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Positivity

On positivity and stability of linear time-varying Volterra equations

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Abstract. Linear time-varying Volterra integro-differential equations of non-convolution type are considered. Positivity is characterized and a sufficient condition for exponential asymptotic stability is presented.

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Nomenclature

$\mathbb{R}_+^{\ell\times q}$		the set of all nonnegative matrices $M = (m_{ij}) \in \mathbb{R}^{\ell\times q}$ with $m_{ij} \geq 0$ for all $i = 1, \dots, \ell$, $j = 1, \dots, q$
$A \geq B$	iff	$a_{ij} \geq b_{ij}$ for all entries of $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{R}^{\ell\times q}$
$A \gg B$	iff	$a_{ij} > b_{ij}$ for all entries of $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{R}^{\ell\times q}$
$C(D, \mathbb{R}^{\ell\times q})$		the vector space of continuous functions $y : D \to \mathbb{R}^{\ell\times q}$, $D \subset \mathbb{R}^p$
$\phi \geq 0$	iff	the function $\phi \in C(J, \mathbb{R}^n)$ is *nonnegative*, i.e. $\phi(t) \in \mathbb{R}^n_+$ for all $t \in J$, $J \subset \mathbb{R}$ an interval
$\lvert \cdot \rvert$	:	a vector norm or induced matrix norm
$\lVert \cdot \rVert$	:	the supremum norm of continuous functions $y : [0,T] \to \mathbb{R}^n$, with respect to a given norm $\lvert \cdot \rvert$ on $\mathbb{R}^n$
$C_0([0,T], \mathbb{R}^n)$	:=	$\{\phi \in C([0,T], \mathbb{R}^n) \mid \phi(T) = 0\}$, $T > 0$
$NBV([0,T], \mathbb{R}^{n\times n})$	:=	$\{\eta \in BV([0,T], \mathbb{R}^{n\times n}) \mid \eta$ is continuous from the left$\}$
Δ	:=	$\{(t,s) \in \mathbb{R}^2 \mid t \geq s \geq 0\}$

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1. Introduction

We consider linear time-varying Volterra integro-differential equations of non-convolution type

$$\dot{x}(t) = A(t)x(t) + \int_0^t B(t,s)x(s)\mathrm{d}s, \quad t \geq 0, \tag{1.1}$$

where

$$A(\cdot) \in C([0,\infty), \mathbb{R}^{n\times n}), \qquad B(\cdot,\cdot) \in C(\Delta, \mathbb{R}^{n\times n}). \tag{1.2}$$

are given.

In Section 2, we characterize positivity of the system (1.1); positivity means, roughly speaking, that for any nonnegative initial condition the corresponding (unique) solution is also nonnegative. The theory of positive systems is based on the theory of nonnegative matrices founded by Perron and Frobenius, as references we mention [2] and [6]. In recent time, problems of positive systems have attracted a lot of attention from researchers, see [9]-[12] and the references therein.

The characterization of positivity, which we present in Theorem 2.4, generalizes a recent result by [12] where A in (1.1) is constant and the equation is of convolution type.

In Section 3, we exploit positivity and present a sufficient condition for exponential asymptotic stability of (1.1) provided $A(\cdot)$ is a diagonal matrix. Theorem 3.2 generalizes a very recent result in [4], namely Theorem 4.2, which is only for scalar equations and the kernel B must be of convolution type.

2. Positivity

Before we state our main theorem, we recall some basic facts on linear time-varying Volterra equations of non-convolution type (1.1) provided, for example, in [5, pp. 221-223]. Let $A(\cdot), B(\cdot,\cdot)$ be as in (1.2). Let $t_0, T \geq 0$ and $\psi \in C([t_0, t_0+T], \mathbb{R}^n)$ be given. Then, there exists a unique solution $x(\,\cdot\,; [t_0, t_0+T], \psi) : [t_0, \infty) \to \mathbb{R}^n$ of the initial value problem

$$\dot{x}(t) = A(t)x(t) + \int_{t_0}^t B(t,s)x(s)\mathrm{d}s, \; t \geq t_0 + T, \quad x(\cdot)\big|_{[t_0,t_0+T]} = \psi(\cdot). \tag{2.1}$$

