem 4.3), there exists $\lambda > 0$ (determined by (4.11)) such that, for all $x_0 \in \mathbb{R}^n$ and $w_0 \in \mathcal{W}_0$,

$$\begin{aligned} &\left\| \Pi_{C_{k,\kappa,\nu}^{\mathrm{Euler}}[h]//P(A,b,c;x^0)} w_0 \right\|_{\mathcal{W},\mathcal{W}} \\ &\quad \le \lambda |x^0| + \left(\beta(k,\kappa,\nu) \frac{1 + h\nu\kappa^{r-1} \sum_{i=1}^{r-1} |k_{i+1}|\, i\eta_p(h,i)}{1 - h\beta(k,\kappa,\nu)\nu\kappa^{r-1} \sum_{i=1}^{r-1} |k_{i+1}|\, i\eta_p(h,i)} \right) \|w_0\|_{\mathcal{W}_0}. \end{aligned}$$

4.5. Exponential stability of $[P(A,b,c;x^0), C_k^{\mathrm{Euler}}[h]]$ with $u_0 \equiv y_0 \equiv 0$. In Proposition 4.1 we have shown that the high-gain derivative feedback controller $C_{k,\kappa,\nu}\colon y_2 \mapsto u_2$ leads to an internally stable system; i.e., (A.5) with $u_0 \equiv y_0 \equiv 0$ gives

$$\exists\, \nu^* \geq 1\ \forall\, \nu \geq \nu^*\ \exists\, \kappa^* \geq 1, \forall\, \kappa \geq \kappa^* \quad : \ \dot z = A_{k,\kappa,\nu} z \quad \text{is exponentially stable.}$$

Now (as in [11], where a more limited class of systems was considered) we will show that an analogous result holds true if a stabilizing derivative feedback controller $C_k\colon y_2 \mapsto u_2$ is replaced by the delay feedback $C_k^{\mathrm{Euler}}[h]\colon y_2 \mapsto u_2$ for $h > 0$ sufficiently small. Exponential stability for a delay differential equation is defined as follows; see, for example, [2, Def. 5.1.1].

DEFINITION 4.6. *Let $h > 0$ and, for $r, n \in \mathbb{N}$ with $r \leq n$, $A_0, \ldots, A_{r-1} \in \mathbb{R}^{n\times n}$. Then the delay initial value problem*

$$\dot x = \sum_{j=0}^{r-1} A_j\, x(t - jh)\,, \qquad x \equiv \varphi \ \textit{on}\ [(1-r)h, 0]\,, \tag{4.12}$$

is said to be exponentially stable *if and only if*

$$\exists\, M, \lambda > 0\ \forall\, t \geq 0\ \forall\, \varphi \in \mathcal{C}_{\mathrm{pw}}([(1-r)h, 0] \to \mathbb{R}^n)\ : \quad |x(t)| \leq M e^{-\lambda t} \max_{s \in [(1-r)h, 0]} |\varphi(s)|\,. \tag{4.13}$$

PROPOSITION 4.7. *Let, for $r, n \in \mathbb{N}$ with $r \leq n$, $(A,b,c) \in \mathcal{P}_{n,r}$, $x^0 \in \mathbb{R}^n$. Consider the signal spaces $\mathcal{U} = \mathcal{Y} = CW_0^{\infty,2}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ and $\mathcal{W} := \mathcal{U} \times \mathcal{Y}$, and choose $k \in \mathbb{R}^{1\times r}$ and $h > 0$ such that*

$$\left\|\Pi_{P(A,b,c;0)//C_k^{\mathrm{Euler}}[h]}\right\|_{\mathcal{W},\mathcal{W}} < \infty\,.$$

Then the delay initial value problem for the closed-loop system $[P(A,b,c;x^0), C_k^{\mathrm{Euler}}[h]]$ given by (4.1), (1.2), and (2.1) with $u_0 \equiv y_0 \equiv 0$ is exponentially stable.

