Ilchmann, Achim ; Kern, Günter *:*

Stabilizability of systems with exponential dichotomy

Zuerst erschienen in:
Systems & Control Letters, 8 (1987) S. 211-220
DOI: 10.1016/0167-6911(87)90029-6

Systems & Control Letters 8 (1987) 211–220
North-Holland

Stabilizability of systems with exponential dichotomy

Achim ILCHMANN *

Institut für Dynamische Systeme, Universität Bremen, D-2800 Bremen 33, West Germany

Günter KERN

Institut für Mathematik, Technische Universität Graz, Kopernikusgasse 24, A-8010 Graz, Austria

Received 2 June 1986

Abstract: In this paper we introduce the concept of controllability into a closed subspace for time-varying linear systems. Various characterizations are given and the dual relation is discussed. This concept is used to present a necessary and sufficient condition for the stabilizability of systems with exponential dichotomy.

Keywords: Stabilizability, Linear time-varying systems, Controllability into a closed subspace, Systems of exponential dichotomy.

1. Introduction

The purpose of this note is to give necessary and sufficient conditions for the stabilizability of systems which possess an exponential dichotomy and to determine a state feedback which stabilizes the system.

For time-varying linear systems of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{1.1a}$$

$$y(t) = C(t)x(t), \tag{1.1b}$$

where $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ are continuous real valued $n \times n$, $n \times m$, $p \times n$ matrices, resp., defined on $\mathbb{R}$. Ikeda et al. [4,5] have shown that there exists a feedback matrix to stabilize (1.1a) if the system is uniformly completely controllable. As far as we know no attempts have been made to weaken this condition, similar to the stabilizability concept for time-invariant systems, see for instance Kwakernaak and Sivan [11]. It seems that a satisfactory weakening can only be given for certain classes of time varying-systems. For those which have an exponential dichotomy this is possible. The idea is simple: The trajectories of a system which possess an exponential dichotomy split (roughly speaking) into two closed subspaces: one which consists of motions exponentially bounded from above, the other of exponentially bounded motions from below. Now to stabilize, it suffices to require that every unstable motion can be controlled, not necessarily to zero, but into a stable one. Thus the concept of complete controllability into a closed subspace will play an important role.

In Section 2 complete controllability into a closed subspace w.r.t. the system (1.1a) is defined. Various characterizations of this concept are given. The dual properties are analyzed in Section 3. Finally in Section 4 we introduce systems which possess an exponential dichotomy and derive a feedback matrix $F(\cdot)$ which stabilizes (1.1a) if it is uniformly completely controllable into the stable subspace.

* This paper was written while the first author was on leave at the Institute of Mathematics, Technical University Graz. The support of the university and the hospitality of the Institute are gratefully acknowledged.

From a practical point of view it is noteworthy that there is a natural generalization of the notion of exponential dichotomy, namely exponential splitting of order n. (See Daleckii and Krein [2].) This means that the vector space of solutions of $\dot{x}(t)=A(t)x(t)$ splits into n-subspaces and the Liapunov exponents of the motions belonging to a subspace are lying in certain (mutually disjoint) intervals. Thus the concept of complete controllability into a closed subspace can be used to develop a concept, similar to the well-known pole-placement problem for time-invariant systems.

Furthermore, there is the possibility to generalize the results of this note to infinite-dimensional linear time-varying systems, for instance $A(\cdot)$ defined on $\mathbb{R}_+$, strongly measurable and locally Bochner integrable with values in $L(X, X)$, X a separable Hilbert space, and $B(\cdot)$ a continuous operator valued matrix from $\mathbb{R}_+$ to $L(Y, X)$, Y a Banach space.

2. Controllability into a closed subspace

Let $X(t)$ be a fundamental matrix of the homogeneous part of (1.1a) and let

$$X(t, t_0)=X(t)X^{-1}(t_0)$$

denote its transition matrix. Assume that mutually complementary projections $P_1 \in \mathbb{R}^{n\times n}$, i.e. $P_1^2=P_1$, and $P_2=I_n-P_1$ are given. Then the vector space $X(\cdot)\mathbb{R}^n$ of free motions can be decomposed into the direct sum

$$X(\cdot)\mathbb{R}^n=\mathscr{V}_1(\cdot)\oplus\mathscr{V}_2(\cdot) \tag{2.1}$$

where

$$\mathscr{V}_i(t):=X(t)P_i\mathbb{R}^n \quad \text{for } i=1, 2.$$

The projections corresponding to the decomposition (2.1) are similar to the projections P_i (see Daleckii and Krein [2], p. 160) and are expressed by

$$P_i(t)=X(t)P_iX^{-1}(t) \quad \text{for } i=1, 2. \tag{2.2}$$

A new controllability concept is introduced w.r.t. the system (1.1a).

