CONSTRUCTION OF CODIMENSION ONE HOMOCLINIC CYCLES

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Abstract. We give an explicit construction of families of D_m-equivariant polynomial vector fields in $\mathbb{R}^4$ possessing a codimension-one homoclinic cycle. The homoclinic cycle consist of m homoclinic trajectories all connected to the equilibrium at the origin. The constructed vector fields can provide a setting for a (numerical) bifurcation study of these homoclinic cycles, in particular for m a multiple of four, where the bifurcations form an open problem.

1. Introduction

Particularly interesting invariant sets for differential equations are heteroclinic networks built from a finite number of equilibria and heteroclinic trajectories connecting these equilibria. Such heteroclinic networks are at the heart of explanations for intermittent time series and also of studies of traveling waves and pattern formation.

In contexts of equivariant differential equations, i.e. differential equations with a symmetry, heteroclinic networks naturally occur robustly or of low codimension. Robust occurrence means that small smooth perturbations of the differential equations with a heteroclinic network admit a heteroclinic network near the one for the unperturbed differential equation. Low codimension similarly means that heteroclinic networks arise in a robust way at isolated parameter values in families of differential equations depending on a small number (the codimension) of parameters.

Of special interest in equivariant differential equations are homoclinic cycles: heteroclinic networks where the heteroclinic trajectories are symmetry related through an element of the symmetry group (precise definitions will be given below). Homburg et al [6] started a bifurcation analysis of codimension one homoclinic cycles. Under open conditions and in a wide range of cases, bifurcation scenarios were established describing how suspended hyperbolic sets appear or disappear in the bifurcation. It turned out that the analysis in [6] breaks down for homoclinic cycles with specific symmetries. The prototype bifurcation where this analysis breaks down arises for homoclinic cycles in differential equations with D_m-symmetry for m a multiple of 4.

The goal of this paper is to provide an explicit construction of polynomial vector fields in $\mathbb{R}^4$ with D_m-symmetry possessing a homoclinic cycle consisting of m homoclinic trajectories. In fact we provide a family of polynomial vector fields unfolding the homoclinic cycle. One purpose of this would be to enable detailed numerical studies, in particular for D_{4n}-equivariant vector fields. The following section describes the precise set-up, contains the necessary background and definitions, and introduces notation. Section 3 specifies the properties (concerning symmetry and unfolding) that the constructed vector fields possess. Section 4 provides the actual construction of the vector fields. Equation (8) is the resulting polynomial vector field for $m = 4$. Note that the formulas depend on coefficients a and b; bounds on them are provided for which the properties in Section 3 hold.

Of interest, but not pursued in this paper, are homoclinic cycles occurring in local bifurcations in systems with symmetry. Matthies [9] studied the Takens-Bogdanov bifurcation with D_3-symmetry and in particular the dynamics near the homoclinic cycle that appears in its unfolding. The homoclinic cycle in [9] consists of three homoclinic trajectories all connected to the equilibrium at the origin. He found a suspended Markov chain appearing in the bifurcation in the neighborhood of the homoclinic cycle. Matthies' study was a motivation for and a special case in the more general study of [6]. Rucklidge [10] studied, also in a setting of Takens-Bogdanov bifurcations and motivated by convection problems, D_4-symmetric homoclinic cycles near an $O(2)$-limit. His considerations are not restricted to a small neighborhood of the homoclinic cycle.

2. D_m-SYMMETRIC CODIMENSION ONE HOMOCLINIC CYCLES

Consider a family of ordinary differential equations (ODEs)

$$\dot{x} = f(x, \lambda), \tag{1}$$

with $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, that is equivariant under the linear action (representation) of a finite group G, [4]:

$$gf(x, \lambda) = f(gx, \lambda), \quad \forall \lambda \in G.$$

Assume that p is a hyperbolic equilibrium of (1) which is a fixed point of the group action i.e. $gp = p$, for all $g \in G$. That means that the isotropy group G_p of p is equal to G,

$$G_q := \{g \in G : gq = q\}.$$

In other words that means that p belongs to the fixed space $\operatorname{Fix} G$ of the group G. More generally, for any subgroup H of G the fixed space of H is defined by

$$\operatorname{Fix} H := \{x \in \mathbb{R}^n : hx = x, \forall h \in H\}.$$

Assume further that γ is a homoclinic trajectory of (1) asymptotic to p. We stipulate that the terms "trajectory" and "solution" are reserved for ODE-flow orbits and that the term "orbit" is reserved for group orbits. Then $G\bar{\gamma}$ is an example of a *homoclinic network* [5]. For $h \in G$, the invariant set

$$\Gamma = \langle h \rangle \bar{\gamma},$$

where $\langle h \rangle$ denotes the cyclic subgroup of G generated by h, is called a *homoclinic cycle*. The homoclinic cycle Γ consists of p and of several homoclinic trajectories to p which are g-images of γ. The number of these homoclinic trajectories coincides with the order of the group $\langle h \rangle$. Note that the homoclinic network $G\bar{\gamma}$ contains $|G/G_\gamma|$ homoclinic trajectories; $G_\gamma := G_q$, $q \in \gamma$, is the isotropy group of γ (the isotropy groups G_q are identical for all $q \in \gamma$).

We denote by D_m the symmetry group of a regular m-gon in the plane. The group D_m can be written as semidirect product of the reflection group $\mathbb{Z}_2(\zeta)$ and a rotation group $\mathbb{Z}_m(\theta_m)$, cf. [3]:

$$D_m = \mathbb{Z}_2(\zeta) \ltimes \mathbb{Z}_m(\theta_m).$$

In this notation the reflection ζ and the rotation θ_m denote the generators of the corresponding subgroups. We write $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$, and we assume that D_m acts on each $\mathbb{R}^2$ absolutely irreducibly, cf. Section 3.1. In this paper we construct D_m-equivariant vector fields f_m in $\mathbb{R}^4$ that possess within $\operatorname{Fix} \langle \zeta \rangle$ a homoclinic trajectory γ_m. Therefore the vector field f_m has a homoclinic cycle

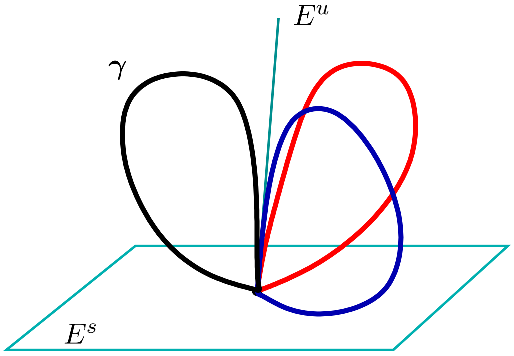


Figure 1. The homoclinic cycle $\Gamma_3 = \langle\theta_3\rangle\bar{\gamma}$; $\gamma \subset \mathrm{Fix}\,\langle\zeta\rangle$.

$\Gamma_m := \langle\theta_m\rangle\bar{\gamma}_m$. Figure 1 visualizes Γ_3. Further we embed this vector field in a family of vector fields $f_m(\cdot,\lambda)$ such that for $\lambda = 0$ the primary cycle does exist, and if λ moves off zero the trajectory γ_m splits up.

In [6] the dynamics near codimension one homoclinic networks is considered. We restrict to homoclinic cycles as introduced above. In this restricted setting, the general assumptions in [6] on the family of differential equations (1) are:

Hypothesis (H1). *(i) The vector fields $f(\cdot,\lambda)$ are equivariant with respect to a finite group G.*

(ii) $\dot{x} = f(x,\lambda)$ has a hyperbolic equilibrium p with real leading stable and unstable eigenvalues μ^s and μ^u, with $0 < -\mu^s < \mathrm{Re}(\mu^u)$. The isotropy group G_p acts absolutely irreducibly on the eigenspace E^s_p corresponding to μ^s.