Proof. For $(A,b,c) \in \mathcal{P}_{n,r}$, $r, n \in \mathbb{N}$ with $r \leq n$ and $h > 0$, the closed-loop system $[P(A,b,c;x^0), C_k^{\mathrm{Euler}}[h]]$ given by (4.1), (1.2), and (2.1) is described by a delay differential equation of the form (4.12) as follows:

$$\left\{\begin{aligned} \dot x(t) &= \left(A + \widetilde A^0\right) x(t) + \sum_{j=1}^{r-1} \widetilde A^j x(t-jh) + b u_0(t) + \sum_{j=0}^{r-1} \widetilde b^j y_0(t-jh)\,, \\ y_1(t) &= c\, x(t)\,, \qquad\qquad\qquad\qquad x \equiv \varphi \text{ on } [(1-r)h, 0]\,, \\ u_1(t) &= u_0(t) + \sum_{j=0}^{r-1} (-1)^j \sum_{i=j}^{r-1} \frac{k_{i+1}}{h^i} \binom{i}{j} (y_0 - y_1)(t - jh), \end{aligned}\right. \tag{4.14}$$

where $\varphi(0) = x^0$ and, in view of (A.2) and (A.3), for $j = 0, \ldots, r-1$,

$$\widetilde A^j := (-1)^{j+1} c A^{r-1} b \sum_{i=j}^{r-1} \frac{k_{i+1}}{h^i} \binom{i}{j} V^{-1} \left[e_r^{(n)} \mid 0_{n\times(n-1)}\right] V\,,$$

$$\widetilde b^j := (-1)^j \sum_{i=j}^{r-1} \frac{k_{i+1}}{h^i} \binom{i}{j} b\,,$$

and the transformation matrix $V \in \mathbb{R}^{n\times n}$ is given by (A.1). Let $G_{P(A,b,c;0)//C_k^{\text{Euler}}[h]} \in \mathbb{R}(s)^{2\times 2}$ denote the transfer function of $\Pi_{P(A,b,c;0)//C_k^{\text{Euler}}[h]}$. Then, [18, Thm. 30] and [3, Thm. 2, section 2.4] yield

$$\sup_{\omega\in\mathbb{R}} \left\| G_{P(A,b,c;0)//C_k^{\text{Euler}}[h]}(i\omega) \right\|_2 = \left\| \Pi_{P(A,b,c;0)//C_k^{\text{Euler}}[h]} \right\|_{\mathcal{W},\mathcal{W}} < \infty \,.$$

Since the denominator of the function $\left(s \mapsto G_{P(A,b,c;0)//C_k^{\text{Euler}}[h]}(s)\right)$ is equal to

$$\det\left(sI - \left(A + \widetilde{A}^0 + e^{-sh}\widetilde{A}^1 + \cdots + e^{-s(r-1)h}\widetilde{A}^{r-1}\right)\right) ,$$

it follows that

$$\forall\ s \in \overline{\mathbb{C}}_+ \ :\ \det\left(sI - \left(A + \widetilde{A}^0 + e^{-sh}\widetilde{A}^1 + \cdots + e^{-s(r-1)h}\widetilde{A}^{r-1}\right)\right) \neq 0\,.$$

Now, [2, Thm. 5.1.5] yields exponential stability of (4.14) with $u_0 \equiv y_0 \equiv 0$. ∎

We conclude this section by noting that for sufficiently large $\kappa, \nu \geq 1$ (determined by Proposition 4.1) and for sufficiently small $h > 0$ (determined by Theorem 4.3) Proposition 4.7 yields that the closed-loop system $[P(A, b, c; x^0), C^{\text{Euler}}_{k,\kappa,\nu}[h]]$ with $u_0 \equiv y_0 \equiv 0$ is exponentially stable.