2.1. Definition. A state $x_0 \in \mathbb{R}^n$ is *controllable at time t_0 into $\mathscr{V}_1$* if there exists a $t_1 \geqslant t_0$ and a continuous control function $u(t)\in\mathbb{R}^m$ with supp $u(\cdot)\subseteq[t_0, t_1]$ such that

$$x(t_1)=X(t_1, t_0)x_0+\int_{t_0}^{t_1}X(t_1, s)B(s)u(s)\,\mathrm{d}s \in \mathscr{V}_1(t_1).$$

If this is true for every $x_0\in\mathbb{R}^n$ we say that (1.1a) is *completely controllable into $\mathscr{V}_1$ at time t_0*. If (1.1a) is completely controllable into $\mathscr{V}_1$ at any time t_0, (1.1a) is called *completely controllable into $\mathscr{V}_1$*.

This definition does not say that every state in $\mathscr{V}_2(t_0)$ can be controlled to zero, but every free motion can be forced in finite time into a free motion of $\mathscr{V}_1$.

As usual let

$$W(t_0, t_1)=\int_{t_0}^{t_1}X(t_0, s)B(s)B^{\mathrm{T}}(s)X^{\mathrm{T}}(t_0, s)\,\mathrm{d}s$$

denote the controllability Gramian of (1.1a). In our approach the induced symmetric map

$$W_2(t_0, t_1)=P_2(t_0)W(t_0, t_1)P_2^{\mathrm{T}}(t_0) \tag{2.3}$$

will become an important tool. Clearly, if $\mathscr{V}_1=\{0\}$ then $W(t_0, t_1)=W_2(t_0, t_1)$ and (1.1a) is completely controllable in the usual sense if $W(t_0, t_1)$ is positive definite. We say $W_2(t_0, t_1)$ is *positive definite on* $P_2^{\mathrm{T}}(t_0)\mathbb{R}^n$ if for all non-trivial vectors $\eta \in P_2^{\mathrm{T}}(t_0)\mathbb{R}^n$ we have $\eta^{\mathrm{T}} W_2(t_0, t_1)\eta > 0$.

2.2. Definition. The system (1.1) is *similar* to

$$\dot{z}(t) = A'(t)z(t) + B'(t)u(t), \tag{2.4a}$$

$$y(t) = C'(t)z(t), \tag{2.4b}$$

if there exists a continuously differentiable invertible matrix $T(\cdot)$ such that $z(t) = T^{-1}(t)x(t)$. If additionally $T(\cdot)$, $T^{-1}(\cdot)$ and $\dot{T}(\cdot)$ are bounded, (1.1) is *kinematically similar* to (2.4).

The following remark shows how the foregoing notions change under similarity. The proof is via direct calculation where

$$W_2'(t_0, t_1) = T^{-1}(t_0)W_2(t_0, t_1)T^{-1\mathrm{T}}(t_0), \qquad P_2'(t) = T^{-1}(t)P_2(t)T(t).$$

2.3. Remark. (i) (1.1a) is completely controllable into $\mathscr{V}_1$ iff (2.4a) is completely controllable into $T^{-1}(\cdot)\mathscr{V}_1(\cdot)$.

(ii) $W_2(t_0, t_1)$ is positive definite on $P_2^{\mathrm{T}}(t_0)\mathbb{R}^n$ iff $W_2'(t_0, t_1)$ is positive definite on $P_2'^{\mathrm{T}}(t_0)\mathbb{R}^n$.

Now we can state the main result of this section which characterizes the concept of controllability into a closed subspace.

2.4. Proposition. *The following are equivalent*:

(i) (1.1a) *is completely controllable into* $\mathscr{V}_1$.

(ii) *Let* $t_0 \in \mathbb{R}$ *be arbitrary. Then every non-trivial solution* $y(\cdot) = X^{\mathrm{T}}(t_0, \cdot)P_2^{\mathrm{T}}(t_0)q$ *of the adjoint equation* $\dot{y}(t) = -A^{\mathrm{T}}(t)y(t)$ *has the property* $y^{\mathrm{T}}(\cdot)B(\cdot) \equiv 0$ *on* $[t_0, \infty)$.

(iii) *For every* $t_0 \in \mathbb{R}$ *there exists some* $t_1 > t_0$ *such that* $W_2(t_0, t_1)$ *is positive definite on* $P_2^{\mathrm{T}}(t_0)\mathbb{R}^n$.

Suppose (1.1a) is completely controllable into $\mathscr{V}_1$ at time t_0. Then condition (iii) implies the existence of a finite time t_1 such that every $x_0 \in \mathbb{R}^n$ at t_0 can be controlled into $\mathscr{V}_1(t_1)$ in time $t_1 - t_0$.

