Shift dynamics near non-elementary T-points with real eigenvalues

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Abstract

We consider non-elementary T-points in reversible systems in $\mathbb{R}^{2n+1}$. We assume that the leading eigenvalues are real. We prove the existence of shift dynamics in the unfolding of this T-point. Furthermore, we study local bifurcations of symmetric periodic orbits occurring in the process of dissolution of the chaotic dynamics.

1 Introduction

T-points, sometimes also referred to as *Bykov cycles*, are heteroclinic cycles connecting two hyperbolic equilibria with different saddle indices. The heteroclinic connections Γ_1 and Γ_2 are assumed to be such that one of them, say Γ_1, breaks up under perturbations while the other one is robust and isolated. The robustness of Γ_2 is due to the transversal intersection of the corresponding stable and unstable manifolds of the equilibria. We refer to the left sketch in Figure 1 for a visualisation.

These kind of cycle were first found in the Lorenz system [1]. Meanwhile T-points have been found to appear in many further applications such as Kuramoto-Sivashinsky systems, electronic oscillators, semiconductor lasers, magneto convection, and travelling waves in reaction-diffusion dynamics. For precise references concerning these applications we refer to [18].

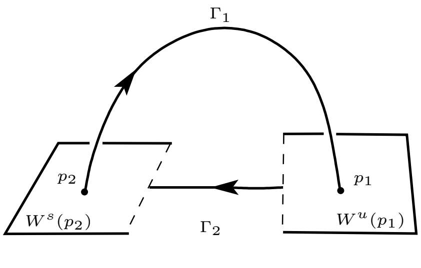

Figure 1: Sketch of a T-point (left) and of a double T-point (right) in $\mathbb{R}^3$ in each case. The "classical" notion T-point implies that the two-dimensional manifolds intersect transversely. Whereas we also allow non-transversal intersections.

Motivated by [1] Bykov studied in a series of papers the dynamics in a neighbourhood of those cycles, cf. e.g. [3] and references therein. More references can also be found in [18] and [14]. It turns out that the complexity of the nonwandering nearby dynamics depends to a large extent on the leading eigenvalues of the equilibria. If the leading eigenvalues of both equilibria are real then the corresponding dynamics is rather tame, whereas shift dynamics occurs in the presence of complex leading eigenvalues [3, 18]. However, in [3] it is also claimed that *double T-points* give rise to shift dynamics also in the case of real leading eigenvalues. Here the notion double T-points refers to two T-points having the non-robust heteroclinic orbit Γ_1 in common. We refer to the right sketch in Figure 1 for a visualisation. Those double T-points will appear generically in the unfolding of a quadratic tangency of the corresponding manifolds $W^u(p_1)$ and $W^s(p_2)$ (notation is chosen according to Figure 1). In order to also capture those cycles we relax the notion T-point to the effect that we admit non-transversal intersections of the manifolds $W^u(p_1)$ and $W^s(p_2)$ along Γ_2. We call the corresponding heteroclinic cycle *degenerate T-point*.

The aim of the paper is to study the dynamics in unfoldings of degenerate T-points. In doing so we demand the underlying vector field to be reversible with respect to a linear involution R. We show that Bykov's result about shift dynamics near double T-points remains true within the reversible setting, cf. Theorem 1.1 below. Moreover we discuss the transition from shift dynamics to "no recurrent dynamics". Our results suggest that this transition is mainly governed by subharmonic bifurcations from branches of periodic orbits.

In what follows we describe the precise setting and present our main results. Concluding this section we comment on the relation of our statements to existing results.

We consider a two-parameter family of vector fields $f : \mathbb{R}^{2n+1} \times \mathbb{R}^2 \to \mathbb{R}^{2n+1}$ ($n \geq 1$), f smooth:

$$\dot{x} = f(x, \mu), \quad \mu = (\mu_1, \mu_2). \tag{1.1}$$

We denote the flow of this vector field by $\{\phi^t_\mu\}$. We assume that the family (1.1) is reversible with respect to a linear involution R, that is

(H 1) $\quad R\, f(x, \mu) = -f(Rx, \mu),$

and we assume that the fixed point space of the involution R is n-dimensional

(H 2) $\dim \operatorname{Fix} R = n$.