5. Example. We now present an example which illustrates Theorem 4.3 and its application within a nonlinear context. Consider the system $P(A, b, c; x^0)\colon u_1 \mapsto y_1$ given by (4.1) with the system matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad c = [1, 0, 0]$$

and initial value $x^0 \in \mathbb{R}^3$. Note that the system has relative degree $r = 2$, positive high-frequency gain $cA^{r-1}b = 1$, and stable zero dynamics and is already in Byrnes–Isidori normal form. An application of $C_{k,\kappa,\nu}$ defined by (4.2) with $k = (1, 1)$ and $\nu = 1$ yields the closed-loop system

(5.1)
$$\left\{\begin{aligned} \begin{pmatrix} \dot{y}_1 \\ \ddot{y}_1 \\ \dot{\eta} \end{pmatrix} &= \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1-\kappa^2 & 1-\kappa & 1 \\ 1 & 0 & -1 \end{bmatrix}}_{=:\,A_\kappa} \begin{pmatrix} y_1 \\ \dot{y}_1 \\ \eta \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \left[u_0 + \kappa^2 y_0 + \kappa \dot{y}_0\right], \quad \begin{pmatrix} y_1 \\ \dot{y}_1 \\ \eta \end{pmatrix}(0) = x^0, \\ u_1 &= u_0 + \kappa^2 y_0 + \kappa \dot{y}_0 - (\kappa^2 y_1 + \kappa \dot{y}_1)\,. \end{aligned}\right.$$

If $\kappa > \sqrt{2}$, then $\operatorname{spec}(A_\kappa) \subset \mathbb{C}_-$. Replacing $C_{k,\kappa,\nu}$ by the appropriate delay feedback controller $C^{\text{Euler}}_{k,\kappa,\nu}[h]$ yields the closed-loop system

(5.2)
$$\left\{\begin{aligned} \begin{pmatrix} \dot{y}_1 \\ \ddot{y}_1 \\ \dot{\eta} \end{pmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 1-\kappa^2-\frac{\kappa}{h} & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{pmatrix} y_1 \\ \dot{y}_1 \\ \eta \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \frac{\kappa}{h} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ \dot{y}_1 \\ \eta \end{pmatrix} (\cdot - h) \\ &\quad + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \left[u_0 + \kappa^2 y_0 + \kappa \frac{y_0 - y_0(\cdot - h)}{h}\right], \qquad \begin{pmatrix} y_1 \\ \dot{y}_1 \\ \eta \end{pmatrix} \equiv \varphi \text{ on } [-h, 0]\,, \\ u_1 &= u_0 + \kappa^2 y_0 + \kappa \frac{y_0 - y_0(\cdot - h)}{h} - \left(\kappa^2 y_1 + \kappa \frac{y_1 - y_1(\cdot - h)}{h}\right), \end{aligned}\right.$$

where $\varphi(0) = x^0$. Theorem 4.3 now guarantees that, for sufficiently small $h > 0$, the delay system (5.2) is $\mathcal{W}_0$-gain stable for signal spaces $\mathcal{W}_0$ of type (A) or (C) defined by (3.2). Moreover, Proposition 4.7 shows that, for $(u_0, y_0) = (0, 0)$, the solution of (5.2) is exponentially stable for sufficiently small $h > 0$.

We now consider the effect of an additional causal and invertible nonlinearity $\Psi\colon \mathcal{U}_0 \to \mathcal{U}_0$. We connect this in a series with the input of the plant, to give the plant operator $P(A, b, c; x^0) \circ \Psi$; see Figure 3.

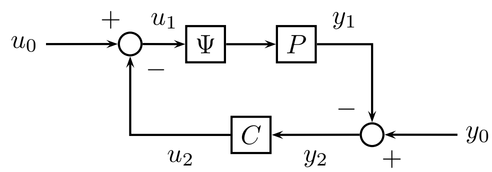

FIG. 3. *The closed-loop system* $[P \circ \Psi, C]$.

Consider the map $\Phi\colon \mathcal{G}_{P(A,b,c;0)} \to \mathcal{W}_0$ defined by

$$\Phi \begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} \Psi^{-1}(u) \\ y \end{pmatrix} \quad \forall \begin{pmatrix} u \\ y \end{pmatrix} \in \mathcal{G}_{P(A,b,c;0)} .$$

Since $\Psi^{-1}(u) \in \mathcal{U}_0$ for all $u \in \mathcal{U}_0$, it follows that $\Phi \in \mathcal{O}_{P(A,b,c;0),P(A,b,c;0)\circ\Psi}$, and hence

$$\vec{\delta}(P(A, b, c; 0), P(A, b, c; 0) \circ \Psi) \leq \|I - \Phi\|_{\mathcal{W}_0,\mathcal{W}_0} \leq \|I - \Psi^{-1}\|_{\mathcal{U}_0,\mathcal{U}_0}.$$

