

Chapter 7

Normal Forms, High-Gain, and Funnel Control for Linear Differential-Algebraic Systems

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We consider linear differential-algebraic m -input m -output systems with positive strict relative degree or proper inverse transfer function; in the single-input single-output case these two disjoint classes make the whole of all linear DAEs without feedthrough term. Structural properties—such as normal forms (i.e., the counterpart to the Byrnes–Isidori form for ODE systems), zero dynamics, and high-gain stabilizability—are analyzed for two purposes: first, to gain insight into the system classes and, second, to solve the output regulation problem by funnel control. The funnel controller achieves tracking of a class of reference signals within a prespecified funnel; this means, in particular, the transient behavior of the output error can be specified and the funnel controller does not incorporate any internal model for the reference signals or any identification mechanism. It is simple in its design. The results are illuminated by position and velocity control of a mechanical system encompassing springs, masses, and dampers.

Nomenclature

\mathbf{N}, \mathbf{N}_0	set of natural numbers, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, set of all integers, resp.
$\mathbf{R}_{\geq 0}$	$= [0, \infty)$
$\mathbf{C}_+(\overline{\mathbf{C}}_+)$	open (closed) set of complex numbers with positive real part, resp.
$\mathbf{C}_-(\overline{\mathbf{C}}_-)$	open (closed) set of complex numbers with negative real part, resp.
$\mathbf{GL}_n(\mathbf{R})$	the group of invertible real $n \times n$ matrices
$\mathbf{R}[s]$	the ring of polynomials with coefficients in \mathbf{R}
$\mathbf{R}(s)$	the quotient field of $\mathbf{R}[s]$
$\mathbf{R}^{n,m}$	the set of $n \times m$ matrices with entries in a ring \mathbf{R}
$\ x\ $	$= \sqrt{x^\top x}$, the Euclidean norm of $x \in \mathbf{R}^n$
$\ M\ $	$= \max \{ \ Mx\ \mid x \in \mathbf{R}^m, \ x\ = 1 \}$, induced norm of $M \in \mathbf{R}^{n,m}$
$\mathcal{C}^\ell(\mathcal{T}; \mathbf{R}^n)$	the set of ℓ -times continuously differentiable functions $f : \mathcal{T} \rightarrow \mathbf{R}^n$
$\mathcal{B}^\ell(\mathcal{T}; \mathbf{R}^n) =$	$\left\{ f \in \mathcal{C}^\ell(\mathcal{T}; \mathbf{R}^n) \mid \frac{d^i}{dt^i} f \text{ is bounded for } i = 0, \dots, \ell \right\}$

7.1 Introduction

We consider linear differential-algebraic systems of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{7.1}$$

where $E, A \in \mathbf{R}^{n,n}$, $B, C^\top \in \mathbf{R}^{n,m}$ are such that the pencil $sE - A \in \mathbf{R}[s]^{n,n}$ is *regular*, i.e., $\det(sE - A) \in \mathbf{R}[s] \setminus \{0\}$; the set of these square (i.e., same number of inputs and outputs) systems is denoted by $\Sigma_{n,m}$, and we write $[E, A, B, C] \in \Sigma_{n,m}$.

The functions $u, y : \mathbf{R} \rightarrow \mathbf{R}^m$ are called *input* and *output* of the system, respectively. A trajectory $(x, u, y) : \mathbf{R} \rightarrow \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^m$ is said to be a *solution* of (7.1) if and only if it belongs to the *behavior* of (7.1):

$$\mathfrak{B}_{(7.1)} := \left\{ (x, u, y) \in \mathcal{C}^1(\mathbf{R}; \mathbf{R}^n) \times \mathcal{C}(\mathbf{R}; \mathbf{R}^m) \times \mathcal{C}(\mathbf{R}; \mathbf{R}^m) \mid \begin{array}{l} (x, u, y) \text{ solves (7.1)} \\ \text{for all } t \in \mathbf{R} \end{array} \right\}.$$

Regularity of the pencil guarantees that for each consistent initial value $x(0) = x^0 \in \mathbf{R}^n$ there exists a unique and global solution of $E\dot{x} = Ax$; see, for example, [15, Section 2.1]. More smoothness for u and y is required for some results.

We also derive frequency domain results for $[E, A, B, C] \in \Sigma_{n,m}$ and its *transfer function*, defined by

$$G(s) = C(sE - A)^{-1}B \in \mathbf{R}(s)^{m,m}.$$

Roughly speaking, and explained in detail in Section 7.1.3, we divide, in the single-input single-output case, the system class $\Sigma_{n,1}$ into the disjoint sets of systems with strictly proper

transfer function and with nonstrictly proper transfer function $g(s) = C(sE - A)^{-1}B \in \mathbf{R}(s)$, respectively. Our generalization for multi-input, multi-output systems $\Sigma_{n,m}$ will treat systems with positive strict relative degree and systems with proper inverse transfer function (see Section 7.1.2 for the definitions), respectively. However, these two disjoint sets do not unify to whole $\Sigma_{n,m}$, as, for example, systems with strictly proper transfer function with a nonconstant vector relative degree are not encompassed.

7.1.1 System Equivalence

We will derive normal forms of systems $[E, A, B, C] \in \Sigma_{n,m}$ belonging to certain classes specified in Section 7.1.3. To this end, recall the following.

Two systems $[E_i, A_i, B_i, C_i] \in \Sigma_{n,m}$, $i = 1, 2$, are called *system equivalent* if and only if

$$\exists W, T \in \mathbf{GL}_n(\mathbf{R}) : \begin{bmatrix} sE_1 - A_1 & B_1 \\ C_1 & 0 \end{bmatrix} = \begin{bmatrix} W & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} sE_2 - A_2 & B_2 \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix};$$

we write

$$[E_1, A_1, B_1, C_1] \stackrel{W, T}{\sim} [E_2, A_2, B_2, C_2]. \quad (7.2)$$

It is easy to see that system equivalence is an equivalence relation on $\Sigma_{n,m}$ and that the transfer function is invariant under system equivalence.

We quote the well-known result by Weierstraß for regular pencils $sE - A$.

Proposition 7.1 (Weierstraß form [6, Theorem XII.3]). *For any regular matrix pencil $sE - A \in \mathbf{R}[s]^{n,n}$, there exist $W, T \in \mathbf{GL}_n(\mathbf{R})$ such that*

$$sE - A = W \begin{bmatrix} sI_{n_s} - A_s & 0 \\ 0 & sN - I_{n_f} \end{bmatrix} T \quad (7.3)$$

for some $A_s \in \mathbf{R}^{n_s \times n_s}$ and nilpotent $N \in \mathbf{R}^{n_f \times n_f}$. The dimensions $n_s, n_f \in \mathbf{N}_0$ are unique, and the matrices A_s and N are unique up to similarity.

The *index of nilpotency* of a nilpotent matrix $N \in \mathbf{R}^{k,k}$ is defined to be the smallest $\nu \in \mathbf{N}$ such that $N^\nu = 0$. It can be shown (see, e.g., [15, Lemma 2.10]) that the index of nilpotency ν of N in (7.3) is uniquely defined by the regular pencil $sE - A$; ν is therefore called the *index* of the pencil $sE - A$ if the nilpotent block is present and the index is $\nu = 0$ if the nilpotent block is absent (i.e., $n_f = 0$).

An immediate result from Proposition 7.1 for systems $[E, A, B, C]$ is the following.

Corollary 7.2 (Decoupled DAE). *Let $[E, A, B, C] \in \Sigma_{n,m}$. Then there exist $W, T \in \mathbf{GL}_n(\mathbf{R})$ such that*

$$[E, A, B, C] \stackrel{W, T}{\sim} \left[\begin{bmatrix} I_{n_s} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} A_s & 0 \\ 0 & I_{n_f} \end{bmatrix}, \begin{bmatrix} B_s \\ B_f \end{bmatrix}, [C_s \quad C_f] \right] \quad (7.4)$$

for some $B_s \in \mathbf{R}^{n_f \times m}$, $B_f \in \mathbf{R}^{n_f \times m}$, $C_s \in \mathbf{R}^{m \times n_s}$, $C_f \in \mathbf{R}^{m \times n_s}$, $A_s \in \mathbf{R}^{n_s \times n_s}$ and nilpotent $N \in \mathbf{R}^{n_f \times n_f}$.

The form (7.4) is interpreted, in terms of the DAE (7.1), as follows:

$$(x, u, y) \in \mathfrak{B}_{(7.1)} \quad \text{if and only if} \quad \begin{pmatrix} x_s(\cdot) \\ x_f(\cdot) \end{pmatrix} := Tx(\cdot)$$

solves the decoupled DAEs

$$\begin{aligned} \dot{x}_s(t) &= A_s x_s(t) + B_s u(t), & N\dot{x}_f(t) &= x_f(t) + B_f u(t), \\ y_s(t) &= C_s x_s(t), & y_f(t) &= C_f x_f(t), \\ y(t) &= y_s(t) + y_f(t). \end{aligned} \tag{7.5}$$

If $(x, u, y) \in \mathfrak{B}_{(7.1)}$ and in addition $u \in \mathcal{C}^{\nu-1}(\mathbf{R}; \mathbf{R}^m)$, then by repeated multiplication of $N\dot{x}_f(t) = x_f(t) + B_f u(t)$ by N from the left, differentiation, and using the identity $N^\nu = 0$, it is easy to see that the solution satisfies

$$x_f(\cdot) = - \sum_{k=0}^{\nu-1} N^k B_f u^{(k)}(\cdot). \tag{7.6}$$

7.1.2 System Properties

Zero Dynamics

An essential concept in this chapter is the *zero dynamics* of system (7.1):

$$\mathcal{ZD}_{(7.1)} := \{ (x, u, y) \in \mathfrak{B}_{(7.1)} \mid y = 0 \}.$$

By linearity of (7.1), $\mathcal{ZD}_{(7.1)}$ is a real vector space.

The zero dynamics of system (7.1) is called *asymptotically stable* if and only if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall (x, u, y) \in \mathcal{ZD}_{(7.1)} \text{ s.t. } \|(x(0), u(0))\| < \delta \forall t \geq 0 : \|(x(t), u(t))\| < \varepsilon$$

and

$$\forall (x, u, y) \in \mathcal{ZD}_{(7.1)} : \lim_{t \rightarrow \infty} (x(t), u(t)) = 0.$$

(Asymptotically stable) zero dynamics is the vector space of those trajectories of the system which are, loosely spoken, not visible at the output (and tend to zero).

Relative Degree

Another fundamental concept is the *relative degree* of the transfer function $G(s) = C(sE - A)^{-1}B$ of (7.1).

A rational matrix function $G(s) \in \mathbf{R}(s)^{m,m}$ is called

$$\begin{aligned} \textit{proper} &\iff \lim_{s \rightarrow \infty} G(s) = D \text{ for some } D \in \mathbf{R}^{m,m}, \\ \textit{strictly proper} &\iff \lim_{s \rightarrow \infty} G(s) = 0, \end{aligned}$$

and we say that the square rational matrix $G(s) \in \mathbf{R}(s)^{m,m}$ has *strict relative degree* $\rho \in \mathbf{Z}$

if and only if

$$\rho = \text{sr deg } G(s) := \sup \left\{ k \in \mathbf{Z} \mid \lim_{s \rightarrow \infty} s^k G(s) \in \mathbf{GL}_m(\mathbf{R}) \text{ and } \lim_{s \rightarrow \infty} s^{k-1} G(s) = 0 \right\}$$

exists. Note that for any $G(s) \in \mathbf{R}(s)^{m,m}$ we have (consider the entries)

$$\lim_{s \rightarrow \infty} s^{k-1} G(s) = 0 \quad \text{for some } k \in \mathbf{Z} \quad \implies \quad \lim_{s \rightarrow \infty} s^{k-i} G(s) = 0 \quad \text{for all } i \in \mathbf{N}.$$

For convenience, we also say that $[E, A, B, C] \in \Sigma_{n,m}$ has strict relative degree ρ if $G(s) = C(sE - A)^{-1}B$ has strict relative degree ρ . Then the matrix

$$\Gamma := \lim_{s \rightarrow \infty} s^\rho G(s) \in \mathbf{GL}_m(\mathbf{R})$$

is called *high-frequency gain matrix*.

If $g(s) = p(s)/q(s)$, for $p(s) \in \mathbf{R}[s]$ and $q(s) \in \mathbf{R}[s] \setminus \{0\}$, is a scalar rational function, then the strict relative degree always exists and coincides with the well-known definition of relative degree:

$$\text{sr deg } g(s) = \text{deg } q(s) - \text{deg } p(s).$$

Stabilizability and Detectability

We recall different concepts of stabilizability and detectability for DAEs (7.1). For brevity, we do not define the concepts in system theoretic terms but recall algebraic characterizations in Proposition 7.3; the latter will be used in our proofs. For system theoretic notions of the concepts see [5, Sections 2 and 3].

Proposition 7.3 (Stabilizability and detectability). *A system (7.1) is*

- (i) *stabilizable* $\iff \text{rank}[sE - A, B] = n \quad \text{for all } s \in \overline{\mathbf{C}}_+$,
- (ii) *detectable* $\iff \text{rank}[sE^\top - A^\top, C^\top] = n \quad \text{for all } s \in \overline{\mathbf{C}}_+$.

Both properties are invariant under system equivalence.

Poles and Zeros

Finally, we recall the definition of transmission zeros and poles of a transfer function; see, for example, [14, Section 6.5].

Let $G(s) \in \mathbf{R}(s)^{m,m}$ with *Smith–McMillan form*

$$U^{-1}(s)G(s)V^{-1}(s) = \text{diag} \left(\frac{\varepsilon_1(s)}{\psi_1(s)}, \dots, \frac{\varepsilon_r(s)}{\psi_r(s)}, 0, \dots, 0 \right) \in \mathbf{R}(s)^{m,m},$$

where $U(s), V(s) \in \mathbf{R}[s]^{m,m}$ are unimodular (i.e., invertible over $\mathbf{R}[s]^{m,m}$), $\text{rank } G(s) = r$ and $\varepsilon_i(s), \psi_i(s) \in \mathbf{R}[s]$ are monic and coprime and satisfy $\varepsilon_i(s) \mid \varepsilon_{i+1}(s)$, $\psi_{i+1}(s) \mid \psi_i(s)$ for all $i = 1, \dots, r-1$. $s_0 \in \mathbf{C}$ is called *transmission zero* of $G(s)$ if $\varepsilon_r(s_0) = 0$ and a *pole* of $G(s)$ if $\psi_1(s_0) = 0$.

