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Stabilizability of linear time-varying systems

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ABSTRACT

For linear time-varying systems with bounded system matrices we discuss the problem of stabilizability by linear state feedback. For example, it is shown that complete controllability implies the existence of a feedback so that the closed-loop system is asymptotically stable. We also show that the system is completely controllable if, and only if, the Lyapunov exponent is arbitrarily assignable by a suitable feedback. For uniform exponential stabilizability and the assignability of the Bohl exponent this property is known. Also, dynamic feedback does not provide more freedom to address the stabilization problem. The unifying tools for our results are two finite (L^2) cost conditions. The distinction of exponential and uniform exponential stabilizability is then a question of whether the finite cost condition is uniform in the initial time or not.

1. Introduction

We consider the linear, time-varying system

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t), \qquad x(t_0) = x^0 \tag{1.1}$$

for measurable and essentially bounded $A : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times n}$ and $B : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times m}$, piecewise continuous input function $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, and initial data $(t_0, x^0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$. Unbounded $A(\cdot)$ and $B(\cdot)$ are only considered in Remarks 3.5 and 3.10. The unique and global solution of (1.1) corresponding to the initial condition $(t_0, x^0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ and the input $u(\cdot)$ is denoted by $x(\cdot; t_0, x^0, u) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$.

For the homogeneous system

$$\dot{x}(t) = A(t)\,x(t), \qquad x(t_0) = x^0 \tag{1.2}$$

with measurable and essentially bounded $A : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times n}$, the unique solution is denoted by $x(\cdot; t_0, x^0) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and its transition matrix by $\Phi_A(\cdot, \cdot)$. It is well known that $\Phi_A(\cdot, t_0)$ is obtained as the unique solution of the initial value problem $\dot{\Phi}(\cdot, t_0) = A(t)\,\Phi(\cdot, t_0)$, $\Phi(t_0, t_0) = I_n$.

In this paper we study the problem of defining conditions under which the system (1.1) is stabilizable by bounded or locally bounded state feedback. In the linear time-invariant case the characterization of stabilizability is well known: the uncontrollable part of the pair (A, B) has to be exponentially stable. In particular, controllability implies stabilizability. In fact, for controllable time-invariant systems it is possible to achieve, by constant static linear feedback, arbitrary exponential decay rates of the closed-loop system.

In the time-varying case problems arise because now there are distinct notions of stability. The strongest of these is uniform asymptotic or, equivalently, uniform exponential stability. This notion corresponds well to uniform controllability, and Ikeda, Maeda and Kodama [1,2] proved among other things the analogue of the time-invariant statement: assuming boundedness of $(A(\cdot), B(\cdot))$ system (1.1) is uniformly controllable if, and only if, it is uniformly exponentially stabilizable by bounded feedback with arbitrary prespecified decay; see [1, Theorem 3]. We note that the 'only if' part of the result is set up as a problem in [3, Problem 14.2-1], based on arguing that if $(A(\cdot), B(\cdot))$ is uniformly controllable then so is $(A(\cdot) + \alpha I, B(\cdot))$ for any $\alpha \in \mathbb{R}$.

For bounded $(A(\cdot), B(\cdot))$, Ravi, Pascoal and Khargonekar prove in their result [4, Lemma 3.3] that uniform exponential stabilizability (not necessarily with arbitrary decay) is equivalent to the existence of a bounded symmetric positive semidefinite solution $P(\cdot)$ to the control Riccati equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\Pi(t) = -A(t)^\top \Pi(t) - \Pi(t)A(t) + \Pi(t)B(t)B(t)^\top \Pi(t) - I_n \quad \text{on } (t_0, \infty). \tag{1.3}$$

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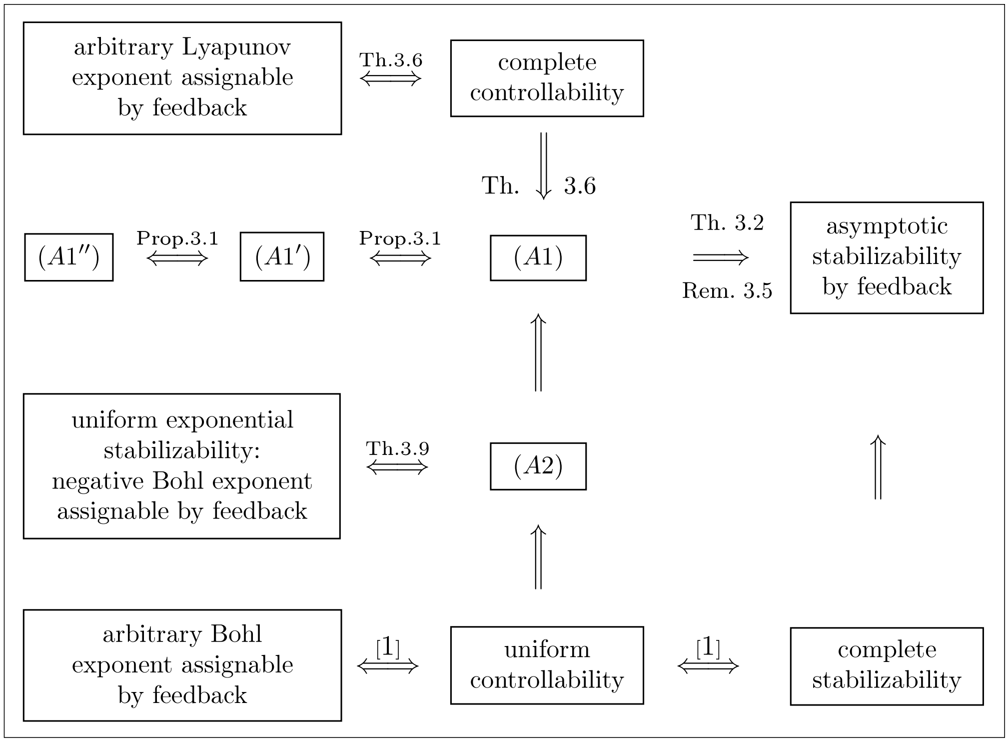

Fig. 1. For each unidirectional implication, there is a counterexample demonstrating it is not bidirectional. Details in the text clarify boundedness assumptions, which are sometimes but not always required.

In this case, the closed-loop system $\dot{x} = [A - BB^\top P]x$ is uniformly exponentially stable. We note that while in [4] only the necessity of the solvability is proved, the sufficiency is straightforward using the arguments provided in [4].

Finally, concerning uniform exponential stabilizability, Rotea and Khargonekar [5] prove that dynamic feedback does not provide more freedom in the sense that if (1.1) is uniformly exponentially stabilizable by dynamic linear feedback, then the same can be achieved by static, time-varying linear feedback.

If we consider merely exponential stability, not necessarily uniform, the picture is not quite as well studied. Ikeda et al. [1] have shown that exponential stabilization with arbitrary time-varying decay (see Remark 3.10 of the present note) is equivalent to complete controllability, even for systems with unbounded data $(A(\cdot), B(\cdot))$. The relation of exponential stabilizability to solutions of the Riccati equation is less well studied and the result similar to [5] is missing.

In the recent paper by Phat and Ha [6] the authors consistently consider the initial time $t_0 = 0$, which fails to capture the issue of uniformity. In [6] the notion of global null controllability in introduced, which is what is called controllability at time $t_0 = 0$ in the present paper, see Section 2.2. It is claimed in [6, Theorem 3.1] that global null controllability is equivalent to complete stabilizability and to the existence of a bounded symmetric positive semidefinite solution of the Riccati equation (1.3). This is incorrect as we show in Example 3.7. However, an easy example is immediate: It is clear that

$$\dot{x} = x + b(t)u, \quad \text{where } b(t) = \begin{cases} 1 & \text{for } t \in [0, 1] \\ 0 & \text{for } t > 1 \end{cases}$$

is controllable at $t_0 = 0$, but not stabilizable by bounded linear feedback, as $b(\cdot)$ vanishes after $t = 1$. Moreover, the corresponding Riccati equation $\dot{p}(t) + 2p(t) - b(t)p(t)^2 + 1 = 0$ does not have bounded positive solutions on $[0, \infty)$. After $t = 1$ the equation becomes linear and any solution of $\dot{p}(t) = -2p(t) - 1$, $t \geq 1$, tends to $-1/2$ as t goes to infinity. Therefore a positive solution does not exist.