(iii) At $\lambda = 0$, there is a codimension one homoclinic cycle Γ generated by a homoclinic trajectory γ asymptotic to p. Stable and unstable manifolds unfold generically with respect to the parameter λ, i.e. a corresponding Melnikov integral is different from zero, cf. [5, Hypothesis 2.2].

The leading eigenvalues are the eigenvalues which are closest to the imaginary axis. Since G_p acts absolutely irreducibly on E^s_p, the eigenvalue μ^s is real.

Note that Hypothesis (H1)(iii) includes the property that the homoclinic trajectory γ is nondegenerate, i.e. along γ the intersection of the tangent spaces of the stable and unstable manifolds of p is one-dimensional:

$$T_{\gamma(0)}W^s(p,0) \cap T_{\gamma(0)}W^u(p,0) = \mathbb{R}\dot{\gamma}(0). \tag{2}$$

As an extensive example in [6, Section 5] homoclinic cycles in D_m-equivariant systems with $m \geq 3$ are discussed. In the course of this treatment several cases are distinguished. Here we focus on one particular case which is characterized by the following assumptions:

Hypothesis (H2). *(i) The hyperbolic equilibrium p has the isotropy group $G_p = D_m$.*

(ii) The eigenspace E^s_p is two-dimensional.

(iii) $G_p = D_m$ acts on E^s_p as D_m.

(iv) The trajectory γ has the isotropy group $G_\gamma = \mathbb{Z}_2(\zeta)$.

Due to Hypothesis (H2)(iv) the cycle Γ consists of m copies of γ. Consequently, recall our restriction to $\mathbb{R}^4$, the unstable eigenspace E^u_p is also two-dimensional. Moreover, p has no strong stable or strong unstable manifolds. Indeed, in [6] a few more assumptions are made. These concern mainly orbit flip and inclination flip conditions. By our restriction to $\mathbb{R}^4$ these assumptions are automatically fulfilled.

Next we review the main theorem of [6] in the context of our hypotheses. For that we have need of topological Markov chains. We simply repeat the definition given in [6]: Let

$$\Sigma_k = \{1, \dots, k\}^{\mathbb{Z}}$$

denote the set of double infinite sequences $\kappa : \mathbb{Z} \to \{1, \dots, k\}$, $i \mapsto \kappa_i$, equipped with the product topology. Let $A = (a_{ij})_{i,j \in \{1,\dots,k\}}$ be a 0-1 matrix, that is $a_{ij} \in \{0, 1\}$. By Σ_A we denote the topological Markov chain defined by A,

$$\Sigma_A = \{\kappa \in \Sigma_k \mid a_{\kappa_i \kappa_{i+1}} = 1\}.$$

Note that the left shift σ operating on Σ_k by $(\sigma\kappa)_i = \kappa_{i+1}$ leaves Σ_A invariant.

Theorem 2.1 ([6], Theorem 1.1)**.** *Consider the differential equations* (1) *in* $\mathbb{R}^4$ *and assume Hypotheses (H1) and (H2). Write* $\gamma_1, \dots, \gamma_m$ *for the homoclinic trajectories that constitute* Γ, $\gamma_i := \theta_m^i \gamma$.

There is an explicit construction of $m \times m$ *matrices* A_- *and* A_+ *with coefficients in* $\{0, 1\}$ *and the nonzero coefficients in mutually disjoint positions, so that the following holds for any generic family unfolding a relative homoclinic cycle as above.*

Take cross sections S_i *transverse to* γ_i *and write* Π_λ *for the first return map on the collection of cross sections* $\cup_{j=1}^m S_j$. *For* $\lambda > 0$ *small enough, there is an invariant set* $\mathcal{D}_\lambda \subset \cup_{j=1}^m S_j$ *for* Π_λ *such that for each* $\kappa \in \Sigma_{A_+}$ *there exists a unique* $x \in D_\lambda$ *with* $\Pi_\lambda^i(x) \in S_{\kappa_i}$. *Moreover,* $(\mathcal{D}_\lambda, \Pi_\lambda)$ *is topologically conjugate to* (Σ_{A_+}, σ). *An analogous statement holds for* $\lambda < 0$ *with* Σ_{A_+} *replaced by* Σ_{A_-}.

This description of the dynamics provides a complete picture of the local nonwandering dynamics near Γ *if and only if*

$$A_+ + A_- = \mathbf{1},$$

where $\mathbf{1}$ *denotes the matrix with all coefficients equal to one.*

The proof relies on Lin's method. It turns out that this procedure imparts a relation between the geometry of Γ and the matrices $A_{+/-}$.

Summarizing the considerations in [6] the equations that determine the trajectories connecting the points $\Pi_\lambda^i(x)$ related to a given sequence (κ_i) read

$$0 = \lambda - e^{2\mu^s(\lambda)\omega_i} \langle \eta^s_{\kappa_{i-1}}(\lambda), \eta^-_{\kappa_i}(\lambda) \rangle + R_i, \qquad i \in \mathbb{Z},$$

where

$$R_i = R_i(\boldsymbol{\omega}, \lambda, \kappa) = O(e^{-2\mu^u(\lambda)\omega_{i+1}}) + O(e^{2\mu^s(\lambda)\omega_i \delta}), \quad \text{for some } \delta > 1.$$

Here $2\omega_i$ are the transition times from $\Pi_\lambda^{i-1}(x) \in \mathcal{S}_{\kappa_{i-1}}$ to $\Pi_\lambda^i(x) \in \mathcal{S}_{\kappa_i}$. The vector $\eta_i^s \in \operatorname{Fix}\langle \theta_m^i \zeta \rangle \cap E_p^s$ is pointing in the direction along which γ_i is approaching p. Note that due to Hypothesis (H1)(ii) and Hypothesis (H2)(iii) the intersection $\operatorname{Fix}\langle \theta_m^i \zeta \rangle \cap E_p^s$ is one-dimensional.

Further, $\eta_j^- \in \operatorname{Fix}\langle \theta_m^j \zeta \rangle$, and η_j^- is orthogonal to E_p^u. The matrices $A_{+/-}$ arise out of the solvability of the bifurcation equations. Define a matrix $M = (m_{ij})$ by $m_{ij} := \operatorname{sgn}\langle \eta_i^s, \eta_j^- \rangle$. With that it follows

$$A_+ = 1/2(M + |M|), \quad A_- = 1/2(M - |M|).$$

Now it is obvious that $A_+ + A_- \neq \mathbf{1}$, if $m_{ij} = 0$ for some i, j. This is the case if m is a multiple of 4: $m = 4n$. Assume for simplicity that E_p^s and E_p^u are orthogonal to each other. Then η_i^s and η_i^- span the same one-dimensional subspace. This situation is depicted in Figure 2 for $n = 1$.

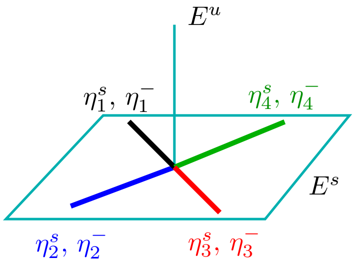


Figure 2. The quantities $\eta_i^{s/-}$ under the assumption of Hypothesis (H2); $m = 4$.

3. Setting

We construct a family of D_4-equivariant polynomial vector fields $f_m : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$, $(x, \lambda) \mapsto f_m(x, \lambda)$ in coordinates $x = (x_1, y_1, x_2, y_2)$. Below we describe the representation of D_4 on $\mathbb{R}^4$ and the properties we demand on the vector fields.