Note that by the transfer function being invariant under system equivalence, this holds true for transmission zeros and poles as well.

7.1.3 System Classes

We introduce the two main system classes investigated in this chapter. In both of them, an explicit feedthrough term is not allowed. However, this assumption is made only for technical reasons. We also explain how systems with feedthrough are related to the two system classes under consideration.

Systems with Positive Strict Relative Degree

Consider the class of DAE systems (7.1) such that $[E, A, B, C] \in \Sigma_{n,m}$ has positive strict relative degree.

If $[E, A, B, C]$ is single-input, single-output, i.e., $m = 1$, then its transfer function

$$g(s) = C(sE - A)^{-1}B = p(s)/q(s)$$

has positive strict relative degree if and only if $\deg q(s) > \deg p(s)$ or, equivalently, $g(s)$ is strictly proper.

Note that for any $G(s) \in \mathbf{R}(s)^{m,m}$ we have

$$G(s) \text{ has positive strict relative degree} \begin{matrix} \implies \\ \not\Leftarrow \\ \text{i.g.} \end{matrix} G(s) \text{ is strictly proper.}$$

An ODE system $[E, A, B, C] = [I, A, B, C]$ has transfer function

$$G(s) = C(sI - A)^{-1}B = CBs^{-1} + CABs^{-2} + CA^2Bs^{-3} + \dots$$

and therefore strict relative degree $\rho \in \mathbf{N}$ if and only if

$$\det(CA^{\rho-1}B) \neq 0 \quad \text{and,} \quad \text{if } \rho > 1, \quad \forall k = 0, \dots, \rho - 2 : CA^k B = 0.$$

Systems with Proper Inverse Transfer Function

The second class of DAE systems (7.1) considered in this chapter are those DAEs $[E, A, B, C] \in \Sigma_{n,m}$ whose transfer function $G(s) = C(sE - A)^{-1}B \in \mathbf{R}(s)^{m,m}$ has *proper inverse* over $\mathbf{R}(s)^{m,m}$; i.e., $G(s)$ is invertible over $\mathbf{R}(s)$ and $\lim_{s \rightarrow \infty} G^{-1}(s) \in \mathbf{R}^{m,m}$.

In [2, Proposition 1.2] we have shown that for any $G(s) \in \mathbf{R}(s)^{m,m}$ the properties of proper inverse transfer function and strict relative degree are related as follows:

$$\text{sr deg } G(s) \leq 0 \begin{matrix} \implies \\ \not\Leftarrow \\ \text{i.g.} \end{matrix} G(s) \text{ has proper inverse.}$$

A zero dynamics form for systems $[E, A, B, C] \in \Sigma_{n,m}$ with transfer function $G(s) \in \mathbf{R}(s)^{m,m}$ whose inverse $G^{-1}(s)$ is proper is studied in Theorem 7.8. Conversely, if a proper transfer function $G^{-1}(s) \in \mathbf{R}(s)^{m,m}$ is given, then this may be realized as an ODE $[I, A, B, C, D]$ with feedthrough term D and it is easily verified that the regular DAE

$$\left[\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} -D & -C \\ B & A \end{bmatrix}, \begin{bmatrix} I \\ 0 \end{bmatrix}, [I, 0] \right] \tag{7.7}$$

without feedthrough term is a realization of $G(s)$.

If $[E, A, B, C]$ is single-input, single-output, then its transfer function $g(s) = C(sE - A)^{-1}B = p(s)/q(s)$ has proper inverse if and only if $\deg q(s) \leq \deg p(s)$ or, equivalently, $g(s)$ is not strictly proper. In this case, there does not exist a realization of $g(s)$ as an ODE system $[I, A, B, C]$ without feedthrough term. The class of single-input single-output DAE systems $\Sigma_{n,1}$ can be decomposed into the disjoint sets of systems with strictly proper transfer function and with nonstrictly proper transfer function $g(s) = C(sE - A)^{-1}B \in \mathbf{R}(s)$, respectively:

$$\begin{aligned} \Sigma_{n,m} &= \{[E, A, B, C] \in \Sigma_{n,1} \mid g(s) = \frac{p(s)}{q(s)}, \deg q(s) > \deg p(s)\} \\ &\dot{\cup} \{[E, A, B, C] \in \Sigma_{n,1} \mid g(s) = \frac{p(s)}{q(s)}, \deg q(s) \leq \deg p(s)\}. \end{aligned}$$

For multi-input, multi-output systems $\Sigma_{n,m}$, generalization to systems with positive strict relative degree and systems with proper inverse transfer function does not allow for such a disjoint union. This is the topic of the following section.

Systems Which Have neither Positive Strict Relative Degree nor Proper Inverse Transfer Function

The class of multi-input, multi-output systems $[E, A, B, C] \in \Sigma_{n,m}$ with transfer function $G(s) = C(sE - A)^{-1}B \in \mathbf{R}(s)^{m,m}$ can be decomposed in the disjoint union of the following three sets:

$$\begin{aligned} \Sigma_{n,m} &= \{[E, A, B, C] \in \Sigma_{n,m} \mid G(s) \text{ has positive strict relative degree}\} \\ &\dot{\cup} \{[E, A, B, C] \in \Sigma_{n,m} \mid G(s) \text{ has proper inverse transfer function}\} \\ &\dot{\cup} \left\{ [E, A, B, C] \in \Sigma_{n,m} \mid \begin{array}{l} G(s) \text{ has neither positive strict relative degree} \\ \text{nor proper inverse transfer function} \end{array} \right\}. \end{aligned}$$

Note that the latter set is, for multi-input, multi-output systems, not empty. It contains in particular systems with a nonconstant vector relative degree and it contains, for example for $m = 2$, the system

$$[E, A, B, C] := \left[\begin{array}{ccc} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right]$$

with transfer function $G(s) = \text{diag}(s, s^{-1}) \in \mathbf{R}(s)^{2,2}$. We stress that $G(s)$ has neither a positive strict relative degree nor a proper inverse. The study of normal forms for this general system class would be very involved, and it is not clear whether funnel control would be possible. Hence it is omitted.

Systems with Feedthrough

So far we have not considered systems with nonzero feedthrough term. In this section we will show that such systems can, by augmenting the state by the feedthrough term, be

rewritten as a system of the form (7.1); the transfer function of the new system coincides with the transfer function of the original system. Furthermore, it is shown that a system with proper transfer function can be rewritten as an ODE with feedthrough.

Remark 7.4 (DAE with feedthrough \longrightarrow DAE without feedthrough). Consider a DAE system with feedthrough of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{7.8}$$

where $E, A \in \mathbf{R}^{n,n}$, $B, C^\top \in \mathbf{R}^{n,m}$, $D \in \mathbf{R}^{m,m}$. Then the behavior $\mathfrak{B}_{(7.8)}$, the zero dynamics $\mathcal{ZD}_{(7.8)}$, and its asymptotic stability can be defined completely analogously as for (7.1); this is omitted.

By introducing an “augmented state” of $x(t)$ and $w(t) = Du(t)$, system (7.8) can be rewritten as a DAE system without feedthrough term, namely

$$\begin{aligned} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} &= \begin{bmatrix} A & 0 \\ 0 & -I \end{bmatrix} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} + \begin{bmatrix} B \\ D \end{bmatrix} u(t), \\ y(t) &= [C \quad I] \begin{pmatrix} x(t) \\ w(t) \end{pmatrix}. \end{aligned} \tag{7.9}$$

Clearly, the behaviors and zero dynamics are related in the following way:

$$\begin{aligned} \mathfrak{B}_{(7.9)} &= \left\{ ((x^\top, w^\top)^\top, u, y) \mid (x, u, y) \in \mathfrak{B}_{(7.8)}, w = Du \right\}, \\ \mathcal{ZD}_{(7.9)} &= \left\{ ((x^\top, w^\top)^\top, u, y) \mid (x, u, y) \in \mathcal{ZD}_{(7.8)}, w = Du \right\}. \end{aligned}$$

The following observations can now be made:

(a) The transfer functions of (7.8) and (7.9) coincide:

$$C(sE - A)^{-1}B + D = [C \quad I] \left(s \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & 0 \\ 0 & -I \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ D \end{bmatrix}.$$

(b) The zero dynamics $\mathcal{ZD}_{(7.8)}$ is asymptotically stable if and only if the zero dynamics $\mathcal{ZD}_{(7.9)}$ is asymptotically stable.

(c) The pencil $s \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & 0 \\ 0 & -I \end{bmatrix}$ is regular if and only if $sE - A$ is regular.

(d) We have

$$\begin{aligned} \det \left[\begin{array}{cc|cc} sE - A & 0 & B & \\ 0 & I & D & \\ \hline C & I & 0 & \end{array} \right] &= \det \begin{bmatrix} sE - A & 0 & B \\ 0 & I & D \\ C & 0 & -D \end{bmatrix} = \det \begin{bmatrix} sE - A & 0 & B \\ 0 & I & 0 \\ C & 0 & -D \end{bmatrix} \\ &= \det \begin{bmatrix} sE - A & B \\ C & -D \end{bmatrix}. \end{aligned}$$

(e) Since, for all $s \in \mathbf{C}$,

$$\begin{aligned} \text{rank} \begin{bmatrix} sE - A & 0 & B \\ 0 & I & D \end{bmatrix} &= \text{rank} \begin{bmatrix} sE - A & B \end{bmatrix} + m, \\ \text{rank} \begin{bmatrix} sE - A & 0 \\ 0 & I \\ C & I \end{bmatrix} &= \text{rank} \begin{bmatrix} sE - A \\ C \end{bmatrix} + m, \end{aligned}$$

it is easily seen that system (7.8) is stabilizable (detectable) if and only if system (7.9) is stabilizable (detectable).

(f) If $E = I$, then (7.8) is an ODE with feedthrough and can be rewritten as a DAE with regular matrix pencil and transfer function $G(s) = C(sI - A)^{-1}B + D$, which coincides with the transfer function of the original ODE system.

Remark 7.5 (DAE with proper transfer function \longrightarrow ODE with feedthrough). For any DAE (7.1) with regular pencil $sE - A \in \mathbf{R}[s]^{n,n}$ and $\lim_{s \rightarrow \infty} C(sE - A)^{-1}B \in \mathbf{R}^{m,m}$ there exists an ODE system with feedthrough

$$\begin{aligned} \dot{z}(t) &= A_s z(t) + B_s u(t), \\ y(t) &= C_s z(t) - C_f B_f u(t) \end{aligned} \quad (7.10)$$

for some $B_s \in \mathbf{R}^{n_f, m}$, $B_f \in \mathbf{R}^{n_f, m}$, $C_s \in \mathbf{R}^{p, n_s}$, $C_f \in \mathbf{R}^{p, n_s}$, $A_s \in \mathbf{R}^{n_s, n_s}$ such that the transfer functions of (7.1) and (7.10) coincide:

$$C(sE - A)^{-1}B = C_s(sI - A_s)^{-1}B_s + C_f B_f.$$

This can be seen as follows: Without restriction of generality, one may assume that (7.1) is in Weierstraß form (see Proposition 7.1) and given by equations in (7.5). Then its transfer function satisfies

$$\begin{aligned} C(sE - A)^{-1}B &= C_f(sN - I_{n_f})^{-1}B_f + C_s(sI_{n_s} - A_s)^{-1}B_s \\ &= - \sum_{i=0}^{\nu-1} C_f N^i B_f s^i + \sum_{i \geq 1} C_s A_s^{i-1} B_s s^{-i} \end{aligned}$$

and the assumption $\lim_{s \rightarrow \infty} C(sE - A)^{-1}B \in \mathbf{R}^{m,m}$ yields $C_f N^i B_f = 0$ for all $i \geq 1$. Now invoking (7.6) and setting $z = x_s$ in (7.10) proves the claim.

7.1.4 Control Objectives

We consider two control strategies with respect to the two system classes: for DAE systems with positive strict relative degree, a constant proportional derivative output feedback is suggested; for DAE systems with proper inverse transfer function, a constant proportional output feedback is considered. These controllers achieve high-gain stabilization and exploit the underlying system properties. However, since no information on the system entries will be required it is unknown how “high” the feedback gain has to be chosen. To

resolve this, we allow for funnel control, that is, a proportional time-varying high-gain error feedback in combination with a filter (the filter “adjusts” the higher relative degree) for DAE systems with positive strict relative degree, and a proportional time-varying high-gain error feedback for DAE systems with proper inverse transfer function.

Constant High-Gain Control

First, constant high-gain proportional output feedback is given by

$$u(t) = -k \cdot p\left(\frac{d}{dt}\right)y(t) \quad \begin{array}{l} \text{for } [E, A, B, C] \in \Sigma_{n,m} \\ \text{with positive strict relative degree,} \end{array} \quad (7.11)$$

$$u(t) = k \cdot y(t) \quad \begin{array}{l} \text{for } [E, A, B, C] \in \Sigma_{n,m} \\ \text{with proper inverse transfer function,} \end{array} \quad (7.12)$$

where $k > 0$ and $p(s) \in \mathbf{R}[s]$ is Hurwitz in (7.11), and $k \in \mathbf{R}$ in (7.12). However, these results are of more theoretical nature to show the consequences of the underlying system properties such as asymptotically stable zero dynamics.

The control objective is output feedback regulation in the sense that the *high-gain controller*, i.e., the time-invariant proportional output-derivative feedback (7.11) or (7.12) respectively, yields an asymptotically stable closed-loop system. Here k is the high-gain parameter and, in general, k has to be large to achieve that the state $x(t)$ decays to zero.

In Section 7.4 we will show that the assumption of asymptotically stable zero dynamics of a system (7.1) which has either positive strict relative degree or proper inverse transfer function implies high-gain stabilizability in the following sense: If the system has positive strict relative degree, $k \gg 1$, and $p(s)$ is Hurwitz with positive leading coefficient, then the high-gain feedback (7.11) or (7.12), respectively, applied to (7.1) forces the output “very quickly” to zero. If the system has proper inverse transfer function, then it is sufficient to assume $|k| \gg 1$ and $p(s) \equiv 1$ to achieve this goal.