Fig. 1 details the results of this paper in view of the various concepts of controllability and stabilizability defined in Sections 2.2 and 2.3. The finite cost conditions (A1) and (A2) require the existence of trajectories that satisfy certain bounds on the L^2-cost; these conditions are the unifying tool in our analysis.

In this paper we extend the existing theory in several directions:

(i) We give a characterization of stabilizability in terms of the finite cost condition (A1). This complements existing characterizations of uniform exponential stabilizability using the solvability of Riccati equations and gives rise to a sufficient uniform finite cost condition (A2) for exponential stabilizability. We also show by example that if $A(\cdot)$ is unbounded, then (A1) is not a sufficient condition for stabilizability, see Remark 3.5.

(ii) We extend [1, Theorem 1]. In particular, it is shown that for bounded $(A(\cdot), B(\cdot))$ complete controllability is equivalent to the fact that using (possibly unbounded) feedback the maximal Lyapunov exponent of the system may be assigned to be below any prespecified constant.

(iii) We show that complete controllability (without necessarily assuming uniform controllability) implies the existence of a (possibly unbounded) feedback so that the closed-loop system is asymptotically stable.

(iv) Concerning the question of dynamic feedback, we provide an alternative proof for the result of [5] for the case of uniform exponential stabilizability and show that a similar statement is true for exponential stabilizability.

The presentation frequently recalls familiar definitions and results in an effort to make the paper self-contained. The paper is organized as follows. In Section 2 we recall the standard definitions of stability, the characterizations of exponential stability via Lyapunov and Bohl exponents, and definitions relating to controllability; we provide definitions of stabilizability, introduce the relevant Riccati equation and formulate an optimal control problem. Section 3 contains the main contributions of the paper. In Section 3.1 we introduce the finite cost condition (A1), give some characterizations and show that this bound implies asymptotic stabilizability using the associated Riccati equation as an intermediate tool.

In Theorem 3.6 it is shown that in the bounded case complete controllability is equivalent to the assignability of arbitrary Lyapunov exponents. Two examples are provided with one discussing the claims of [6] and a further one discussing the relation of bounded exponential stabilizability and uniform exponential stabilizability. In Section 3.2 the L^2-cost bound (A1) is strengthened to a uniform finite cost condition (A2), and it is shown that this yields a characterization of uniform exponential stabilizability. Section 4 provides a new proof for the result of Rotea and Khargonekar and a similar statement for the case of exponential stabilization.

2. Preliminaries

We denote the space of column vectors with real entries by $\mathbb{R}^n$ and for $x \in \mathbb{R}^n$ the Euclidean norm is $\|x\| := \sqrt{x^\top x}$. For an interval $I \subset \mathbb{R}$ the set of essentially bounded $f : I \to \mathbb{R}^{n\times m}$ is denoted by $L^\infty(I; \mathbb{R}^{n\times m})$, $L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n\times m})$ is the set $f : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times m}$, such that on any compact interval $I \subset R_{\geq 0}$, the restriction $f_I \in L^\infty(I; \mathbb{R}^{n\times m})$, and $L^2(I; \mathbb{R}^m)$ is the set of integrable functions $f : I \to \mathbb{R}^m$ such that $\int_I \|f(\tau)\|^2 \mathrm{d}\tau < \infty$. For real symmetric matrices we frequently use the semi-order induced by the cone of positive definite matrices and we denote by $P \succ Q$ (resp. $P \succeq Q$) the fact that $P - Q$ is positive definite (resp. positive semi-definite).

In the present section we collect definitions of well-known concepts and their characterizations; they will be used in Sections 3 and 4.

2.1. Stability, Lyapunov exponents, and Bohl exponents

We are interested in the stabilization of system (1.1). We briefly recall several distinct notions of stability. The zero solution of system (1.2) is called

$$\textit{attractive} :\iff \forall\, t_0 \geq 0\, \forall\, x^0 \in \mathbb{R}^n : \lim_{t\to\infty} x(t; t_0, x^0) = 0,$$

$$\textit{stable} :\iff \forall\, \varepsilon > 0\, \forall\, t_0 \geq 0\, \exists\, \delta > 0\, \forall\, x^0 \in \mathbb{R}^n : \|x^0\| < \delta,\quad t \geq t_0 \Rightarrow \|x(t; t_0, x^0)\| < \varepsilon,$$

$$\textit{asymptotically stable} :\iff \text{(1.2) is attractive and stable},$$

$$\textit{exponentially stable} :\iff \exists\, M \geq 1\, \exists\, \beta > 0\, \forall\, t \geq 0 : \quad \|\Phi_A(t,0)\| \leq M\, \mathrm{e}^{-\beta t},$$

$$\textit{uniformly exp. stable} :\iff \exists\, M \geq 1\, \exists\, \beta > 0\, \forall\, t \geq t_0 \geq 0 : \quad \|\Phi_A(t,t_0)\| \leq M\, \mathrm{e}^{-\beta(t-t_0)}.$$

It is well-known, see for example [7, Proposition 3.3.2], that for linear systems attractivity of (1.2) implies asymptotic stability. Also, uniform exponential stability implies exponential stability, which in turn implies asymptotic stability. The converse of the latter statements is false.

The concepts regarding exponential stability can be characterized in terms of the *Lyapunov exponent*, defined by

$$k_L(A) := \inf\Big\{\omega \in \mathbb{R} \,|\, \exists\, M_\omega \geq 0\, \forall\, t \geq 0 : \quad \|\Phi_A(t,0)\| \leq M_\omega\, \mathrm{e}^{\omega t}\Big\} \in \mathbb{R}.$$

We then have that (1.2) is exponentially stable if, and only if, $k_L(A) < 0$. The *Bohl exponent* is defined by

$$k_B(A) := \inf\Big\{\omega \in \mathbb{R} \,|\, \exists\, M_\omega \geq 0\, \forall\, t \geq t_0 \geq 0 : \quad \|\Phi_A(t,t_0)\| \leq M_\omega\, \mathrm{e}^{\omega(t-t_0)}\Big\} \in \mathbb{R};$$

it gives rise to the characterization

$$\text{(1.2) is uniformly exponentially stable} \iff k_B(A) < 0. \tag{2.1}$$

See Daleckiĭ and Kreĭn [8, Section III.4] and Hinrichsen and Pritchard [7, Section 3.3]; there you also find that Lyapunov and Bohl exponents are finite if A is bounded.

2.2. Controllability

Kalman [9] introduced the following notions: System (1.1) is called

$$\textit{controllable at } t_0 \geq 0 :\iff \forall\, x^0 \in \mathbb{R}^n\, \exists\, t_1 \geq t_0\ \exists\, u \in L^2([t_0,t_1]; \mathbb{R}^m) : x(t_1; t_0, x^0, u) = 0,$$

$$\textit{completely controllable} :\iff \forall\, (t_0, x^0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n\, \exists\, t_1 \geq t_0\ \exists\, u \in L^2([t_0,t_1]; \mathbb{R}^m) : x(t_1; t_0, x^0, u) = 0.$$

These concepts can be characterized (see Kalman [9, Proposition (5.2)]) in terms of the *controllability Gramian* defined by

$$W(t_0,t_1) := \int_{t_0}^{t_1} \Phi_A(t_0,s)B(s)B(s)^\top \Phi_A(t_0,s)^\top \mathrm{d}s, \quad t_1 \geq t_0 \geq 0 \tag{2.2}$$

as follows:

$$\text{(1.1) is controllable at } t_0 \iff \exists\, t_1 \geq t_0 : W(t_0,t_1) \succ 0,$$

$$\text{(1.1) is completely controllable} \iff \forall\, t_0 \geq 0\, \exists\, t_1 \geq t_0 : \quad W(t_0,t_1) \succ 0.$$

The latter leads to the more restrictive notion of *uniform controllability* of (1.1); see Kalman [9, Definition (5.13)]. In our case, where $A(\cdot)$, $B(\cdot)$ are bounded, this may be formulated as

$$\exists\, \alpha_1, \alpha_2, T > 0 \quad \forall\, t \geq 0 : \alpha_1 I_n \prec W(t, t+T) \prec \alpha_2 I_n. \tag{2.3}$$

Note that the inequality on the right hand side is automatic because of the assumed boundedness of $A(\cdot)$ and $B(\cdot)$.