That is, $x(\cdot) := x(\,\cdot\,; [t_0, t_0+T], \psi)$ is continuous on $[t_0, \infty)$ and continuously differentiable on $[t_0+T, \infty)$ such that $x(\cdot)$ fulfills the initial condition $x(\cdot)\big|_{[t_0,t_0+T]} = \psi(\cdot)$ and satisfies the differential equation in (2.1) for all $t \geq t_0 + T$.

In the following we write, for simplicity, $x(\,\cdot\,; T, \phi)$ instead of $x(\cdot\,; [0,T], \phi)$ if $t_0 := 0$.

Remark 2.1. It is easy to check that $x(\cdot) := x(\,\cdot\,; [t_0, t_0+T], \psi)$ is the solution of (2.1) if, and only if, $y(\cdot) := x(\cdot + t_0)$ is the solution of the initial value problem

$$\dot{y}(t) = A(t+t_0)y(t) + \int_0^t B(t+t_0, s+t_0)y(s)\mathrm{d}s, \; t \geq T, \quad y(\cdot)\big|_{[0,T]} = \psi(\cdot + t_0). \tag{2.2}$$

Definition 2.2. *Equation* (1.1) *is said to be* positive *if, and only if, for any* $T \geq 0$ *and any nonnegative* $\phi \in C([0,T],\mathbb{R}^n)$*, the solution* $x(\,\cdot\,;T,\phi)$ *of the initial value problem* (2.1) *(with* $t_0 := 0$*) is nonnegative.*

Definition 2.3 ([7]). *A matrix* $A \in \mathbb{R}^{n\times n}$ *is called a Metzler matrix if, and only if, all off-diagonal elements of* $A = (a_{ij})$ *are nonnegative, i.e.* $a_{ij} \geq 0$ *for all* $i \neq j$*.*

We are now ready to state our main result on positivity of (1.1) in terms of the system matrices $A(\cdot)$ and $B(\cdot,\cdot)$.

Theorem 2.4. *Equation* (1.1) *is positive if, and only if,* $A(t)$ *is a Metzler matrix for every* $t \geq 0$ *and* $B(t,s) \geq 0$ *for all* $(t,s) \in \Delta$*.*

Remark 2.5. In view of Definition 2.2, Remark 2.1 and Theorem 2.4, it is easy to see that the following statements are equivalent:

(i) Equation (1.1) is positive;

(ii) $A(t)$ is a Metzler matrix for every $t \geq 0$ and $B(t,s) \geq 0$ for all $(t,s) \in \Delta$;

(iii) For any $t_0, T \geq 0$ and any nonnegative $\psi \in C([t_0, t_0+T],\mathbb{R}^n)$ the solution $x(\,\cdot\,;[t_0,t_0+T],\psi)$ of (2.1) is nonnegative.

Theorem 2.4 generalizes the recent result [12, Th. 3.7]) where (1.1) is considered with constant $A(\cdot) = A$, kernel of convolution type $B(t,s) = B(t-s)$ and $B(\cdot) \in L^1(\mathbb{R}_+,\mathbb{R}^{n\times n})$.

To prove Theorem 2.4, we first collect some well-known facts (see e.g. [3] or [5, pp. 221-223] on time-varying Volterra equations of non-convolution type (1.1) and prove some technical lemmata.

For $A(\cdot), B(\cdot,\cdot)$ as in (1.2), the matrix initial value problem

$$\frac{\partial R(t,s)}{\partial t} = A(t)R(t,s) + \int_s^t B(t,u)R(u,s)\mathrm{d}u, \qquad R(s,s) = I_n \tag{2.3}$$

has a unique solution $R(\cdot,\cdot)\colon \Delta \to \mathbb{R}^{n\times n}$, called the *resolvent* of equation (1.1); this solution is continuously differentiable.