An application of Theorem 2.1 then shows that if the nonlinearity Ψ is sufficiently mild, i.e., if $\|I-\Psi^{-1}\|_{\mathcal{U}_0,\mathcal{U}_0} < \|\Pi_{C_{k,\kappa,\nu}//P(A,b,c;0)}\|^{-1}_{\mathcal{W}_0,\mathcal{W}_0}$, then $[P(A, b, c; 0) \circ \Psi, C_{k,\kappa,\nu}]$ is also gain stable. Consequently, Theorem 3.1 establishes, for sufficiently small $h > 0$, that $[P(A, b, c; 0) \circ \Psi, C^{\text{Euler}}_{k,\kappa,\nu}[h]]$ is $\mathcal{W}_0$-gain stable for signal spaces $\mathcal{W}_0$ of type (A) or (C) defined by (3.2).

For example, in the setting of signal space (A), we consider $\Psi\colon \mathcal{U}_0 \to \mathcal{U}_0$ defined by $\Psi(u)(t) = \psi(u(t))$ for the memoryless function $\psi\colon \mathbb{R} \to \mathbb{R}$, $\psi(s) = s + \varepsilon \sin(s)$, $0 < \varepsilon \leq 1$. ψ is bijective and sector bounded; i.e., $a_\varepsilon s \leq \psi(s) \leq b_\varepsilon s$, $a_\varepsilon, b_\varepsilon > 0$, with the property that $a_\varepsilon, b_\varepsilon \to 1$ as $\varepsilon \to 0$. It follows that there exists an inverse ψ^{-1} which is necessarily sector bounded, i.e., $\frac{1}{b_\varepsilon} s \leq \psi^{-1}(s) \leq \frac{1}{a_\varepsilon} s$, and hence the bounded inverse $\Psi^{-1}\colon \mathcal{U}_0 \to \mathcal{U}_0$ is given by $\Psi^{-1}(u)(t) = \psi^{-1}(u(t))$, and

$$\|I - \Psi^{-1}\| \leq \sup_{s\in\mathbb{R}} \frac{|s - \psi^{-1}(s)|}{|s|} \leq 1 - \max\left\{\frac{1}{a_\varepsilon}, \frac{1}{b_\varepsilon}\right\} \to 0 \qquad \text{as } \varepsilon \to 0.$$

Hence for sufficiently small $\varepsilon > 0$ and $h > 0$, $[P(A, b, c; x^0) \circ \Psi, C^{\text{Euler}}_{k,\kappa,\nu}[h]]$ is $\mathcal{W}_0$-gain stable.

We illustrate the effect of initial conditions in the following simulation study, where we have chosen $\kappa = 2$ and $\varepsilon = 1$. The simulations are computed with the MATLAB solver `dde23` for delay differential equations; see also [17]. We illustrate the state variables $y_1, \dot{y}_1, \eta$ of the undisturbed ($u_0 = y_0 = 0$) closed-loop systems for $t \in [0, 20]$. Figure 4(a) shows an exponentially convergent solution of (5.1) with initial conditions $(y_1(0), \dot{y}_1(0), \eta(0)) = {x^0}^T = (1, 1, 1)$ and $(u_0, y_0) = (0, 0)$. Figures 4(b), (c), (d) illustrate $[P(A, b, c; x^0) \circ \Psi, C^{\text{Euler}}_{k,\kappa,\nu}[h]]$ with $(u_0, y_0) = (0, 0)$ and with initial function

$$\left(y_1|_{[-h,0]}, \dot{y}_1|_{[-h,0]}, \eta|_{[-h,0]}\right) \equiv \varphi^T \equiv (1, 1, 1)$$

for Euler step sizes $h = 0.1, 0.5, 1$, respectively.