Cheng [10] (see also Rugh [11, p. 261]) introduced the *weighted controllability Gramian* as a tool for the stabilization problem. It is defined by

$$W_\alpha(t_0,t_1) := \int_{t_0}^{t_1} \mathrm{e}^{4\alpha(t_0-s)} \Phi_A(t_0,s)B(s)B(s)^\top \Phi_A(t_0,s)^\top \mathrm{d}s, \quad t_1 \geq t_0 \geq 0,\ \alpha > 0, \tag{2.4}$$

and is seen to be the controllability Gramian of
$\dot{x} = [A(t) + 2\alpha I]x + B(t)u$.

2.3. Stabilizability

We now introduce the main concepts of stabilizability under investigation; they all correspond to a notion of stability. In particular, we call (1.1)

$$\textit{asymptotically stabilizable} :\iff \exists\, F \in L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^{m\times n}) : \quad \dot{x} = [A + BF]x \text{ is asymp. stable},$$

$$\textit{exponentially stabilizable} :\iff \exists\, \alpha > 0\ \exists\, F \in L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^{m\times n}) : k_L(A+BF) < -\alpha,$$

$$\textit{uniformly exponentially stabilizable} :\iff \exists\, \alpha > 0\ \exists\, F \in L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^{m\times n}) : \ k_B(A+BF) < -\alpha.$$

In case that in any of the above we may choose $F \in L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^{m\times n})$, we say that the system may be *(exponentially etc.) stabilized by bounded feedback*.

Complete controllability and uniform controllability are frequently brought into relation with the notion of *complete stabilizability*. The first use of this term appears to be in the paper of Ikeda et al. [1]. For time-invariant systems this is understood as the possibility to assign arbitrary decay rates (and this is indeed the problem studied in [12]). For time-varying system the very general definition of [1] is discussed in Remark 3.10.

2.4. Stabilizability versus controllability

Ikeda et al. [1, Theorem 3] have shown for bounded $(A(\cdot), B(\cdot))$ that

(1.1) is uniformly controllable
$$\iff \forall\,\alpha \in \mathbb{R}\,\exists\,F \in L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^{m\times n}) : k_B(A+BF) < -\alpha.$$

A more constructive version with the aid of (2.4) is by Cheng [10], see also Rugh [11, Theorem 14.7]:

(1.1) is uniformly controllable, T as in (2.3)
$$\implies \forall\,\alpha > 0 : k_B\left(A - BB^\top W_\alpha(t, t+T)^{-1}\right) < -\alpha.$$

2.5. Optimal control

To study the stabilization problem, we consider, for system (1.1) with initial data $(t_0, x^0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ and control input $u \in L^2([t_0,\infty); \mathbb{R}^m)$, the *finite-time cost on* $[t_0, t_1)$

$$J(t_1; t_0, x^0, u) := \int_{t_0}^{t_1} \left[\|x(\tau; t_0, x^0, u)\|^2 + \|u(\tau)\|^2\right] \mathrm{d}\tau, \qquad (2.5)$$

and the *infinite-time cost on* $[t_0, \infty)$

$$J(\infty; t_0, x^0, u) := \int_{t_0}^{\infty} \left[\|x(\tau; t_0, x^0, u)\|^2 + \|u(\tau)\|^2\right] \mathrm{d}\tau \in [0, \infty]. \qquad (2.6)$$

The *value function* associated to (2.5) is given by

$$V(t_1; t_0, x^0) := \inf_{u\in L^2(t_0,t_1)} J(t_1; t_0, x^0, u). \qquad (2.7)$$

The following proposition summarizes some of the well-known properties of the relationship between the optimal control problem and the differential Riccati equation

$$\frac{\mathrm{d}}{\mathrm{d}t}P(t) = -A(t)^\top P(t) - P(t)A(t) + P(t)B(t)B(t)^\top P(t) - I_n. \quad (2.8)$$

Proposition 2.1 *([13, Theorem 37, p. 364]). Consider system (1.1) on the interval $[t_0, t_1]$ for $0 \leq t_0 \leq t_1 < \infty$. Then the differential Riccati equation (2.8) with final condition $P(t_1) = 0$ has a unique symmetric solution*

$$P(\cdot; t_1, t_0) : [t_0, t_1] \to \mathbb{R}^{n\times n}.$$

This solution has the following properties:

(i) *The feedback*

$$u(t) = -B(t)^\top P(t; t_1, t_0)x(t)$$

yields, for all $x^0 \in \mathbb{R}^n$, the unique optimal solution $(u(\cdot), x(\cdot; t_0, x^0, u))$ to the problem (2.5), and (2.7).

(ii)
$$\forall\, t \in [t_0, t_1] \quad \forall\, x^0 \in \mathbb{R}^n : \quad V(t_1; t, x^0) = (x^0)^\top P(t; t_1, t_0)\, x^0. \quad (2.9)$$

(iii) *In view of* (ii) *and (2.5),*

$$\forall\, t \in [t_0, t_1) : P(t; t_1, t_0) \succ 0.$$

(iv) *From (2.9) we conclude: $\forall\, t_0 \leq t \leq t_1 < t_2$: $P(t; t_1, t_0) \preceq P(t; t_2, t_0)$.*

3. Controllability, stabilizability, and finite cost conditions

In this section we derive, in terms of the cost, a sufficient condition for asymptotic stabilizability and also show that the system (1.1) is completely controllable if, and only if, the Lyapunov exponent is arbitrarily assignable by a suitable feedback. We then concentrate on uniform exponential stabilizability and show that a strengthened condition in terms of the cost holds if, and only if, the Bohl exponent is arbitrarily assignable by a suitable feedback.

3.1. Stabilizability

We begin our discussion of the stabilization problem by introducing a condition for the infinite time cost that can be achieved for the optimal control problem (2.6). This criterion turns out to be already close to stabilizability.

Consider system (1.1). The following finite cost condition is essential in our setup

(A1) $\forall\, t_0 \geq 0\, \exists\, C(t_0) \geq 0\, \forall\, x^0 \in \mathbb{R}^n\, \exists\, u \in L^2([t_0, \infty); \mathbb{R}^m)$: $J(\infty; t_0, x^0, u) \leq C(t_0)\, \|x^0\|^2$.

The concept introduced in (A1) is closely related to what has been called "optimizable" in an infinite-dimensional context. Indeed, the equivalent notion (A1′) defined below is exactly this concept as introduced in [14].

Proposition 3.1. *Consider system (1.1). Then the following assumptions are each equivalent to* (A1)*:*

(A1′) $\forall\,(t_0, x^0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n\, \exists\, u \in L^2([t_0,\infty); \mathbb{R}^m)$: $J(\infty; t_0, x^0, u) < \infty$;

(A1″) $\forall\,(t_0, x^0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n\, \exists\, u \in L^2([t_0,\infty); \mathbb{R}^m) : x(\cdot; t_0, x^0, u) \in L^2([t_0,\infty); \mathbb{R}^n)$ *and* $\lim_{t\to\infty} x(t; t_0, x^0, u) = 0$.

Proof. The implications (A1) ⇒ (A1′) and (A1″) ⇒ (A1′) are trivial.