Moreover, for $f \in C(\mathbb{R}_+,\mathbb{R}^n)$ and fixed $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, the initial value problem

$$\dot{x}(t) = A(t)x(t) + \int_{t_0}^t B(t,s)x(s)\mathrm{d}s + f(t), \quad x(t_0) = x_0, \quad t \geq t_0 \tag{2.4}$$

has a unique solution which may be represented by the variation of constants formula

$$x(t) = R(t,t_0)x_0 + \int_{t_0}^t R(t,s)f(s)\mathrm{d}s, \quad t \geq t_0. \tag{2.5}$$

Lemma 2.6. *For any initial interval* $[0,t_0]$*, where* $t_0 \geq 0$*, consider for* (1.1) *a sequence of initial functions* $(\phi_k)_{k\in\mathbb{N}}$ *in* $C([0,t_0],\mathbb{R}^n)$ *satisfying*

(i) $(\phi_k)_{k\in\mathbb{N}}$ *is bounded,*

(ii) *there exists a measurable function* $\phi\colon [0,t_0] \to \mathbb{R}^n$ *such that* $\lim_{k\to\infty}\phi_k(t_0) = \phi(t_0)$ *and* $\lim_{k\to\infty}\phi_k(t) = \phi(t)$ *for almost all* $t \in [0,t_0]$*.*

Then

$$\lim_{k\to\infty} x(t;t_0,\phi_k) = R(t,t_0)\phi(t_0) + \int_{t_0}^{t} R(t,s)\left(\int_0^{t_0} B(s,u)\phi(u)\mathrm{d}u\right)\mathrm{d}s \quad \forall t \ge t_0. \tag{2.6}$$

Proof. For a given $k \in \mathbb{N}$, setting

$$x_0 := \phi_k(t_0) \qquad \text{and} \qquad f(t) := \int_0^{t_0} B(t,u)\phi_k(u)\mathrm{d}u \quad \forall t \ge t_0$$

in (2.4) yields, by invoking (2.5), that

$$x(t;t_0,\phi_k) = R(t,t_0)\phi_k(t_0) + \int_{t_0}^{t} R(t,s)\left(\int_0^{t_0} B(s,u)\phi_k(u)\mathrm{d}u\right)\mathrm{d}s \quad \forall t \ge t_0.$$

Since R and B are continuous and the assertions (i) and (ii) hold, we may apply the Lebesgue dominated convergence theorem to arrive at (2.6). □

Corollary 2.7. *If* (1.1) *is positive, then the resolvent satisfies* $R(t,s) \ge 0$ *for all* $(t,s) \in \Delta$.

Proof. Fix $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n_+$ and consider, for $k \in \mathbb{N}$, the continuous function

$$\phi_k\colon [0,t_0] \to \mathbb{R}^n, \quad t \mapsto \phi_k(t) = \begin{cases} 0 & \text{if } t \in [0, t_0 - 1/k] \\ ktx_0 + (1-kt_0)x_0 & \text{if } t \in [t_0 - 1/k, t_0]. \end{cases}$$

Then we have, for all $t \in [0,t_0]$,

$$\lim_{k\to\infty} \phi_k(t) = \phi(t) := \begin{cases} 0 & \text{if } t \in [0,t_0) \\ x_0 & \text{if } t = t_0. \end{cases}$$

Since ϕ so defined is measurable, Lemma 2.6 yields, for all $t \ge t_0$,

$$\begin{aligned}\lim_{k\to\infty} x(t;t_0,\phi_k) &= \lim_{k\to\infty}\left[R(t,t_0)\phi_k(t_0) + \int_{t_0}^{t} R(t,s)\left(\int_0^{t_0} B(s,u)\phi_k(u)\mathrm{d}u\right)\mathrm{d}s\right] \\ &= R(t,t_0)x_0.\end{aligned}$$