Funnel Control

The *control objective* is output feedback regulation in the sense that the *funnel controller*, applied to any system $[E, A, B, C] \in \Sigma_{n,m}$, achieves tracking of the output of any reference signal $y_{\text{ref}} \in \mathcal{B}^{\nu+1}(\mathbf{R}_{\geq 0}; \mathbf{R}^m)$, where ν is the index of the $sE - A$, with prespecified transient behavior:

$$\forall t > 0: \|e(t)\| < 1/\varphi(t).$$

The transient behavior is prespecified by a funnel boundary $1/\varphi(\cdot)$ given by

$$\varphi \in \Phi^\mu := \left\{ \varphi \in \mathcal{B}^\mu(\mathbf{R}_{\geq 0}, \mathbf{R}) \mid \varphi(0) = 0, \varphi(s) > 0 \forall s > 0 \text{ and } \liminf_{s \rightarrow \infty} \varphi(s) > 0 \right\},$$

with which we associate, see Figure 7.1, the *performance funnel*

$$\mathcal{F}_\varphi := \left\{ (t, e) \in \mathbf{R}_{\geq 0} \times \mathbf{R}^m \mid \varphi(t)\|e\| < 1 \right\}. \quad (7.13)$$

Note that there is no need that the funnel be monotone. However, the funnel is not allowed to “shrink” to zero as $t \rightarrow \infty$. But the minimal distance (at infinity) between

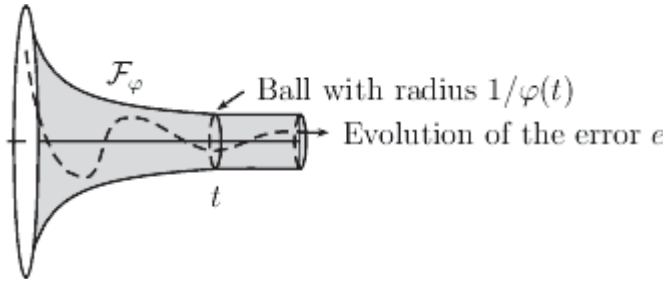


Figure 7.1. Error evolution in the funnel \mathcal{F}_φ with “width ∞ ” at $t = 0$, i.e., $\varphi(0) = 0$.

the funnel boundary and zero, i.e., $\lambda = \liminf_{s \rightarrow \infty} \varphi(s)^{-1}$, must be positive albeit can be arbitrarily small; from a practical point of view, this is not a limitation.

The funnel controller takes two forms. For DAE systems $[E, A, B, C] \in \Sigma_{n,m}$ with positive strict relative degree the higher degree is an obstacle; in (7.11) we have used derivative feedback while now we will incorporate a filter. This idea goes back to ODEs; it is shown in [11] that funnel control is feasible if a filter is incorporated in the feedback. This filter is constructed as follows;

$$\dot{z}(t) = \underbrace{\begin{bmatrix} -I_m & I_m & 0 & \cdots & 0 & 0 \\ 0 & -I_m & I_m & \cdots & 0 & 0 \\ 0 & 0 & -I_m & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -I_m & I_m \\ 0 & 0 & 0 & \cdots & 0 & -I_m \end{bmatrix}}_{=:F_\rho} z(t) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix}}_{=:G_\rho} u(t), \quad z(0) = z^0 \quad (7.14)$$

with initial data $z^0 \in \mathbf{R}^{(\rho-1)m}$. The feedback law is defined recursively by the C^∞ -functions

$$\begin{aligned} \gamma_1 : \quad & \mathbf{R} \times \mathbf{R}^m \rightarrow \mathbf{R}^m, \\ & (k, e) \mapsto k e, \\ \gamma_2 : \quad & \mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^m \rightarrow \mathbf{R}^m, \\ & (k, e, z_1) \mapsto \gamma_1(k, e) + \|D\gamma_1(k, e)\|^2 k^4 (1 + \|z_1\|^2)(z_1 + \gamma_1(k, e)) \end{aligned}$$

and, for $i = 3, \dots, \rho$,

$$\begin{aligned} \gamma_i : \quad & \mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^{(i-1)m} \rightarrow \mathbf{R}^m, \quad (k, e, (z_1, \dots, z_{i-1})) \\ \mapsto & \gamma_{i-1}(k, e, (z_1, \dots, z_{i-2})) + \|D\gamma_{i-1}(k, e, (z_1, \dots, z_{i-2}))\|^2 k^4 (1 + \|(k, e, (z_1, \dots, z_{i-1}))\|^2) \\ & \times (z_{i-1} + \gamma_{i-1}(k, e, (z_1, \dots, z_{i-2}))), \end{aligned}$$

where D denotes the derivative (Jacobian matrix). For a lengthy discussion of the intuition for the filter see [11]. Now the funnel controller (with filter (7.14)) for systems

$[E, A, B, C] \in \Sigma_{n,m}$ with positive strict relative degree $\rho \in \mathbf{N}$ takes the form

$$\begin{cases} u(t) &= -\gamma_\rho(k(t), e(t), z(t)), & e(t) &= y(t) - y_{\text{ref}}(t), \\ \dot{z}(t) &= F_\rho z(t) + G_\rho u(t), & k(t) &= \frac{1}{1 - \varphi(t)^2 \|e(t)\|^2}. \end{cases} \quad (7.15)$$

The funnel controller for systems $[E, A, B, C] \in \Sigma_{n,m}$ with proper inverse transfer function is much simpler: For appropriately chosen gain parameter $\hat{k} \in \mathbf{R} \setminus \{0\}$ it is

$$\begin{cases} u(t) &= -k(t)e(t), & e(t) &= y(t) - y_{\text{ref}}(t), \\ k(t) &= \frac{\hat{k}}{1 - \varphi(t)^2 \|e(t)\|^2}. \end{cases} \quad (7.16)$$

In Section 7.5 we will show that the assumption of asymptotically stable zero dynamics of a system (7.1) which has either positive strict relative degree or proper inverse transfer function implies feasibility of the funnel controller. In view of the fact that such systems are high-gain stabilizable (see Section 7.4), intuitively we may believe that if $\|e(t)\|$ is close to the funnel boundary $\varphi(t)^{-1}$, then the high-gain $k(t)$ forces $\|e(t)\|$ away from the funnel boundary. This is the essential property to allow for funnel control of these systems: $k(t)$ is designed in such a way that it is large if the error $\|e(t)\|$ is close to the funnel boundary $\varphi(t)^{-1}$, hence avoiding contact.

The funnel controller is a high-gain adaptive controller which is not high-gain and not adaptive: Certainly, the gain $k(t)$ in (7.15) and (7.16) takes high values if $k(t)$ is “close” to the funnel boundary; but it is by no means monotone and exploits the high-gain property of the system class (i.e., asymptotically stable zero dynamics) only if necessary. The gain $k(t)$ is adapted by the output, more precisely by the distance of the norm of the output error to the funnel boundary.

7.1.5 Literature and Outline

This chapter is based on the following precursors: The Byrnes–Isidori form for ODE systems with strictly proper transfer function is derived in [12]; see also [13, Section 5.1]. The zero dynamics form for DAE systems with proper inverse transfer function is derived in [2]. The concept of funnel control is introduced in [10]; for a further discussion of funnel control see the survey article [9].

In this chapter we consider two classes of linear multi-input, multi-output DAE systems: systems with positive strict relative degree and systems with proper inverse transfer function; in the single-input single-output case, these two disjoint classes are the whole of all linear DAEs without feedthrough term. In Section 7.2, zero dynamics forms for these system classes are derived; they are the counterpart to the Byrnes–Isidori form for ODE systems. The normal forms may be interesting in their own right and are here essential for later proofs of funnel control. In Section 7.3, parameterization of the zero dynamics and characterization of asymptotically zero dynamics in terms of a matrix condition, the zero dynamics form, and the transfer function are derived. In Section 7.4, it is shown that a system with asymptotically stable zero dynamics is high-gain stabilizable with constant gain (in case of proper inverse transfer function) and constant gain in combination with a filter (in case of positive strict relative degree of the transfer function). In Section 7.5,

we are ready to present two funnel controllers for the two system classes. These funnel controllers achieve tracking of a reference signal (belonging to a rather large class) within a prespecified funnel; this means in particular that transient behavior of the output error can be prespecified and the funnel controller does not incorporate any internal model but is simple in its design. These results are illustrated in Section 7.6 by the position control and velocity control of a mechanical system consisting of springs, masses, and dampers. The proofs are delegated to Section 7.7.

7.2 Normal Forms

The notion “normal form” means that it is weaker than “canonical form”: Recall that if a set (of systems) may be divided into equivalence classes and if each orbit (i.e., equivalence class) has a unique representative, then the mapping from the set to each representative within the orbit is called a *canonical form*. The Byrnes–Isidori form for ODE state-space systems with strictly proper transfer function (see [9] and also [13, Section 5.1]) is not a canonical form. However, the freedom left within the nonzero entries of the Byrnes–Isidori form can be specified and is not significant; this justifies the notion “normal form.”

In the present section, certain normal forms, the so-called zero dynamics forms, are studied for two classes of DAE systems with positive strict relative degree and with proper inverse transfer function. They are interesting in their own right since they give insight into the properties of the systems class; we will use them to derive results on zero dynamics and on high-gain output feedback and funnel control.

We first present a zero dynamics form (7.17) for descriptor systems with positive strict relative degree; see Section 7.1.3.

Theorem 7.6 (Zero dynamics form for systems with positive strict relative degree). *Suppose $[E, A, B, C] \in \Sigma_{n,m}$ has strict relative degree $\rho \in \mathbf{N}$. Then there exist $W, T \in \mathbf{GL}_n(\mathbf{R})$ such that*

$$[E, A, B, C] \stackrel{W, T}{\sim} [\hat{E}, \hat{A}, \hat{B}, \hat{C}],$$

with

$$s\hat{E} - \hat{A} = \left[\begin{array}{cccccc|cc} sI_m & -I_m & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & sI_m & -I_m & 0 & & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & sI_m & -I_m & 0 & 0 & 0 \\ -R_1 & -R_2 & \cdots & -R_{\rho-1} & sI_m - R_\rho & -S & 0 & 0 \\ -P & 0 & \cdots & 0 & 0 & sI_\mu - Q & 0 & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & sN_c - I_{n_c} & sN_{c\bar{c}} \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & sN_{\bar{c}} - I_{n_{\bar{c}}} \end{array} \right],$$

$$\hat{B} = [0 \quad 0 \quad \cdots \quad 0 \quad \Gamma^\top \quad 0 \mid B_c^\top \quad 0]^\top,$$

$$\hat{C} = [I_m \quad 0 \quad \cdots \quad 0 \quad 0 \quad 0 \mid 0 \quad C_{\bar{c}}],$$

(7.17)

where, for some $n_c, n_{\bar{c}} \in \mathbf{N}_0$, and $\mu = n - n_c - n_{\bar{c}} - \rho m$,

$$\begin{aligned} \Gamma &= \lim_{s \rightarrow \infty} s^\rho C(sE - A)^{-1} B \in \mathbf{G}\mathbf{I}_m(\mathbf{R}) \text{ is the high frequency gain matrix,} \\ S &\in \mathbf{R}^{m, \mu}, P \in \mathbf{R}^{\mu, m}, Q \in \mathbf{R}^{\mu, \mu}, [R_1, \dots, R_\rho] \in \mathbf{R}^{m, \rho m}, \\ B_c &\in \mathbf{R}^{n_c, m}, C_{\bar{c}} \in \mathbf{R}^{m, n_{\bar{c}}}, N_{c\bar{c}} \in \mathbf{R}^{n_c, n_{\bar{c}}}, \\ N_c &\in \mathbf{R}^{n_c, n_c}, N_{\bar{c}} \in \mathbf{R}^{n_{\bar{c}}, n_{\bar{c}}} \text{ are nilpotent, and } \text{rank}[N_c, B_c] = n_c. \end{aligned} \tag{7.18}$$

The entries R_1, \dots, R_ρ are unique, system $[I, Q, P, S]$ is unique up to system equivalence $\hat{T}^{-1} \hat{T}$, and the matrices N_c and $N_{\bar{c}}$ are unique up to similarity. If E is invertible, then $n_c = n_{\bar{c}} = 0$; this means only the upper left block in \hat{E} is present. The form (7.17) diagrammed in Figure 7.2 is called zero dynamics form of (7.1). The transfer function satisfies, where $R_{\rho+1} := -I$,

$$C(sE - A)^{-1} B = - \left[\sum_{i=1}^{\rho+1} R_i s^{i-1} + S(sI_{n-n_c-n_{\bar{c}}-\rho m} - Q)^{-1} P \right]^{-1} \Gamma. \tag{7.19}$$

The proof is given in Section 7.7.

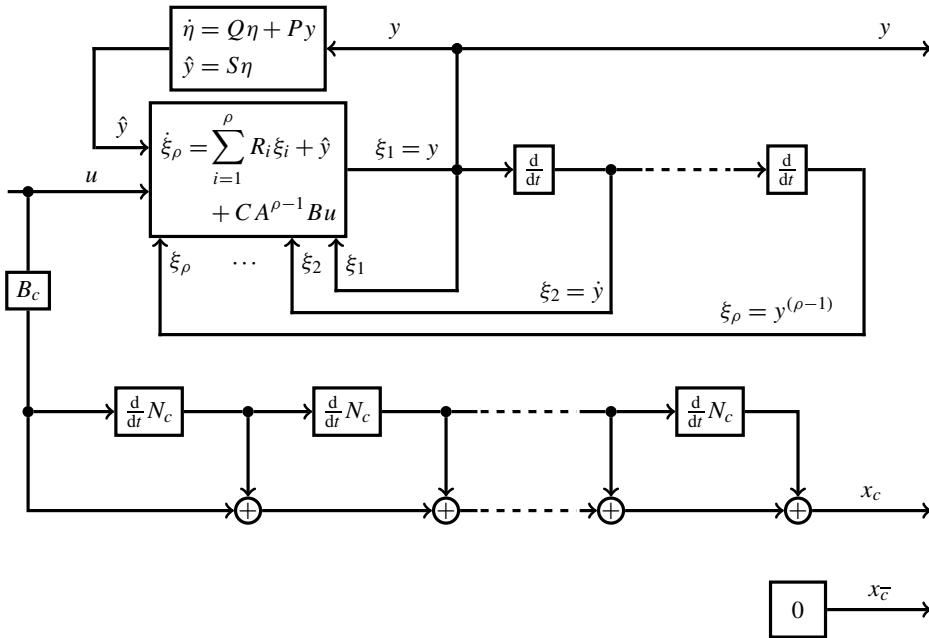


Figure 7.2. Zero dynamics form for systems with positive strict relative degree.