We refer to [31], [15], [25] or [6] for detailed information concerning reversible systems.

We aim to study the dynamics in the neighbourhood of a *symmetric degenerate T-point*. For that we assume:

(H 3) At $\mu = 0$ there exists a heteroclinic cycle Γ between two hyperbolic equilibria of saddle type p_1 and p_2 with different saddle indices.

Without loss of generality we assume that $f(p_i, \mu) = 0$, $i = 1, 2$ for sufficiently small $|\mu|$. This can be fixed with a (local) smooth change of coordinates.

The saddle index of an equilibrium is the dimension of the unstable manifold. Throughout this paper we denote the stable manifold of the equilibrium p_i at parameter(s) μ by $W^s(p_i, \mu)$. For brevity, we also denote $W^s(p_i, 0)$ by $W^s(p_i)$. In the same manner we use $W^u(p_i, \mu)$ and $W^u(p_i)$ to denote the corresponding unstable manifolds.

Let

$$\Gamma = \Gamma_1 \cup \Gamma_2,$$

where $\Gamma_1 = \{q_1(t) : t \in \mathbb{R}\}$ denotes the heteroclinic solution connecting p_2 to p_1, i.e. $\Gamma_1 \subset W^u(p_2) \cap W^s(p_1)$, and similarly $\Gamma_2 = \{q_2(t) : t \in \mathbb{R}\}$ denotes the heteroclinic orbit connecting p_1 to p_2, i.e. $\Gamma_2 \subset W^u(p_1) \cap W^s(p_2)$. We also assume that

(H 4) $\dim T_{q_2(0)} W^s(p_2) = \dim T_{q_2(0)} W^u(p_1) = n + 1$.

For the orbit Γ_1 we assume

(H 5) $\dim(T_{q_1(0)} W^s(p_1) \cap T_{q_1(0)} W^u(p_2)) = 1$.

Due to the last two hypotheses the heteroclinic orbit Γ_1 will generically break up while moving μ away from zero. The heteroclinic orbit Γ_2 however will generically be robust, or in other words, generically $W^u(p_1)$ and $W^s(p_2)$ will intersect transversely along Γ_2. In this case the cycle Γ is called a T-point, cf. [12]. Here however we assume that $W^u(p_1)$ and $W^s(p_2)$ have a further common tangent in addition to the vector field direction:

(H 6) $\dim(T_{q_2(0)} W^s(p_2) \cap T_{q_2(0)} W^u(p_1)) = 2$.

This induces the notion degenerate T-point. Finally, the assumption that the heteroclinic cycle Γ is symmetric, that is

(H 7) $R\Gamma = \Gamma$,

justifies the notion symmetric degenerate T-point. Together with Hypothesis (H 4) the symmetry of Γ implies that the fixed points p_1 and p_2 are non-symmetric but lie in the same group orbit, that is $R(p_1) = p_2$, and it implies that both Γ_1 and Γ_2 are symmetric heteroclinic orbits.

Hence we may assume that

$$q_i(0) \in \operatorname{Fix} R, \quad i \in \{1, 2\}.$$

Further, let $\langle \cdot, \cdot \rangle$ be an R-invariant scalar product, cf. also Remark 2.6 below, and let

$$Y_i := \{f(q_i(0), 0)\}^\perp, \quad i = 1, 2. \tag{1.2}$$

With that we construct the cross-sections Σ_1 and Σ_2 as follows

$$\Sigma_i := q_i(0) + Y_i, \quad i = 1, 2.$$

By construction we find that $\mathrm{Fix}\, R \subset Y_i$, $i = 1, 2$. For the details regarding this as well as the following explanations we refer to Section 2. Further, Hypothesis (H 6) gives rise to define

$$U := T_{q_2(0)} W^u(p_1) \cap T_{q_2(0)} W^s(p_2) \cap Y_2.$$

According to Hypothesis (H 6) the manifolds $W^u(p_1)$ and $W^s(p_2)$ have a common tangent along U in Σ_2 which, according to the symmetry of Γ_2, belongs either to $\mathrm{Fix}\, R$ or to $\mathrm{Fix}\,(-R)$. In the present paper we assume:

(H 8) The cycle Γ is *non-elementary*, i.e.