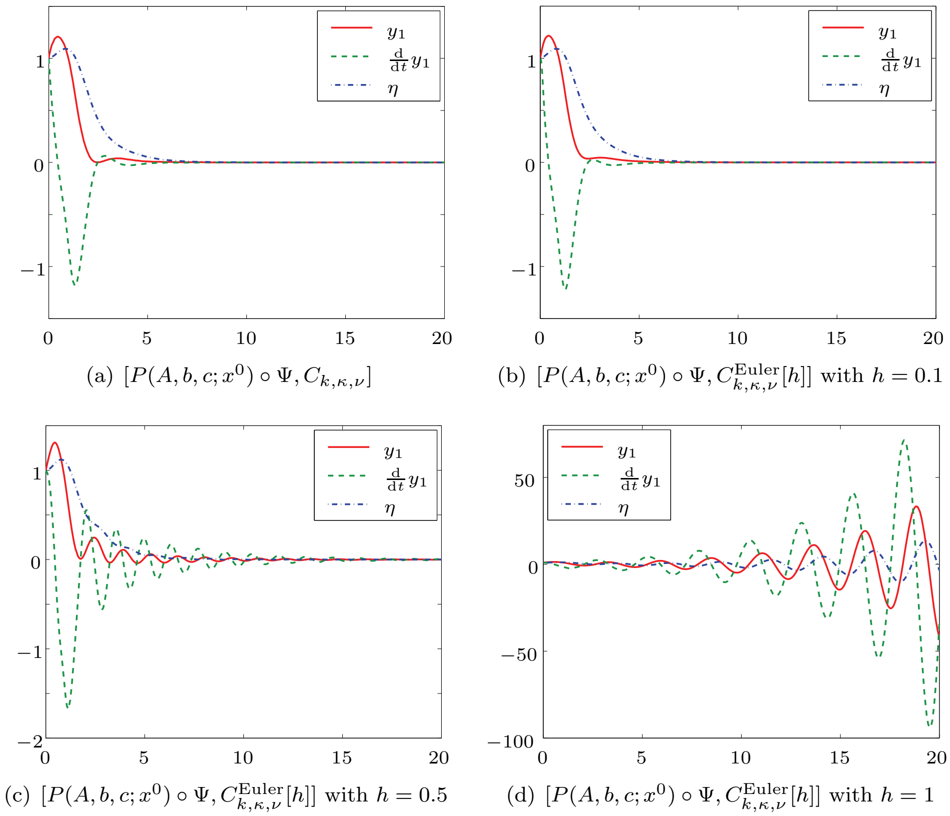

FIG. 4. *Output derivative feedback and output delay feedback simulations.*

6. Conclusion. We have established conditions under which a memoryless derivative feedback controller can be replaced by a controller in which the derivatives are replaced by their Euler approximations for suitably small step sizes. The global nonlinear result requires regularity assumptions on the external disturbances which act on both the input and the output channels. We have considered linear systems in some detail to show how these regularity assumptions can be related to structural assumptions on the plant (e.g., relative degree) and to the order of the controller.

It is perhaps surprising that rigorous statements justifying such approximations do not appear in the literature; however, we note that alternative means of deriving similar results, for example, using the techniques of approximate identities [7] (as drawn to our attention by a reviewer of this paper), are certainly feasible and are worthy of further study and of comparison to the above presented techniques.

We also note that the companion paper [6] to this contribution derives significant extensions of the results presented here, in both the regional and global contexts, by introducing sampling schemes and by developing a common framework which also includes high gain observer reconstructions of the state.

Appendix. Proof for exponential stability of $[P(A,b,c;x^0), C_{k,\kappa,\nu}]$. The proof of Proposition 4.1 is based on the following Byrnes–Isidori normal form.

LEMMA A.1. *Let, for $r,n \in \mathbb{N}$ with $r \le n$, $(A,b,c) \in \mathcal{P}_{n,r}$. For $\mathcal{C} := \left[c/cA/\dots/cA^{r-1}\right]$, $\mathcal{B} := \left[b, Ab, \dots, A^{r-1}b\right]$, $\mathcal{N} \in \mathbb{R}^{n\times(n-r)}$ with* $\operatorname{im}\mathcal{N} = \ker\mathcal{C}$, *and*