Finally, by positivity of (1.1), we may conclude $\lim_{k\to\infty} x(t;t_0,\phi_k) = R(t,t_0)x_0 \ge 0$, and since $x_0 \in \mathbb{R}^n_+$ and $t_0 \ge 0$ are arbitrary, it follows that $R(t,s) \ge 0$ for all $(t,s) \in \Delta$. □

Lemma 2.8. *Let* $T > 0$ *and* $\eta \in NBV([0,T],\mathbb{R}^{n\times n})$. *Then the linear operator*

$$L \;:\; C_0([0,T],\mathbb{R}^n) \to \mathbb{R}^n, \qquad \phi \mapsto L\phi = \int_0^T \mathrm{d}[\eta(\theta)]\phi(\theta)$$

is positive (i.e. $L\phi \ge 0$ *for every nonnegative* $\phi \in C_0([0,T],\mathbb{R}^n)$*) if, and only if,* η *is increasing (i.e. every entry function* η_{ij} *satisfies, for all* t_1,t_2 *with* $0 \le t_1 \le t_2 \le T$, $\eta_{ij}(t_2) \ge \eta_{ij}(t_1)$*).*

Proof. Let $\eta \in NBV([0,T],\mathbb{R}^{n\times n})$ be increasing. By definition of the Riemann-Stieltjes integrals,

$$L\phi = \lim_{d(P)\to 0} \sum_{k=1}^{p} (\eta(\theta_k) - \eta(\theta_{k-1}))\phi(\zeta_k) \geq 0, \qquad \forall \phi \in C_0([0,T],\mathbb{R}^n), \quad \phi \geq 0.$$

Therefore, L is positive.

Conversely, assume that L is positive on $C_0([0,T],\mathbb{R}^n)$. We show that $\eta(\cdot) = (\eta_{ij}(\cdot)) \in NBV([0,T],\mathbb{R}^{n\times n})$ is an increasing scalar function for every $i,j \in \{1,2,\dots,n\}$. Since L is positive, it is easy to see that the functional

$$L_{ij} : C_0([0,T],\mathbb{R}) \to \mathbb{R}, \quad \psi \mapsto L_{ij}\psi := \int_0^T \psi(\theta)\mathrm{d}[\eta_{ij}(\theta)]$$

is also positive for every $i,j \in \{1,2,\dots,n\}$. Fix

$$\theta_1, \theta_2 \in (0,T) \text{ with } \theta_1 < \theta_2 \quad \text{and} \quad k \in \mathbb{N} \ \text{ with } \ k > \max\left\{\frac{1}{\theta_1}, \frac{1}{\theta_2}, \frac{1}{\theta_2 - \theta_1}\right\}$$

and consider the continuous function

$$\psi_k : [0,T] \to \mathbb{R}, \quad \theta \mapsto \psi_k(\theta) := \begin{cases} 0 & \text{if } \ \theta \in [0, \theta_1 - \frac{1}{k}] \\ k\theta + 1 - k\theta_1 & \text{if } \ \theta \in (\theta_1 - \frac{1}{k}, \theta_1] \\ 1 & \text{if } \ \theta \in (\theta_1, \theta_2 - \frac{1}{k}] \\ -k\theta + k\theta_2 & \text{if } \ \theta \in (\theta_2 - \frac{1}{k}, \theta_2] \\ 0 & \text{if } \ \theta \in (\theta_2, T]. \end{cases}$$

Since ψ_k is continuous on $[0,T]$, it follows from a standard property of the Riemann-Stieltjes integral, see e.g. [13, p.109], that

$$\int_0^T \psi_k(\theta)\mathrm{d}[\eta_{ij}(\theta)] = \left(\int_0^{\theta_1 - \frac{1}{k}} + \int_{\theta_1 - \frac{1}{k}}^{\theta_1} + \int_{\theta_1}^{\theta_2 - \frac{1}{k}} + \int_{\theta_2 - \frac{1}{k}}^{\theta_2} + \int_{\theta_2}^{T}\right) \psi_k(\theta)\mathrm{d}[\eta_{ij}(\theta)].$$

