Hinrichsen, D. ; Ilchmann, Achim ; Pritchard, A. J.:

A stability radius for time-varying linear systems

Aus:
A stability radius for time-varying linear systems

D. Hinrichsen  A. Ilchmann  A. J. Pritchard

Notation

\( \mathbb{R}_+ = \{ z \in \mathbb{R} | z \geq 0 \} \)
\( \mathbb{C}_- = \{ z \in \mathbb{C} | \text{Re} z < 0 \} \)
\( \sigma(A) \) spectrum of \( A \in \mathbb{C}^{n \times n} \)
\( GL_n(\mathbb{C}) \) the set of all invertible matrices \( T \in \mathbb{C}^{n \times n} \)
\( \| x \| \) Euclidean norm of \( x \in \mathbb{C}^n \)
\( \| D \| \) induced operator norm for \( D \in \mathbb{C}^{m \times p} \)
\( \| D(\cdot) \|_{L_\infty} = \sup_{t_0 < t < t_1} \{ \| D(t) \| \} \) for \( D(\cdot) \in PC ((t_0, t_1); \mathbb{C}^{m \times p}) \)

\[ L_2(t_0, t_1; \mathbb{C}^m) \] space of functions \( u : (t_0, t_1) \rightarrow \mathbb{C}^m \) s. t. \( t \mapsto ||u(t)||^2 \) is integrable over \( (t_0, t_1) \)

\[ PC((t_0, t_1); \mathbb{C}^{n \times m}) \] set of piecewise continuous matrix functions \( D(\cdot) : (t_0, t_1) \rightarrow \mathbb{C}^{n \times m} \)

\[ PC_b((t_0, t_1); \mathbb{C}^{n \times m}) \] set of all bounded matrix functions in \( PC((t_0, t_1); \mathbb{C}^{n \times m}) \)

\[ PC^1((t_0, t_1); GL_n(\mathbb{C})) \] set of all piecewise continuously differentiable functions \( D(\cdot) : (t_0, t_1) \rightarrow GL_n(\mathbb{C}) \)

\[ C^1(t_0, t_1; \mathbb{C}^{n \times m}) \] set of all continuously differentiable \( D(\cdot) : (t_0, t_1) \rightarrow \mathbb{C}^{n \times m} \)
1 Introduction

In recent years problems of robust stability have received a good deal of attention. Most of the work on time-invariant linear systems – including the successful $H^\infty$ approach (see [4], [12]) – is based on transform techniques. However, in [7], [8] a state space approach via the concept of stability radius is proposed. In the present paper this approach is extended to a time-varying setting.

Consider a nominal system of the form

$$\dot{x}(t) = A(t)x(t), \quad t \geq 0 \tag{1.1}$$

where $A(\cdot) \in PC(\mathbb{R}_+^+, \mathbb{C}^{n \times n})$. Assume that (1.1) is exponentially stable, i.e. there exist $M, \omega > 0$ so that

$$\|\phi(t, s)\| \leq Me^{-\omega(t-s)} \quad \text{for all } t \geq s \geq 0 \tag{1.2}$$

where $\phi(t, s)$ denotes the transition matrix of (1.1). Many authors (see [1], [2], [3], [5], [10]) have determined bounds $\delta > 0$ so that exponential stability of the disturbed system

$$\dot{x}(t) = [A(t) + D(t)]x(t), \quad t \geq 0 \tag{1.3}$$

is preserved whenever

$$||D(\cdot)||_{L_\infty} < \delta \quad \text{for } D(\cdot) \in PC(\mathbb{R}_+^+, \mathbb{C}^{n \times n}). \tag{1.4}$$