$$V = \begin{bmatrix} \mathcal{C} \\ \left(\mathcal{N}^T\mathcal{N}\right)^{-1}\mathcal{N}^T\left[I_{n-r} - \mathcal{B}\left(\mathcal{C}\mathcal{B}\right)^{-1}\mathcal{C}\right]\end{bmatrix}, \qquad V^{-1} = \left[\mathcal{B}\left(\mathcal{C}\mathcal{B}\right)^{-1}, \mathcal{N}\right], \tag{A.1}$$

the coordinate transformation

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix}(t) := Vx(t) \tag{A.2}$$

converts (A,b,c), given by (4.1), into the normal form $(VAV^{-1}, Vb, cV^{-1}) \in \mathcal{P}_{n,r}$:

$$\left\{\begin{array}{l} \dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \dots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \dots & 0 & 1 & 0\\ R_1 & \dots & & R_r & S\\ T & 0 & \dots & 0 & Q\end{bmatrix}}_{=VAV^{-1}}\begin{pmatrix}\xi\\ \eta\end{pmatrix} + \underbrace{\begin{pmatrix}0\\ \vdots\\ 0\\ cA^{r-1}b\\ 0\end{pmatrix}}_{=Vb}u_1, \quad \begin{pmatrix}\xi\\ \eta\end{pmatrix}(0) = Vx^0 \in \mathbb{R}^n,\\ \qquad y_1 = \xi_1 \;=\; \underbrace{(1,0,\dots,0)}_{=cV^{-1}}\begin{pmatrix}\xi\\ \eta\end{pmatrix},\\ \qquad T \in \mathbb{R}^{(n-r)\times 1},\ Q \in \mathbb{R}^{(n-r)\times(n-r)},\ S \in \mathbb{R}^{1\times(n-r)},\ R_1,\dots,R_r \in \mathbb{R}. \end{array}\right. \tag{A.3}$$

Proof. See [10, Lem. 3.5]. ∎

Proof of Proposition 4.1. By (2.1) and (4.2),

$$u_1(t) = -\sum_{i=0}^{r-1}\nu\kappa^{r-i}k_{i+1}\left[y_1^{(i)}(t) - y_0^{(i)}(t)\right] + u_0(t)\,. \tag{A.4}$$

The closed-loop system (4.1), (A.4) is equivalent to (A.3), (A.4). Setting $\zeta_i = \kappa^{-i+1}\xi_i$ for $i = 1,\dots,r$ yields

$$\begin{aligned} \dot\zeta_i &= \kappa^{-i+1}\dot\xi_i = \kappa^{-i+1}\xi_{i+1} = \kappa\zeta_{i+1} \qquad \text{for } i = 1,\dots,r-1\,,\\ \dot\zeta_r &= \kappa^{-r+1}\dot\xi_r\\ &= \kappa^{-r+1}((R_1 - \vartheta\nu k_1\kappa^r)\xi_1 + \dots + (R_{r-1} - \vartheta\nu k_{r-1}\kappa^2)\xi_{r-1} + (R_r - \vartheta\nu k_r\kappa)\xi_r)\\ &\quad + \kappa^{-r+1}S\eta\\ &= \kappa\left(\left(\frac{R_1}{\kappa^r} - \vartheta\nu k_1\right)\zeta_1 + \dots + \left(\frac{R_{r-1}}{\kappa^2} - \vartheta\nu k_{r-1}\right)\zeta_{r-1} + \left(\frac{R_r}{\kappa} - \vartheta\nu k_r\right)\zeta_r\right)\\ &\quad + \kappa^{-r+1}S\eta\,. \end{aligned}$$

Thus the scaling

$$\begin{pmatrix}\zeta\\ \eta\end{pmatrix} = U_\kappa\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \qquad U_\kappa := \operatorname{diag}\left(1,\kappa^{-1},\dots,\kappa^{-r+1},1,\dots,1\right),$$