This gives, for all $k \in \mathbb{N}$ sufficiently large,

$$\int_{\theta_1 - \frac{1}{k}}^{\theta_1} \psi_k(\theta)d[\eta_{ij}(\theta)] + \eta_{ij}\left(\theta_2 - \frac{1}{k}\right) - \eta_{ij}(\theta_1) + \int_{\theta_2 - \frac{1}{k}}^{\theta_2} \psi_k(\theta)\mathrm{d}[\eta_{\mathrm{ij}}(\theta)] \geq 0.$$

Taking into account that η_{ij} is continuous from the left at θ_1, θ_2 and letting $k \to \infty$, we have $\eta_{ij}(\theta_2) \geq \eta_{ij}(\theta_1)$ for every $\theta_1, \theta_2 \in (0,T), \theta_2 \geq \theta_1$. In case of $\theta_1 = 0 < \theta_2 < T$, a similar argument gives $\eta_{ij}(\theta_2) \geq \eta_{ij}(\theta_1)$. Finally, since η_{ij} is continuous from the left at T, we have $\eta_{ij}(T) \geq \eta_{ij}(\theta)$ for all $\theta \in [0,T]$. This completes the proof. □

We are now in the position to prove Theorem 2.4.

Proof of Theorem 2.4. Assume that (1.1) is positive. Fix $s \geq 0$. It follows from (2.3) that

$$A(s) = \left.\frac{\partial R(t,s)}{\partial t}\right|_{t=s} = \lim_{t\downarrow s} \frac{R(t,s) - I_n}{t-s}.$$

Since by Corollary 2.7, $R(t,s) \geq 0$ for all $(t,s) \in \Delta$, we conclude that $A(s)$ is a Metzler matrix.

We now show that $B(t,s) \geq 0$ for all $(t,s) \in \Delta$. Fix $T \geq 0$ and $\phi \in C_0([0,T],\mathbb{R}^n), \phi \geq 0$. Then $x(\cdot) := x(\cdot;T,\phi)$ satisfies

$$\begin{aligned} 0 \leq \lim_{t\downarrow T} \frac{x(t)}{t-T} = \lim_{t\downarrow T} \frac{x(t)-x(T)}{t-T} \;&=\; \dot{x}(T) \\ = A(T)\phi(T) + \int_0^T B(T,s)\phi(s)\mathrm{d}s \;&=\; \int_0^T B(T,s)\phi(s)\mathrm{d}s. \end{aligned}$$

Thus, for $\eta(s) := \int_0^s B(T,\theta)\mathrm{d}\theta$, $s \in [0,T]$,

$$\int_0^T B(T,s)\phi(s)\mathrm{d}s = \int_0^T \mathrm{d}[\eta(\mathrm{s})]\phi(s) \geq 0, \quad \forall \phi \in C_0([0,T],\mathbb{R}^n),\ \phi \geq 0.$$

By Lemma 2.8, $\eta(\cdot)$ is increasing on $[0,T]$. This implies that $B(T,s) \geq 0$ for $s \in [0,T]$. Since $T \geq 0$ is arbitrary, we have $B(t,s) \geq 0$ for all $(t,s) \in \Delta$.

Conversely, assume $A(t)$ is a Metzler matrix for every $t \geq 0$ and $B(t,s) \geq 0$ for all $(t,s) \in \Delta$. Fix $T \geq 0$ and $\phi \in C([0,T],\mathbb{R}^n)$ with $\phi \geq 0$. We prove that $x(t;T,\phi) \geq 0$ for all $t \geq T$.