Remark 7.7 (Zero dynamics form for DAEs). An immediate consequence of Theorem 7.6 is the simplified representation of system $[E, A, B, C] \in \Sigma_{n,m}$: If v denotes the index of $sE - A$, then a trajectory satisfies

$$(x, u, y) \in \mathfrak{B}_{(7.1)} \cap \left(\mathcal{C}^1(\mathbf{R}; \mathbf{R}^n) \times \mathcal{C}^{v-1}(\mathbf{R}; \mathbf{R}^m) \times \mathcal{C}^\rho(\mathbf{R}; \mathbf{R}^m) \right)$$

if and only if $Tx = (y^\top, \dot{y}^\top, \dots, y^{(\rho-1)\top}, \eta^\top, x_c^\top, x_{\bar{c}}^\top)^\top \in \mathcal{C}^1(\mathbf{R}; \mathbf{R}^n)$ fulfills

$$\begin{aligned}
 y^{(\rho)}(t) &= \sum_{i=1}^{\rho} R_i y^{(i-1)}(t) + S\eta(t) + \Gamma u(t), \\
 \dot{\eta}(t) &= P y(t) + Q \eta(t), \\
 x_c(t) &= -\sum_{i=0}^{v-1} N_c^i B_c u^{(i)}(t), \\
 x_{\bar{c}}(t) &= 0.
 \end{aligned} \tag{7.20}$$

The second zero dynamics form is for systems $[E, A, B, C] \in \Sigma_{n,m}$, where the transfer function $C(sE - A)^{-1}B$ has proper inverse; see Section 7.1.3. This is the counterpart to (7.17).

Theorem 7.8 (Zero dynamics form for systems with proper inverse transfer function [2, Theorem 2.3]). *Suppose $[E, A, B, C] \in \Sigma_{n,m}$ is such that $C(sE - A)^{-1}B$ has proper inverse. Then there exist $W, T \in \mathbf{GL}_n(\mathbf{R})$ such that*

$$[E, A, B, C] \stackrel{W, T}{\sim} [\hat{E}, \hat{A}, \hat{B}, \hat{C}],$$

with

$$s\hat{E} - \hat{A} = \begin{bmatrix} -A_{11} & -A_{12} & 0 & sE_{\bar{c}} \\ -A_{21} & sI_{n_2} - Q & 0 & 0 \\ sE_c & 0 & sN_c - I_{n_c} & sN_{c\bar{c}} \\ 0 & 0 & 0 & sN_{\bar{c}} - I_{n_{\bar{c}}} \end{bmatrix}, \quad \hat{B} = \hat{C}^\top = \begin{bmatrix} I_m \\ 0_{n_2, m} \\ 0_{n_c, m} \\ 0_{n_{\bar{c}}, m} \end{bmatrix}, \tag{7.21}$$

where, for some $n_2, n_c, n_{\bar{c}} \in \mathbf{N}_0$,

$$\begin{aligned}
 A_{11} &\in \mathbf{R}^{m, m}, \quad A_{12} \in \mathbf{R}^{m, n_2}, \quad A_{21} \in \mathbf{R}^{n_2, m}, \quad Q \in \mathbf{R}^{n_2, n_2}, \\
 E_c &\in \mathbf{R}^{n_c, m}, \quad E_{\bar{c}} \in \mathbf{R}^{m, n_{\bar{c}}}, \quad N_{c\bar{c}} \in \mathbf{R}^{n_c, n_{\bar{c}}}, \\
 N_c &\in \mathbf{R}^{n_c, n_c}, \quad N_{\bar{c}} \in \mathbf{R}^{n_{\bar{c}}, n_{\bar{c}}} \text{ are nilpotent, and } \text{rank}[N_c, E_c] = n_c.
 \end{aligned} \tag{7.22}$$

The form (7.21) diagrammed in Figure 7.3 is called zero dynamics form of (7.1). Furthermore, the following hold:

- (i) $N_c^v = 0$ and $N_{\bar{c}}^v = 0$, where v denotes the index of the pencil $sE - A$.
- (ii) The transfer function satisfies

$$C(sE - A)^{-1}B = -\left(A_{11} + A_{12}(sI_{n_2} - Q)^{-1}A_{21}\right)^{-1}.$$

- (iii) $\text{sr deg}(C(sE - A)^{-1}B) = 0$ if and only if $A_{11} \in \mathbf{GL}_m(\mathbf{R})$.

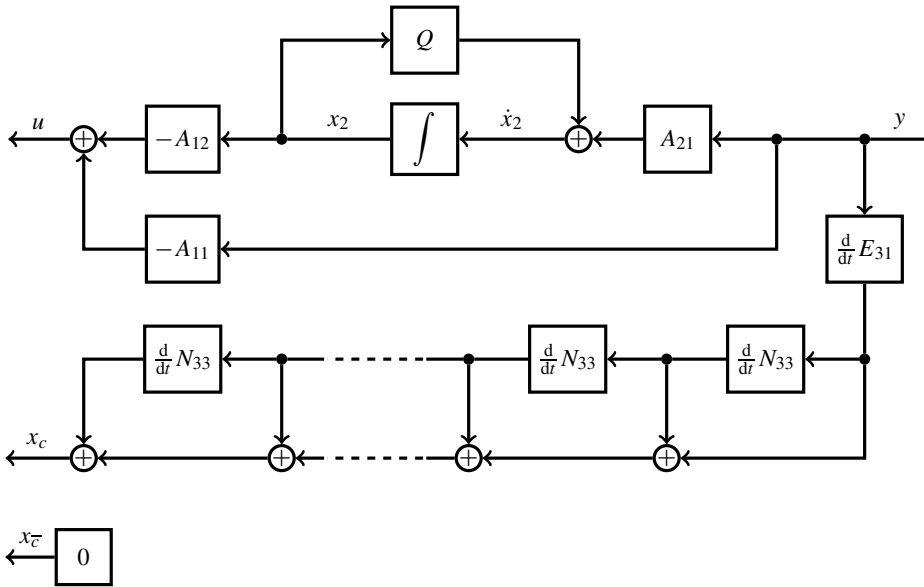


Figure 7.3. Zero dynamics form for systems with proper inverse transfer function.

- (iv) $\text{sr deg}(C(sE - A)^{-1}B) = \rho < 0$ if and only if $A_{11} = 0$ and $\text{sr deg}(A_{12}(sI_{n_2} - Q)^{-1}A_{21}) = -\rho$.

For uniqueness we have the following:

- (v) If $[E, A, B, C]$, $[\hat{E}, \hat{A}, \hat{B}, \hat{C}] \in \Sigma_{n,m,m}$ are both in zero dynamics form (7.21) and

$$[\hat{E}, \hat{A}, \hat{B}, \hat{C}] \stackrel{W,T}{\sim} [E, A, B, C] \quad \text{for some } W, T \in \mathbf{GL}_n(\mathbf{R}), \quad (7.23)$$

then there exist $W_{ii} \in \mathbf{GL}_{n_i}(\mathbf{R})$, $i \in \{2, c, \bar{c}\}$, $W_{c\bar{c}} \in \mathbf{R}^{n_c \times n_{\bar{c}}}$ such that

$$W = T^{-1} = \begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & W_{22} & 0 & 0 \\ 0 & 0 & W_{cc} & W_{c\bar{c}} \\ 0 & 0 & 0 & W_{\bar{c}\bar{c}} \end{bmatrix}. \quad (7.24)$$

- (vi) The dimensions $n_2, n_c, n_{\bar{c}}$ are unique; the matrices $N_c, N_{\bar{c}}, Q$ are unique up to similarity, so in particular the spectrum of Q is unique; and $A_{11} = -\lim_{s \rightarrow \infty} (C(sE - A)^{-1}B)^{-1}$.

The proof is given in [2, Theorem 2.3].

Remark 7.9 (Zero dynamics form for DAE). An immediate consequence of Theorem 7.8 is the representation as a DAE: If v is the index of $sE - A$, then a trajectory satisfies

$$(x, u, y) \in \mathfrak{B}_{(7.1)} \cap \left(\mathcal{C}^1(\mathbf{R}; \mathbf{R}^n) \times \mathcal{C}^1(\mathbf{R}; \mathbf{R}^m) \times \mathcal{C}^v(\mathbf{R}; \mathbf{R}^m) \right)$$

if and only if $Tx = (y, x_2^\top, x_c^\top, x_{\bar{c}}^\top)^\top \in \mathcal{C}^1(\mathbf{R}; \mathbf{R}^n)$ fulfills

$$\begin{cases} 0 &= A_{11}y(t) + A_{12}x_2(t) + u(t), \\ \dot{x}_2(t) &= A_{21}y(t) + Qx_2(t), \\ x_c(t) &= \sum_{i=0}^{v-1} N_c^i E_c y^{(i+1)}(t), \\ x_{\bar{c}}(t) &= 0. \end{cases} \tag{7.25}$$

Remark 7.10 (ODE systems with feedthrough). For ODE systems with feedthrough, i.e., (7.8) with $E = I$, and strict relative degree zero, we have $D \in \mathbf{GL}_m(\mathbf{R})$. In this case, by using the transcription of (7.8) as a DAE system (7.9), the zero dynamics form (7.25) can be achieved without changing the coordinates of the state, namely

$$\begin{cases} 0 &= -D^{-1}y(t) + D^{-1}Cx(t) + u(t), \\ \dot{x}(t) &= BD^{-1}y(t) + (A - BD^{-1}C)x(t). \end{cases} \tag{7.26}$$

In the notation of (7.25), the quantities n_c and $n_{\bar{c}}$ are then trivial, while $n_2 = n$ and $A_{11} = -D^{-1}$, $A_{12} = D^{-1}C$, $A_{21} = A - BD^{-1}C$, and $Q = A - BD^{-1}C$.

7.3 Zero Dynamics

Before we characterize asymptotically stable zero dynamics, we show that both zero dynamics forms allow us to parameterize the zero dynamics.

Remark 7.11 (Parameterization of zero dynamics).

- (i) Suppose that $[E, A, B, C] \in \Sigma_{n,m}$ has positive strict relative degree and let v be the index of $sE - A$. Then the zero dynamics are given, in terms of the matrices in (7.17), by

$$\mathcal{ZD}_{(7.1)} = \left\{ \left(\left(\begin{array}{c} 0_{\rho m} \\ e^{Q \cdot} \eta^0 \\ \sum_{i=0}^{v-1} N_c^i B_c \Gamma^{-1} S Q^i e^{Q \cdot} \eta^0 \\ 0_{n_{\bar{c}}} \end{array} \right), -\Gamma^{-1} S e^{Q \cdot} \eta^0, 0 \right) \middle| \eta^0 \in \mathbf{R}^\mu \right\}. \tag{7.27}$$

- (ii) Suppose that $[E, A, B, C] \in \Sigma_{n,m}$ has proper inverse transfer function. Then the zero dynamics are given, in terms of (7.21), by

$$\mathcal{ZD}_{(7.1)} = \left\{ \left(\left(\begin{array}{c} 0_m \\ e^{Q \cdot} x_2^0 \\ 0_{n_c} \\ 0_{n_{\bar{c}}} \end{array} \right), -A_{12} e^{Q \cdot} x_2^0, 0 \right) \middle| x_2^0 \in \mathbf{R}^{n_2} \right\}. \tag{7.28}$$

We are now in a position to characterize asymptotically stable zero dynamics of systems $[E, A, B, C] \in \Sigma_{n,m}$ with positive strict relative degree or proper inverse transfer function.

Theorem 7.12 (Asymptotically stable zero dynamics). *Consider a system $[E, A, B, C] \in \Sigma_{n,m}$ and suppose it has positive strict relative degree or proper inverse transfer function. Then the following statements are, in terms of the system properties defined in Section 7.1.2, equivalent:*

(i) *the zero dynamics of system (7.1) are asymptotically stable ;*

(ii)

$$\forall s \in \overline{\mathbf{C}}_+ : \det \begin{bmatrix} sE - A & B \\ C & 0 \end{bmatrix} \neq 0;$$

(iii) *system (7.1) is minimum phase, i.e.,*

(a) *(7.1) is stabilizable,*

(b) *(7.1) is detectable,*

(c) *$C(sE - A)^{-1}B$ does not have any transmission zeros in $\overline{\mathbf{C}}_+$;*

(iv) *$\sigma(Q) \subseteq \mathbf{C}_-$, where Q is in the zero dynamics form (7.17) or (7.21), respectively.*

The proof is given in Section 7.7.

Condition (ii) in Theorem 7.12 is a convenient mathematical frequency domain characterization which leads to condition (iv); the latter is the important property to show constant high-gain stabilizability in Section 7.4. Condition (iii) is the traditional definition of a minimum phase system. Conditions (ii) and (iii) are very similar to what is known for asymptotically stable zero dynamics of ODE systems; see [13, Section 4.3].

The following two remarks, although not exploited in the next section, may be worth knowing in their own right.

Remark 7.13 (Zero dynamics and behavior).