converts (A.3), (A.4) into

$$
\text{(A.5)}\quad \begin{cases}
\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\zeta\\ \eta\end{pmatrix} = \underbrace{\left[\begin{array}{c|c} \kappa\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ \frac{R_1}{\kappa^r}-\vartheta\nu k_1 & \dots\ \frac{R_{r-1}}{\kappa^2}-\vartheta\nu k_{r-1} & & \frac{R_r}{\kappa}-\vartheta\nu k_r \end{bmatrix} & \begin{matrix}0\\ \vdots\\ 0\\ \kappa^{-r+1}S\end{matrix} \\ \hline T \quad 0 \quad \dots \quad 0 & Q \end{array}\right]}_{=U_\kappa(VAV^{-1}+Vb(\nu k\mid 0))U_\kappa^{-1}=:\widehat{A}_{k,\kappa,\nu}}\begin{pmatrix}\zeta\\ \eta\end{pmatrix} \\
\qquad + \kappa^{-r+1}\vartheta\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ 0\end{pmatrix}\left[\sum_{i=0}^{r-1}\nu\kappa^{r-i}k_{i+1}y_0^{(i)} + u_0\right], \\
y_1 = (1,0,\dots,0)\begin{pmatrix}\zeta\\ \eta\end{pmatrix}, \\
u_1 = -\sum_{i=0}^{r-1}\nu\kappa^{r-i}k_{i+1}\left[y_1^{(i)} - y_0^{(i)}\right] + u_0, \\
\begin{pmatrix}\zeta\\ \eta\end{pmatrix}(0) = \begin{pmatrix}\zeta^0\\ \eta^0\end{pmatrix} = U_\kappa V x^0,
\end{cases}
$$

where $\vartheta = cA^{r-1}b$, $R_1,\dots,R_r \in \mathbb{R}$, $S \in \mathbb{R}^{1\times(n-r)}$, $T \in \mathbb{R}^{n-r}$, $Q \in \mathbb{R}^{(n-r)\times(n-r)}$ with $\operatorname{spec}(Q) \subset \mathbb{C}_-$, and $V \in \mathbb{R}^{n\times n}$ with $\det V \neq 0$.

In view of $u_0 \equiv y_0 \equiv 0$ and the equivalence of (4.1), (4.2), (2.1), and the closed-loop equations (A.5), it remains to show that

$$
\text{(A.6)}\quad \exists\,\nu^* \ge 1\ \forall\,\nu \ge \nu^*\ \exists\,\kappa^* \ge 1\ \exists\,M > 0\ \exists\,\alpha > 0\ \ \forall\,\kappa \ge \kappa^*\ \forall\,t \ge 0\ :\quad \left|e^{\widehat{A}_{k,\kappa,\nu}t}\right| \le M e^{-\alpha t}.
$$

Since $k_r > 0$, $\vartheta > 0$ and $s \mapsto p(s) = \sum_{i=0}^{r-1} k_{i+1}\, s^i$ is Hurwitz, it follows from the root-locus [15, section 5] that $r-1$ roots of $q_\nu(s) = s^r + \nu\vartheta p(s) = 0$ converge to the roots of $p(s) = 0$ as $\nu \to \infty$ and that the remaining root of q_ν is real and diverges to $-\infty$ as $\nu \to \infty$. Thus there exists $\nu^* > 0$ such that the polynomial q_ν is Hurwitz for all $\nu \ge \nu^*$. Let $\nu \ge \nu^*$, and choose the positive definite matrices $N_\zeta = N_\zeta^T \in \mathbb{R}^{r\times r}$ and $N_\eta = N_\eta^T \in \mathbb{R}^{(n-r)\times(n-r)}$ solving

$$
\text{(A.7)}\quad N_\zeta\begin{bmatrix} 0_{(r-1)\times 1} & I_{r-1} \\ -\vartheta\nu k_1 & \dots\ -\vartheta\nu k_r\end{bmatrix} + \begin{bmatrix} 0_{(r-1)\times 1} & I_{r-1} \\ -\vartheta\nu k_1 & \dots\ -\vartheta\nu k_r\end{bmatrix}^T N_\zeta = -I_r, \quad N_\eta Q + Q^T N_\eta = -I_{n-r}.
$$