For the proof of Proposition 2.4 we need a lemma:

2.5. Lemma. *For arbitrary* $t_0 < t_1$ *the following are equivalent*:

(i) $W_2(t_0, t_1)$ *is positive definite on* $P_2^{\mathrm{T}}(t_0)\mathbb{R}^n$.

(ii) im $W_2(t_0, t_1) = P_2(t_0)\mathbb{R}^n$.

(iii) *The map* $\varphi: \mathscr{C}^m \to P_2(t_0)\mathbb{R}^n$,

$$u(\cdot) \mapsto \int_{t_0}^{t_1} X(t_0)P_2X^{-1}(s)B(s)u(s)\,\mathrm{d}s$$

satisfies im $\varphi = P_2(t_0)\mathbb{R}^n$.

$\mathscr{C}^m$ denotes the vector space of continuous m-vector functions. We delete the proof, it can be carried out in the spirit of Knobloch and Kappel [9], p. 103.

Proof of Proposition 2.4. To simplify the proof we introduce a further condition:

(ii′) For every $t_0 \in \mathbb{R}$ there exists a $t_1 > t_0$ such that $y(\cdot) = X^{\mathrm{T}}(t_0, \cdot)P^{\mathrm{T}}(t_0)q \neq 0$ implies $y^{\mathrm{T}}(t)B(t) = 0$ for all $t \in [t_0, t_1]$.

Because of Remark 2.3 we assume without loss of generality $A(\cdot) \equiv 0$, $X(\cdot) = I_n$ and proceed as follows: (i) ⇔ (iii) ⇒ (ii′) ⇒ (iii).

(i) ⇒ (iii): For $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ there exist $t_1 \geq t_0$ and an input vector $u(\cdot)$ such that

$$x_1 = x_0 + \int_{t_0}^{t_1} B(s)u(s)\,\mathrm{d}s \in P_1\mathbb{R}^n.$$

Then

$$P_2(x_1 - x_0) = -P_2x_0 = \int_{t_0}^{t_1} P_2B(s)u(s)\,\mathrm{d}s.$$

By Lemma 2.5 this yields condition (iii).

(iii) ⇒ (i): It suffices to determine for an arbitrary pair $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ some $t_1 > t_0$ and control functions $u(\cdot)$ such that

$$x(t_1) = P_1 x_0 + P_2 x_0 + \int_{t_0}^{t_1} B(s) u(s)\, \mathrm{d}s \in P_1 \mathbb{R}^n.$$

By assumption there exists $t_1 > t_0$ such that $W_2(t_0, t_1)$ is positive definite on $P_2^{\mathrm{T}} \mathbb{R}^n$. Defining

$$u(t) = \begin{cases} -B^{\mathrm{T}}(t) W_2^{-1}(t_0, t_1) P_2 x_0 & \text{for } t_0 \leqslant t \leqslant t_1, \\ 0 & \text{for } t > t_1, \end{cases}$$

gives

$$P_2\left(P_2 x_0 + \int_{t_0}^{t_1} B(s) u(s)\, \mathrm{d}s \right) = P_2 x_0 - \int_{t_0}^{t_1} P_2 B(s) B^{\mathrm{T}}(s) P_2^{\mathrm{T}}\, \mathrm{d}s\, W_2^{-1}(t_0, t_1) P_2 x_0 = 0.$$

This proves $x(t_1) \in P_1 \mathbb{R}^n$.

(iii) ⇒ (ii): By contradiction. Assume that some $P_2^{\mathrm{T}} q \neq 0$ satisfies

$$y^{\mathrm{T}}(t) B(t) = q^{\mathrm{T}} P_2 B(t) = 0 \quad \text{for all } t \geqslant t_0.$$

Then

$$q^{\mathrm{T}} W_2(t_0, t_1) q = \int_{t_0}^{t_1} q^{\mathrm{T}} P_2 B(s) B^{\mathrm{T}}(s) P_2^{\mathrm{T}} q\, \mathrm{d}s = 0 \quad \text{for all } t_1 > t_0$$

which contradicts (iii).

(ii) ⇒ (ii′): Analogously to Knobloch and Kwakernaak [10] p. 33.

(ii′) ⇒ (iii): It suffices to prove that for arbitrary $t_0 < t_1$ the implication

$$y(\cdot) = P_2^{\mathrm{T}} q \neq 0 \quad \Rightarrow \quad y^{\mathrm{T}}(t) B(t) \neq 0 \quad \text{for some } t \in [t_0, t_1]$$

implies that $W_2(t_0, t_1)$ is positive definite on $P_2^{\mathrm{T}} \mathbb{R}^n$. The proof is immediate by contradiction. □

To complete this section we show under the assumption that A and B are real analytic matrices, complete controllability into a closed subspace can be characterized in terms of a controllability matrix or a coprime relation. We introduce some notations:

$\mathscr{A}$:= the set of real analytic functions,

$\mathscr{M}$:= the set of real meromorphic functions.