3.1. **Representations of the group D_m.** For information on group theory in dynamical systems contexts we refer to [3]. For our purpose only the two-dimensional absolutely irreducible representation $\vartheta_m : D_m \to GL(2, \mathbb{R})$ is of interest [8], where

$$\zeta := \vartheta_m(\zeta) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \theta_m := \vartheta_m(\theta_m) = \begin{pmatrix} \cos(\frac{2\pi}{m}) & \sin(\frac{2\pi}{m}) \\ -\sin(\frac{2\pi}{m}) & \cos(\frac{2\pi}{m}) \end{pmatrix}.$$

On $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$, the state space of the desired vector field, the group D_m acts as $\vartheta_m + \vartheta_m$

$$(\vartheta_m + \vartheta_m)(g)((x_1, y_1), (x_2, y_2)) \equiv (\vartheta_m(g)(x_1, y_1), \vartheta_m(g)(x_2, y_2)),$$

for all $g \in D_m$, $(x_i, y_i) \in \mathbb{R}^2$, $i = 1, 2$. For this representation we have

$$\operatorname{Fix} \mathbb{Z}_2(\zeta) := \{(x_1, 0, x_2, 0) : x_i \in \mathbb{R}, i = 1, 2\}. \tag{3}$$

If m is even, this fixed point space is invariant under $\theta_m^{m/2}$.

3.2. **Demands on the vector field f_m.** We construct f_m with properties formulated in Properties (P1)–(P4) below.

Property (P1). *$f_m(\cdot, \lambda)$ is D_m-equivariant with $m \in \mathbb{N}$, $m \geq 3$, where D_m acts on $\mathbb{R}^2 \times \mathbb{R}^2$ as $\vartheta_m + \vartheta_m$.*

Let $f_m(\cdot,\lambda)$ satisfy Property (P1), then the mapping $f_m(\cdot,\lambda)$ leaves $\operatorname{Fix}\mathbb{Z}_2(\zeta)$ invariant:

$$\zeta f_m(x_1,0,x_2,0,\lambda) = f_m(\zeta(x_1,0,x_2,0),\lambda) = f_m(x_1,0,x_2,0,\lambda).$$

In other words, the vector field $f_m(\cdot,\lambda)$ is tangent to $\operatorname{Fix}\mathbb{Z}_2(\zeta)$, and therefore this fixed point space is invariant under the flow of $f_m(\cdot,\lambda)$. Denote the restriction of f_m to $\operatorname{Fix}\mathbb{Z}_2(\zeta)$ by $\hat{f}_m$:

$$\hat{f}_m = f_m\big|_{\operatorname{Fix}\mathbb{Z}_2(\zeta)}.$$

If m is even, the vector field $\hat{f}_m$ is $\mathbb{Z}_2(\theta_m^{m/2})$-equivariant. Note that $\theta_m^{m/2}$ then acts on $\operatorname{Fix}\mathbb{Z}_2(\zeta)$ as $-id$.

Property (P2). *$f_m(\cdot,\lambda)$ has a hyperbolic equilibrium $p=0$.*

Obviously the isotropy group of $p=0$ is equal to D_m: $G_p = D_m$.

Property (P3). *$f_m(\cdot,0)$ has in $\operatorname{Fix}\mathbb{Z}_2(\zeta)$ a homoclinic trajectory γ_m asymptotic to p.*

The trajectory γ_m has the isotropy subgroup $G_{\gamma_m} = \mathbb{Z}_2(\zeta)$. It is also a homoclinic trajectory of $\hat{f}_m(\cdot,0)$. If m is even, then by Property (P2), the vector field $\hat{f}_m(\cdot,0)$ has two homoclinic trajectories, γ_m and $-\gamma_m$.

Let μ^s and μ^u denote the stable and unstable eigenvalues, respectively of p restricted to $\operatorname{Fix}\mathbb{Z}_2(\zeta)$, and let e^s and e^u be corresponding eigenvectors. Due to the equivariance of f_m we find $\theta_m D_1 f_m(0,0) = D_1 f_m(0,0)\theta_m$, and therefore $\theta_m e^{s/u}$ are also eigenvectors belonging to $\mu^{s/u}$. This yields that $D_1 f(p,0)$ has two real eigenvalues $\mu^s < 0 < \mu^u$, both of geometric multiplicity two. According to Property (P1) the group D_m acts absolutely irreducibly on the stable and unstable subspaces E^s and E^u.

Property (P4). *Within* $\operatorname{Fix}\,\mathbb{Z}_2(\zeta)$ *the homoclinic trajectory γ splits up with non-zero speed at $\lambda=0$.*

With that we have the following lemma:

Lemma 3.1. *Any vector field f_m on $\mathbb{R}^4$ which satisfies Properties (P1)–(P4) satisfies Hypotheses (H1)(i),(ii), Hypothesis (H1)(iii) restricted to* $\operatorname{Fix}\,\mathbb{Z}_2(\zeta)$ *(which means that the nondegeneracy condition* (2) *is not automatically satisfied) and (H2).*

By Properties (P1)–(P4) the leading eigenvalue is not yet determined. In the constructions below we introduce freely selectable coefficients a and b. In our construction we can choose these coefficients in such a way that the stable eigenvalue is the leading one (cf. Hypothesis (H1)(ii)). Note that (2) does not follow from Properties (P1)–(P4) – this condition is solely satisfied within $\operatorname{Fix}\,\mathbb{Z}_2(\zeta)$. However, we may assume that (2) is also satisfied in the full space since this condition describes the generic situation. In Section 4.3 we state a range of coefficients for which the constructed vector fields also satisfy condition (2).

4. Construction of D_m-equivariant vector fields in $\mathbb{R}^4$

We build the desired D_m-equivariant family of vector fields f_m in several steps. First we construct a single vector field $\hat{f}_m$ in $\mathbb{R}^2$ possessing a homoclinic trajectory to a hyperbolic equilibrium. In doing so we follow the idea of Sandstede [12] – we construct $\hat{f}_m$ in such a way that a (generalized) Cartesian leaf forms a homoclinic trajectory.

Next we embed the vector field $\hat{f}_m$ in a one-parameter family such that by changing the family parameter λ (off the critical value) the homoclinic trajectory splits up with non-zero speed.

In the final step we extend this family into $\mathbb{R}^4$ and end up with a family as stated in Section 3.

4.1. **Basic construction in $\mathbb{R}^2$.**

Sandstede used in [12] the Cartesian leaf to construct a vector field in $\mathbb{R}^2$ having a homoclinic trajectory. Here we use the slightly modified curves

$$\mathcal{C}_m(x_1, x_2) := x_1^2(1 - x_1^{m-2}) - x_2^2.$$

Note that $\mathcal{C}_3^{-1}(0)$ is the Cartesian leaf, and $\mathcal{C}_4^{-1}(0)$ is a lemniscate, cf. Figure 3. For any odd or even m the curves $\mathcal{C}_m^{-1}(0)$ resemble those for $m = 3$ or $m = 4$, respectively.