Fix $T_1 > T$. Since $A(\cdot)$ is continuous on $[0,T_1]$ and $A(t)$ is a Metzler matrix for every $t \geq 0$, we may choose $r > 0$ such that $rI_n + A(t) \geq 0$ for all $t \in [0,T_1]$. Consider

$$z\colon [0,T_1] \to \mathbb{R}^n, \quad t \mapsto z(t) := e^{rt}x(t;T,\phi).$$

Then z satisfies

$$\dot{z}(t) = (A(t) + rI_n)z(t) + \int_0^t e^{r(t-s)}B(t,s)z(s)\mathrm{d}s \quad \forall t \in [T,T_1]. \tag{2.7}$$

It remains to consider two cases:

(i) Assume $\phi(T) \gg 0$. We show that $z(t) \geq 0$ for all $t \in [T,T_1]$. Seeking a contradiction, suppose

$$T_0 = \inf\{t \in [T,T_1] |\ z(t) \ngeq 0\} \in [T,T_1].$$

Then by continuity $z(T_0) \geq 0$ and so (2.7) yields

$$\begin{aligned} z(T_0) &= z(T) + \int_T^{T_0} \dot{z}(\tau)\mathrm{d}\tau \\ &= \phi(T) + \int_T^{T_0} \left((A(\tau)+rI_n)z(\tau) + \int_0^\tau e^{r(\tau-s)}B(\tau,s)z(s)\mathrm{d}s \right) \mathrm{d}\tau \geq \phi(T) \gg 0. \end{aligned}$$

By continuity, there exists $\epsilon > 0$ such that $z(t) \gg 0$ for all $t \in [T_0, T_0+\epsilon)$. However, this contradicts the definition of T_0; whence $z(t) \geq 0$ for all $t \in [T,T_1]$. Since $T_1 \geq T$ is arbitrary, we have $z(t) \geq 0$ for all $t \geq T$ and therefore, $x(t) \geq 0$ for all $t \geq T$.

(ii) Assume $\phi(T) \geq 0$. Then $\phi_k := \phi + (1/k)e$, where $e = (1, 1, \dots, 1)^T \in \mathbb{R}^n$ and $k \in \mathbb{N}$, yields $\phi_k(T) \gg 0$. Now (2.6) together with Part (i) gives $\lim_{k\to\infty} x(t; T, \phi_k) = x(t; T, \phi) \geq 0$, $\forall\, t \geq T$. This completes the proof of the theorem. □

3. Exponential asymptotic stability

In this section, we give a sufficient condition for exponential asymptotic stability of (1.1) which is defined as follows.

Definition 3.1. *Equation* (1.1) *is said to be* exponentially asymptotically stable *if, and only if,*

$$\exists M, \alpha > 0 \forall t_0, \quad T \geq 0, \quad \forall \phi \in C([t_0, t_0 + T], \mathbb{R}^n), \quad \forall t \geq t_0 + T:$$
$$|x(t; [t_0, t_0 + T], \phi)| \;\leq\; M e^{-\alpha(t - t_0 - T)} \|\phi\|,$$

where $x(\,\cdot\,; [t_0, t_0 + T], \phi)$ denotes the unique solution of the initial value problem (2.1).

The above definition of exponential asymptotic stability is uniform in time t. In particular, if t_0 is fixed and is equal to 0 then it reduces to the notion of exponential asymptotic stability used in [4], [8].

Theorem 3.2. *Suppose $A(\cdot) \in C([0, \infty), \mathbb{R}^{n\times n})$ and $B(\cdot, \cdot) \in C(\Delta, \mathbb{R}^{n\times n})$ satisfy*

(i) $A(\cdot) = \mathrm{diag}\{a_1(\cdot), \dots, a_n(\cdot)\}$,

(ii) $\exists\, \alpha > 0\ \forall\, (t, t_0) \in \Delta\ :\ A(t) + \int_{t_0}^{t} B(t, s)\, e^{-\int_s^t A(u)\mathrm{d}u} \mathrm{d}s\ \leq -\alpha I_n$,

(iii) $\sup_{s\geq 0} \int_0^\infty e^{\alpha\tau} |B(\tau + s, s)|\, \mathrm{d}\tau < \infty$.

Then positivity of (1.1) *implies exponential asymptotic stability of* (1.1).

Remark 3.3. (i) Theorem 3.2 presents a sufficient condition for a diagonal matrix $A(\cdot)$ only. However, this generalizes a recent result by [4] (namely, Theorem 4.2) which is only for scalar equations and the kernel B must be of convolution type.