The differential operator on $\mathscr{M}$ is denoted by D: $\mathscr{M} \to \mathscr{M}$, $f \mapsto \dot{f}$ and

$$\mathscr{M}[\mathrm{D}] := \left\{ \sum_{i=0}^{k} f_i \mathrm{D}^i \,\middle|\, f_i \in \mathscr{M},\ 0 \leqslant i \leqslant k,\ k \in \mathbb{N} \right\}$$

denotes a skew-polynomial ring with multiplication rule

$$\mathrm{D} f = f \mathrm{D} + \dot{f}.$$

For details see Ilchmann et al. [6].

If A and B are $n-k$ times differentiable we define the operators

$$(\mathrm{D} - A(t))^0 = I_n,$$

$$(\mathrm{D} - A(t))(P_2(t) B(t)) = (P_2(t) B(t))^{\cdot} - A(t) P_2(t) B(t),$$

$$(\mathrm{D} - A(t))^i (P_2(t) B(t)) = (\mathrm{D} - A(t))^{i-1}((\mathrm{D} - A(t))(P_2(t) B(t)) \quad \text{for } i \geqslant 1.$$

2.6. Proposition. *Let A and B be real analytic matrices and* rk $P_1 = k$. *Then the following statements are equivalent*:

(i) (1.1a) *is completely controllable into* $\mathscr{V}_1$.

(ii) $\operatorname{rk}[P_2(t)B(t),\ldots,(\mathrm{D}-A(t))^{n-k}(P_2(t)B(t))] = n-k$ *for all* $t \in \mathbb{R} \setminus N$. *N is a discrete set.*

(iii) *There exist* $U(\mathrm{D}) \in \mathscr{M}[D]^{n\times n}$, $V(\mathrm{D}) \in \mathscr{M}[\mathrm{D}]^{m\times n}$ *such that for all* $t \in \mathbb{R}$,

$$P_2(t) = (\mathrm{D}-A(t))\cdot U(\mathrm{D}) + P_2(t)B(t)\cdot V(\mathrm{D}).$$

These characterizations generalize complete controllability in the usual sense. Suppose $P_2 = I_n$. Then condition (ii) was proved by Silverman and Meadows [14]; in the time invariant case it was derived by Kalman [8]; condition (iii) was shown by Ilchmann et al. [6]; and for time-invariant systems it is known as the left-coprime condition on $(sI - A)$ and B, see Rosenbrock [12].

The following is an immediate consequence of the proof of Proposition 2.6.

2.7. Proposition. *Suppose the matrices A and B are* $n-k-1$ *and* $n-k$ *times continuously differentiable, respectively. Then the system* (1.1a) *is completely controllable at time* t_0 *into* $\mathscr{V}_1$ *if there exists some* $t_1 > t_0$ *such that*

$$\operatorname{rk}\left[P_2(t)B(t),\ldots,(\mathrm{D}-A(t))^{n-k}(P_2(t)B(t))\right] = n-k$$

for a set of points dense in $[t_0, t_1]$.

Proof of Proposition 2.6. We have: If (1.1a) and (2.4a) are similar via T and T is l-times differentiable, then

$$T(t)(\mathrm{D}I_n - A(t))^i(B(t)) = (\mathrm{D}-A'(t))^i(B'(t)) \quad \text{for } 0 \leqslant i \leqslant l$$

(see Ilchmann [7], Lemma 3.1). Furthermore

$$X^{-1}(t)(\mathrm{D}-A(t))X(t) = \mathrm{D}.$$

Hence it suffices to prove the proposition for $A(\cdot) \equiv 0$. Let $S \in \mathbb{R}^{n\times n}$ be an invertible matrix such that

$$S^{-1}P_1 S = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}.$$

Define

$$F(t) := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f_{k+1}(t) \\ \vdots \\ f_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-k} \end{bmatrix} S^{-1}B(t) \in \mathbb{R}^{n\times m},$$

$$\bar{F}(t) := \begin{bmatrix} f_{k+1}(t) \\ \vdots \\ f_n(t) \end{bmatrix} \in \mathbb{R}^{(n-k)\times m}.$$

We first prove that for arbitrary $t_0 < t_1$ the following are equivalent:

(α) $W_2(t_0, t_1)$ is positive definite on $P_2^{\mathrm{T}}\mathbb{R}^n$.

(β) $\operatorname{rk}\int_{t_0}^{t_1}\bar{F}(s)\bar{F}^{\mathrm{T}}(s)\,\mathrm{d}s = n-k$.

(γ) The row-vector functions $f_{k+1}(t),\ldots,f_n(t)$ are linearly independent on $[t_0, t_1]$.