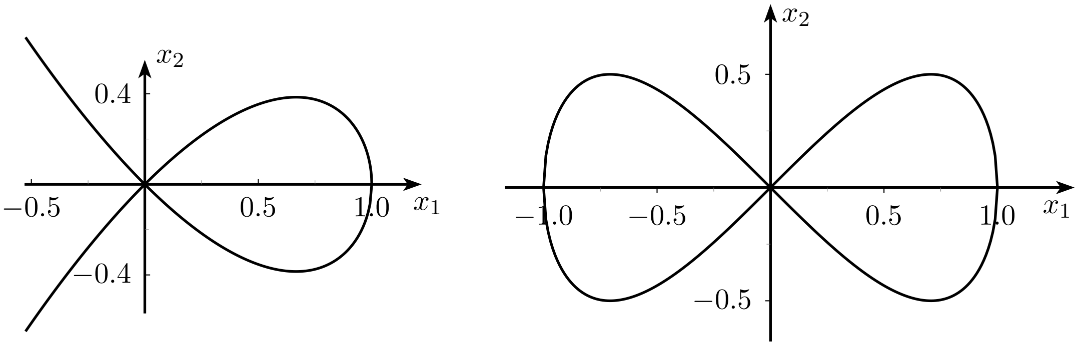


Figure 3. The curves $\mathcal{C}_3(x_1, x_2) = 0$ (left) and $\mathcal{C}_4(x_1, x_2) = 0$ (right).

The zero level set $\mathcal{C}_m^{-1}(0)$ of $\mathcal{C}_m$ is invariant under the flow of a given vector field $\hat{f}$ if and only if

$$\left\langle \nabla\mathcal{C}_m(x_1, x_2), \hat{f}(x_1, x_2)\right\rangle = 0, \quad \forall (x_1, x_2) \in \mathcal{C}_m^{-1}(0). \tag{4}$$

Lemma 4.1. *Let $m \geq 3$ and $a, b \in \mathbb{R}\backslash\{0\}$, $a^2 < b^2$. The vector field*

$$\hat{f}_m(x_1, x_2) := \left(\begin{array}{l} ax_1 + bx_2 - ax_1^{m-1} \\ bx_1 + ax_2 - b\frac{m}{2}x_1^{m-1} - a\frac{m}{2}x_1^{m-2}x_2 \end{array} \right)$$

has a homoclinic trajectory $\hat{\gamma}_m$ which is a subset of $\mathcal{C}_m^{-1}(0) \cap \{x_1 > 0\}$.

The expression of $\hat{f}_m$ follows from a general polynomial ansatz of degree $m - 1$ plugged into Equation (4). Here we confine to showing that the vector field has a homoclinic trajectory to $(x_1, x_2) = (0, 0)$.

Proof. First we show that $\hat{f}_m$ satisfies (4): let $(x_1, x_2) \in \mathcal{C}_m^{-1}(0)$. Using

$$\nabla\mathcal{C}_m(x_1, x_2) = \begin{pmatrix} 2x_1 - mx_1^{m-1} \\ -2x_2 \end{pmatrix},$$

compute

$$\begin{aligned}\left\langle \nabla\mathcal{C}_m(x_1,x_2), \hat{f}_m(x_1,x_2)\right\rangle &= a((2x_1 - mx_1^{m-1})(x_1 - x_1^{m-1}) - 2x_2(x_2 - \tfrac{m}{2}x_1^{m-2}x_2)) \\ &= a(2\underbrace{(x_1^2(1-x_1^{m-2}) - x_2^2)}_{=\mathcal{C}_m(x_1,x_2)} - mx_1^{m-2}\underbrace{(x_1^2 - x_1^m - x_2^2)}_{=\mathcal{C}_m(x_1,x_2)}) \\ &= 0.\end{aligned}$$

We must verify that $\hat{f}_m(x_1,x_2) \neq 0$ for all $(x_1,x_2) \in \mathcal{C}_m^{-1}(0)$, $x_1 > 0$: the first component $\hat{f}_m^1$ evaluated at those points equals

$$\hat{f}_m^1(x_1, \pm x_1\sqrt{1-x_1^{m-2}}) = x_1\sqrt{1-x_1^{m-2}}\left(a\sqrt{1-x_1^{m-2}} \pm b\right)$$

and becomes zero for $x_1 = 0$, $x_1^{m-2} = 1$ or $x_1^{m-2} = 1 - (b/a)^2$. With the assumption $a^2 < b^2$ the right-hand side $1-(b/a)^2$ of the last equation is negative. Hence, if m is even, this equation has no real solution. If m is odd the only real solution is negative. Further, the second equation, $x_1^{m-2} = 1$, implies $|x_1| = 1$. But the second component $\hat{f}_m^2(1,0) = b(1-m/2)$ is different from zero, since $b \neq 0$. □

Remark 4.1. Because of $a^2 < b^2$ the equilibrium $(0,0)$ is a saddle point with eigenvalues $a+b$ and $a-b$. If one imposes $0 < a^2 < b^2$ then $|a+b| \neq |a-b|$. This implies that the vector field $\hat{f}_m$ is neither Hamiltonian nor reversible. □

Remark 4.2. Let m be even. The vector field $\hat{f}_m$ is equivariant with respect to $\mathbb{Z}_2(\theta_2)$, where θ_2 acts on $\mathbb{R}^2$ as $-id$. Consequently, $\theta_2(\hat{\gamma}_m)$ is also a homoclinic orbit of $\hat{f}_m$ asymptotic to $(x_1,x_2) = (0,0)$. The both orbits $\hat{\gamma}_m$ and $\theta_2(\hat{\gamma}_m)$ together with the equilibrium $(0,0)$ form the "figure-eight" drawn in the right panel of Figure 3. □

Remark 4.3. We can find an analytic solution for the homoclinic trajectory $\hat{\gamma}_m = (\hat{\gamma}_m^1, \hat{\gamma}_m^2)$ by choosing the ansatz

$$\hat{\gamma}_m^1(t) = (1-u(t)^2)^{\frac{1}{m-2}}, \qquad \hat{\gamma}_m^2(t) = -\hat{\gamma}_m^1(t)u(t)$$

with $u : (-\infty,\infty) \to (-1,1)$. Then u satisfies the initial value problem

$$\dot{u} = \frac{m-2}{2}(b-au)(1-u^2), \qquad u(0) = 0.$$

that can be solved by separation of variables. We obtain the inverse function of $u = H(t)$ by

$$u \mapsto t = H^{-1}(u) = \frac{2a\ln(1-\frac{a}{b}u) - (a+b)\ln(1-u) - (a-b)\ln(1+u)}{(m-2)(b^2-a^2)}.$$

□

Next we add a perturbation term to the vector field $\hat{f}_m$ that splits up the homoclinic trajectory $\hat{\gamma}_m$ with non-zero speed (at $\lambda = 0$), and obtain the family of vector fields $\hat{f}_m : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$.

Lemma 4.2. *Let $m \geq 3$ and $a, b \in \mathbb{R}\backslash\{0\}$, $a^2 < b^2$. Consider the family of vector fields*

$$\hat{f}_m(x,\lambda) := \begin{pmatrix} ax_1 + bx_2 - ax_1^{m-1} \\ bx_1 + ax_2 - b\frac{m}{2}x_1^{m-1} - a\frac{m}{2}x_1^{m-2}x_2 \end{pmatrix} + \lambda\nabla\mathcal{C}_m(x_1,x_2).$$

In accordance with Lemma 4.1 this vector field has the homoclinic trajectory $\hat{\gamma}_m$ for $\lambda = 0$. This homoclinic trajectory splits up as λ moves off zero. Moreover, let for λ close to zero $d(\lambda)$ denote *the distance of the stable and unstable manifolds of the equilibrium* $(0,0)$*, measured in a direction perpendicular to* $\hat{f}_m(\hat{\gamma}_m(0),0)$*. The derivative* $d'(0)$ *is different from zero.*