These bounds are conservative. Our problem is to determine a sharp upper bound. We call this bound the (complex)\(^1\) stability radius and define it by

$$r_C(A) = \inf \{||D(\cdot)||_{L_\infty} | D \in PC_b(\mathbb{R}_+^+, \mathbb{C}^{n \times n})$$

$$\text{and (1.3) is not exponentially stable} \} \tag{1.5}$$

We also consider the case where $A$ is subjected to structured perturbations, so that the perturbed system is

$$\dot{x}(t) = [A(t) + B(t)D(t)C(t)]x(t), \quad t \geq 0 \tag{1.6}$$

where $D(\cdot) \in PC_b(\mathbb{R}_+^+, \mathbb{C}^{n \times p})$ is an unknown bounded time-varying disturbance matrix and $B(\cdot) \in PC(\mathbb{R}_+^+, \mathbb{C}^{n \times m})$, $C(\cdot) \in PC(\mathbb{R}_+^+, \mathbb{C}^{p \times n})$ are given "scaling matrices" defining the "structure" of the perturbation. Then the structured stability

\(^1\)The real stability radius is defined analogously. In spite of its prime importance it is not studied here, since even in the time-invariant setup only rudimentary results are available.
radius is

\[ r_c(A; B, C) = \inf \{ \| D(\cdot) \|_{L_\infty} | D \in PC_b(\mathbb{R}_+, \mathbb{C}^{n \times p}) \}
\]

and (1.6) is not exponentially stable \(^{(1.7)}\)

In the unstructured case \( r_c(A) \) is simply the distance of (1.1) from the set of not exponentially stable systems with respect to the \( L_\infty \)-norm.

**Remark 1.1** The following properties are easily obtained:

(a) \( r_c(A) = 0 \iff \) (1.1) is not exponentially stable
(b) \( r_c(\alpha A) = \alpha r_c(A) \) for all \( \alpha \geq 0 \)
(c) \( A(\cdot) \mapsto r_c(A) \) is continuous on \( PC_b(\mathbb{R}_+, \mathbb{C}^{n \times n}) \)

## 2 Bohl exponent and Bohl transformation

For the stability behaviour of (1.1) the number

\[ k_B(A) := \inf \{ -w \in \mathbb{R} | \exists M_w > 0 : \]

\[ t \geq s \geq 0 \implies \| \phi(t, s) \| \leq M_w e^{-w(t-s)} \} \]

introduced by Bohl [1] is useful. We call \( k_B(A) \) the **Bohl exponent** of (1.1). It is possible that \( k_B(A) = \pm \infty \). The following properties are easily seen.

**Proposition 2.1** Let \( A(\cdot) \in PC(\mathbb{R}_+, \mathbb{C}^{n \times n}) \). Then

(a) \( k_B(A) < 0 \iff \) (1.1) is exponentially stable

(b) If \( A(\cdot) \equiv A \in \mathbb{C}^{n \times n} \) then

\[ k_B(A) = \max_{i \in \mathbb{R}} Re \lambda_i(A), \quad \text{where } \lambda_i(A) \text{ are the eigenvalues of } A. \]

(c) In the scalar case, i.e. \( n = 1 \), we have

\[ r_c(A) = -k_B(A) \]

(d) For the matrix case only an inequality is valid:

\[ r_c(A) \leq -k_B(A) \]
Remark 2.2 We want to emphasize that $k_B(A)$ may be a bad indicator for the robustness margin of (1.1). Consider

$$A_k = \begin{bmatrix} k & k^3 \\ 0 & k \end{bmatrix}, \quad D_k = k^{-1} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad \text{for } k \in \mathbb{N}.$$ 

Then $\lim_{k \to \infty} k_B(A_k) = -\infty$. However, $\sigma(A_k + D_k) = \{\frac{1}{k}, \frac{1}{k} - 2k\}$ although $\lim_{k \to \infty} \|D_k\| = 0$. Thus $\lim_{k \to \infty} r_C(A_k) = 0$.

The following properties of the Bohl exponent can be found in [3].