(A1′) ⇒ (A1): Let $(t_0, x^0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ and set

$$x^0 = \sum_{i=1}^n x_i^0\, e_i \quad \text{for the canonical vectors } e_1, \dots, e_n \in \mathbb{R}^n.$$

Then Assumption (A1′) yields

$$\forall\, i = 1, \dots, n\, \exists\, u_i \in L^2([t_0,\infty); \mathbb{R}^m) : J(\infty; t_0, e_i, u_i) < \infty.$$

Now by linearity of (1.1) we conclude, for $u(\cdot) := \sum_{i=1}^n x_i^0\, u_i(\cdot)$, that

$$x(t; t_0, x^0, u) = \sum_{i=1}^n x_i^0 \left[\Phi(t, t_0)\, e_i + \int_{t_0}^t \Phi(t, \tau)B(\tau)\, u_i(\tau)\, \mathrm{d}\tau\right] = \sum_{i=1}^n x_i^0\, x(t; t_0, e_i, u_i)$$

and therefore, using the inequality $(\sum_{k=1}^n a_k)^2 \leq n(\sum_{k=1}^n a_k^2)$, we arrive at

$$\begin{aligned} J(\infty; t_0, x^0, u) &\overset{(2.6)}{=} \int_{t_0}^\infty \left[\|x(\tau; t_0, x^0, u)\|^2 + \|u(\tau)\|^2\right] \mathrm{d}\tau \\ &= \int_{t_0}^\infty \left[\left\|\sum_{i=1}^n x_i^0\, x(t; t_0, e_i, u_i)\right\|^2 + \left\|\sum_{i=1}^n x_i^0 u_i(\tau)\right\|^2\right] \mathrm{d}\tau \\ &\leq n \max_{i\in\{1,\dots,n\}} |x_i^0|^2 \sum_{i=1}^n \int_{t_0}^\infty \left[\|x(t; t_0, e_i, u_i)\|^2 + \|u_i(\tau)\|^2\right] \mathrm{d}\tau \\ &\leq n \sum_{i=1}^n J(\infty; t_0, e_i, u_i)\, \|x^0\|^2 =: C(t_0)\, \|x^0\|^2. \end{aligned}$$

(A1′) ⇒ (A1″): If $J(\infty; t_0, x^0, u) < \infty$, then $x \in L^2([t_0,\infty); \mathbb{R}^n)$, $u \in L^2([t_0,\infty); \mathbb{R}^m)$, and by boundedness of $A(\cdot), B(\cdot)$ we have

$$\dot{x}(\cdot) = A(\cdot)\, x(\cdot) + B(\cdot)\, u(\cdot) \in L^2([t_0,\infty); \mathbb{R}^n).$$

This implies $\lim_{t\to\infty} x(t) = 0$ by Barbălat's lemma, see e.g. [7, Lemma 2.3.9]. The proof is complete. □

The next theorem is the first main result of the present paper. We derive a sufficient criterion for asymptotic stabilizability, see (A1″), based on the L^2-cost bound (A1).

Theorem 3.2. *For system (1.1). We have the implications*

$$(\text{A1}) \Longrightarrow (3.1) \Longrightarrow (3.2) \Longrightarrow (3.3)$$

where, with $C(t_0)$, $t_0 \geq 0$ given by (A1),

$$\left.\begin{array}{l}\exists \text{ a function } \gamma : (0,\infty) \to (0,\infty) : \forall\, 0 \leq t_0 < t_1 \leq \infty \\ \forall x^0 \in \mathbb{R}^n : \gamma(t_1 - t_0)\,\|x^0\|^2 \leq V(t_1; t_0, x^0) \leq C(t_0)\,\|x^0\|^2\end{array}\right\} \tag{3.1}$$

$$\left.\begin{array}{l}\forall\, t_0 \geq 0\ \exists \text{ solution } \varPi(\cdot, t_0) : [t_0,\infty) \to \mathbb{R}^{n\times n} \text{ of } (2.8) \\ \quad \text{with initial condition } \varPi(t_0,t_0) := \lim_{t_1\to\infty} P(t_0; t_1, t_0), \\ \quad \text{where } P(\cdot; t_1, t_0) \text{ is from Proposition 2.1 and this} \\ \quad \text{solution is unique, symmetric and positive definite} \\ \quad \text{for all } t \geq t_0;\ \text{further, for all } t \geq t_0, \\ \quad \varPi(t,t_0) = \lim_{t_1\to\infty} P(t; t_1, t_0)\end{array}\right\}. \tag{3.2}$$

$$\left.\begin{array}{l}\text{With } \varPi(\cdot) := \varPi(\cdot, t_0) \text{ as defined in (3.2), the closed-loop} \\ \quad \text{system } \dot{x} = [A - BB^\top \varPi]\,x \text{ is asymptotically stable.} \\ \quad \text{In particular, } B(\cdot)^\top \varPi(\cdot) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^{m\times n})\end{array}\right\}. \tag{3.3}$$

For the proof of Theorem 3.2, we record the following lemma, a proof of which can be found, for example, in [7, Lemma 3.3.4].

Lemma 3.3. *The transition matrix of the homogeneous system (1.2) satisfies, for $a := \|A\|_{L^\infty(0,\infty)}$, and all $0 \leq s \leq t$*

$$\left.\begin{array}{l}\mathrm{e}^{-a(t-s)} \leq \|\varPhi_A(t,s)\| \leq \mathrm{e}^{a(t-s)} \\ \mathrm{e}^{-a(t-s)} \leq \|\varPhi_A(s,t)\|^{-1} \leq \mathrm{e}^{a(t-s)}.\end{array}\right\} \tag{3.4}$$

Proof of Theorem 3.2. (A1) $\Rightarrow$ (3.1): For $x^0 = 0$ there is nothing to show, so assume $x^0 \neq 0$. Set $a := \|A\|_{L^\infty(0,\infty)}$, $b := \|B\|_{L^\infty(0,\infty)}$. We derive, for all $t \in [t_0, t_1]$ and all $u \in L^2([t_0,t_1]; \mathbb{R}^m)$, and in view of the convolution inequality [7, Proposition A.3.14],

$$\begin{aligned}\|x(t; t_0, x^0, u)\| &\geq \|\varPhi(t,t_0)\,x^0\| - \left\|\int_{t_0}^{t} \varPhi(t,\tau)B(\tau)\,u(\tau)\,\mathrm{d}\tau\right\| \\ &\overset{(3.4)}{\geq} \mathrm{e}^{-a(t-t_0)}\,\|x^0\| - \int_{t_0}^{t} \mathrm{e}^{a[t-\tau]} b\,\|u(\tau)\|\,\mathrm{d}\tau \\ &\geq \mathrm{e}^{-a(t-t_0)}\,\|x^0\| - b\,\|\mathrm{e}^{a\cdot}\|_{L^1(0,t-t_0)}\,\|u\|_{L^2(t_0,t)} \\ &=: \varLambda(t-t_0, u).\end{aligned} \tag{3.5}$$

Note that $\varLambda(r,u)$ is strictly decreasing in $r > 0$. Define

$$T := \min\{t_1,\ t_0 + (\log 2)/2a\} \tag{3.6}$$
$$R(r) := \min\{r,\ (\log 2)/2a\}, \quad r \geq 0$$
$$t_u := \min\left\{t \geq t_0 \mid \varLambda(t - t_0, u) = \|x^0\|/2\right\}.$$

Then

$$J(t_1; t_0, x^0, u) \geq J(T; t_0, x^0, u) \overset{(3.5)}{\geq} \int_{t_0}^{\min\{T,t_u\}} \left[\frac{1}{4}\|x^0\|^2 + \|u(\tau)\|^2\right] \mathrm{d}\tau.$$

We will now derive a lower bound for the expression on the right hand side and consider two cases:

(i) If $u \in L^2([t_0,t_1])$ is such that $t_u \geq T$ we have

$$J(T; t_0, x^0, u) \geq \frac{T - t_0}{4}\,\|x^0\|^2 = \frac{R(t_1 - t_0)}{4}\,\|x^0\|^2. \tag{3.7}$$

(ii) If $t_u < T$, then $\varLambda(t_u - t_0, u) = \frac{1}{2}\,\|x^0\|$ yields

$$\begin{aligned}\|u\|^2_{L^2(t_0,t_u)} &\overset{(3.5)}{=} \left(\frac{\mathrm{e}^{-a[t_u-t_0]} - \frac{1}{2}}{b\,\|\mathrm{e}^{a\cdot}\|_{L^1(0,t_u-t_0)}}\right)^2 \|x^0\|^2 \\ &\overset{(3.6)}{\geq} \frac{\left(\sqrt{\frac{1}{2}} - \frac{1}{2}\right)^2}{b^2\,\|\mathrm{e}^{a\cdot}\|^2_{L^1(0,T-t_0)}}\,\|x^0\|^2\end{aligned} \tag{3.8}$$

we obtain from (3.7), and (3.8) that for all $u \in L^2([t_0,t_1]; \mathbb{R}^m)$, $x^0 \in \mathbb{R}^n$

$$\begin{aligned}J(t; t_0, x^0, u) &\geq \min\left\{\frac{R(t_1 - t_0)}{4},\ \frac{\left(\sqrt{\frac{1}{2}} - \frac{1}{2}\right)^2}{b^2\,\|\mathrm{e}^{a\cdot}\|^2_{L^1(0,R(t_1-t_0))}}\right\}\|x^0\|^2 \\ &=: \gamma(t_1 - t_0)\|x^0\|^2.\end{aligned}$$

This proves the lower bound of the claim.