Proof. The verification that the perturbation splits the homoclinic trajectory in the described way can be done by using the Melnikov integral. It can be shown that, cf. [5],

$$d'(0) = \int\limits_{-\infty}^{\infty} \left\langle \eta(t), D_\lambda \hat{f}_m(\hat{\gamma}_m(t),0) \right\rangle dt,$$

where $\eta(t)$ solves the adjoint variational equation $\dot{v} = -[D_x \hat{f}_m(\hat{\gamma}_m(t),0)]^T v$ with $|\eta(0)| = 1$; $\eta(0) \perp \hat{\gamma}_m(0)$. Therefore

$$\eta(t) = \phi(t) \begin{pmatrix} -\hat{f}_m^2(\hat{\gamma}_m(t),0) \\ \hat{f}_m^1(\hat{\gamma}_m(t),0) \end{pmatrix}$$

for a scalar function ϕ. Simple calculations show that the function ϕ solves

$$\dot{\phi} = -\operatorname{div}(\hat{f}_m)(\hat{\gamma}_m(t),0)\phi.$$

Combining these results yields

$$d'(0) = \int\limits_{-\infty}^{\infty} \phi(t) \left\langle \begin{pmatrix} -\hat{f}_m^2(\hat{\gamma}_m(t),0) \\ \hat{f}_m^1(\hat{\gamma}_m(t),0) \end{pmatrix}, D_\lambda \hat{f}_m(\hat{\gamma}_m(t),0) \right\rangle dt.$$

By construction the scalar product within this integral is always positive or negative, and as a solution of a scalar linear differential equation $\phi(t)$ does not change sign. Hence $d'(0) \neq 0$. ☐

Remark 4.4. If m is even the entire family $\hat{f}_m(\cdot,\lambda)$ is equivariant with respect to the representation of $\mathbb{Z}_2(\theta_2)$ which is given in Remark 4.2. Consequently, both homoclinic trajectories $\hat{\gamma}_m$ and $\theta_2(\hat{\gamma}_m)$ split up as λ moves off zero.

Denote the stable and unstable eigenvalues by μ^s and μ^u respectively. Let $a > 0$, then $|\mu^s| < \mu^u$. Applying a first return map, cf. [5], yields the bifurcation diagram depicted in Figure 4. In particular this diagram reveals for which parameter values which periodic orbits do exist. ☐

4.2. **Extending the vector field to $\mathbb{R}^4$.**

We construct the vector field $f_m = (f_m^1, f_m^2, f_m^3, f_m^4)^T$ such that "$f_m|_{\mathrm{Fix}\ \mathbb{Z}_2(\zeta)} = \hat{f}_m$", or more precisely

$$f_m(x_1,0,x_2,0,\lambda) = (\hat{f}_m^1(x_1,x_2,\lambda),0,\hat{f}_m^2(x_1,x_2,\lambda),0)^T. \tag{5}$$

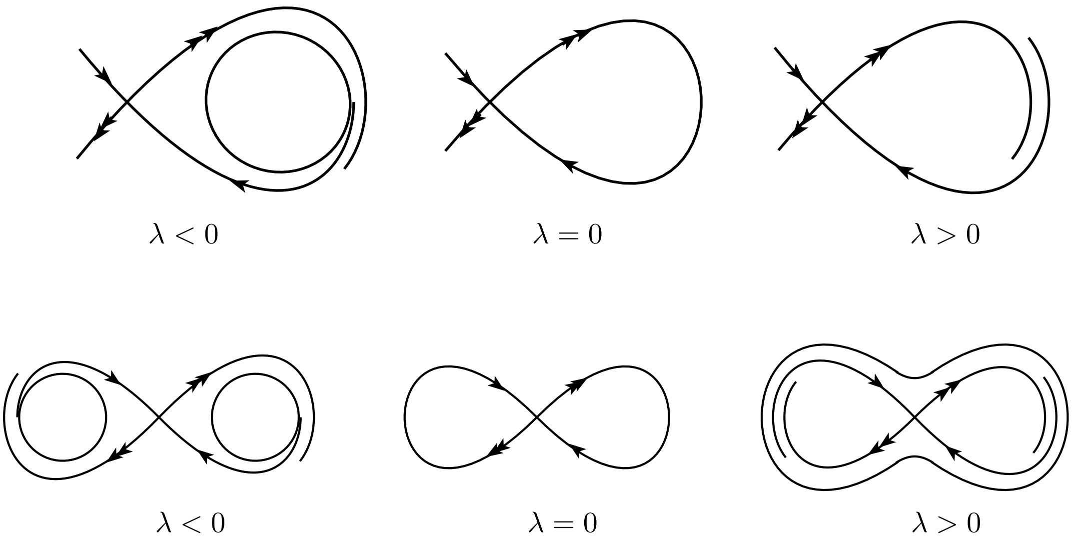
We extend the perturbed vector field to $\mathbb{R}^4$ (see Theorem 4.1) by using a set of generators for D_m-equivariant vector fields. Such generators can be found in a paper by Matthies [9] for the case $m = 3$ and in a paper by Lari-Lavassani, Langford, Huseyin and Gatermann [8] for $m = 3, 4$.

4.2.1. *D_m-equivariant vector fields in $\mathbb{R}^4$.*

In this section we first describe the structure of D_m-equivariant polynomial vector fields in $\mathbb{R}^4$. To this end we recall the following definitions and results from [1, 4].

Let G be a finite group acting on the vector space $\mathbb{R}^n$. A (polynomial) function $s : \mathbb{R}^n \to \mathbb{R}$ is called *G-invariant* with respect to the given representation of the group if

$$s = s \circ g \quad \forall g \in G.$$



$\lambda < 0$ $\lambda = 0$ $\lambda > 0$

$\lambda < 0$ $\lambda = 0$ $\lambda > 0$

Figure 4. Bifurcation diagram of f_m, $a > 0$: m is odd (top); m is even (bottom).

The set $\mathcal{R}_G$ of G-invariant polynomials forms a ring. This ring is finitely generated, i.e. there is a finite set $s_1, \cdots, s_k$ of G-invariant polynomials, the *generators* for the ring $\mathcal{R}_G$, such that each $s \in \mathcal{R}_G$ is of the form

$$s(x) = B(s_1(x), ..., s_k(x)),$$

where $B : \mathbb{R}^k \to \mathbb{R}$ is polynomial. Further, the set of equivariant polynomial vector fields forms a module $\mathcal{M}_G$ over the ring $\mathcal{R}_G$. This module is finitely generated, i.e. there exists a set $\{h_1, \cdots, h_l\}$ of G-equivariant polynomial vector fields such that each $f \in \mathcal{M}_G$ can be written as

$$f(x) = \sum_{i=1}^{l} B_i(s_1(x), ..., s_k(x))h_i(x) \quad \forall x, \tag{6}$$

where B_i are polynomials.

Lari-Lavassani et al. [8] present a generating set for D_3- and D_4-equivariant vector fields $f : \mathbb{R}^4 \to \mathbb{R}^4$, where D_m acts as $\vartheta_m + \vartheta_m$. Matthies [9] also presents a generating set for D_3-equivariant vector fields. Unlike Lari-Lavassani et al., he considered complex vector fields $f_{\mathbb{C}} : \mathbb{C}^2 \to \mathbb{C}^2$. In what follows we identify $\mathbb{R}^2$ with $\mathbb{C}$ since it seems to be adequate to work with complex coordinates in the context of the D_m representations under consideration.