Then the derivative of

$$
t \mapsto V(t) \ :=\ \frac{1}{2}\,\zeta(t)^T N_\zeta \zeta(t) + \frac{1}{2}\,\eta(t)^T N_\eta \eta(t)
$$

along the solution of

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\zeta\\ \eta\end{pmatrix}(t) = \widehat{A}_{k,\kappa,\nu}\begin{pmatrix}\zeta\\ \eta\end{pmatrix}(t)
$$

yields, for all $t \ge 0$, and omitting the argument t for brevity,

$$
\begin{aligned}
\dot V(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\zeta^T N_\zeta \zeta + \frac{1}{2}\eta^T N_\eta \eta\right) \\
&= \zeta^T N_\zeta \left(\kappa \begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & & 1 \\ \frac{R_1}{\kappa^r} - \vartheta\nu k_1 & \dots & \frac{R_{r-1}}{\kappa^2} - \vartheta\nu k_{r-1} & & \frac{R_r}{\kappa} - \vartheta\nu k_r \end{bmatrix} \zeta + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \kappa^{-r+1} S\eta \end{bmatrix} \right) \\
&\quad + \eta^T N_\eta (Q\eta + T\zeta_1) \\
&\overset{(A.7)}{\le} -\frac{\kappa}{2}|\zeta|^2 + \kappa\zeta N_\zeta \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \\ \frac{R_1}{\kappa^r} & \dots & \frac{R_r}{\kappa} \end{bmatrix} \zeta + \frac{1}{\kappa^{r-1}} |N_\zeta|\,|S|\,|\zeta|\,|\eta| - \frac{1}{2}|\eta|^2 + |N_\eta T|\,|\eta|\,|\zeta_1| \\
&\overset{\kappa\ge 1}{\le} -\frac{\kappa}{2}|\zeta|^2 + |N_\zeta|\,|(R_1,\dots,R_r)|\,|\zeta|^2 + \frac{1}{\kappa^{r-1}}|N_\zeta|\,|S|\,|\zeta|^2 + \frac{1}{\kappa^{r-1}}|N_\zeta|\,|S|\,|\eta|^2 \\
&\quad -\frac{1}{2}|\eta|^2 + \frac{1}{4}|\eta|^2 + 4|N_\eta T|\,|\zeta_1|^2 \\
&\le -\Big(\frac{\kappa}{2} - |N_\zeta|\,|(R_1,\dots,R_r)| - |N_\zeta|\,|S| - 4|N_\eta T|\Big)|\zeta|^2 - \left(\frac{1}{4} - \frac{|N_\zeta|\,|S|}{\kappa^{r-1}}\right)|\eta|^2 ,
\end{aligned}
$$

and so, for

$$
\begin{aligned}
\kappa^* &:= \max\left\{ \frac{1}{4} + 2(|N_\zeta|\,|(R_1,\dots,R_r)| - |N_\zeta|\,|S| - 4|N_\eta T|), (8|N_\zeta|\,|S|)^{-r+1} \right\} , \\
\alpha &:= \min\left\{ \frac{1}{8|N_\zeta|}, \frac{1}{8|N_\eta|} \right\} ,
\end{aligned}
$$

we conclude, for all $t \ge 0$ and $\kappa \ge \kappa^*$,

$$
\dot V(t) \le -\frac{1}{8}|\zeta(t)|^2 - \frac{1}{8}|\eta(t)|^2 \le -\frac{1}{8|N_\zeta|}\zeta(t)^T N_\zeta \zeta(t) - \frac{1}{8|N_\eta|}\eta(t)^T N_\eta \eta(t) \le -\alpha V(t) ,
$$

and hence

$$
\forall\, t \ge t_0,\ \forall\, t_0 \ge 0\ :\ \left| \begin{pmatrix} \zeta(t) \\ \eta(t) \end{pmatrix} \right| \le \exp\left(-\alpha(t-t_0)\right) \sqrt{ \frac{\max\operatorname{spec}\begin{bmatrix} N_\zeta & 0 \\ 0 & N_\eta \end{bmatrix}}{\min\operatorname{spec}\begin{bmatrix} N_\zeta & 0 \\ 0 & N_\eta \end{bmatrix}} } \left| \begin{pmatrix} \zeta(t_0) \\ \eta(t_0) \end{pmatrix} \right| .
$$

This proves (A.6) and completes the proof of the proposition. ∎

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