Proposition 2.3 Let $A(\cdot) \in PC(\mathbb{R}_+, \mathbb{C}^{n \times n})$. Then

(a) $k_B(A)$ is finite if $A(\cdot)$ is bounded.
(b) $k_B(A)$ is finite iff $\sup_{0 \leq |t-s| \leq 1} \|\phi(t, s)\| < \infty$.
(c) If $k_B(A) < \infty$ then

$$k_B(A) = \lim_{t \to \infty} \sup_{s, t-s \to \infty} \frac{\log \|\phi(t, s)\|}{t-s}.$$ 

We now analyse the effect of time-varying linear coordinate transformations

$$z(t) = T(t)^{-1} x(t), \quad T(\cdot) \in PC^1(\mathbb{R}_+, GL_n(\mathbb{C})) \quad (2.2)$$

on the system (1.1) which yields

$$\dot{z}(t) = \dot{A}(t) z(t), \quad \text{where } \dot{A} = T^{-1} A T - T^{-1} \dot{T} \quad (2.3)$$

These transformations will not, in general, preserve exponential stability. Therefore we introduce the set of Bohl transformations $\mathcal{B}_n$, i.e. the set of all $T(\cdot) \in PC^1(\mathbb{R}_+, GL_n(\mathbb{C}))$ such that

$$\inf \{ \varepsilon \in \mathbb{R} | \exists M_\varepsilon > 0 : \forall t, s \geq 0 \Rightarrow \|T(t)^{-1} \cdot T(s)\| \leq M_\varepsilon e^{\varepsilon |t-s|} \} = 0 \quad (2.4)$$

Remark 2.4 It is obvious that

(a) the set $\mathcal{B}_n$ forms a group with respect to (pointwise) multiplication
(b) $\mathcal{B}_n$ contains the group of Lyapunov transformations, i.e. all $T(\cdot) \in PC^1(\mathbb{R}_+, GL_n(\mathbb{C}))$ so that $T(\cdot), T(\cdot)^{-1}, T(\cdot)$ are bounded,
(c) $k_B(A) = k_B(T^{-1} A T - T^{-1} \dot{T})$ for all $T \in \mathcal{B}_n$.
The following proposition shows that even in the time-invariant case a similarity transformation may drastically change the stability radius.

**Proposition 2.5** [7] If $A \in \mathbb{C}^{n \times n}$ with $\sigma(A) \subset \mathbb{C}_-$ then $\{r_C(T^{-1}AT); T \in GL_n(\mathbb{C})\}$ is equal to the interval $(0, -\max_{i \in \mathbb{N}} \text{Re} \lambda_i(A))$ with possible exception of the right extremum.

In the scalar case we can prove

**Proposition 2.5** If

$$\dot{x}(t) = a(t)x(t), \quad a(\cdot) \in PC([0, \infty), \mathbb{C}) \quad (2.5)$$

has a *strict* Bohl exponent, i.e.

$$k_B(a) = \lim_{s,t \to -\infty} \frac{\log \|\phi(t, s)\|}{t - s}$$

then there exists $\Theta \in \mathcal{B}_1$ so that $z(t) = \Theta(t)^{-1}x(t)$ converts (2.5) into

$$\dot{z}(t) = k_B(a)z(t)$$

### 3 The perturbation operator

In the time-invariant setup, where $(A, B, C) \in \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times m} \times \mathbb{C}^{p \times n}$, the structured stability radius can be characterized via the convolution operator

$$L_0 : L^2(0, \infty; \mathbb{C}^m) \rightarrow L^2(0, \infty; \mathbb{C}^p)$$

$$u(\cdot) \mapsto \left(t \mapsto \int_0^t Ce^{A(t-s)}Bu(s)ds\right) \quad (3.1)$$

as follows.