(ii) Exponential stability of linear Volterra integro-differential systems of convolution type is also studied by [1]. There different conditions on the kernel $B(\cdot)$ and integrability of the solution are given to guarantee exponential stability.

(iii) We show how Theorem 3.2 is related to Murakami's results [8, Th. 1 and 3]. Consider, as Murakami does, scalar linear Volterra equations of the form (1.1) with $A(\cdot) \equiv a \in \mathbb{R}$ and $B(t, s) := b(t - s) \in C([0, \infty), \mathbb{R})$ and $b(t) \geq 0$ for all $t \geq 0$.

Then Assumptions (ii)-(iii) of Theorem 3.2 reduce to

$$a + \int_0^\infty e^{-as} b(s)\mathrm{d}s < 0. \tag{3.1}$$

Now [8, Th. 1] says that if

$$\int_0^\infty e^{\alpha s}|b(s)|\mathrm{ds} < \infty \tag{3.2}$$

for some $\alpha > 0$, and (1.1) is uniformly asymptotically stable, then it is exponentially asymptotically stable.

Furthermore, [8, Th. 3] says that if $b(\cdot) \in L_1(\mathbb{R}_+)$ and it does not change sign on $\mathbb{R}_+$ and (1.1) is uniformly asymptotically stable, then it is exponentially asymptotically stable if, and only if, (3.2) holds true.

For the remainder consider the case $b(t) \geq 0$ for all $t \geq 0$. Then uniform asymptotic stability of (1.1) is equivalent to

$$a + \int_0^{+\infty} b(s)\mathrm{ds} < 0, \tag{3.3}$$

see [12, Th. 4.6], and finally (3.3) and (3.2) imply that

$$a + \int_0^\infty e^{\alpha_1 s} b(s)\mathrm{ds} < 0$$

for some $\alpha_1 \in [0, \alpha]$, which is closely related to (3.1).

Proof of Theorem 3.2. In the following the matrix norms are assumed to be operator norms induced by monotonic vector norms; this implies that

$$\forall P, Q \in \mathbb{R}^{\ell \times q} \quad \text{with} \quad 0 \leq P \leq Q \implies |P| \leq |Q|, \tag{3.4}$$

see e.g. [12].

Fix $t_0 \geq 0$ and let R denote the resolvent of (2.3).

Step 1: We show

$$R(s, t_0) \leq e^{-\int_s^t A(\mu)\mathrm{d}\mu} R(t, t_0) \ =: \ G(t), \quad \forall s \in [t_0, t] \ \forall t \geq t_0. \tag{3.5}$$

Differentiation of G and use of assertion (i) and the resolvent equation (2.3) yields, for all $\tau \in [s, t]$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau} G(\tau) &= \frac{\partial}{\partial \tau} e^{-\int_s^\tau A(\mu)\mathrm{d}\mu} \cdot R(\tau, t_0) + e^{-\int_s^\tau A(\mu)\mathrm{d}\mu} \cdot \frac{\partial}{\partial \tau} R(\tau, t_0) \\
&= -e^{-\int_s^\tau A(\mu)\mathrm{d}\mu} \ A(\tau) \ R(\tau, t_0) \\
&\quad + \ e^{-\int_s^\tau A(\mu)\mathrm{d}\mu} \ \left[A(\tau) R(\tau, t_0) + \int_{t_0}^\tau B(\tau, u) R(u, t_0)\mathrm{d}u \right] \\
&= e^{-\int_s^\tau A(\mu)\mathrm{d}\mu} \ \int_{t_0}^\tau B(\tau, u) R(u, t_0)\mathrm{d}u.
\end{aligned} \tag{3.6}$$