The coordinates on $\mathbb{C}^2$ we denote by $z = (v, w)$ where $v = x_1 + iy_1$ and $w = x_2 + iy_2$. With the isomorphism $\mathcal{I} : \mathbb{C}^2 \to \mathbb{R}^4$

$$\mathcal{I} : \mathbb{C}^2 \to \mathbb{R}^4, \qquad (v, w) = (x_1 + iy_1, x_2 + iy_2) \mapsto (x_1, y_1, x_2, y_2),$$

we obtain a complex vector field $f_{\mathbb{C}}$ by

$$f_{\mathbb{C}} := \mathcal{I}^{-1} \circ f \circ \mathcal{I}.$$

The vector field $f_{\mathbb{C}}$ is equivariant with respect to the complex representation $(\vartheta_m + \vartheta_m)_{\mathbb{C}}$ of the group D_m defined by

$$g_{\mathbb{C}} := \mathcal{I}^{-1} \circ g \circ \mathcal{I}, \quad g \in D_m.$$

In particular, the corresponding complex representations of ζ and θ_m read

$$\zeta_{\mathbb{C}}(v,w) = (\bar{v},\bar{w}), \qquad \theta_{m,\mathbb{C}}(v,w) = (e^{i2\pi/m}v, e^{i2\pi/m}w). \tag{7}$$

In the following we present sets of D_m-invariant functions and D_m-equivariant vector fields. In the cases $m = 3$ and $m = 4$ these are generator sets for the corresponding ring $\mathcal{R}_{D_m}$ or the module $\mathcal{M}_{D_m}$, respectively, [8, 9].

Lemma 4.3. *Assume D_m acts on $\mathbb{C}^2$ as defined in* (7).

(i) The functions

$$s_0(v,w) = v\bar{v}, \quad s_1(v,w) = w\bar{w}, \quad s_2(v,w) = v\bar{w} + \bar{v}w$$

$$t_j(v,w) = v^j w^{m-j} + \bar{v}^j \bar{w}^{m-j}, \quad j \in \{0,\dots,m\}$$

are D_m-invariant polynomials on $\mathbb{C}^2$.

(ii) The mappings

$$g_0(v,w) = (v,0), \quad g_1(v,w) = (0,v), \quad g_2(v,w) = (w,0), \quad g_3(v,w) = (0,w),$$

$$k_j(v,w) = (\bar{v}^j \bar{w}^{m-1-j}, 0), \quad h_j(v,w) = (0, \bar{v}^j \bar{w}^{m-1-j}), \quad j \in \{0,\dots,m-1\}.$$

are D_m-equivariant polynomial mappings $\mathbb{C}^2 \to \mathbb{C}^2$.

Proof. The invariance or equivariance of the given functions or mappings, respectively, can be verified by straightforward calculations. □

4.2.2. *The vector field f_m.* We use the mappings presented in Lemma 4.3 to extend the vector field $\hat{f}_m$ to the desired vector field f_m in $\mathbb{R}^4$. In the course of this we use representations of these vector fields in complex coordinates.

In complex coordinates the fixed space of $\mathbb{Z}_2(\zeta_{\mathbb{C}})$ reads, cf. (3) and (7),

$$\operatorname{Fix}\mathbb{Z}_2(\zeta_{\mathbb{C}}) := \{(x_1,x_2) : x_i \in \mathbb{R}, i = 1,2\}.$$

According to (5) the $\mathbb{R}^2$-vector field $\hat{f}_m$ can be seen as vector field on $\mathbb{Z}_2(\zeta) \subset \mathbb{R}^4$ (we denote this vector field again by $\hat{f}_m$). The related vector field $\hat{f}_{m,\mathbb{C}}$ reads (in complex coordinates)

$$\hat{f}_{m,\mathbb{C}}(x_1,x_2,\lambda) = (\hat{f}^1_m(x_1,x_2,\lambda), \hat{f}^2_m(x_1,x_2,\lambda)).$$

Crucial for the intended extension is the observation that the mappings (vector fields) in Lemma 4.3(ii) leave $\operatorname{Fix}\mathbb{Z}_2(\zeta_{\mathbb{C}})$ invariant, and the polynomials in Lemma 4.3(i) are real-valued. Further we find for $(v,w) \in \operatorname{Fix}\mathbb{Z}_2(\zeta_{\mathbb{C}})$, i.e. for $(v,w) = (x_1,x_2)$,

$$g_0(x_1,x_2) = (x_1,0), \quad g_1(x_1,x_2) = (0,x_1), \quad g_2(x_1,x_2) = (x_2,0), \quad g_3(x_1,x_2) = (0,x_2),$$

$$k_{m-1}(x_1,x_2) = (x_1^{m-1},0), \quad h_{m-1}(x_1,x_2) = (0,x_1^{m-1}), \quad h_{m-2}(x_1,x_2) = (0,x_1^{m-2}x_2).$$

Consequently, the vector field $\hat{f}_{m,\mathbb{C}}$ can be represented by means of just these vector fields:

$$\begin{aligned}\hat{f}_{m,\mathbb{C}}(x_1,x_2,\lambda) &= a\left(g_0 + g_3 - k_{m-1} - \tfrac{m}{2}h_{m-2}\right)(x_1,x_2) + b\left(g_1 + g_2 - \tfrac{m}{2}h_{m-1}\right)(x_1,x_2) \\ &\quad + \lambda\left(2g_0 - mk_{m-1} - 2g_3\right)(x_1,x_2).\end{aligned}$$

Theorem 4.1. *The vector field*

$$\begin{aligned} f_{m,\mathbb{C}}(v,w,\lambda) &= a\left(g_0+g_3-k_{m-1}-\tfrac{m}{2}h_{m-2}\right)(v,w)+b\left(g_1+g_2-\tfrac{m}{2}h_{m-1}\right)(v,w) \\ &\quad +\lambda\left(2g_0-mk_{m-1}-2g_3\right)(v,w) \end{aligned}$$

is equivariant with respect to the complex representation $(\vartheta_m+\vartheta_m)_{\mathbb{C}}$.

Moreover, the vector field $f_m := \mathcal{I}\circ f_{m,\mathbb{C}}\circ\mathcal{I}^{-1}$ *satisfies Properties (P1)–(P4).*

Proof. The first part of the theorem is an immediate consequence of the above representation of $\hat{f}_{m,\mathbb{C}}$.

Further, by construction the vector field f_m leaves Fix $\mathbb{Z}_2(\zeta)$ invariant and its restriction to this fixed space coincides with $\hat{f}_m$. So the properties concerning the homoclinic trajectory γ_m follow from the considerations in Section 4.1. □

Remark 4.5. Not any vector field $\hat{f}_{\mathbb{C}}$ on $\mathrm{Fix}\,\mathbb{Z}_2(\zeta_{\mathbb{C}})$ can be extended to an $\vartheta_m\times\vartheta_m$-equivariant vector field $f_{\mathbb{C}}$ on $\mathbb{C}^2$: if m is even, the components of all vector fields in Lemma 4.3 are monomials of odd degree, whereas all functions in Lemma 4.3 are homogeneous polynomials of even degree. Hence, the components of all polynomial vector fields $f_{\mathbb{C}}$ that can be generated by those functions and vector fields, cf. (6), are sums of homogeneous polynomials of odd degree. This must be true already for the restricted vector field $\hat{f}_{\mathbb{C}}$.

Recall that for $m=4$, Lemma 4.3 presents a set of generators for the module $\mathcal{M}_{D_m}$. □

For the real vector fields f_m the polynomial structure is getting more and more complicated as m increases. For that reason we confine ourselves to present the representation of f_4,

$$\begin{aligned} f_4(x,\lambda) &= \begin{pmatrix} ax_1 & +bx_2 & -ax_1^3+3ax_1y_1^2 & \\ ay_1 & +by_2 & +3ax_1^2y_1-ay_1^3 & \\ bx_1 & +ax_2 & -2bx_1^3+6bx_1y_1^2 & -2ax_1^2x_2+2ay_1^2x_2+4ax_1y_1y_2 \\ by_1 & +ay_2 & +6bx_1^2y_1-2by_1^3 & +4ax_1y_1x_2+2ay_2(x_1^2-y_1^2) \end{pmatrix} \\ &\quad +\lambda\begin{pmatrix} 2x_1 & -4x_1^3+12x_1y_1^2 \\ 2y_1 & +12x_1^2y_1-4y_1^3 \\ -2x_2 & \\ -2y_2 & \end{pmatrix}. \end{aligned} \tag{8}$$

4.3. **Verification of the nondegeneracy condition** (2)**.** We show that for $|a|\ll|b|$ the constructed vector field f_m satisfies the nondegeneracy condition (2). The perturbation analysis below that shows this yields a strong restriction on the parameters a and b. We do not believe the estimates to be optimal and we expect that condition (2) is still satisfied for many times greater ratios $|a|/|b|$. However this is not covered by our analysis.