Suppose Assumption (A1) holds. For u given as in (A1) we conclude, for all $t \in [t_0, t_1]$ and all $x^0 \in \mathbb{R}^n$,

$$\begin{aligned}(x^0)^\top P(t; t_1, t_0)\,x^0 &\overset{(2.9)}{=} V(t_1; t, x^0) \overset{(2.7)}{\leq} J(t_1; t, x^0, u) \\ &\leq J(\infty; t, x^0, u) \overset{(\text{A1})}{\leq} C(t)\,\|x^0\|^2,\end{aligned} \tag{3.9}$$

which proves the upper bound of the claim.

(3.1) $\Rightarrow$ (3.2): Fix $t_0 \geq 0$. Then we have for all $x^0 \in \mathbb{R}^n$

$$\begin{aligned}\gamma(t_1 - t_0)\,\|x^0\|^2 &\overset{(3.1)}{\leq} (x^0)^\top P(t; t_1, t_0)\,x^0 \\ &\overset{(2.9)}{=} V(t_1; t, x^0) \overset{(3.1)}{\leq} C(t)\,\|x^0\|^2;\end{aligned}$$

and therefore, since $P(t; \cdot, t_0)$ is monotonically non-decreasing by Proposition 2.1(iv), we conclude existence, positive definiteness, and symmetry of

$$\varPi(t, t_0) := \lim_{t_1\to\infty} P(t; t_1, t_0).$$

As the monotone limit of solutions of (2.8), $\varPi(\cdot, t_0)$ solves the Riccati equation (2.8) with initial condition $\varPi(t_0, t_0) = \lim_{t_1\to\infty} P(t_0; t_1, t_0)$. Finally, uniqueness follows from standard theorems on ordinary differential equations.

(3.2) $\Rightarrow$ (3.3): We now consider the closed-loop system

$$\dot{x} = [A(t) - B(t)B(t)^\top \varPi(t, 0)]\,x \tag{3.10}$$

on $[0,\infty)$, which corresponds to the control input $u(\cdot) := -B(\cdot)^\top \varPi(\cdot, 0)x(\cdot)$. The derivative of the positive definite function $(t,x) \mapsto x^\top \varPi(t,0)x$ along any solution $x(\cdot)$ of (3.10) satisfies, for all $t \geq 0$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t)^\top \varPi(t,0)x(t)\right) &= x(t)^\top\Big[A(t)^\top \varPi(t,0) + \varPi(t,0)A(t) \\ &\qquad - 2\varPi(t,0)B(t)B(t)^\top \varPi(t,0) + \frac{\mathrm{d}}{\mathrm{d}t}\varPi(t,0)\Big]x(t) \\ &\overset{(2.8)}{=} x(t)^\top\left[-I_n - \varPi(t,0)B(t)B(t)^\top \varPi(t,0)\right]x(t),\end{aligned}$$

and integration of the latter over $[0, t_1]$ yields

$$\begin{aligned}-x(0)^\top \varPi(0,0)x(0) &\leq x(t_1)^\top \varPi(t_1,0)x(t_1) - x(0)^\top \varPi(0,0)x(0) \\ &= -\int_0^{t_1} \left[\|x(\tau)\|^2 + \|B(\tau)^\top \varPi(\tau,0)x(\tau)\|^2\right] \mathrm{d}\tau \\ &\leq -\int_0^{t_1} \|x(\tau)\|^2\,\mathrm{d}\tau\end{aligned}$$

and so that $x(0)^\top\, \varPi(0,0)\,x(0) \geq \|x\|_{L^2(0,t_1)}$; and similarly $x(0)^\top\, \varPi(0,0)\,x(0) \geq \|u\|_{L^2(0,t_1)}$.

Since the above two inequalities hold true for all $t_1 \geq 0$, it follows that $x(\cdot) \in L^2([0,\infty),\mathbb{R}^n)$ and $u(\cdot) \in L^2([0,\infty),\mathbb{R}^m)$. As $A(\cdot)$ and $B(\cdot)$ are essentially bounded by assumption, it follows that

$$\dot{x}(\cdot) = [A(\cdot)x(\cdot) + B(\cdot)u(\cdot)] \in L^2([0,\infty),\mathbb{R}^n).$$

Hence, $x(\cdot)$ is an L^2-function with derivative in $L^2([0,\infty),\mathbb{R}^n)$ and we may apply [7, Lemma 2.3.9] to conclude that $\lim_{t\to\infty} x(t) = 0$.

If $(t_0, x^0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ and $x(\cdot; t_0, x^0)$ is the unique solution to

$$\dot{x} = [A(t) - B(t)B(t)^\top \Pi(t,0)]x, \qquad x(t_0) = x^0,$$

then $x(t; t_0, x^0) = x(t; 0, x(0; t_0, x^0))$, $t \geq t_0$ and by the previous findings it follows that $\lim_{t\to\infty} x(t; t_0, x^0) = 0$. Hence the closed-loop system is attractive; and for linear systems this implies asymptotic stability, see [7, Proposition 3.3.2]. Finally, we have $B(\cdot)^\top \Pi(\cdot) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^{m\times n})$ as $B(\cdot)$ is essentially bounded and $\Pi(\cdot)$ is continuous. This completes the proof of the theorem. $\square$

We note the following immediate corollary.

Corollary 3.4. *If system (1.1) given by $(A(\cdot), B(\cdot))$ is such that for some $\lambda > 0$ the shifted system $(A(\cdot) - \lambda I, B(\cdot))$ satisfies* (A1), *then (1.1) is exponentially stabilizable by locally bounded feedback.*

Proof. By Theorem 3.2 there exists an asymptotically stabilizing feedback F for $(A(\cdot) - \lambda I, B(\cdot))$. Hence, $k_L(A - \lambda I + BF) \leq 0$ and so $k_L(A + BF) \leq -\lambda$. $\square$

Remark 3.5. (i) Note that in Theorem 3.2 it cannot be expected that the closed-loop system in (3.3) has a negative Lyapunov exponent: consider $\dot{x} = \frac{-1}{1+t}x$, which clearly satisfies (A1) but $k_L(A - BB^\top P) < 0$ is not the case.

Also, it is not necessary that (A1) holds in order that a system is stabilizable, as in general, asymptotically stable linear time-varying systems need not have trajectories that are square integrable. An example in point is $\dot{x} = \frac{-1}{2(1+t)}x$, which has solutions $x(t; t_0, x^0) = \sqrt{(t_0+1)/(t+1)}\,x(t_0)$ for $t \geq t_0 \geq 0$.

(ii) Furthermore, it has to be pointed out that Theorem 3.2 is strictly a result for bounded (A, B). The proof uses boundedness in an essential manner in the application of Barbălat's lemma. And indeed, for unbounded (A, B) the cost condition (A1) does not imply stabilizability. To see this consider, for some $\varepsilon > 0$, a C^∞-function $z : (-\varepsilon, \infty) \to (0, \infty)$, so that $z(\cdot) \in L^2([0,\infty))$ and $z(t)$ does not converge for $t \to \infty$. Defining $a(t) := \dot{z}(t)/z(t)$ for $t \geq 0$, it follows by construction that z solves $\dot{x} = a(t)x$ and (A1) is satisfied. But the system is not asymptotically stable and cannot be stabilized. Note that by construction $a(\cdot)$ is unbounded. $\square$

The much stronger condition of arbitrarily assigning the Lyapunov exponent by state feedback is equivalent to complete controllability and treated, by invoking Theorem 3.2, in the next main result.

Theorem 3.6. *System (1.1) is completely controllable if, and only if, for all $\lambda > 0$ there exists an $F \in L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^{m\times n})$ such that $k_L(A+BF) \leq -\lambda$. Both conditions imply* (A1)*; and* (A1) *does not, in general, imply complete controllability of (1.1).*

Proof. We proceed in several steps.

Step 1: We show that complete controllability implies (A1): Choose, for initial data (t_0, x^0), a $t_1 \geq t_0$ and $u(\cdot)$ such that $x(t_1; t_0, x^0, u) = 0$. Define

$$\tilde{u}(t) := \begin{cases} u(t), & t \in [t_0, t_1) \\ 0, & t \geq t_1. \end{cases} \tag{3.11}$$

Then $x(t; t_0, x^0, u) = 0$ for $t \geq t_1$, and so $J(\infty; t_0, x^0, \tilde{u}) = J(t_1; t_0, x^0, u) < \infty$. This proves (A1′) and the claim follows from Proposition 3.1. The converse of Step 1 is obviously false, as the simple example $\dot{x} = -x$ shows.