(i) *Suppose that $[E, A, B, C] \in \Sigma_{n,m}$ has positive strict relative degree. Then the behavior $\mathfrak{B}_{(7.1)}$ can be decomposed, in terms of the transformation matrix T from Theorem 7.6, into a direct sum of the zero dynamics and a summand as*

$$\mathfrak{B}_{(7.1)} = \mathcal{Z}\mathcal{D}_{(7.1)} \oplus \mathcal{R},$$

where

$$\mathcal{R} := \left\{ \begin{array}{l|l} (x, u, y) \in \mathcal{C}^1(\mathbf{R}; \mathbf{R}^n) & (x, u, y) \text{ solves (7.1) and} \\ \times \mathcal{C}(\mathbf{R}; \mathbf{R}^m) \times \mathcal{C}(\mathbf{R}; \mathbf{R}^m) & [0_{\mu, \rho m}, I_{\mu}, 0_{\mu, n_c}, 0_{\mu, n_{\bar{c}}}]Tx(0) = 0 \end{array} \right\}.$$

In terms of (7.27), the representation is immediate from

$$\begin{aligned} (x(\cdot), u(\cdot), y(\cdot)) &= \left(T^{-1} \begin{pmatrix} 0_{\rho m, \rho m} \\ e^{Q \cdot} \eta^0 \\ \sum_{i=0}^{v-1} N_c^i B_c \Gamma^{-1} S Q^i e^{Q \cdot} \eta^0 \\ 0_{n_{\bar{c}}, n_{\bar{c}}} \end{pmatrix}, -\Gamma^{-1} S e^{Q \cdot} \eta^0, 0 \right) \\ &+ \left(x(\cdot) - T^{-1} \begin{pmatrix} 0_{\rho m, \rho m} \\ e^{Q \cdot} \eta^0 \\ \sum_{i=0}^{v-1} N_c^i B_c \Gamma^{-1} S Q^i e^{Q \cdot} \eta^0 \\ 0_{n_{\bar{c}}, n_{\bar{c}}} \end{pmatrix}, u(\cdot) + \Gamma^{-1} S e^{Q \cdot} \eta^0, y(\cdot) \right) \end{aligned}$$

for any $(x, u, y) \in \mathfrak{B}_{(7.1)}$, where $\eta^0 = [0_{\mu, \rho m}, I_{\mu}, 0_{\mu, n_c}, 0_{\mu, n_{\bar{c}}}] T x(0)$.

- (ii) Suppose that $[E, A, B, C] \in \Sigma_{n, m}$ has proper inverse transfer function. Then the behavior $\mathfrak{B}_{(7.1)}$ can be decomposed, in terms of the transformation matrix T from Theorem 7.8, into a direct sum of the zero dynamics and a summand as

$$\mathfrak{B}_{(7.1)} = \mathcal{ZD}_{(7.1)} \oplus \mathcal{R},$$

where

$$\mathcal{R} := \left\{ \begin{array}{l|l} (x, u, y) \in \mathcal{C}^1(\mathbf{R}; \mathbf{R}^n) & (x, u, y) \text{ solves (7.1) and} \\ \times \mathcal{C}(\mathbf{R}; \mathbf{R}^m) \times \mathcal{C}(\mathbf{R}; \mathbf{R}^m) & [0_{n_2, m}, I_{n_2}, 0_{n_2, n_c}, 0_{n_2, n_{\bar{c}}}] T x(0) = 0 \end{array} \right\}.$$

This fact is shown in [2, Remark 3.3].

Finally, we show that the zero dynamics carries, in a certain sense, the structure of a dynamical system.

Remark 7.14 (Zero dynamics are a dynamical system). Let v be the index of the pencil $sE - A$. The transition map of system (7.1) is defined, in terms of Proposition 7.1, as

$$\begin{aligned} \varphi : \mathbf{R} \times \mathbf{R} \times \mathbf{R}^n \times \mathcal{C}^{v-1}(\mathbf{R}; \mathbf{R}^m) &\rightarrow \mathbf{R}^n \\ (t, t_0, x^0, u(\cdot)) &\mapsto T^{-1} \begin{bmatrix} e^{A_s(t-t_0)} & 0 \\ 0 & 0 \end{bmatrix} T x^0 \\ &+ \int_{t_0}^t T^{-1} \begin{bmatrix} e^{A_s(t-\tau)} & 0 \\ 0 & 0 \end{bmatrix} W^{-1} B u(\tau) d\tau \\ &- \sum_{k=0}^{v-1} T^{-1} \begin{bmatrix} 0 & 0 \\ 0 & N^k \end{bmatrix} W^{-1} B u^{(k)}(t). \end{aligned}$$

It has been shown in [3, Proposition 2.20] that for any $(t_0, x^0, u) \in \mathbf{R} \times \mathbf{R}^n \times \mathcal{C}^{v-1}(\mathbf{R}; \mathbf{R}^m)$ the map $t \mapsto x(t) := \varphi(t, t_0, x^0, u(\cdot))$ solves the IVP

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x^0 \tag{7.29}$$

if and only if

$$x^0 \in \mathcal{V}_{t_0, u(\cdot)} := \left\{ x \in \mathbf{R}^n \mid x + \sum_{i=0}^{v-1} T^{-1} \begin{bmatrix} 0 & 0 \\ 0 & N^i \end{bmatrix} W^{-1} Bu^{(i)}(t_0) \in \text{im } T^{-1} \begin{bmatrix} I_{n_s} \\ 0 \end{bmatrix} \right\}.$$

Therefore, consistency of the initial value x^0 depends on the initial time t_0 and the input $u(\cdot)$. The output map of system (7.1) is defined by

$$\eta : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^m, \quad (t, x, u) \mapsto Cx.$$

It is readily verified that the structure $(\mathbf{R}, \mathbf{R}^m, \mathcal{C}^{v-1}(\mathbf{R}; \mathbf{R}^m), \mathbf{R}^n, \mathbf{R}^m, \varphi, \eta)$, where $\varphi : \mathcal{D}_\varphi \rightarrow \mathbf{R}^n$ is the restriction of the transition map (by abuse of notation we write the same symbol) on

$$\mathcal{D}_\varphi := \left\{ (t, t_0, x^0, u) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^n \times \mathcal{C}^{v-1}(\mathbf{R}; \mathbf{R}^m) \mid \begin{array}{l} x^0 \in \mathcal{V}_{t_0, u(\cdot)}, \\ C\varphi(\cdot; t_0, x^0, u(\cdot)) = 0 \end{array} \right\},$$

is an \mathbf{R} -linear time-invariant dynamical system as defined in [7, Definitions 2.1.1, 2.1.24, 2.1.26].

Next let $u \in \mathcal{C}^{v-1}(\mathbf{R}; \mathbf{R}^m)$. As a consequence of uniqueness and global existence of the solution of the IVP (7.29) for $x^0 \in \mathcal{V}_{0, u(\cdot)}$ (see again [3, Proposition 2.20]), the map

$$\Psi : \mathcal{D}_{\varphi, 0} \rightarrow \mathcal{Z}\mathcal{D}_{(7.1)}, \quad (0, 0, x^0, u(\cdot)) \mapsto (\varphi(\cdot; 0, x^0, u(\cdot)), u(\cdot), C\varphi(\cdot; 0, x^0, u(\cdot)))$$

is well defined, where

$$\mathcal{D}_{\varphi, 0} := \left\{ (0, 0, x^0, u) \in \mathcal{D}_\varphi \right\} \subset \mathcal{D}_\varphi.$$

Most importantly, if $[E, A, B, C]$ has positive strict relative degree or proper inverse transfer function, then Ψ is an isomorphism: it is surjective since a preimage of $(x, u, 0) \in \mathcal{Z}\mathcal{D}_{(7.1)}$ is $(0, 0, x(0), u(\cdot)) \in \mathcal{D}_{\varphi, 0}$ (note that $u \in \mathcal{C}^{v-1}(\mathbf{R}; \mathbf{R}^m)$ by Remark 7.11), it is injective by uniqueness of the solution of the IVP (7.29). In this sense, we may say that $\mathcal{Z}\mathcal{D}_{(7.1)}$ is a dynamical system.

7.4 Stabilization by Constant High-Gain Output Feedback

Next we investigate whether asymptotically stable zero dynamics is a sufficient (or even necessary) condition for high-gain stabilizability in the sense that the feedback (7.11) or (7.12), respectively, applied to (7.1) yields an asymptotically stable closed-loop system.

Theorem 7.15 (High-gain stabilizability). Consider a system $[E, A, B, C] \in \Sigma_{n,m}$. Then the following statements hold:

- (i) Suppose $[E, A, B, C]$ has strict relative degree $\rho \in \mathbf{N}$ and positive definite high frequency gain matrix $\Gamma \in \mathbf{R}^{m,m}$ (cf. (7.18)). Let $p(s) = \sum_{i=0}^{\rho-1} p_i s^i \in \mathbf{R}[s]$ be Hurwitz

and $p_{\rho-1} > 0$. Then

$$\mathcal{ZD}_{(7.1)} \text{ is asympt. stable} \begin{array}{l} \implies \\ \not\leftarrow \\ \text{i.g.} \end{array} \left\{ \begin{array}{l} \exists k^* \geq 0 \forall k \geq k^* : \\ \text{"(7.1) \& (7.11)" is asympt. stable.} \end{array} \right.$$

(ii) Suppose $[E, A, B, C]$ has proper inverse transfer function. Then

$$\mathcal{ZD}_{(7.1)} \text{ is asympt. stable} \begin{array}{l} \implies \\ \not\leftarrow \\ \text{i.g.} \end{array} \left\{ \begin{array}{l} \exists k^* \geq 0 \forall k \in \mathbf{R}, |k| \geq k^* \forall \text{sln. } x(\cdot) \text{ of} \\ \text{"(7.1) \& (7.12)": } \lim_{t \rightarrow \infty} x(t) = 0. \end{array} \right.$$

The proof is given in Section 7.7.

Remark 7.16 (High-gain stabilizability).

(i) In case of strict relative degree one, the feedback law (7.11) reduces to the proportional output feedback $u(t) = -k y(t)$. If the system has higher relative degree, (7.11) incorporates a compensator $p(s)$ (and thus derivative feedback) to achieve a relative degree one system.

For ODE systems, the result is proven in [4] for relative degree one. By using the form (7.17), this result can be generalized to differential-algebraic systems with positive strict relative degree by using the same techniques.

(ii) Consider $[E, A, B, C]$ with proper inverse transfer function. It may be surprising that the sign of k in (7.12) is irrelevant and, furthermore, no compensator is required.

In view of Remark 7.9, the closed-loop system " $u(t) = ky(t)$ & (7.1)" is equivalent to

$$\begin{aligned} -(A_{11} + kI_m)y(t) &= A_{12}x_2(t), \\ \dot{x}_2(t) &= Qx_2(t) + A_{21}y(t), \\ x_c(t) &= \sum_{i=0}^{v-1} N_c^i E_c y^{(i+1)}(t) \end{aligned}$$

and, if $|k| > \|A_{11}\|$, equivalent to

$$\begin{aligned} y(t) &= -(A_{11} + kI_m)^{-1} A_{12}x_2(t), \\ \dot{x}_2(t) &= [Q - A_{21}(A_{11} + kI_m)^{-1} A_{12}] x_2(t), \\ x_c(t) &= \sum_{i=0}^{v-1} N_c^i E_c y^{(i+1)}(t). \end{aligned} \tag{7.30}$$

Note that $x_2 \in \mathcal{C}^\infty(\mathbf{R}; \mathbf{R}^{n_2})$ yields $y \in \mathcal{C}^\infty(\mathbf{R}; \mathbf{R}^m)$ and so the algebraic equation for x_c is well defined. Note also that asymptotic stability of the zero dynamics is equivalent to $\sigma(Q) \subseteq \mathbf{C}_-$ by Theorem 7.12. Since

$$\lim_{k \rightarrow \pm\infty} \sigma \left(Q - A_{21}(A_{11} + kI_m)^{-1} A_{12} \right) = \sigma(Q),$$

the assumptions “ $|k|$ sufficiently large” and “(7.1) has asymptotically stable zero dynamics” yield exponential decay of $x_2(\cdot)$, and therefore $x_c(\cdot)$ and $y(\cdot)$ decay exponentially, too.

7.5 Funnel Control

In this section we will show that the two funnel controllers (7.15) and (7.16) achieve output tracking of a rather general reference signal within a prespecified funnel.

Theorem 7.17 (Funnel control). *Suppose that $[E, A, B, C] \in \Sigma_{n,m}$ has asymptotically stable zero dynamics, and let v be the index of $sE - A$. Let $\varphi \in \Phi^{v+1}$ define a performance funnel \mathcal{F}_φ . The initial data are an arbitrary consistent initial value $x^0 \in \mathbf{R}^n$ and a reference signal $y_{\text{ref}} \in \mathcal{B}^{v+1}(\mathbf{R}_{\geq 0}; \mathbf{R}^m)$.*

- (i) *If $[E, A, B, C]$ has positive strict relative degree $\rho \in \mathbf{N}$ and positive definite high frequency gain matrix $\Gamma = \lim_{s \rightarrow \infty} s^\rho C(sE - A)^{-1}B$, then the application of the funnel controller (7.15) to (7.1) yields a closed-loop IVP with precisely one maximal continuously differentiable solution $x : [0, \omega) \rightarrow \mathbf{R}^n$ and this solution is global (i.e., $\omega = \infty$), and all functions x, z, k, u are bounded.*
- (ii) *If $[E, A, B, C]$ has proper inverse transfer function and the initial gain $k(0) = \hat{k} \in \mathbf{R}$ satisfies $|\hat{k}| > \lim_{s \rightarrow \infty} \|G^{-1}(s)\|$, then the application of the funnel controller (7.16) to (7.1) yields a closed-loop IVP with precisely one maximal continuously differentiable solution $x : [0, \omega) \rightarrow \mathbf{R}^n$ and this solution is global (i.e., $\omega = \infty$), and all functions x, k, u are bounded.*

Most importantly, in both cases (i) and (ii), the tracking error $e(\cdot) = Cx(\cdot) - y_{\text{ref}}(\cdot)$ satisfies

$$\exists \varepsilon > 0 \forall t > 0 : \|e(t)\| \leq \varphi(t)^{-1} - \varepsilon \tag{7.31}$$

(that means $e(\cdot)$ evolves within the performance funnel \mathcal{F}_φ and is uniformly bounded away from the boundary) and for the same ε the gain is bounded by

$$\forall t_0 > 0 : \sup_{t \geq t_0} |k(t)| \leq \frac{|\hat{k}|}{1 - (1 - \varepsilon \lambda_{t_0})^2}, \tag{7.32}$$

where $\lambda_{t_0} := \inf_{t \geq t_0} \varphi(t) > 0$ for all $t_0 > 0$. $\hat{k} = 1$ in case of (i).

The proof is given in Section 7.7.

Remark 7.18.