(δ) $\mathrm{rk}[\bar{F}(t),\ldots,\bar{F}^{(n-k)}(t)] = \mathrm{rk}[P_2B(t),\ldots,(P_2B(t))^{(n-k)}] = n-k$ for a set of points dense in $[t_0, t_1]$.

(α) $\Leftrightarrow$ (β): We have

$$S^{-1}W_2(t_0, t_1)S = \int_{t_0}^{t_1} F(s)F^{\mathrm{T}}(s)\,\mathrm{d}s.$$

Since

$$F(t)F^{\mathrm{T}}(t) = \begin{bmatrix} 0_{k\times k} & 0_{k\times(n-k)} \\ 0_{(n-k)\times k} & \bar{F}(t)\bar{F}^{\mathrm{T}}(t) \end{bmatrix},$$

the equivalence is obvious.

(β) $\Leftrightarrow$ (γ): A consequence of Gram's criterium, cf. Gantmacher [3], p. 247.

(γ) $\Leftrightarrow$ (δ): Follows by Silverman and Meadows [14], Lemma 3 and from the facts

$$\mathrm{rk}\big[\bar{F}(t),\ldots,\bar{F}^{(n-k)}(t)\big] = \mathrm{rk}\big[F(t),\ldots,F^{(n-k)}(t)\big],$$

$$SF^{(i)}(t) = \big(P_2B(t)\big)^{(i)} \quad \text{for } i=0,\ldots,n-k.$$

(i) $\Leftrightarrow$ (ii): By Proposition 2.4 and the equivalence (α) $\Leftrightarrow$ (δ) we conclude that (1.1a) is completely controllable into $\mathscr{V}_1$ at time t_0 iff there exists a $t_1 > t_0$ such that (δ) is valid. The equivalence (i) $\Leftrightarrow$ (ii) now follows from elementary properties of real analytic functions.

(ii) $\Leftrightarrow$ (iii): Since

$$\mathrm{rk}\big[\bar{F}(t),\ldots,\bar{F}^{(n-k)}(t)\big] = \mathrm{rk}\big[P_2B(t),\ldots,(P_2B(t))^{(n-k)}\big],$$

Theorem 6.4 in [6] implies that (ii) is equivalent to the existence of some $\hat{U} \in \mathscr{M}[\mathrm{D}]^{(n-k)\times(n-k)}$, $\hat{V} \in \mathscr{M}[\mathrm{D}]^{m\times(n-k)}$ such that

$$I_{n-k} = \mathrm{D}I_{n-k}\cdot\hat{U} + \bar{F}\cdot\hat{V}.$$

This equation is valid iff

$$S\begin{bmatrix} 0 & 0 \\ 0 & I_{n-k} \end{bmatrix}S^{-1} = S\begin{bmatrix} \mathrm{D}I_k & 0 \\ 0 & \mathrm{D}I_{n-k} \end{bmatrix}S^{-1}\cdot S\begin{bmatrix} 0 \\ \hat{U} \end{bmatrix}S^{-1} + S\begin{bmatrix} 0 \\ \bar{F} \end{bmatrix}\cdot\hat{V}S^{-1}.$$

Note that

$$P_2 = S\begin{bmatrix} 0 & 0 \\ 0 & I_{n-k} \end{bmatrix}S^{-1}, \qquad \mathrm{D}I_n = S\begin{bmatrix} \mathrm{D}I_k & 0 \\ 0 & \mathrm{D}I_{n-k} \end{bmatrix}S^{-1}, \qquad P_2B(t) = S\begin{bmatrix} 0 \\ \bar{F}(t) \end{bmatrix}$$

and the proof is complete. □

3. Duality between controllability into a subspace and reconstructibility w.r.t. a subspace

In this section we sketch the dual concept to controllability into a subspace. The dual system of (1.1) is defined by

$$\dot{x}(t) = A^{\mathrm{T}}(-t)x(t) + C^{\mathrm{T}}(-t)u(t), \tag{3.1a}$$

$$y(t) = B^{\mathrm{T}}(-t)x(t). \tag{3.1b}$$

3.1. Definition. The state $x_1 \in \mathbb{R}^n$ is *reconstructible w.r.t.* $\mathscr{V}_2$ *at time* t_0 if there exists $t_{-1} < t_0$ such that

$$C(t)X(t, t_{-1})x_1 = 0 \quad \text{for all } t \in [t_{-1}, t_0] \quad \text{and} \quad x_1 \in P_2^{\mathrm{T}}(t_{-1})\mathbb{R}^n$$

implies $x_1 = 0$.

(1.1) is called *completely reconstructible w.r.t.* $\mathscr{V}_2$ if every $x_1 \in \mathbb{R}^n$ is reconstructible w.r.t. $\mathscr{V}_2$ at any time $t_0 \in \mathbb{R}$.