Step 2: Assume that (1.1) is completely controllable and note that by (2.2) this is true if, and only if, $(A + \lambda I, B)$ is completely controllable for every $\lambda \in \mathbb{R}$. Fix $\lambda > 0$. By Step 1 the system $(A + \lambda I, B)$ satisfies (A1) and so Theorem 3.2 yields the existence of some $F \in L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^{m\times n})$ such that $k_L(A + \lambda I + BF) \leq 0$. Now it is straightforward to see that $k_L(A + BF) \leq -\lambda$.

Step 3: Finally, assume that for every $\lambda < 0$ we may achieve $k_L(A + BF) \leq -\lambda$. We have, by definition of $W(\cdot,\cdot)$,

$$\forall\, 0 \leq t_0 \leq t_1 \leq t_2 : \ker W(t_0, t_2) \subset \ker W(t_0, t_1),$$

and since $\ker W(t_0, t)$ is a finite dimensional space, it is easily verified that

(1.1) is not completely controllable

$$\Leftrightarrow \exists\, t_0 \geq 0 : \bigcap_{t\geq t_0} \ker W(t_0, t) \neq \{0\}.$$

Now assume that (1.1) is not completely controllable and

$$\text{choose } t_0 \geq 0 \quad\text{and}\quad x^0 \in \bigcap_{t\geq t_0} \ker W(t_0, t) \quad\text{such that } \|x^0\| = 1.$$

Let $F(\cdot) \in L^\infty([t_0,\infty), \mathbb{R}^{m\times n})$ be arbitrary and set $u(\cdot) := F(\cdot)\Phi_{A+BF}(\cdot, t_0)x^0$ and $z(\cdot) := \int_{t_0}^{\cdot} \Phi_A(t_0, s)B(s)u(s)\,\mathrm{d}s$. Then the unique solution $x_F(\cdot)$ of

$$\dot{x} = [A + BF]x, \qquad x(t_0) = x^0$$

satisfies by variation of constants that

$$\begin{aligned} x_F(t) &= \Phi_A(t, t_0)x^0 + \int_{t_0}^t \Phi_A(t, s)B(s)u(s)\,\mathrm{d}s \\ &= \Phi_A(t, t_0)\left[x^0 + z(t)\right]. \end{aligned} \tag{3.12}$$

We now use the well-known fact, [15, Hilfssatz 3.1], that for all $t \geq t_0$ we have $z(t) \in \operatorname{im} W(t_0, t)$ and briefly summarize the arguments provided in [15] for the veracity of that claim:

It is clear that

$$\operatorname{im} W(t_0, t) \subset \mathcal{R} := \left\{ \int_{t_0}^t \Phi_A(t_0, s)B(s)u(s)ds \,|\, u(\cdot) \in L^1(t_0, t) \right\}.$$

As $W(t, t_0)$ is symmetric, it follows that equality holds if, and only if, the intersection $\ker W(t_0, t) \cap \mathcal{R} = \{0\}$. Now $x \in \ker W(t_0, t)$ if, and only if, $B^\top(s)\Phi_A(t_0, s)^\top x = 0$ almost everywhere on $[t_0, t]$. So if $x \in \ker W(t_0, t) \cap \mathcal{R}$, we have for a suitable $u(\cdot)$

$$\|x\|^2 = x^\top x = \int_{t_0}^t x^\top \Phi_A(t_0, s)B(s)u(s)ds = 0,$$

and this proves the claim.

Continuing with the proof, note that since $x^0 \in \ker W(t_0, t)$ for all $t \geq t_0$ and $W(t_0, t)$ is diagonalizable by orthogonal transformation, we conclude

$$\begin{aligned} \forall\, t \geq t_0 : \|x^0 + z(t)\| &= \sqrt{\|x^0\|^2 + \|z(t)\|^2} \\ &= \sqrt{1 + \|z(t)\|^2}. \end{aligned} \tag{3.13}$$

Thus

$$\begin{aligned} k_L(A + BF) &\overset{(3.12)}{\geq} \limsup_{t\to\infty} \frac{1}{t} \log \|\Phi_A(t, t_0)\left(x^0 + z(t)\right)\| \\ &\geq \limsup_{t\to\infty} \frac{1}{t} \log\left(\|\Phi_A(t_0, t)\|^{-1} \sqrt{1 + \|z(t)\|^2} \right) \\ &= \limsup_{t\to\infty} \frac{1}{t} \left(\log \|\Phi_A(t_0, t)\|^{-1} + \frac{1}{2}\log\left(1 + \|z(t)\|^2\right) \right) \\ &\geq \limsup_{t\to\infty} \frac{1}{t} \log \|\Phi_A(t_0, t)\|^{-1} \overset{(3.4)}{\geq} -\|A\|_\infty, \end{aligned}$$

where the first inequality is a consequence of the definition of the Lyapunov exponent and the last inequality uses Lemma 3.3.

It follows that without the assumption of complete controllability it is not possible to assign arbitrarily negative Lyapunov exponents. □

We conclude this section with an example showing that various implications between controllability, stabilizability by bounded feedback, and existence of positive and bounded solutions of the Riccati equation (2.8) cannot be expected.

Example 3.7. Consider $a(\cdot) \equiv 1, b : t \mapsto (t+1)^{-1}$, so that we have

$$\dot{x}(t) = x(t) + \frac{1}{t+1}\, u(t), \quad t \geq 0. \tag{3.14}$$

(1) First note that the system (3.14) is completely controllable by immediate calculation. This implies that (A1′) is satisfied because we can steer any initial condition (t_0, x^0) to the origin in finite time. By Proposition 3.1, (A1) holds.

(2) Next we show that the system (3.14) cannot be expected to be stabilizable by a bounded feedback. Let $u(\cdot) = f(\cdot)\,x(\cdot)$ and suppose that $f(\cdot)$ is bounded. Then $\lim_{t\to\infty} a(t) + b(t)f(t) = \lim_{t\to\infty} 1 + f(t)/(t+1) = 1$ and hence the closed-loop system satisfies, for large t, $\dot{x}(t) = [a(t) + b(t)f(t)]x(t) \geq x(t)/2$, and so it is unstable and, in particular, the feedback is not stabilizing.

(3) It is easy to see that the associated control Riccati equation (2.8), i.e.

$$\dot{p}(t) + 2p(t) - \frac{1}{(t+1)^2}\, p(t)^2 + 1 = 0$$

to the system (3.14) does not have any positive bounded solution.

(4) In particular, system (3.14) provides a counterexample to [6, Theorem 3.1] where the authors claim the equivalence of (i) "global null-controllability", (ii) the existence of a bounded positive solution to the associated control Riccati equation, and (iii) the possibility of assigning arbitrary Lyapunov exponents by bounded feedback. We have seen that complete controllability holds for the present example. This yields (i); however, (ii) and (iii) fail.

Example 3.8. In this example we show that it may be possible to assign arbitrary Lyapunov exponents by bounded feedback without the possibility of uniform exponential stabilization. To this end, let

$$T_0 := 0, \qquad T_{k+1} := T_k + 2(k+1) = k(k+1), \quad k \in \mathbb{N}_0$$

and consider $\dot{x} = x + b(t)u(t)$ for

$$b(t) = \begin{cases} 1, & t \in [T_k, T_k + k) \\ 0, & t \in [T_k + k, T_k + 2k). \end{cases}$$

So for any feedback $u(t) = f(t)\,x(t)$ with $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R})$ we have

$$\forall\, k \in \mathbb{N} : \Phi_f(T_{k+1}, T_k + k) = \exp \int_{T_k+k}^{T_{k+1}} [a(\tau) + b(\tau)f(\tau)]\,\mathrm{d}\tau = \mathrm{e}^k$$

and therefore it is impossible to find a uniform constant $M \geq 0$ and $\lambda < 1$ such that, for all $k \in \mathbb{N}_0$, we have $|\Phi_f(T_{k+1}, T_k + k)| = \mathrm{e}^k \leq M\mathrm{e}^{\lambda k}$. This shows that the Bohl exponent of the closed-loop system is at least equal to 1.