The condition (2) is equivalent to the fact that the adjoint of the variational equation along γ_m,

$$\dot{\psi} = -\left(D_{(x,y)}f_m(\gamma_m(t),0)\right)^T\psi, \tag{9}$$

has (up to multiples) only one solution which is bounded on $\mathbb{R}$. According to the construction, see also in the proof of Lemma 4.2, one such bounded solution lies in Fix $\mathbb{Z}_2(\zeta)$. With that said it remains to show that (9) has no bounded solution outside of Fix $\mathbb{Z}_2(\zeta)$. Recall that exactly those solutions of (9) are bounded on $\mathbb{R}$ which start in the orthogonal complement of the sum of the tangent spaces of the stable and unstable manifolds of the equilibrium $p = 0$ along γ_m.

As before we drop the dependence of the vector field on a and b in our notation.

With $H_i : \mathbb{R}^4 \to \mathbb{R}^4$, $H_i = \big(H_i^1,\, H_i^2,\, H_i^3,\, H_i^4\big)^T$ defined by

$$\begin{aligned}
H_1(x,0) :=& \big(0,\, a + (m-1)ax_1^{m-2},\, 0,\, b + \frac{m(m-1)}{2}bx_1^{m-2} + \frac{m(m-2)}{2}ax_1^{m-3}x_2\big)^T, \\
H_2(x,0) :=& \big(0,\, b,\, 0,\, a + \frac{m}{2}ax_1^{m-2}\big)^T,
\end{aligned}$$

we write

$$f_m(x,y,0) = \big(\hat{f}_m^1(x,0), 0, \hat{f}_m^2(x,0), 0\big)^T + y_1 H_1(x,y) + y_2 H_2(x,y).$$

Note that the y-components of $\gamma_m(t)$ are zero. Therefore the Jacobian of f_m at $\gamma_m(t)$ reads

$$\begin{aligned}
D_{(x,y)} f_m(\gamma_m(t),0) =& D_{(x,y)} \big(\hat{f}_m^1(\gamma_m(t),0), 0, \hat{f}_m^2(\gamma_m(t),0), 0\big)^T \\
&+ H_1(\gamma_m(t))\,(0,1,0,0) + H_1(\gamma_m(t))\,(0,0,0,1).
\end{aligned}$$

Within Fix $\mathbb{Z}_2(\zeta)$ a solution w of (9) is bounded on $\mathbb{R}$ if and only if $w(t) \perp \gamma_m(t)$ for all t. Using the inner unit normal ν of $\mathcal{C}_m^{-1}(0) \cap \{x_1 > 0\}$ within Fix $\mathbb{Z}_2(\zeta)$ we decompose a bounded solution w of (9) as follows:

$$w(t) = w_1(t)\nu(t) + w_2(t)(0,1,0,0)^T + w_4(t)(0,0,0,1)^T,$$

where $\nu(t) = (\nu^1(t),\, 0\,, \nu^2(t),\, 0)^T$, $|\nu(t)| \equiv 1$, $\langle \gamma_m(t), \nu(t)\rangle = 0$. With that we find

$$\dot{w}(t) = \dot{w}_1(t)\nu(t) + w_1(t)\dot{\nu}(t) + \dot{w}_2(t)(0,1,0,0)^T + \dot{w}_4(t)(0,0,0,1)^T.$$

We plug this expression into (9) and take the inner product with $(0,1,0,0)^T$ or $(0,0,0,1)^T$ to get differential equations for w_2 or w_4, respectively. Here we take into consideration that, because of $\langle \nu(t), \nu(t)\rangle \equiv 1$, the derivative $\dot{\nu}(t)$ is perpendicular to $\nu(t)$. Moreover, $\dot{\nu}(t)$ is also perpendicular to $(0,1,0,0)^T$ and $(0,0,0,1)^T$. Further we exploit that $H_i^1(\gamma_m(t)) \equiv H_i^3(\gamma_m(t)) \equiv 0$, $i = 1,2$. Thus it follows

$$\begin{aligned}
\dot{w}_2 =& - H_1^2(\gamma_m(t))w_2 - H_1^4(\gamma_m(t))w_4, \\
\dot{w}_4 =& - H_2^2(\gamma_m(t))w_2 - H_2^4(\gamma_m(t))w_4.
\end{aligned}$$

In more detail this equation is written as

$$\begin{pmatrix} \dot{w}_2 \\ \dot{w}_4 \end{pmatrix} = -\Big(aA(t) + bB(t)\Big)\begin{pmatrix} w_2 \\ w_4 \end{pmatrix}, \tag{10}$$

with

$$\begin{aligned}
A(t) &= \begin{pmatrix} 1 + (m-1)\big(\gamma_m^1(t)\big)^{m-2} & \frac{m(m-2)}{2}\big(\gamma_m^1(t)\big)^{m-3}\gamma_m^2(t) \\ 0 & 1 + \frac{m}{2}\big(\gamma_m^1(t)\big)^{m-2} \end{pmatrix}, \\
B(t) &= \begin{pmatrix} 0 & 1 + \frac{m(m-1)}{2}\big(\gamma_m^1(t)\big)^{m-2} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{m(m-1)}{2}\big(\gamma_m^1(t)\big)^{m-2} \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{11}$$

Instead of (10) we first consider the truncated equation

$$\begin{pmatrix} \dot{w}_2 \\ \dot{w}_4 \end{pmatrix} = -bB(t) \begin{pmatrix} w_2 \\ w_4 \end{pmatrix}. \tag{12}$$

With $v := w_4$ this equation can be rewritten as second order equation

$$\ddot{v} = b^2\Big(1 + \frac{m(m-1)}{2}\big(\gamma_m^1(t)\big)^{m-2}\Big)v. \tag{13}$$

This equation can be treated as a similar problem in [13, Lemma 2.2, Lemma 2.3]. Recall that $\gamma_m^1(t) > 0$ for all $t \in \mathbb{R}$. Suppose (13) has a nontrivial bounded solution v. Then both v and $\dot{v}$ are square-integrable over $\mathbb{R}$, moreover v decays exponentially fast as $t \to \pm\infty$. Keeping this in mind we find by multiplying (13) by v and integrating that

$$0 > -\int_{-\infty}^{\infty} \big(\dot{v}(t)\big)^2\, dt = \int_{-\infty}^{\infty} b^2\Big(1 + \frac{m(m-1)}{2}\big(\gamma_m^1(t)\big)^{m-2}\Big)\big(v(t)\big)^2\, dt > 0.$$

This contradicts the assumption of a bounded solution.

Summarizing, Equation (12) has no bounded solution. Further, $B(t)$ can be seen as an exponentially decaying perturbation of a hyperbolic matrix. By the roughness theorem [2, Lecture 4, Proposition 1], this equation has exponential dichotomies on both $\mathbb{R}^+$ and $\mathbb{R}^-$. Altogether this implies that (12) has an exponential dichotomy on $\mathbb{R}$.