- (i) *Minimum-phase systems (7.8), with transfer function $G(s) = C(sE - A)^{-1}B + D$ that has either a proper inverse or positive strict relative degree, also allow funnel control in the sense of Theorem 7.17. This follows from the artificial extension (7.9).*
- (ii) *Consistency of the initial value $x^0 \in \mathbf{R}^n$ means that the closed-loop system (7.1), (7.15) or (7.1), (7.16), respectively, with $x(0) = x^0$ has a solution $x : [0, \omega) \rightarrow \mathbf{R}^n$ for some $\omega \in (0, \infty]$. In practice, consistency of the initial state of the “unknown” system should be satisfied as far as the DAE $[E, A, B, C]$ is the correct model.*

- (iii) In case of Theorem 7.17(ii), the assumption on the initial gain $k(0) = \hat{k}$ reduces to $\hat{k} > 0$ if $\lim_{s \rightarrow \infty} G^{-1}(s) = 0$; the latter means that the transfer function cannot be realized by an ODE system with feedthrough; see [2, Remark 6.3].
- (iv) In case of ODE systems (7.8), $E = I$, with invertible feedthrough term, the transfer function $G(s) = C(sI - A)^{-1}B + D$ has proper inverse and the assumption on the initial gain $k(0) = \hat{k}$ is equivalent to $|\hat{k}| > \|D^{-1}\|$.
- (v) Theorem 7.17 is on system class $\Sigma_{n,m}$. A careful inspection of the proof of the theorem shows that one actually needs only that the nominal system can be brought into the form (7.20) or (7.25), respectively. Thus in particular, the pencil $sE - A$ does not have to be regular; what is important is that $\sigma(Q) \subseteq \mathbb{C}_-$.

7.6 Application: Position and Velocity Control of a Mechanical System with Springs, Masses, and Dampers

7.6.1 The Mechanical Model

For purposes of illustration, we consider a mechanical system, see Figure 7.4, with springs, masses, and dampers with single-input spatial distance between the two masses and single-output position of one mass. We are indebted to Professor P.C. Müller (BU Wuppertal) for suggesting this example to us. The masses m_1, m_2 , damping constants d_1, d_2 , and spring constants c_1, c_2 are all assumed to be positive. As output $y(t) = z_2(t)$ we take the position of the mass m_2 . In the first example, the input $u(t) = z_2(t) - z_1(t)$ is the spatial distance between the masses m_1 and m_2 , whereas, in the second example, the input is the relative velocity between the masses m_1 and m_2 , i.e., $u(t) = \dot{z}_2(t) - \dot{z}_1(t)$.

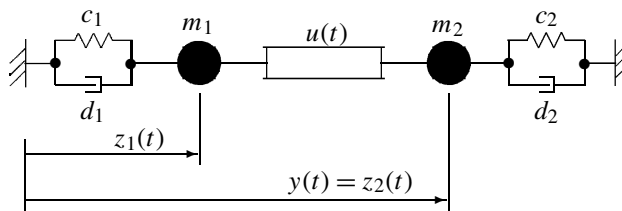


Figure 7.4. Mass-spring-damper system.

Position Control

With input $u(t) = z_2(t) - z_1(t)$, the mechanical system in Figure 7.4 may be modeled by the second order DAE

$$\begin{aligned}
 m_1 \ddot{z}_1(t) + d_1 \dot{z}_1(t) + c_1 z_1(t) - \lambda(t) &= 0, \\
 m_2 \ddot{z}_2(t) + d_2 \dot{z}_2(t) + c_2 z_2(t) + \lambda(t) &= 0, \\
 z_2(t) - z_1(t) &= u(t), \\
 y(t) &= z_2(t),
 \end{aligned} \tag{7.33}$$

where $\lambda(\cdot)$ is a constraint force viewed as a variable. Defining the state

$$x(t) = (z_1(t), \dot{z}_1(t), z_2(t), \dot{z}_2(t), \lambda(t))^T, \tag{7.34}$$

model (7.33) may be rewritten as the linear differential-algebraic input-output system (7.1) for

$$sE - A = \begin{bmatrix} s & -1 & 0 & 0 & 0 \\ c_1 & sm_1 + d_1 & 0 & 0 & -1 \\ 0 & 0 & s & -1 & 0 \\ 0 & 0 & c_2 & sm_2 + d_2 & 1 \\ -1 & 0 & 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}^T. \tag{7.35}$$

We may immediately see that the pencil $sE - A$ is regular and has index $\nu = 3$. The transfer function

$$G(s) = C(sE - A)^{-1}B = \frac{m_1s^2 + d_1s + c_1}{(m_1 + m_2)s^2 + (d_1 + d_2)s + (c_1 + c_2)}$$

has proper inverse: $\lim_{s \rightarrow \infty} G^{-1}(s) = (m_1 + m_2)/m_1$. The zero dynamics of (7.35) is asymptotically stable: setting $y(\cdot) = 0$ in (7.33) yields $z_2(\cdot) = 0, \lambda(\cdot) = 0, z_1(\cdot) = -u(\cdot)$ and $m_1 \dot{z}_1(t) + d_1 z_1(t) + c_1 z_1(t) = 0$ for all $t \geq 0$; positivity of $m_1, d_1,$ and c_1 then gives $\lim_{t \rightarrow \infty} \dot{z}_1(t) = \lim_{t \rightarrow \infty} z_1(t) = 0$. Summarizing, system (7.35) satisfies the assumptions of Theorem 7.17(ii).

Velocity Control

With input $u(t) = \dot{z}_2(t) - \dot{z}_1(t)$, the mechanical system in Figure 7.4 may, analogous to position control, be modeled by the second order DAE

$$\begin{aligned} m_1 \ddot{z}_1(t) + d_1 \dot{z}_1(t) + c_1 z_1(t) - \lambda(t) &= 0, \\ m_2 \ddot{z}_2(t) + d_2 \dot{z}_2(t) + c_2 z_2(t) + \lambda(t) &= 0, \\ \dot{z}_2(t) - \dot{z}_1(t) &= u(t), \\ y(t) &= z_2(t). \end{aligned} \tag{7.36}$$

Defining the state as in (7.34), the model (7.36) may be rewritten as the linear differential-algebraic input-output system (7.1) for

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -c_1 & -d_1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -c_2 & -d_2 & -1 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix} \quad \text{and } E, B, C \text{ as in (7.35)}. \tag{7.37}$$

We may immediately see that the pencil $sE - A$ is regular and has index $\nu = 2$. The transfer

function

$$G(s) = C(sE - A)^{-1}B = \frac{m_1 s^2 + d_1 s + c_1}{(m_1 + m_2)s^3 + (d_1 + d_2)s^2 + (c_1 + c_2)s}$$

has strict relative degree one: $\lim_{s \rightarrow \infty} s \cdot G(s) = m_1/(m_1 + m_2)$. Similar to Section 7.6.1, we may see that the zero dynamics of (7.37) is asymptotically stable, whence we are in the situation of Theorem 7.17(i).

7.6.2 Simulations

In both examples, as reference signal $y_{\text{ref}} : \mathbf{R}_{\geq 0} \rightarrow \mathbf{R}$, we take the first component of the solution of the following IVP for the Lorenz system:

$$\begin{aligned} \dot{\xi}_1(t) &= 10(\xi_2(t) - \xi_1(t)), & \xi_1(0) &= 5, \\ \dot{\xi}_2(t) &= 28\xi_1(t) - \xi_1(t)\xi_3(t) - \xi_2(t), & \xi_2(0) &= 5, \\ \dot{\xi}_3(t) &= \xi_1(t)\xi_2(t) - \frac{8}{3}\xi_3(t), & \xi_3(0) &= 5. \end{aligned} \quad (7.38)$$

This may be viewed as a rather academic choice; however, it is well known (see, for example, [16, Appendix C]) that the Lorenz system is chaotic (and thus the reference signal is rather “wild”), and the unique global solution of (7.38) is bounded with bounded derivative on the positive real axis (and thus our assumptions on the class of reference signals are satisfied). The solution of (7.38) is depicted in Figure 7.5. The funnel \mathcal{F}_φ is determined by the function

$$\varphi : \mathbf{R}_{\geq 0} \rightarrow \mathbf{R}_{\geq 0}, \quad t \mapsto 0.5 te^{-t} + 2 \arctan t. \quad (7.39)$$

Note that this prescribes an exponentially (exponent 1) decaying funnel in the transient phase $[0, T]$, where $T \approx 3$, and a tracking accuracy quantified by $\lambda = 1/\pi$ thereafter; see, e.g., Figure 7.6(d).

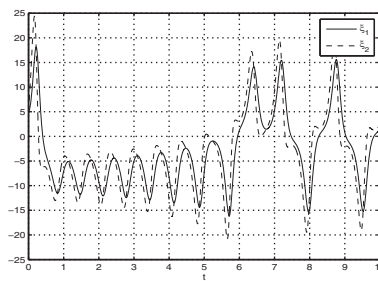


Figure 7.5. Solution of the Lorenz system (7.38).

Spring and damping constants, masses, and their initial positions are chosen, for the simulations, as

$$\begin{aligned} m_1 &= 1, & m_2 &= 3, & c_1 &= 2, & c_2 &= 1, & d_1 &= 3, & d_2 &= 5, \\ z_1(0) &= 101, & z_2(0) &= 21, & \text{and} & & \hat{k} &= 5. \end{aligned} \quad (7.40)$$

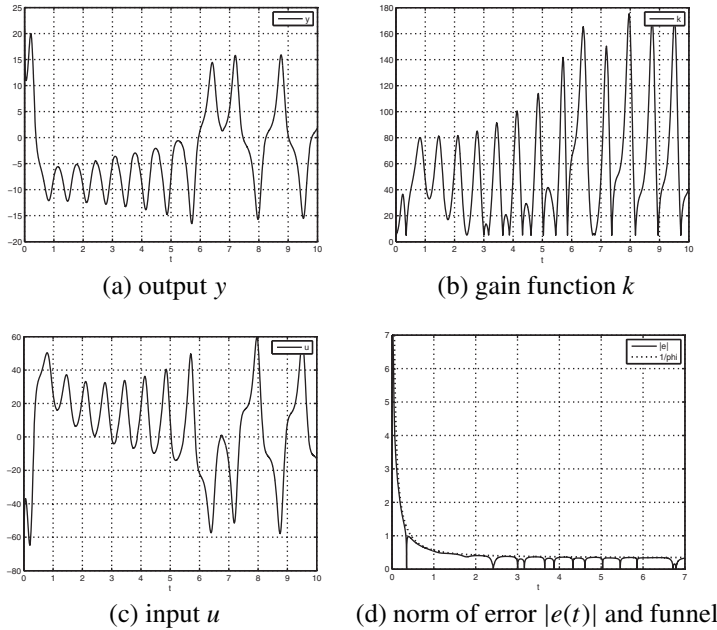


Figure 7.6. Position control: Simulation of the funnel controller (7.16) with funnel boundary specified in (7.39) and reference signal $y_{\text{ref}}(\cdot) = \xi_1(\cdot)$ given in (7.38) applied to the mechanical model (7.33) with data (7.40).

For position control, straightforward calculations show that the closed-loop system (7.16), (7.33) has uniquely determined initial velocities $\dot{z}_1(0)$, $\dot{z}_2(0)$ as well as initial constraint force $\lambda(0)$ and that the initialization is consistent.

In the case of velocity control we further choose the initial velocities

$$\dot{z}_1(0) = -11, \quad \dot{z}_2(0) = -3 \tag{7.41}$$

and clearly there is a unique initial constraint force $\lambda(0)$ and the initialization of (7.15), (7.37) is consistent.

All numerical simulations are performed by MATLAB.

Position Control

Since

$$\hat{k} = 5 > 4 = \lim_{s \rightarrow \infty} G^{-1}(s),$$

all assumptions of Theorem 7.17(ii) are satisfied and we may apply the funnel controller (7.16) with funnel boundary specified in (7.39) and reference signal $y_{\text{ref}}(\cdot) = \xi_1(\cdot)$ given in (7.38).

The simulations over the time interval $[0, 10]$ are depicted in Figure 7.6: Figure 7.6(a) shows the output $y(\cdot)$ tracking the rather “vivid” reference signal $y_{\text{ref}}(\cdot)$ within the funnel

shown in Figure 7.6(d). Note that the input $u(\cdot)$ in Figure 7.6(c) as well as the gain function $k(\cdot)$ in Figure 7.6(b) have spikes at those times t when the norm of the error $\|e(t)\|$ is “close” to the funnel boundary $\varphi(t)^{-1}$; this is due to rapid change of the reference signal. We stress that the gain function $k(\cdot)$ is nonmonotone.

Velocity Control

Since the system has relative degree one with positive high frequency gain $\Gamma = m_1/(m_1 + m_2) = 1/4$, all assumptions of Theorem 7.17(i) are satisfied and we may apply the funnel controller (7.15) with funnel boundary specified in (7.39) and reference signal $y_{\text{ref}}(\cdot) = \xi_1(\cdot)$ given in (7.38).

The simulations over the time interval $[0, 10]$ are depicted in Figure 7.7: Figure 7.7(a) shows the output $y(\cdot)$ tracking the reference signal $y_{\text{ref}}(\cdot)$; the error within the funnel is depicted in Figure 7.7(d). Note that, due to the rather “academic choice” of the example, the input $u(\cdot)$ (in Figure 7.7(c)) and the gain function $k(\cdot)$ (in Figure 7.7(b)) both take considerable larger values than for position control. Another reason for this behavior is that we have kept the funnel as tight as for position control simulated in Figure 7.6, and the velocity exhibits a very “vivid” behavior which causes the error to approach the funnel boundary faster, resulting in the high values of the gain function.