**Proposition 3.1** [8] If $\sigma(A) \subset \mathbb{C}_-$ and $G(s) := C(sI_n - A)^{-1}B$ then

$$\rho_C(A, B, C) = \left\{ \begin{array}{ll} \|L_0\|^{-1} = \left[\max_{w \in \mathbb{R}} \|G(iw)\|\right]^{-1} & \text{if } G \neq 0 \\ \infty & \text{if } G = 0 \end{array} \right.$$
\[ L^\Sigma_{t_0} : L_2(t_0, \infty; \mathbb{C}^m) \rightarrow L_2(t_0, \infty; \mathbb{C}^p), \quad t_0 \geq 0 \]  
\[ u(\cdot) \mapsto (t \mapsto \int_{t_0}^{t} C(t)\phi(t,s)B(s)u(s)ds) \]  

associated with

\[ \Sigma = (A, B, C) \in PC(\mathbb{R}_+, \mathbb{C}^{n \times n}) \times PC_b(\mathbb{R}_+, \mathbb{C}^{n \times m}) \times PC_b(\mathbb{R}_+, \mathbb{C}^{p \times n}), \quad k_B(A) < 0 \]  

Basic properties of \( L^\Sigma_{t_0} \) are summarized in the following

**Proposition 3.2** [6]

(a) \( L^\Sigma_{t_0} \) is a bounded operator

(b) \( t_0 \mapsto \|L^\Sigma_{t_0}\| \) is monotonically decreasing on \( \mathbb{R}_+ \)

(c) \( \|L^\Sigma_{t_1}\| = \|L^\Sigma_{t_0}\| \) for all \( t_0, t_1 \in \mathbb{R}_+ \) if \( A, B, C \) are periodic with a common period

(d) \( \|L^\Sigma_{t_0}\|^{-1} \leq r_C(A; B, C) \)

(e) For the unstructured case, i.e. \( B(\cdot) = C(\cdot) = 1 \), if \( M, w > 0 \) satisfy (1.2) then

\[ \frac{w}{M} \leq \|L^\Sigma_{t_0}\|^{-1} \leq \lim_{t_0 \to -\infty} \|L^\Sigma_{t_0}\|^{-1} \leq r_C(A) \]

As opposed to the time-invariant case \( \|L^\Sigma_{t_0}\|^{-1} \) or \( \lim_{t_0 \to -\infty} \|L^\Sigma_{t_0}\|^{-1} \) do not necessarily coincide with \( r_C(A; B, C) \). Even in the simple case when \( a(\cdot) \in PC(\mathbb{R}_+, \mathbb{R}) \) is periodic and \( b = c = 1 \), we have worked out an example in [6] for which

\[ \|L^\Sigma_{t_0}\|^{-1} = \|L^\Sigma_{t_1}\|^{-1} < r_C(a) \text{ for all } t_0, t_1 \in \mathbb{R}_+ \]

However note that scalar Bohl transformations \( \Theta \in \mathcal{B}_1 \) do not change the stability radius but will change the norm of the perturbation operator. Let

\[ \Sigma_\Theta := (A - \Theta \otimes I_n; B, C) \]

By using Proposition 2.5 and 3.1 one can show

**Proposition 3.3** Suppose \( a(\cdot) \in PC(\mathbb{R}_+, \mathbb{C}) \) has a strict Bohl exponent \( k_B(a) < 0 \) and \( b, c \in \mathbb{C} \). Then there exists a \( \Theta \in \mathcal{B}_1 \) such that

\[ r_C(a; b, c) = \|L^\Sigma_{t_0}\|^{-1} \text{ for all } t_0 \geq 0. \]
For the matrix case we have the following

**Conjecture 3.4** Suppose \( \Sigma = (A, B, C) \) satisfies (3.3), then

\[
 r_C(A; B, C) = \sup_{t \in \mathbb{B}_1} \left\{ \lim_{t_0 \to -\infty} \|L_{t_0}^{t_0}\|^{-1} \right\}
\]

4 The associated parametrized differential Riccati equation

In the time-invariant setup another useful characterization of \( r_C(A; B, C) \) is possible via the parametrized algebraic Riccati equation, \( \text{ARE}_\rho \)

\[
 A^* P + PA - \rho C^* C - PBB^* P = 0, \quad \rho \in \mathbb{R}
\]

**Proposition 4.1** [8] Suppose \( (A, B, C) \in \mathbb{F}^{n \times n} \times \mathbb{C}^{n \times m} \times \mathbb{C}^{p \times m} \) and \( \sigma(A) \subset \mathbb{C}_- \):