On the other hand, it is easy to see that with the choice of $f(\cdot) = -1-2\lambda$ for any $\lambda \in \mathbb{R}$, the closed-loop system is $\dot{x} = [1+b(t)f(t)]x$ and

$$\forall\, k \in \mathbb{N}_0 : \Phi_f(T_k + 2k, T_k) = \mathrm{e}^{(-1-2\lambda)k}\mathrm{e}^k = \mathrm{e}^{-\lambda\, 2k}.$$

Now it is easy to derive that the Lyapunov exponent is $k_L\big(1 + b(t)f(t)\big) = -\lambda$.

3.2. Uniform exponential stabilizability

In view of Theorem 3.2, the reason that the construction of the feedback does not result in a bounded stabilizing feedback lies in the specific t_0-dependence of the bound $C(t_0)$ in (A1). Indeed, in (3.9) we have seen that $\Pi(t)$ is bounded by $C(t)$. It would then seem reasonable to strengthen (A1) to the uniform finite cost condition

(A2) $\exists\, C \geq 0\ \forall\, (t_0, x^0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n\ \exists\, u \in L^2([t_0, \infty); \mathbb{R}^m) :$ $J(\infty; t_0, x^0, u) \leq C\,\|x^0\|^2.$

To show the equivalence of (A2) and uniform exponential stabilizability, we note that, via application of Theorem 3.2, we have that (A2) implies (3.1) with $C(t_0) = C$ for all $t_0 \geq 0$; in other words: (A2) implies

$$\begin{aligned} &\exists \text{ a function } \gamma : (0, \infty) \to (0, \infty)\ \forall\, 0 \leq t_0 < t_1 \leq \infty \\ &\quad \forall\, x^0 \in \mathbb{R}^n : \gamma(t_1 - t_0)\,\|x^0\|^2 \leq V(t_1; t_0, x^0) \leq C\,\|x^0\|^2. \end{aligned} \tag{3.15}$$

We are now in a position to show that (A2) is equivalent to uniform exponential stabilizability.

Theorem 3.9. *For any system (1.1) we have:*

(A2) $\Longleftrightarrow$ *(1.1) is uniformly exponentially stabilizable by bounded feedback.*

Proof. $\Leftarrow$ Let $F(\cdot) \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^{m\times n})$ be uniformly exponentially stabilizing. Denote, for $(t_0, x^0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$, the unique solution of the initial value problem $\dot{x} = [A + BF]x, x(t_0) = x^0$, by $x_F(\cdot; t_0, x^0) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and set

$$u(\cdot) = F(\cdot)\,x_F(\cdot; t_0, x^0).$$

By uniform exponential stability, there exist $M \geq 1, \beta > 0$ such that

$$\forall\, (t_0, x^0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n\ \forall\, t \geq t_0 : \|x_F(t; t_0, x^0)\| \leq M\,\mathrm{e}^{-\beta(t-t_0)}\|x^0\|,$$

and therefore, $\|u(t)\| \leq \|F\|_{L^\infty(0,\infty)} M\,\mathrm{e}^{-\beta(t-t_0)}\|x^0\|$, and (A2) clearly holds.

$\Rightarrow$ Suppose (A2) holds, let $\Pi(\cdot, 0)$ be the solution of (2.8) defined by (3.2), and define $F(t) = -B(t)^\top \Pi(t, 0)$. As

$$\Pi(t, 0) = \lim_{t_1\to\infty} P(t; t_1, 0) \succeq P(t; t+1, 0),$$

it follows from (A2) and with $\alpha := \gamma(1)$, where $\gamma(\cdot)$ is given by (3.15), that

$$\forall\, t \geq 0 : \alpha\, I_n \preceq \Pi(t, 0) \preceq C\, I_n. \tag{3.16}$$

This shows that $F \in L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^{m\times n})$.

Following the calculations in the proof of (3.2), we see that the derivative of $x(t)^\top \Pi(t, 0)x(t)$ along any solution $x(\cdot)$ of (3.10) satisfies, for all $t \geq 0$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(x(t)^\top \Pi(t, 0)x(t)\right) &\leq -\|x(t)\|^2 \\ &\overset{(3.16)}{\leq} -\frac{1}{C}\left(x(t)^\top \Pi(t, 0)x(t)\right) \end{aligned}$$

and integration of the latter over $[t_0, t]$ yields for all $t \geq t_0$ that

$$\begin{aligned} \|x(t)\|^2 &\overset{(3.16)}{\leq} \frac{1}{\alpha} x(t)^\top \Pi(t, 0)x(t) \\ &\leq \frac{1}{\alpha}\mathrm{e}^{-(t-t_0)/C} x(t_0)^\top \Pi(t_0, t_0)x(t_0) \\ &\overset{(3.16)}{\leq} \frac{C}{\alpha}\mathrm{e}^{-(t-t_0)/C}\|x(t_0)\|^2. \end{aligned}$$

This shows uniform exponential stability of $\dot{x} = [A + BF]x$. □

Remark 3.10. There are some results known for the case that $(A(\cdot), B(\cdot))$ are unbounded and we briefly comment on these.

(i) If boundedness is not assumed, it is still true that uniform controllability, as defined in [9, Definition (5.13)], implies that arbitrary Bohl exponents may be assigned, see [1, Theorem 2]. The converse has no hope of being true, as in the unbounded case a system without inputs can have Bohl exponent equal to $-\infty$, so the property that the Bohl exponent is below any real bound does not really say much about the system. A further example may be found in [1, Example 3].

(ii) Again in the case of unbounded $(A(\cdot), B(\cdot))$, a characterization of complete controllability is possible using the concept of *complete stabilizability* as introduced in [1]. The latter property is satisfied if arbitrary varying exponential decay rates can be achieved by feedback. More precisely, it is required that for any continuous function $(t, t_0) \mapsto \delta(t, t_0)$ there exists a feedback F and constants $a(t_0)$, $t_0 \geq 0$ such that the closed-loop system satisfies

$$\|\Phi_F(t, t_0)\| \leq a(t_0)\, \mathrm{e}^{-\delta(t,t_0)}, \quad \forall\, t \geq t_0.$$

Ikeda et al. [1] prove for possibly unbounded $(A(\cdot), B(\cdot))$ that system (1.1) is completely controllable if, and only if, it is completely stabilizable. □

We now derive a sufficient condition for (A2) that is related to 'uniform stabilizability' as introduced in [16] for the discrete-time case to show the existence of a uniformly exponentially stabilizing feedback. The condition below is a slight modification of this and also interpreted for continuous time systems. Loosely speaking, condition (3.17) says that, if on a given interval a trajectory is not decaying, then it is possible to control the system sufficiently. Consider

$$\begin{aligned}&\exists\; T \geq S > 0\; \exists\, d \in [0, 1)\; \exists\, c > 0\; \forall\, t \geq 0\; \forall\, x \in \mathbb{R}^n :\\ &\quad \|\Phi_A(t+T, t)x\| \geq d\|x\| \Rightarrow x^\top W(t, t+S)x \geq c\|x\|^2. \end{aligned} \tag{3.17}$$

Proposition 3.11. (3.17) *implies* (A2).

Proof. For reasons of space, we only give an outline. Suppose that (3.17) holds for given constants S, T, d, c. As $S \leq T$ by assumption, we have $W(t, t+S) \preceq W(t, t+T)$, and thus we may assume that $S = T$.

Fix $x^0 \in \mathbb{R}^n$, $t_0 \geq 0$ and define $t_k = t_0 + kT$ for $k = 0, 1, \ldots.$

For each k we choose an orthogonal transformation $T(k)$ such that

$$T(k)^\top W(t_k, t_{k+1})\, T(k) = \begin{bmatrix} V_1(k) & 0 \\ 0 & V_2(k) \end{bmatrix} \quad \text{and}$$
$$V_1(k) \succeq cI, \qquad V_2(k) \prec cI,$$

with $0 \leq \dim V_1(k) = n - \dim V_2(k) \leq n$. Then it may be shown, for any $k \in \mathbb{N}$, that

$$\begin{aligned}&\|\Phi_A(t_{k+1}, t_k)\, x(t_k)\| \geq d\; \|x(t_k)\| \Longrightarrow\\ &\quad \left\| \Phi_A(t_{k+1}, t_k) \left(T(k) \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} T(k)^\top x(t_k) \right) \right\| < d\; \|x(t_k)\|\, . \end{aligned} \tag{3.18}$$

Next we design a control $u_k(\cdot)$ on $[t_k, t_{k+1})$ such that

$$\|x(t_{k+1}; t_k, x(t_k), u_k)\| < d\; \|x(t_k)\|\, . \tag{3.19}$$

If the presupposition in (3.18) does not hold, then put $u_k \equiv 0$ and (3.19) is trivially satisfied. If the presupposition in (3.18) holds, then put

$$u_k(\tau) = -B(\tau)^\top \Phi_A(t_k, \tau)^\top T(k) \begin{bmatrix} V_1(k)^{-1} & 0 \\ 0 & 0 \end{bmatrix} T(k)^\top x(t_k), \qquad \tau \in [t_k, t_{k+1})$$

and a straightforward computation using (3.18) shows (3.19).