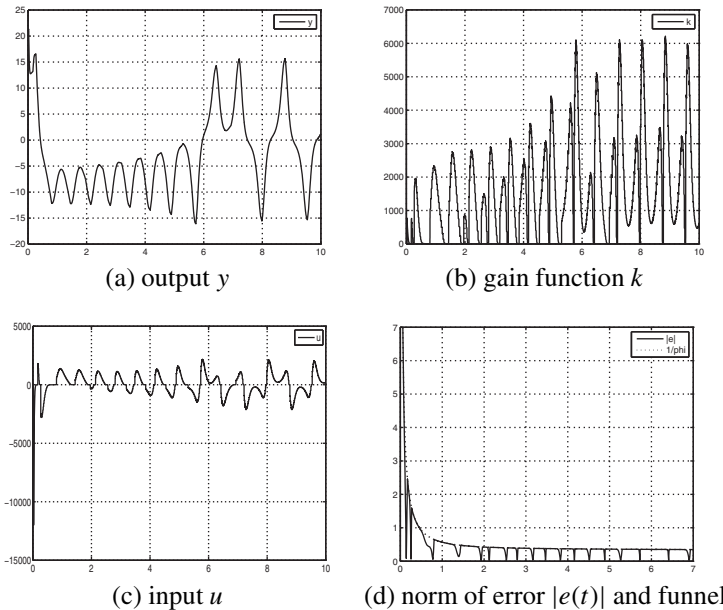


Figure 7.7. Velocity control: Simulation of the funnel controller (7.15) with funnel boundary specified in (7.39) and reference signal $y_{\text{ref}}(\cdot) = \xi_1(\cdot)$ given in (7.38) applied to the mechanical model (7.36) with data (7.40), (7.41).

7.7 Appendix: Proofs

This section contains the proofs of the results presented in Sections 7.1–7.5. First we prove a preliminary lemma.

Lemma 7.19. *Suppose $[E, A, B, C] \in \Sigma_{n,m}$ has strictly proper transfer function $G(s) = C(sE - A)^{-1}B$. Then there exist $W, T \in \mathbf{GL}_n(\mathbf{R})$ such that*

$$[E, A, B, C] \stackrel{W, T}{\sim} \left[\begin{bmatrix} I_{n_s} & 0 & 0 \\ 0 & N_c & N_{c\bar{c}} \\ 0 & 0 & N_{\bar{c}} \end{bmatrix}, \begin{bmatrix} A_s & 0 & 0 \\ 0 & I_{n_{fc}} & 0 \\ 0 & 0 & I_{n_{f\bar{c}}} \end{bmatrix}, \begin{bmatrix} B_s \\ B_{fc} \\ 0 \end{bmatrix}, [C_s \quad 0 \quad C_{f\bar{c}}] \right] \tag{7.42}$$

for some $A_s \in \mathbf{R}^{n_s, n_s}$, $B_s \in \mathbf{R}^{n_s, m}$, $C_s \in \mathbf{R}^{m, n_s}$, $N_c \in \mathbf{R}^{n_{fc}, n_{fc}}$, $N_{c\bar{c}} \in \mathbf{R}^{n_{fc}, n_{f\bar{c}}}$, $N_{\bar{c}} \in \mathbf{R}^{n_{f\bar{c}}, n_{f\bar{c}}}$, $B_{fc} \in \mathbf{R}^{n_{fc}, m}$, and $C_{f\bar{c}} \in \mathbf{R}^{m, n_{f\bar{c}}}$, where $N_c, N_{\bar{c}}$ are nilpotent and $\text{rank}[N_c, B_{fc}] = n_{fc}$. The dimensions $n_s, n_{fc}, n_{f\bar{c}} \in \mathbf{N}_0$ are unique, and the matrices $A_s, N_c,$ and $N_{\bar{c}}$ are unique up to similarity. Furthermore, the system $[E, A, B, C]$ has strict relative degree $\rho > 0$ if and only if

$$\det(C_s A_s^{\rho-1} B_s) \neq 0 \quad \text{and, if } \rho > 1, \quad \forall k = 0, \dots, \rho - 2 : C_s A_s^k B_s = 0. \tag{7.43}$$

Proof. We proceed in several steps.

Step 1: We show that there exist $W, T \in \mathbf{GL}_n(\mathbf{R})$ such that (7.42) holds true. Corollary 7.2 yields (7.4) for some $W_1, T_1 \in \mathbf{GL}_n(\mathbf{R})$. It follows from [5, Section 2-1] that the system $[N, I_{n_f}, B_f, C_f]$ may be decomposed into controllability form so that, for some $T_2 \in \mathbf{GL}_{n_f}(\mathbf{R})$,

$$[N, I_{n_f}, B_f, C_f] \stackrel{T_2^{-1}, T_2}{\sim} \left[\begin{bmatrix} N_c & N_{c\bar{c}} \\ 0 & N_{\bar{c}} \end{bmatrix}, \begin{bmatrix} I_{n_{fc}} & 0 \\ 0 & I_{n_{f\bar{c}}} \end{bmatrix}, \begin{bmatrix} B_{fc} \\ 0 \end{bmatrix}, [C_{fc}, C_{f\bar{c}}] \right],$$

where $N_c \in \mathbf{R}^{n_{fc}, n_{fc}}$, $N_{\bar{c}} \in \mathbf{R}^{n_{f\bar{c}}, n_{f\bar{c}}}$, $N_{12} \in \mathbf{R}^{n_{fc}, n_{f\bar{c}}}$, $B_{fc} \in \mathbf{R}^{n_{fc}, m}$, $C_{fc} \in \mathbf{R}^{m, n_{fc}}$, and $C_{f\bar{c}} \in \mathbf{R}^{m, n_{f\bar{c}}}$ such that $N_c, N_{\bar{c}}$ are nilpotent and $\text{rank}[N_c, B_{fc}] = n_{fc}$.

We show that $C_{fc} = 0$: Since the transfer function is invariant under system equivalence we have, using $(sN_c - I_{n_f})^{-1} = -I_{n_f} - sN - s^2N^2 - \dots - s^{v-1}N^{v-1}$,

$$G(s) = C(sE - A)^{-1}B = C_s(sI_{n_s} - A_s)^{-1}B_s - \sum_{k=0}^{v-1} s^k C_{fc} N_c^k B_{fc},$$

and since $G(s)$ is strictly proper, it follows that $C_{fc} N_c^i B_{fc} = 0$ for $i = 1, \dots, v - 1$. The nilpotency of N_c gives $C_{fc} N_c^{v-1} [N_c, B_{fc}] = 0$, whence $C_{fc} N_c^{v-1} = 0$. Repeating this

argument $\nu - 1$ times, we obtain $C_{fc} = 0$. Setting $W := W_1 \begin{bmatrix} I & 0 \\ 0 & T_2^{-1} \end{bmatrix}$ and $T := \begin{bmatrix} I & 0 \\ 0 & T_2 \end{bmatrix} T_1$, we obtain (7.42).

Step 2: We show that the dimensions $n_s, n_{fc}, n_{f\bar{c}} \in \mathbf{N}_0$ are unique and that the matrices A_s , N_c , and $N_{\bar{c}}$ are unique up to similarity: Assume that

$$\begin{aligned} & \left[\begin{bmatrix} I_{n_{s1}} & 0 & 0 \\ 0 & N_{c1} & N_{c\bar{c}1} \\ 0 & 0 & N_{\bar{c}1} \end{bmatrix}, \begin{bmatrix} A_{s1} & 0 & 0 \\ 0 & I_{n_{fc1}} & 0 \\ 0 & 0 & I_{n_{f\bar{c}1}} \end{bmatrix}, \begin{bmatrix} B_{s1} \\ B_{fc1} \\ 0 \end{bmatrix}, [C_{s1} \ 0 \ C_{f\bar{c}1}] \right] \\ & \stackrel{W,T}{\sim} \left[\begin{bmatrix} I_{n_{s2}} & 0 & 0 \\ 0 & N_{c2} & N_{c\bar{c}2} \\ 0 & 0 & N_{\bar{c}2} \end{bmatrix}, \begin{bmatrix} A_{s2} & 0 & 0 \\ 0 & I_{n_{fc2}} & 0 \\ 0 & 0 & I_{n_{f\bar{c}2}} \end{bmatrix}, \begin{bmatrix} B_{s2} \\ B_{fc2} \\ 0 \end{bmatrix}, [C_{s2} \ 0 \ C_{f\bar{c}2}] \right]. \end{aligned}$$

Proposition 7.1 implies that $n_{s1} = n_{s2}$ as well as the similarity of A_{s1} and A_{s2} . Proposition 7.1 also yields the existence of $T_{11} \in \mathbf{R}^{n_{fc1}, n_{fc2}}$, $T_{12} \in \mathbf{R}^{n_{fc1}, n_{f\bar{c}2}}$, $T_{21} \in \mathbf{R}^{n_{f\bar{c}1}, n_{fc2}}$, and $T_{22} \in \mathbf{R}^{n_{f\bar{c}1}, n_{f\bar{c}2}}$ such that

$$\begin{aligned} T &= \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in \mathbf{GL}_{n_f}(\mathbf{R}), \\ \begin{bmatrix} N_{c1} & N_{c\bar{c}1} \\ 0 & N_{\bar{c}1} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} &= \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} N_{c2} & N_{c\bar{c}2} \\ 0 & N_{\bar{c}2} \end{bmatrix}, \begin{bmatrix} B_{fc1} \\ 0 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} B_{fc2} \\ 0 \end{bmatrix}. \end{aligned}$$

Therefore, $0 = T_{21} B_{fc2}$ and $T_{21} N_{c2} = N_{\bar{c}1} T_{21}$.

Hence, for $k = 1, \dots, \nu - 1$, we have $T_{21} N_{c2}^k B_{fc2} = N_{\bar{c}1}^k T_{21} B_{fc2} = 0$, and so $T_{21} N_{c2}^{\nu-1} [N_{c2} B_{fc2}] = 0$, whence $T_{21} N_{c2}^{\nu-1} = 0$. Repeating this argument $\nu - 1$ times, we obtain $T_{21} = 0$. Then $T \in \mathbf{GL}_{n_f}(\mathbf{R})$ yields $n_{fc2} \leq n_{fc1}$. By reversing the roles of the above matrices, we analogously obtain $n_{fc1} \leq n_{fc2}$ and thus $n_{fc1} = n_{fc2}$, $n_{f\bar{c}1} = n_{f\bar{c}2}$. This shows that T_{11} and T_{22} are square. Together with $T \in \mathbf{GL}_{n_f}(\mathbf{R})$, we obtain $T_{11} \in \mathbf{GL}_{n_{fc}}(\mathbf{R})$ and $T_{22} \in \mathbf{GL}_{n_{f\bar{c}}}(\mathbf{R})$. Hence N_{c1}, N_{c2} and $N_{\bar{c}1}, N_{\bar{c}2}$ are similar, respectively.

Step 3: We show that $[E, A, B, C]$ has strict relative degree $\rho > 0$ if and only if (7.43) holds.

This is an immediate consequence of the fact that, due to Step 1, the transfer function has the representation $G(s) = C(sE - A)^{-1}B = C_s(sI_{n_s} - A_s)^{-1}B_s$. This completes the proof of the lemma. \square

Proof of Theorem 7.6. We proceed in several steps.

Step 1: We show that there exist $W, T \in \mathbf{GL}_n(\mathbf{R})$ such that $[E, A, B, C] \stackrel{W,T}{\sim} [\hat{E}, \hat{A}, \hat{B}, \hat{C}]$ for $[\hat{E}, \hat{A}, \hat{B}, \hat{C}]$ as in (7.17).

Since a positive strict relative degree implies that $G(s)$ is strictly proper, we may apply Lemma 7.19 to obtain (7.42) for some $W_1, T_1 \in \mathbf{GL}_n(\mathbf{R})$. Furthermore, (7.43) holds and hence we may transform $[I, A_s, B_s, C_s]$ into Byrnes–Isidori form (see [12, Lemma

Since the transfer function is invariant under system equivalence, letting $\bar{n} = n - n_{fc} - n_{f\bar{c}}$ we have

$$\begin{aligned}
 C(sE - A)^{-1}B &= \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} \\
 &= [I_m \quad 0 \quad \cdots \quad 0] \left(sI_{\bar{n}} - \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_m & 0 & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_m & 0 \\ R_1 & R_2 & \cdots & R_{\rho-1} & R_\rho & S \\ P & 0 & \cdots & 0 & 0 & Q \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \Gamma \\ 0 \end{bmatrix} \\
 &\quad + [0 \quad C_{f\bar{c}}] \begin{bmatrix} sN_c - I_{n_{fc}} & sN_{c\bar{c}} \\ 0 & sN_{\bar{c}} - I_{n_{f\bar{c}}} \end{bmatrix}^{-1} \begin{bmatrix} B_{fc} \\ 0 \end{bmatrix} \\
 &= [I_m \quad 0 \quad \cdots \quad 0] \left(sI_{\bar{n}} - \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_m & 0 & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_m & 0 \\ R_1 & R_2 & \cdots & R_{\rho-1} & R_\rho & S \\ P & 0 & \cdots & 0 & 0 & Q \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \Gamma \\ 0 \end{bmatrix} \\
 &\stackrel{(7.45)}{=} X_1(s) \stackrel{(7.46), R_{\rho+1}=-I}{=} - \left[\sum_{i=1}^{\rho+1} R_i s^{i-1} + S(sI_{n-n_{fc}-n_{f\bar{c}}-\rho m} - Q)^{-1}P \right]^{-1} \Gamma.
 \end{aligned}$$

This proves (7.19). Finally,

$$\begin{aligned}
 \Gamma &= \lim_{s \rightarrow \infty} - \left[\sum_{i=1}^{\rho+1} R_i s^{i-1} + S(sI_{n-n_{fc}-n_{f\bar{c}}-\rho m} - Q)^{-1}P \right] G(s) \\
 &= - \sum_{i=1}^{\rho} R_i \lim_{s \rightarrow \infty} s^{i-1} G(s) + \lim_{s \rightarrow \infty} s^\rho G(s) - \lim_{s \rightarrow \infty} S(sI_{n-n_{fc}-n_{f\bar{c}}-\rho m} - Q)^{-1}P G(s) \\
 &= \lim_{s \rightarrow \infty} s^\rho G(s),
 \end{aligned}$$

and the proof of the theorem is complete. \square

Proof of Theorem 7.12. (i) \Leftrightarrow (iv): This is immediate from Remark 7.11.

For the remainder of the proof we consider only systems with positive strict relative degree; the proof for systems which have a proper inverse transfer function is in [2, Theorem 5.4]. In view of invariance of the transfer function under system equivalence we may assume, without loss of generality, that $[E, A, B, C]$ is in the form (7.17).