(a) If \( -\infty < \rho < r_C^2(A; B, C) \) then there exists a unique stabilizing Hermitian solution \( P_\rho \) of \( \text{ARE}_\rho \), that is a solution \( P_\rho = P^*_\rho \) which satisfies

\[
 \sigma(A - BB^* P_\rho) \subset \mathbb{C}_-.
\]

If \( \rho = r_C^2 \) then there exists a unique Hermitian solution \( P_{r_C} \) of \( \text{ARE}_{r_C} \) having the property \( \sigma(A - BB^* P_{r_C}) \subset \mathbb{C}_- \).

(b) If there exists a Hermitian solution \( P_\rho \) of \( \text{ARE}_\rho \) then necessarily

\[
 \rho \leq r_C^2(A; B, C).
\]

Guided by this result we study, in the time-varying setting, the parametrized differential Riccati equation, \( \text{DRE}_\rho \)

\[
 \dot{P}(t) + A^*(t)P(t) + P(t)A(t) - \rho C^*(t)C(t) - P(t)B(t)B^*(t)P(t) = 0, \quad t \geq t_0
\]

associated with the system

\[
 \begin{cases}
 \dot{x}(t) = A(t)x(t) + B(t)u(t), & x(t_0) = x_0 \in \mathbb{C}^n \\
 y(t) = C(t)x(t), & t \geq t_0 \geq 0
\end{cases}
\]

(4.1)

Throughout this section we assume \( \Sigma = (A, B, C) \) satisfies (3.3).

Kalman [9] and Reghis and Megan [11] among others, have studied differential Riccati equations, however their results cannot be applied to \( \text{DRE}_\rho \) if \( \rho > 0 \).

Just as in the time-invariant case we consider the parametrized optimal control problem \( \text{OCP}_\rho \):
Minimize over $u \in L_2(t_0, t_1; \mathbb{C}^m)$

$$J_p(x_0, (t_0, t_1), u(\cdot)) = \int_{t_0}^{t_1} \left( ||u(s)||^2 - \rho ||y(s)||^2 \right) ds$$

where $y(\cdot)$ is defined via (4.1) and $\rho \in \mathbb{R}$. If $\rho < 0$ this is the usual linear quadratic regulator problem LQR, whereas in our situation $\rho > 0$, so that the state penalty is negative. To consider a cost functional with negative state penalty is quite natural in this context since we are concerned with a minimum norm destabilization problem while the classical LQR problem is concerned with stabilization.

The analysis of the $\text{DRE}_p$ and its relation to the $\text{OCP}_p$ is quite involved, details may be found in [6]. Here we only state the main results.

**Proposition 4.2** (finite time) If $\rho < \|L^*\|^{-2}$, $0 \leq t_0 < t_1 < \infty$, then

(a) there exists a unique Hermitian solution $P^t(\cdot)$ of $\text{DRE}_p$

(b) $P^t(t) \leq 0$ (resp. $\geq 0$) for all $t \in [t_1, t_0]$ if $\rho \geq 0$ (resp. $\rho \leq 0$).

(c) the minimal cost of $\text{OCP}_p$ is

$$\inf_{u \in L_2(t_0, t_1; \mathbb{C}^m)} J_p(x_0, (t_0, t_1), u(\cdot)) = < x_0, P^t_0(t_0)x_0 >$$

(d) the optimal control is given by

$$u(t) = -B^*(t)P^t(t)x(t)$$

where $x(\cdot)$ solves $\dot{x}(t) = [A - BB^*P^t](t)x(t), \ x(t_0) = x_0$.

The next proposition is obtained by studying what happens if $t_1 \to \infty$.