Now define a control $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$ by stacking together the controls u_k designed for (3.19) and consider the solution $x(\cdot) := x(\cdot; t_0, x^0, u)$. Then the superposition property of linear ordinary differential equations gives

$$\forall\, k \in \mathbb{N} : \|x(t_{k+1})\| < d^k\, \|x^0\|. \tag{3.20}$$

From the definition of $u_k(\cdot)$, the boundedness of $A(\cdot)$ and $B(\cdot)$, and as $\|V_1(k)^{-1}\| \leq c^{-1}$ we see that

$$\begin{aligned} \|u_k\|_{L^\infty(t_k, t_{k+1})} &\overset{\text{Lemma 3.3}}{\leq} \|B(\cdot)\|_\infty \frac{\mathrm{e}^{aT}}{c} \|x(t_k)\| \\ &\overset{(3.20)}{<} \|B(\cdot)\|_\infty \frac{\mathrm{e}^{aT}}{c} d^k \|x^0\|, \end{aligned} \tag{3.21}$$

and thus a longish but straightforward calculation yields

$$\|u(\cdot)\|^2_{L^2(0,\infty)} < T \left(\frac{\|B(\cdot)\|_\infty \mathrm{e}^{aT}}{c} \right)^2 \frac{1}{1-d^2} \|x^0\|^2.$$

Similarly,

$$\|x(\cdot)\|^2_{L^2(0,\infty)} < \sum_{k=0}^{\infty} \Big[3T\, \mathrm{e}^{2aT}\, \|x(t_k)\|^2 + 3\, T \left(T\, \mathrm{e}^{aT} \|B(\cdot)\|_\infty \|u_k\|_{L^\infty(t_k, t_{k+1})} \right)^2 \Big]$$

and by (3.20) and (3.21) there exists a constant $\tilde{C} > 0$ such that $\|x(\cdot)\|^2_{L^2(0,\infty)} \leq \tilde{C}\, \|x^0\|^2$. This shows (A2). □

4. Dynamic feedback

In [5] it is shown that if system (1.1) is uniformly exponentially stabilizable by linear dynamic state feedback, then it is uniformly exponentially stabilizable. The proof relies on a clever manipulation of the Lyapunov function of the closed-loop system and is of interest in its own right. Here we are able to give a shorter and conceptually simpler proof, based on the previous results. Furthermore, we extend the statement to the case of exponential stabilizability.

A dynamic feedback for system (1.1) is given by a system

$$\begin{aligned} \dot{z}(t) &= K(t)z(t) + L(t)x(t), \qquad z(0) = z^0 \\ u(t) &= M(t)z(t) + N(t)x(t), \end{aligned} \tag{4.1}$$

where $z^0 \in \mathbb{R}^p$, $K \in L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^{p\times p})$, $L \in L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^{p\times n})$, $M \in L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^{m\times p})$, $N \in L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^{m\times n})$. This yields the closed-loop system

$$\begin{aligned} \dot{x}(t) &= [A(t) + B(t)N(t)]\, x(t) + B(t)M(t)z(t), \quad x(0) = x^0 \\ \dot{z}(t) &= K(t)z(t) + L(t)x(t), \qquad\qquad z(0) = z^0. \end{aligned} \tag{4.2}$$

Theorem 4.1. *Consider system* (1.1)*.*

(i) *If there exists a dynamic feedback of the form* (4.1) *such that the closed-loop* (4.2) *is uniformly exponentially stable, then* (1.1) *is uniformly exponentially stabilizable by linear static, time-varying, bounded feedback.*

(ii) *If there exists a dynamic feedback of the form (4.1) such that the closed-loop (4.2) is exponentially stable, then (1.1) is exponentially stabilizable by linear static, time-varying, possibly unbounded feedback.*

Proof. (i) If the closed-loop system (4.2) is uniformly exponentially stable, then there exist constants $D \geq 1, \beta > 0$ such that, for all initial conditions $(t_0, x^0, z^0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^p$ and for all $t \geq t_0$, we have

$$\|x(t; t_0, x^0, z^0)\|^2 + \|z(t; t_0, x^0, z^0)\|^2 \leq D^2 \, \mathrm{e}^{-2\beta(t-t_0)} \left(\|x^0\|^2 + \|z^0\|^2\right). \quad (4.3)$$

We obtain in particular for the choice $z^0 = 0$ a bound only depending on $\|x^0\|$ on the right hand side. Writing

$$u(t) := N(t)\, x(t; t_0, x^0, 0) + M(t)\, z(t; t_0, x^0, 0) \quad (4.4)$$

and $\eta = \max\{\|N\|_\infty, \|M\|_\infty\}$ it follows that

$$\forall\, t \geq 0 : \|u(t)\| \leq \eta D \, \mathrm{e}^{-\beta(t-t_0)} \|x^0\|. \quad (4.5)$$

The combination of (4.3) and (4.5) shows that (A2) is satisfied. Now the claim follows from Theorem 3.9.

(ii) If the closed-loop system (4.2) is exponentially stable, there exist constants $D(t_0) \geq 1, \beta > 0$ such that, for all initial conditions $(x^0, z^0) \in \mathbb{R}^n \times \mathbb{R}^p$ and for all $t \geq t_0$, we have

$$\|x(t; t_0, x^0, z^0)\|^2 + \|z(t; t_0, x^0, z^0)\|^2 \leq D(t_0)^2 \, \mathrm{e}^{-2\beta(t-t_0)} \left(\|x^0\|^2 + \|z^0\|^2\right). \quad (4.6)$$

In particular for $\lambda \in (0, \beta)$, the solution (x_λ, z_λ) of the λ-shifted system

$$\dot{x}(t) = [A(t) + \lambda I + B(t)N(t)]\; x(t) + B(t)M(t)z(t) \quad (4.7)$$

$$\dot{z}(t) = [K(t) + \lambda I]\; z(t) + L(t)x(t) \quad (4.8)$$

satisfies

$$\|x_\lambda(t; t_0, x^0, z^0)\|^2 + \|z_\lambda(t; t_0, x^0, z^0)\|^2 \leq D(t_0)^2 \, \mathrm{e}^{-2(\beta-\lambda)(t-t_0)} \left(\|x^0\|^2 + \|z^0\|^2\right). \quad (4.9)$$

This shows that the system corresponding to $(A(\cdot) + \lambda I, B(\cdot))$ is exponentially stabilizable by dynamic feedback. A calculation as in (4.4), and (4.5) shows that the corresponding input $u_\lambda(\cdot) = N(t)x_\lambda(\cdot) + M(t)z_\lambda(\cdot)$ satisfies

$$\forall\, t \geq 0 : \|u_\lambda(t)\| \leq \eta D(t_0)\, \mathrm{e}^{-(\beta-\lambda)(t-t_0)} \|x^0\|. \quad (4.10)$$

By (4.9) and (4.10) it can be seen that $(A(\cdot)+\lambda I, B(\cdot))$ satisfies (A1′). By Proposition 3.1, (A1) is satisfied, and so by Theorem 3.2 there exists an asymptotically stabilizing static state feedback $F_\lambda(\cdot)$ for this system. If follows that

$$\dot{x}(t) = [A(t) + B(t)F_\lambda(t)]\, x(t)$$

is exponentially stable with maximal Lyapunov exponent less than or equal to $-\lambda$. □

5. Conclusions

In this paper we have filled some gaps in the stabilizability theory of linear time-varying systems. In particular, for the question of characterizing stabilizability in the case that systems are not completely controllable or completely controllable but not uniformly so, criteria for stabilizability are provided.

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