(ii) \Leftrightarrow (iv): First note that Schur's complement formula (see, for example, [7, Lemma A.1.17]) yields, for all $X \in \mathbf{R}^{k,m}$, $Y \in \mathbf{R}^{k,k}$, $Z \in \mathbf{R}^{m,m}$,

$$\det \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix} = \det \begin{bmatrix} 0 & I_k \\ I_m & 0 \end{bmatrix} \cdot \det \begin{bmatrix} Z & 0 \\ X & Y \end{bmatrix} = (-1)^{km} \det Z \cdot \det Y. \quad (7.47)$$

This and the fact that $\det(sN - I_k) = (-1)^k$ for any nilpotent $N \in \mathbf{R}^{k,k}$ yield

$$\begin{aligned} & \det \begin{bmatrix} sE - A & B \\ C & 0 \end{bmatrix} \\ &= \det \left[\begin{array}{cccccc|cc} sI_m & -I_m & & & & & 0 & \\ & \ddots & \ddots & & & & \vdots & \\ & & sI_m & -I_m & & & 0 & \\ -R_1 & \dots & -R_{\rho-1} & sI_m - R_\rho & -S & & \Gamma & \\ -P & 0 & \dots & 0 & sI_\mu - Q & & 0 & \\ 0 & 0 & & \dots & 0 & sN_c - I_{n_c} & sN_{c\bar{c}} & B_c \\ 0 & 0 & & \dots & 0 & 0 & sN_{\bar{c}} - I_{n_{\bar{c}}} & 0 \\ \hline I_m & 0 & \dots & 0 & 0 & 0 & C_{\bar{c}} & 0 \end{array} \right] \\ &= (-1)^m \det \left[\begin{array}{cccccc|cc} 0 & -I_m & & & & & sI_m & \\ \vdots & \ddots & \ddots & & & & & \\ 0 & & sI_m & -I_m & & & & \\ \Gamma & \dots & -R_{\rho-1} & sI_m - R_\rho & -S & & -R_1 & \\ 0 & 0 & \dots & 0 & sI_\mu - Q & & -P & \\ B_c & 0 & 0 & \dots & 0 & sN_c - I_{n_c} & sN_{c\bar{c}} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & sN_{\bar{c}} - I_{n_{\bar{c}}} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & C_{\bar{c}} & I_m \end{array} \right] \\ &= (-1)^m \det \begin{bmatrix} sN_{\bar{c}} - I_{n_{\bar{c}}} & 0 \\ C_{\bar{c}} & I_m \end{bmatrix} \cdot \det(sN_c - I_{n_c}) \cdot \det(sI_\mu - Q) \\ &\quad \cdot \det \begin{bmatrix} 0 & -I_m & & & \\ 0 & sI_m & -I_m & & \\ \vdots & & \ddots & \ddots & \\ 0 & & & sI_m & -I_m \\ \Gamma & -R_2 & \dots & -R_{\rho-1} & sI_m - R_\rho \end{bmatrix} \\ &\stackrel{(7.47)}{=} (-1)^{m+n_c+n_{\bar{c}}+(\rho-1)(m^2+m)} \det(sI_\mu - Q) \det \Gamma. \end{aligned}$$

This proves the claim.

(iii) \Leftrightarrow (iv): Set

$$[A_s, B_s, C_s] = \left[\begin{array}{c} \left[\begin{array}{cccccc} 0 & I_m & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_m & 0 & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_m & 0 \\ R_1 & R_2 & \cdots & R_{\rho-1} & R_\rho & S \\ P & 0 & \cdots & 0 & 0 & Q \end{array} \right], \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ \Gamma \\ 0 \end{array} \right], [I_m \ 0 \ \cdots \ 0] \end{array} \right].$$

We immediately see that

- (a) $\forall s \in \mathbf{C} : \text{rank}[sE - A, B] = \text{rank}[sI_{n_s} - A_s, B_s] + n_c + n_{\bar{c}}$,
- (b) $\forall s \in \mathbf{C} : \text{rank}[sE^\top - A^\top, C^\top] = \text{rank}[sI_{n_s} - A_s^\top, C_s^\top] + n_c + n_{\bar{c}}$,
- (c) $G(s) = C(sE - A)^{-1}B = C_s(sI_{n_s} - A_s)^{-1}B_s$,

where (c) is proved similar to Step 3 in the proof of Theorem 7.6. Therefore, the equivalence of (iii) and (iv) has been reduced to the ODE system $[I_{n_s}, A_s, B_s, C_s]$. Now the claim follows from [8, Proposition 2.1.2]. This completes the proof of the theorem. \square

For the proof of Theorem 7.15 a lemma is required.

Lemma 7.20 (High-gain lemma). Consider, for $\Gamma \in \mathbf{GL}_m(\mathbf{C})$, $\tilde{R} \in \mathbf{C}^{m,m}$, $\tilde{S}^\top, \tilde{P} \in \mathbf{C}^{n-m,m}$, $\tilde{Q} \in \mathbf{C}^{n-m,n-m}$, the parameterized matrix

$$A_\kappa := \begin{bmatrix} \tilde{R} - \kappa\Gamma & \tilde{S} \\ \tilde{P} & \tilde{Q} \end{bmatrix}, \quad \kappa \geq 0.$$

Denote the spectra of Γ and \tilde{Q} by

$$\sigma(\Gamma) = \{\gamma_1, \dots, \gamma_m\} \subseteq \mathbf{C} \setminus \{0\} \quad \text{and} \quad \sigma(\tilde{Q}) = \{q_{m+1}, \dots, q_n\} \subseteq \mathbf{C}.$$

Then there exist $z_1, \dots, z_m \in \mathbf{C}$ and $\hat{\theta} > 0$ with the following property: For all $\varepsilon > 0$ and all $\theta \in (0, \hat{\theta})$ there exist $r \geq 0$ and $\kappa^* \geq 1$ such that, with a suitable enumeration of the eigenvalues $\lambda_1(A_\kappa), \dots, \lambda_n(A_\kappa)$ of A_κ , we have, for all $\kappa \geq \kappa^*$,

- (i) $B(z_i - \kappa\gamma_i, r + \kappa\theta) \cap B(0, 1/\varepsilon) = \emptyset$ for $i = 1, \dots, m$,
- (ii) $\lambda_i(A_\kappa) \in \bigcup_{j=1}^m B((z_j - \kappa\gamma_j, r + \kappa\theta))$ for $i = 1, \dots, m$,
- (iii) $\lambda_i(A_\kappa) \in \bigcup_{j=m+1}^n B((q_j, \varepsilon))$ for $i = m + 1, \dots, n$,

where $B(z, \varepsilon) = \{w \in \mathbf{C} \mid |z - w| < \varepsilon\}$ denotes the ball of radius ε around z in \mathbf{C} .

Proof. Let

$$\hat{\theta} := \frac{1}{4} \min\{|\gamma_1|, \dots, |\gamma_m|\} > 0 \tag{7.48}$$

We now show that the center of the balls $B(z_i - \kappa \gamma_i, r + \kappa \theta)$, $i = 1, \dots, m$, tends, as $\kappa \rightarrow \infty$, to infinity at a faster pace than its radius. To this end note that using (7.48) gives

$$|z_i - \kappa \gamma_i| \geq ||z_i| - 4\kappa \hat{\theta}|$$

and for $\kappa > (r + |z_i|)/\hat{\theta}$ we have $|z_i| < \kappa \hat{\theta}$ and hence

$$|z_i - \kappa \gamma_i| > 3\kappa \hat{\theta} - |z_i| > 2\kappa \hat{\theta} + r > \kappa \hat{\theta} + (r + \kappa \theta).$$

Therefore,

$$|z_i - \kappa \gamma_i| - (\kappa \theta + r) > \kappa \hat{\theta},$$

which implies that $B(z_i - \kappa \gamma_i, r + \kappa \theta) \cap B(0, \kappa \hat{\theta}) = \emptyset$. Choosing

$$\kappa^* > \max \{1/(\varepsilon \hat{\theta}), (r + |z_1|)/\hat{\theta}, \dots, (r + |z_m|)/\hat{\theta}\}$$

we obtain assertion (i). Since $\gamma_i \neq 0$ for all $i = 1, \dots, m$, we may now choose $\kappa^* \geq 1$ sufficiently large so that

$$\forall \kappa \geq \kappa^* : \bigcup_{i=1}^m B(z_i - \kappa \gamma_i, r + \kappa \theta) \cap \bigcup_{j=m+1}^n B(q_j, \varepsilon) = \emptyset.$$

We are now in a position to apply Gershgorin's disks, see [7, Theorem 4.2.19], to deduce (ii) and (iii). This completes the proof of the lemma. \square

Proof of Theorem 7.15.

(i) We prove “ \Rightarrow ”. By Theorem 7.6, $[E, A, B, C]$ is equivalent to a system in the form (7.20). We introduce the “new states”

$$\xi := \frac{1}{p_{\rho-1}} \cdot p \left(\frac{d}{dx} \right) y, \quad \chi := \left(y^\top, \dot{y}^\top, \dots, \left(y^{(\rho-2)} \right)^\top, \eta^\top \right)^\top$$

and observe that

$$\begin{aligned} \dot{\xi}(t) &= \tilde{R} \xi(t) + \tilde{S} \chi(t) + \Gamma u(t), \\ \dot{\chi}(t) &= \tilde{P} \xi(t) + \tilde{Q} \chi(t), \end{aligned} \tag{7.50}$$

where \tilde{R}, \tilde{S} are matrices of appropriate size and

$$\tilde{P} = [0, \dots, 0, I_m, 0]^\top, \quad \tilde{Q} = \begin{bmatrix} \hat{A} & 0 \\ \hat{P} & Q \end{bmatrix},$$

with $\hat{A} = \begin{bmatrix} 0 & I & & & \\ & \ddots & \ddots & & \\ & & 0 & & I \\ -\frac{p_0}{p_{\rho-1}} I & \dots & -\frac{p_{\rho-3}}{p_{\rho-1}} I & -\frac{p_{\rho-2}}{p_{\rho-1}} I \end{bmatrix}, \quad \hat{P} = [P, 0, \dots, 0].$

Note that \hat{A} is Hurwitz since $p(s)$ is Hurwitz. The feedback (7.15) reads in the new coordinates as

$$u(t) = -k p\left(\frac{d}{dt}\right)y(t) = -k p_{\rho-1}\xi(t), \tag{7.51}$$

and therefore the application of (7.15) to $[E, A, B, C]$ results, in terms of the new system (7.50), in the closed-loop system

$$\frac{d}{dt} \begin{pmatrix} \xi(t) \\ \chi(t) \end{pmatrix} = \underbrace{\begin{bmatrix} \tilde{R} - kp_{\rho-1}\Gamma & \tilde{S} \\ \tilde{P} & \tilde{Q} \end{bmatrix}}_{=: A_k} \begin{pmatrix} \xi(t) \\ \chi(t) \end{pmatrix}. \tag{7.52}$$

Note that the closed-loop system (7.1), (7.11) is asymptotically stable if (7.52) is asymptotically stable. This is due to the fact that $p(s)$ is Hurwitz and if y decays exponentially so do all derivatives of y and, by (7.20), also u and all derivatives of u , which finally gives that x_c decays exponentially.

We show that (7.52) is asymptotically stable. Note that \tilde{Q} is Hurwitz since Q is Hurwitz. Therefore, by $p_{\rho-1} > 0$, Γ positive definite and $\sigma(\tilde{Q}) \subseteq \mathbf{C}_-$, we may apply Lemma 7.20 to conclude that

$$\exists k^* \geq 0 \forall k \geq k^* : \sigma(A_k) \subseteq \mathbf{C}_-.$$

This proves the claim.

To see that “ \Leftarrow ” does not hold in general, consider system (7.1) for

$$E = I, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [1, 0], \tag{7.53}$$

which is in zero dynamics form (7.17) and has strict relative degree one. We may observe that $Q = 0$ and therefore the zero dynamics of (7.53) are not asymptotically stable. However, the closed-loop system (7.53), (7.11) where $p(s) \equiv p_0 > 0$ takes the form

$$\dot{x}(t) = A_k x(t) = \begin{bmatrix} -kp_0 & 1 \\ -1 & 0 \end{bmatrix} x(t),$$

which is, since $\sigma(A_k) = \{-p_0k/2 \pm \sqrt{(p_0k)^2/4 - 1}\}$, asymptotically stable for all $k > 0$. This shows that $[E, A, B, C]$ is high-gain stabilizable.

(ii) The proof of “ \Rightarrow ” has been carried out in Remark 7.16(ii). It remains to show that “ \Leftarrow ” does not hold in general. Consider system (7.1) for

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{7.54}$$

which is in zero dynamics form (7.21) with $n_1 = n_2 = 2, n_c = n_{\bar{c}} = 0$ and

$$A_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

Since Q has an eigenvalue at the origin, Theorem 7.12 yields that the zero dynamics are not asymptotically stable. The asymptotic stability of the closed-loop system “(7.54) & (7.12)” is determined by the spectrum of the matrix

$$Q - A_{21}(kI_2 + A_{11})^{-1}A_{12} = \begin{bmatrix} 0 & 1/k \\ -1/k & -1 \end{bmatrix},$$

and this satisfies, for all $k \in \mathbf{R} \setminus \{0\}$,

$$\sigma(Q - A_{21}(kI_2 + A_{11})^{-1}A_{12}) = \left\{ -1/2 \pm \sqrt{1/4 - 1/k^2} \right\} \subseteq \mathbf{C}_-.$$

This shows that $[E, A, B, C]$ is high-gain stabilizable and completes the proof of the theorem. \square

Proof of Theorem 7.17.

(i) Without restriction of generality, one may consider $[E, A, B, C]$ in the form (7.20). Ignoring the bottom two algebraic equations in (7.20), the claim in (i) and the bound on e follow from [11, Theorem 2]. Since γ_ρ is a \mathcal{C}^∞ -function, it is easy to see that u is $(\nu - 1)$ -times continuously differentiable and all of these derivatives are bounded functions. Therefore, x_c and \bar{x}_c in (7.20) are bounded functions. It remains to show the bound on k in (7.32): This follows from the following inequalities, which hold for all $t > 0$:

$$k(t) = \hat{k} + k(t)\varphi(t)^2 \|e(t)\|^2 \stackrel{(7.31)}{\leq} \hat{k} + k(t)\varphi(t)^2 (\varphi(t)^{-1} - \varepsilon)^2 = \hat{k} + k(t)(1 - \varphi(t)\varepsilon)^2.$$

(ii) This result is shown in [2, Theorem 6.2]. \square

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