By assertion (i), Theorem 2.4 and Corollary 2.7 we may conclude that

$$\forall t_0 \leq s \leq \tau \leq t \quad \forall t_0 \leq u \leq \tau : \ e^{-\int_s^\tau A(\mu)\mathrm{d}\mu} \geq 0, \quad B(\tau, u) \geq 0, \quad R(u, t_0) \geq 0,$$

and (3.6) yields

$$\frac{\mathrm{d}}{\mathrm{d}\tau} G(\tau) \geq 0 \quad \forall \tau \in [s,t].$$

Therefore,

$$\forall\ t_0 \leq s \leq t: \ R(s,t_0) = G(s) \leq G(t) = e^{-\int_s^t A(\mu)\mathrm{d}\mu} R(t,t_0).$$

Step 2: We show that

$$0 \leq R(t,t_0) \leq e^{-\alpha(t-t_0)} I_n \quad \forall (t,t_0) \in \Delta. \tag{3.7}$$

The first inequality of (3.7) follows from Corollary 2.7. An elementary calculation yields, for all $t \geq t_0$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\alpha(t-t_0)} R(t,t_0)\right) &\overset{(2.3)}{=} \alpha e^{\alpha(t-t_0)} R(t,t_0) \\
&\quad + \ e^{\alpha(t-t_0)} \left[A(t)R(t,t_0) + \int_{t_0}^t B(t,u)R(u,t_0)\mathrm{d}u\right] \\
&\overset{(3.5)}{\leq} e^{\alpha(t-t_0)} \left[\alpha I_n + A(t) + \int_{t_0}^t B(t,u) e^{-\int_u^t A(\mu)\mathrm{d}\mu}\mathrm{d}u\right] R(t,t_0) \\
&\overset{\text{(ii)}}{\leq} 0.
\end{aligned}$$

Step 3: We finally have for any solution $x(\,\cdot\,;[t_0,t_0+T],\phi)$ of (2.1) and for all $t \geq t_0 + T$,

$$\begin{aligned}
|x(t;[t_0,t_0+T],\phi)| &\overset{(2.5)}{\leq} |R(t,t_0+T)|\ |\phi(t_0+T)| \\
&\qquad + \int_{t_0+T}^t |R(t,\tau)| \int_{t_0}^{t_0+T} |B(\tau,s)|\ |\phi(s)|\mathrm{d}s\ \mathrm{d}\tau \\
&\overset{(3.4)-(3.7)}{\leq} e^{-\alpha(t-t_0-T)}\ \|\phi\| \\
&\qquad + \int_{t_0+T}^t e^{-\alpha(t-\tau)} \int_{t_0}^{t_0+T} |B(\tau,s)|\mathrm{d}s\ \mathrm{d}\tau\ \|\phi\| \\
&= e^{-\alpha(t-t_0-T)}\ \|\phi\| \\
&\qquad + \int_{t_0}^{t_0+T} e^{-\alpha(t-s)} \int_{t_0+T}^t e^{\alpha(\tau-s)}|B(\tau,s)|\mathrm{d}\tau\ \mathrm{d}s\ \|\phi\| \\
&\overset{\text{(iii)}}{\leq} \left[e^{-\alpha(t-t_0-T)} + K\int_0^{t_0+T} e^{\alpha(s-t)}\mathrm{d}s\right]\ \|\phi\|,
\end{aligned}$$

where

$$K := \sup_{s\geq 0}\int_0^\infty e^{\alpha\tau}|B(\tau+s,s)|\mathrm{d}\tau,$$

and therefore

$$\begin{aligned}|x(t;[t_0,t_0+T],\phi)| &\leq \left(e^{-\alpha(t-t_0-T)}+\frac{K}{\alpha}(e^{-\alpha(t-t_0-T)}-e^{-\alpha t})\right)\|\phi\|\\ &= \left(1+\frac{K}{\alpha}(1-e^{-\alpha(t_0+T)})\right)e^{-\alpha(t-t_0-T)}\|\phi\|\\ &\leq \left(1+\frac{K}{\alpha}\right)e^{-\alpha(t-t_0-T)}\|\phi\|, \qquad t\geq t_0+T.\end{aligned}$$

This completes the proof. □

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