Analogous results to that of Section 2 can be derived by use of the matrix

$$H_2(t_{-1}, t_0) = P_2(t_{-1}) \int_{t_{-1}}^{t_0} X^{\mathrm{T}}(s, t_{-1}) C^{\mathrm{T}}(s) C(s) X(s, t_{-1}) \, \mathrm{d}s P_2^{\mathrm{T}}(t_{-1}).$$

As an expected result we state without proof the following dual relation.

3.2. Proposition. *The system* (1.1) *is completely controllable into* $\mathscr{V}_1$ *iff its dual system* (3.1) *is completely reconstructible w.r.t.* $\mathscr{V}_2$.

It is straightforward to define the concepts of complete reachability from $\mathscr{V}_2$ and complete observability w.r.t. $\mathscr{V}_1$ and to prove its dual relationships.

4. Stabilizability of systems with exponential dichotomy

In this section we consider systems of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \geq 0, \tag{4.1}$$

where $A(\cdot)$, $B(\cdot)$ are continuous real valued $n \times n$, $n \times m$ matrices, resp. Additionally it will be assumed that its homogeneous part possesses an exponential dichotomy, i.e. there exist positive constants K, L, α, β such that

$$\begin{aligned} &| X(t) P_1 x_0 | \leq K \, \mathrm{e}^{-\alpha(t-s)} | X(s) P_1 x_0 | && \text{for } t \geq s, \\ &| X(t) P_2 x_0 | \leq L \, \mathrm{e}^{-\beta(s-t)} | X(s) P_2 x_0 | && \text{for } s \geq t, \\ &| P_1(t) | \leq M && \text{for } t \geq 0, \end{aligned} \tag{4.2}$$

for all $x_0 \in \mathbb{R}^n$. Using the notation of Section 2, $\mathscr{V}_1$ and $\mathscr{V}_2$ consist of the motions exponentially bounded from above and below, respectively. Clearly, every time-invariant system $\dot{x}(t) = Ax(t)$ with A having no eigenvalues on the imaginary axis has an exponential dichotomy. For a discussion of exponential dichotomy see, for instance, Coppel [1]. We want to determine a feedback matrix $F(\cdot)$ such that the closed-loop system

$$\dot{x}(t) = (A + BF)(t)x(t), \quad t \geq 0,$$

is uniformly asymptotically stable. It is well known that for many control problems of time-varying systems, uniformity constraints are necessary, cf. Kalman [8]. In our approach this becomes as follows:

4.1. Definition. The system (4.1) is *uniformly completely controllable into* $\mathscr{V}_1$ if there exist positive constants σ, a, b such that

$$aI_n \leq W_2(t, t+\sigma) \leq bI_n \quad \text{on } P_2^{\mathrm{T}}(t)\mathbb{R}^n \text{ for all } t \geq 0. \tag{4.3}$$

If (4.1) is bounded, i.e. $|A(t)| \leq c$ and $|B(t)| \leq c$ for some $c > 0$, the second inequality in (4.3) is automatically satisfied. See Lemma 1 in Silverman and Anderson [13].

A straightforward calculation shows that uniform complete controllability into a subspace is invariant under kinematic similarity. To stabilize system (4.1) we have to introduce the following matrix:

$$\tilde{W}_2(t, t+\sigma) = X(t) P_2 \int_t^{t+\sigma} X^{-1}(s) B(s) B^{\mathrm{T}}(s) X^{-1\mathrm{T}}(s) \, \mathrm{e}^{-2\alpha'(t-s)} \, \mathrm{d}s P_2^{\mathrm{T}} X^{\mathrm{T}}(t) \tag{4.4}$$

for some $\alpha' > \alpha$. It is easily proved that (4.3) implies the following inequalities:

$$a\,\mathrm{e}^{-2\sigma\alpha'} I_n \leqslant \tilde{W}_2(t,\, t+\sigma) \leqslant b\,\mathrm{e}^{2\sigma\alpha'} I_n \quad \text{on } P_2^{\mathrm{T}}(t)\mathbb{R}^n \tag{4.5}$$

and

$$b^{-1}\,\mathrm{e}^{-2\sigma\alpha'} I_n \leqslant \tilde{W}_2^{-1}(t,\, t+\sigma) \leqslant a^{-1}\,\mathrm{e}^{2\sigma\alpha'} I_n \quad \text{on } P_2(t)\mathbb{R}^n. \tag{4.6}$$

Now we can state the main result: if (4.1) is uniformly completely controllable and $B(\cdot)$ satisfies a certain boundedness condition, then a feedback $F(\cdot)$ can be determined such that all motions of the closed-loop system

$$\dot{x}(t) = (A(t) + B(t)F(t))x(t), \quad t \geqslant 0, \tag{4.7}$$

are as stable as the stable solutions of (4.1). More precisely:

4.2. Proposition. *Let the system* (4.1) *satisfy* (4.3) *and*

$$\left| P_1(t)B(t)(P_2(t)B(t))^{\mathrm{T}} \right| \leqslant c \quad \textit{for all } t \geqslant 0 \tag{4.8}$$

for some $c > 0$. *Then with the feedback matrix*

$$F(t) := -\tfrac{1}{2} B^{\mathrm{T}}(t)\tilde{W}_2^{-1}(t,\, t+\sigma)P_2(t)$$

the motions of the closed-loop system (4.7) *fulfil*

$$|x(t)| \leqslant K'\,\mathrm{e}^{-\alpha(t-s)}|x(s)| \quad \textit{for } t \geqslant s \geqslant 0. \tag{4.9}$$

For a bounded system (4.1), uniform complete controllability into $\mathscr{V}_1$ is also necessary for stabilizability.

4.3. Proposition. *Let* (4.1) *be a bounded system. Then there exists a bounded feedback* $F(\cdot)$ *such that the trajectories of*

$$\dot{x}(t) = (A(t) + B(t)F(t))x(t), \quad t \geqslant 0,$$

are uniformly asymptotically stable iff (4.1) *is uniformly completely controllable into* $\mathscr{V}_1$.

Proof of Proposition 4.2. Since P_1 is kinematically similar to $\left[\begin{smallmatrix} I_k & 0 \\ 0 & 0 \end{smallmatrix}\right]$ we assume without loss of generality, that P_1 is of this form. Coppel [1] has proved that a system of exponential dichotomy is always reducible by a kinematic similarity transformation to the form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix} u(t)$$

with fundamental matrix

$$X(t) = \begin{bmatrix} X_1(t) & 0 \\ 0 & X_2(t) \end{bmatrix}.$$

Hence

$$\tilde{W}_2(t,\, t+\sigma) = \begin{bmatrix} 0 & 0 \\ 0 & V_2(t,\, t+\sigma) \end{bmatrix},$$

where

$$V_2(t,\, t+\sigma) = \int_t^{t+\sigma} X_2(t,\, s)B_2(s)B_2^{\mathrm{T}}(s)X_2^{\mathrm{T}}(t,\, s)\,\mathrm{e}^{-2\alpha'(t-s)}\,\mathrm{d}s.$$

The feedback law becomes

$$u(t)=F(t)x(t)=-\tfrac{1}{2}B_2^{\mathrm{T}}(t)V_2^{-1}(t,\ t+\sigma)x_2(t)$$

where the closed-loop system is of the form

$$\dot{x}_1(t)=A_1(t)x_1(t)-\tfrac{1}{2}B_1(t)B_2^{\mathrm{T}}(t)V_2^{-1}(t,\ t+\sigma)x_2(t), \tag{4.10}$$

$$\dot{x}_2(t)=\left[A_2(t)-\tfrac{1}{2}B_2(t)B_2^{\mathrm{T}}(t)V_2^{-1}(t,\ t+\sigma)\right]x_2(t). \tag{4.11}$$

Ikeda et al. [5], Theorem 3.1, proved that the free motions of (4.11) are uniformly asymptotically bounded; more precisely,

$$|x_2(t)|\leqslant c_1\ \mathrm{e}^{-\alpha'(t-s)}|x_2(s)|\quad \text{for } t\geqslant s\geqslant 0 \tag{4.12}$$

for some $c_1>0$.

Using Variation of Constants, (4.10) becomes

$$x_1(t)=X_1(t,\ s)x_1(s)-\tfrac{1}{2}\int_s^t X_1(t,\ \tau)B_1(\tau)B_2^{\mathrm{T}}(\tau)V_2^{-1}(\tau,\ \tau+\sigma)x_2(\tau)\ \mathrm{d}\tau. \tag{4.13}$$

Let $c_2:=\frac{1}{2}Kca^{-1}c_1\ \mathrm{e}^{2\sigma\alpha'}$ and apply (4.2), (4.8), (4.6) and (4.12) to (4.13). Then

$$|x_1(t)|\leqslant K\ \mathrm{e}^{-\alpha(t-s)}|x_1(s)|+c_2\int_s^t K\ \mathrm{e}^{-\alpha(t-\tau)}\ \mathrm{e}^{-\alpha'(\tau-s)}|x_2(s)|\ \mathrm{d}\tau. \tag{4.14}$$

Because $\alpha'>\alpha$ we obtain

$$\begin{aligned}|x_1(t)|&\leqslant K\ \mathrm{e}^{-\alpha(t-s)}|x_1(s)|+\frac{c_2}{\alpha'-\alpha}\left[\mathrm{e}^{-\alpha(t-s)}-\mathrm{e}^{-\alpha'(t-s)}\right]|x_2(s)|\\ &\leqslant c_3\ \mathrm{e}^{-\alpha(t-s)}\left[|x_1(s)|+\left(1-\mathrm{e}^{(\alpha-\alpha')(t-s)}\right)|x_2(s)|\right]\\ &\leqslant c_3\ \mathrm{e}^{-\alpha(t-s)}\left[|x_1(s)|+|x_2(s)|\right]\end{aligned} \tag{4.15}$$

where $c_3:=\max\{K,\ c_2/(\alpha'-\alpha)\}$.