**Proposition 4.3** (infinite time) If $\rho < \|L^*\|^{-2}$, $t_0 \geq 0$, then

(a) $P^+(t) = \lim_{t_1 \to \infty} P^t_1(t)$ exists for all $t \geq t_0$ and yields a bounded Hermitian solution of $\text{DRE}_p$;

(b) $P^+(\cdot)$ is the only solution so that $k_B(A - BB^*P^+) < 0$;

(c) for any other bounded Hermitian solution $Q(\cdot) \in C^1(t_0, \infty; \mathbb{C}^{n \times n})$, $t_0 \geq t_0$ of $\text{DRE}_p$ we have $Q(t) \leq P^+(t)$ for all $t \geq t_0$;

(d) the minimal cost is

$$\inf_{u \in L_2(t_0, \infty; \mathbb{C}^m)} J_p(x_0, (t_0, \infty), u(\cdot)) = < x_0, P^+(t_0)x_0 >$$

(e) the optimal control is

$$u(t) = -B^*(t)P^+(t)x(t), \ t \geq t_0$$

where $x(\cdot)$ solves

$$\dot{x}(t) = [A - BB^*P^+(t)]x(t), \ x(t_0) = x_0, \ t \geq t_0$$
As a partial converse of Proposition 4.3 we have

**Proposition 4.4** If $Q(t) \in C^1((t_0, \infty); \mathbb{C}^{n \times n})$ is a bounded Hermitian solution of $DRE_{\rho}$ on $(t_0, \infty)$ then necessarily $\rho \leq ||L_{t_0}^e||^{-2}$.

While the previous two propositions yield a complete characterization of the norm $||L_{t_0}^e||$ in terms of the associated parametrized differential Riccati equation, they do not provide a full generalization of Proposition 4.1 to the time-varying case. To find a complete characterization of the stability radius $r_C(A;B,C)$ for time-varying systems is an open problem.

## 5 Robust Lyapunov functions and nonlinear perturbations

The following proposition shows how solutions of the parametrized differential Riccati equation $DRE_{\rho}$ can be used to construct robust Lyapunov functions for the system (1.1).

**Proposition 5.1** Suppose $0 < \rho < ||L_{t_0}^e||^{-2}$. If $P_\rho(\cdot)$ solves $DRE_{\rho}$ then

$$V(t,x) := - < x, P_\rho(t)x >, \quad t \geq t_0, \ x \in \mathbb{C}^n$$

is a common Lyapunov function for all perturbed systems

$$\dot{x}(t) = [A + BCD](t)x(t), \quad t \geq t_0, \ x(t_0) = x_0$$

with $||D(\cdot)||^2_{L_\infty} < \rho$.

If $(A,B,C)$ are constant matrices and $\sigma(A) \subset \mathbb{C}^-$ a similar result to Proposition 5.1 holds true for $||D(\cdot)||_{L_\infty} < r_C(A;B,C)$, see [8].

Using the above Lyapunov function it is possible to extend our robustness analysis to nonlinear perturbations of the form $\Delta(t) = B(t)N(C(t)x,t)$ so that the perturbed system is

$$\dot{x}(t) = A(t)x(t) + B(t)N(C(t)x(t)), \quad x(t_0) = x_0$$

(5.1)

where $(A,B,C)$ satisfy (3.3) and $N : \mathbb{R}^p \times \mathbb{R}_+ \rightarrow \mathbb{R}^m$ is continuously differentiable. We assume $N(0,t) = 0$ so that 0 is an equilibrium state of (5.1). The following result shows that no nonlinear perturbation with global gain smaller than $||L_{t_0}^e||^{-1}$ can destabilize the system.

**Proposition 5.2** Suppose $\gamma < ||L_{t_0}^e||^{-1}$ and

$$||N(y,t)|| \leq \gamma ||y|| \quad \text{for all} \quad t \geq t_0, \ y \in \mathbb{C}^p$$

Then the origin is globally exponentially stable for the system (5.1).

It is not clear whether an analogous statement holds (over a suitable time interval $(t_0, \infty)$) if the gain of the nonlinear perturbation is strictly less than $r_C(A;B,C)$. 

\(\text{Time-varying linear systems}\quad 289\)
References