Finally, by [7, Theorem 3.2], Equation (10) still has an exponential dichotomy for a satisfying

$$|a| \sup_{t\in\mathbb{R}} \|A(t)\| \left(\frac{K_1}{\alpha} + \frac{K_2}{\beta}\right) < 1. \tag{14}$$

Here $\|\cdot\|$ is the operator norm, and K_1, K_2, α and β are constants related to the exponential dichotomy of the truncated equation (12):

$$\begin{array}{lcll} \|\Phi(t,\tau)P(\tau)\| & \leq & K_1 e^{-\alpha(t-\tau)}, & t \geq \tau, \\ \|\Phi(\tau,t)(id - P(t))\| & \leq & K_2 e^{\beta(\tau-t)}, & t \geq \tau, \end{array}$$

where $\Phi(\cdot,\cdot)$ denotes the transition matrix of (12), and $P(t)$ are the projections related to the exponential dichotomy of this equation.

In what follows we give an estimate of the constant a such that (14) holds true. To that end we need estimates of the constants K_1, K_2, α and β. First we consider exponential dichotomies of (12) on subintervals $[t_0^+, \infty)$ and $(-\infty, t_0^-]$ with corresponding constants $\tilde{K}_1^+$, $\tilde{K}_2^+$ and $\tilde{K}_1^-$, $\tilde{K}_2^-$, respectively. Somewhat lengthy computations which we don't include here, show that for

$$\begin{array}{lcl} t_0^+ & = & \frac{1}{(m-2)\mu^s} \ln\left(\frac{-2(m-2)\mu^s}{3m(m-1)|b|}\right) = \frac{1}{(m-2)(a-|b|)} \ln\left(\frac{2(m-2)(|b|-a)}{3m(m-1)|b|}\right), \\ t_0^- & = & \frac{1}{(m-2)\mu^u} \ln\left(\frac{2(m-2)\mu^u}{3m(m-1)|b|}\right) = \frac{1}{(m-2)(a+|b|)} \ln\left(\frac{2(m-2)(a+|b|)}{3m(m-1)|b|}\right), \end{array}$$

one may take constants $\tilde{K}_1^\pm = \tilde{K}_2^\pm = \frac{9}{2}$ and $\alpha = \beta = |b|$. We indicate the approach to these computations. As in (11), treat $B(t)$ as a perturbation of the constant hyperbolic matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Observe that the perturbation term tends to zero exponentially fast as t tends to $\pm\infty$. The values of the constants are now derived from an inspection of the proof of the roughness statement as in [11, Lemma 1.1], incorporating the exponential decay of the perturbation term.

By [2, Lecture 2] all constants $K_{1/2}^{\pm}$ satisfying

$$K_1^{\pm} \geq \tilde{K}_1^{\pm} N_{\pm}^2 e^{\pm|b|t_0^{\pm}}, \qquad K_2^{\pm} \geq \tilde{K}_2^{\pm} N_{\pm}^2 e^{\pm|b|t_0^{\pm}}$$

with

$$N_{\pm} = e^{\pm \int_0^{t_0^{\pm}} |b| \|B(\tau)\| d\tau} \tag{15}$$

are suitable constants for the exponential dichotomy of (12) on $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively. The right-hand side in (15) can be estimated by

$$e^{\pm \int_0^{t_0^{\pm}} |b| \|B(\tau)\| d\tau} \leq e^{\pm|b|t_0^{\pm}} e^{\pm \frac{m(m-1)}{2}|b| \int_0^{\pm\infty} (\gamma_m^1(\tau))^{m-2} d\tau}.$$

Using the representation of γ_m^1 given in Remark 4.3 we find

$$\begin{aligned}
\int_{-\infty}^{0} \left(\gamma_m^1(\tau)\right)^{m-2} d\tau &= -\int_0^1 H^{-1}\left(-\operatorname{sgn}(b)\sqrt{1-s}\right) ds &&= \frac{2\ln(1+\frac{a}{|b|})}{a(m-2)}, \\
\int_0^{\infty} \left(\gamma_m^1(\tau)\right)^{m-2} d\tau &= \int_0^1 H^{-1}\left(\operatorname{sgn}(b)\sqrt{1-s}\right) ds &&= \frac{-2\ln(1-\frac{a}{|b|})}{a(m-2)}.
\end{aligned}$$

For $a > 0$ we have

$$\int_{-\infty}^{0} \left(\gamma_m^1(\tau)\right)^{m-2} d\tau \leq \int_0^{\infty} \left(\gamma_m^1(\tau)\right)^{m-2} d\tau \quad \text{and} \quad -t_0^- \leq t_0^+.$$

For $a < 0$ the relation signs are inversed.

For our further analysis we assume $a > 0$, and define

$$K(a,b) := \frac{9}{2} e^{3|b|t_0^+} e^{m(m-1)|b| \int_0^{\infty} \left(\gamma_m^1(\tau)\right)^{m-2} d\tau} = \frac{9}{2} e^{\frac{3|b|}{(m-2)(a-|b|)} \ln\left(\frac{2(m-2)(|b|-a)}{3m(m-1)|b|}\right)} e^{\frac{-2m(m-1)|b|}{a(m-2)} \ln(1-\frac{a}{|b|})}.$$

According to our above considerations we may choose

$$K_1^{\pm} = K_2^{\pm} = K(a,b).$$

Further, using $|\gamma_m^i(t)| \leq 1$, $i = 1, 2$, we find $\sup_{t\in\mathbb{R}} \|A(t)\| \leq \frac{m^2}{\sqrt{2}}$. Summarizing, from (14) with $\alpha = \beta = |b|$, $K_1 = K_2 = K(a,b)$ and the above estimate of $\|A(t)\|$ we obtain

$$a \leq \frac{|b|}{\sqrt{2} m^2 K(a,b)}. \tag{16}$$

First we realize that $K(a,b)$ decreases monotonically as $a \to 0$. Consequently we find that for $a \leq \frac{|b|}{r}$, $r \gg 1$

$$\frac{|b|}{\sqrt{2} m^2 K(\frac{|b|}{r}, b)} \leq \frac{|b|}{\sqrt{2} m^2 K(a,b)}.$$

Hence, any $a > 0$ with

$$a \leq |b| \min\left\{\frac{1}{r}, \frac{1}{\sqrt{2} m^2 K(\frac{|b|}{r}, b)}\right\}, \quad r \gg 1$$

satisfies the inequality (16). Indeed $K(\frac{|b|}{r}, b) = \frac{9}{2} e^{\frac{-3r}{(m-2)(r-1)} \ln(\frac{2(m-2)}{3m(m-1)}(1-\frac{1}{r}))} e^{\frac{-2m(m-1)r \ln(1-\frac{1}{r})}{(m-2)}}$ does not depend on b.

Consider the cases $m = 3$ and $m = 4$. With $r = 1000$ we then find in case

$$\begin{aligned} m = 3: &\qquad a \le |b| \cdot 0.145 \cdot 10^{-9}, \\ m = 4: &\qquad a \le |b| \cdot 0.221 \cdot 10^{-8}. \end{aligned}$$

In order to assess the quality of this estimate we remark that $\frac{|b|}{\sqrt{2}m^2\mathcal{K}}$, where

$$\mathcal{K} := \lim_{a \to +0} K(a,b) = \frac{9}{2}\left(\frac{3m(m-1)}{2(m-2)}\right)^{\frac{3}{m-2}} e^{\frac{2m(m-1)}{m-2}},$$

is an upper bound for a. For $m = 3$ we find $\frac{1}{\sqrt{2}m^2\mathcal{K}} \approx 0.147 \cdot 10^{-9}$, and for $m = 4$ we find $\frac{1}{\sqrt{2}m^2\mathcal{K}} \approx 0.223 \cdot 10^{-8}$.

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