Finally, (4.12) and (4.15) give (4.9). □

Proof of Proposition 4.3. Clearly, $F(\cdot)$ defined by (4.9) is bounded if (4.1) is bounded and the uniform complete controllability condition into $\mathscr{V}_1$ holds. To prove that uniform complete controllability into $\mathscr{V}_1$ is also necessary, note that Silverman and Anderson [13] have shown that

$$|W(t_1,\ t_2)|^2\leqslant k\quad \text{for all } t_2>t_1\geqslant 0$$

is valid for some $k>0$ if (4.1) is bounded.

Thus the right inequality in (4.3) holds and it remains to prove the left inequality. This can be done in the same way as with Ikeda et al. [4], the necessity part of the proof of Theorem 3. □

The results of this section also hold for a slight generalization of exponential dichotomy, we can assume α and β in (4.2) to be arbitrary real numbers.

Without proof we state further results.

4.4. Remark. Suppose (1.1) is uniformly completely reachable from $\mathscr{V}_2$ (defined in an analogous way). Then there exists a feedback $F(\cdot)$ such that the motions of the closed-loop system

$$\dot{x}(t)=(A+BF)(t)x(t),\quad t\geqslant 0,$$

are exponentially bounded from below, i.e. for some $\tilde{K}>0$,

$$|x(t)|\leqslant \tilde{K}\ \mathrm{e}^{-\beta(s-t)}|x(s)|,\quad s\geqslant t\geqslant 0.$$

Similar to [5] it can be shown that if the real numbers α, β satisfy $-\beta < -\alpha$ and additionally (4.1) is uniformly completely controllable into $\mathscr{V}_1$ and uniformly completely reachable from $\mathscr{V}_2$, there exist a feedback $F(\cdot)$ and an estimation $H(\cdot)$ such that the motions of the total closed-loop system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A(t) & B(t)F(t) \\ H(t)C(t) & A(t) - H(t)C(t) + B(t)F(t) \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}$$

fulfil

$$c_1 \left| \begin{matrix} x(s) \\ z(s) \end{matrix} \right| \mathrm{e}^{-\beta(t-s)} \leqslant \left| \begin{matrix} x(t) \\ z(t) \end{matrix} \right| \leqslant c_2 \left| \begin{matrix} x(s) \\ z(s) \end{matrix} \right| \mathrm{e}^{-\alpha(t-s)}$$

for some c_1, $c_2 > 0$.

References

[1] W.A. Coppel, *Dichotomies in Stability Theory*, Lecture Notes in Mathematics No. 629 (Springer, Berlin–New York, 1978).
[2] J.L. Daleckii and M.G. Krein, *Stability of Solutions of Differential Equations in Banach Spaces* (AMS, Providence, RI, 1974).
[3] F.R. Gantmacher, *The Theory of Matrices*, Vol. 1 (Chelsea, New York, 1959).
[4] M. Ikeda, H. Maeda and S. Kodama, Stabilization of linear systems, *SIAM J. Control* **10** (1972) 716–729.
[5] M. Ikeda, H. Maeda and S. Kodama, Estimation and feedback in linear time-varying systems: a deterministic theory, *SIAM J. Control* **13** (1975) 304–326.
[6] A. Ilchmann, I. Nürnberger and W. Schmale, Time-varying polynomial matrix systems, *Internat. J. Control* **40** (1984) 329–362.
[7] A. Ilchmann, Time-varying linear systems and invariants of system equivalence, *Internat. J. Control* **42** (1985) 759–790.
[8] R.E. Kalman, Contributions to the theory of optimal control, *Bul. Soc. Mat. Mexico* **5** (1960) 102–119.
[9] H.W. Knobloch and F. Kappel, *Gewöhnliche Differentialgleichungen* (Teubner, Stuttgart, 1974).
[10] H.W. Knobloch and H. Kwakernaak, *Lineare Kontrolltheorie* (Springer, Berlin, 1985).
[11] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems* (Wiley, New York, 1972).
[12] H.H. Rosenbrock, *State Space and Multivariable Theory* (Nelson, London, 1970).
[13] L.M. Silverman and B.D.O. Anderson, *Controllability, Observability and Stability of Linear Systems*, *SIAM J. Control* **6** (1968) 121–130.
[14] L.M. Silverman and H.E. Meadows, Controllability and observability in time-variable linear systems, *SIAM J. Control* **5** (1967) 64–73.