(i) $U \subset \mathrm{Fix}\, R$

(ii) $W^s(p_2)$ and $W^u(p_1)$ have a quadratic tangency along U.

The assumption (H 8)(ii) excludes further degeneracies which could be caused by a vanishing second order term. We refer to Figure 2 for a sketch of a T-point in $\mathbb{R}^3$ satisfying Hypotheses (H 4)-(H 8).

If alternatively it is assumed that $U \subset \mathrm{Fix}\,(-R)$, then the cycle Γ is called *elementary*. The notions *non-elementary* and *elementary* are borrowed from the context of symmetric homoclinic orbits, cf. [31, 15]. However, in this paper we only consider non-elementary T-points.

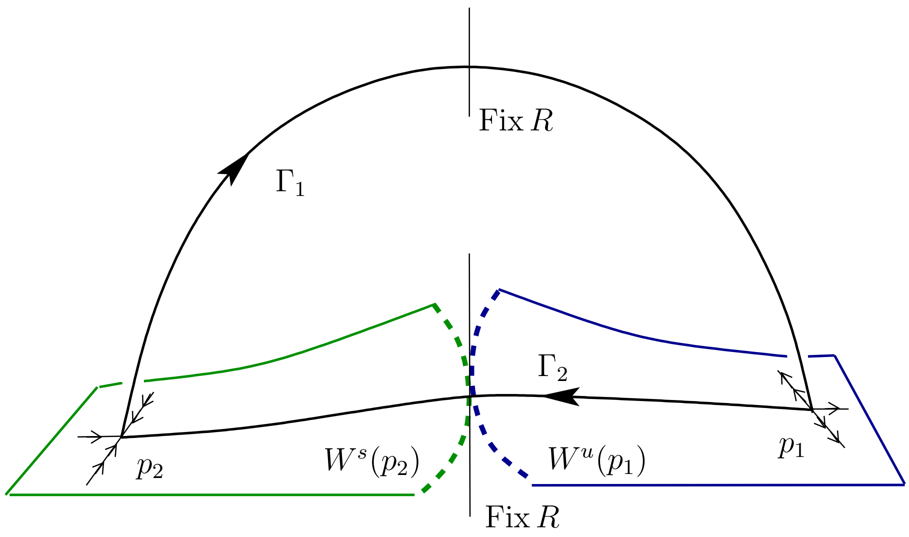

Figure 2: Sketch of an example of a non-elementary symmetric degenerate T-point heteroclinic cycle in $\mathbb{R}^3$ between two real saddles.

Under the above hypotheses both Γ_1 and Γ_2 have codimension one with respect to parameter unfoldings. We use the parameter μ_1 to unfold the splitting of Γ_1, and we use the parameter μ_2 to unfold the splitting of Γ_2. The first row in Figure 3 shows the unfolding of the quadratic tangency of $W^u(p_1)$ and $W^s(p_2)$, cf. Hypothesis (H 8)(ii). The situation depicted in the right panel (of this row) is referred to as a *double T-point*. By this we denote a situation where $W^s(p_2)$ and $W^u(p_1)$ intersect within two different isolated heteroclinic orbits directly connecting the equilibria p_1 and p_2 (without following Γ_1). We want to emphasise that here the double T-point appears in the unfolding of a degenerate T-point. Further we want to note that the representation in Figure 3 suggests that the quadratic tangency of the degenerate T-point can be unfolded by μ_2. The corresponding justification is given in Section 2.1, where we also explain that the splitting of $W^s(p_1)$ and $W^u(p_2)$ can be controlled only by μ_1.

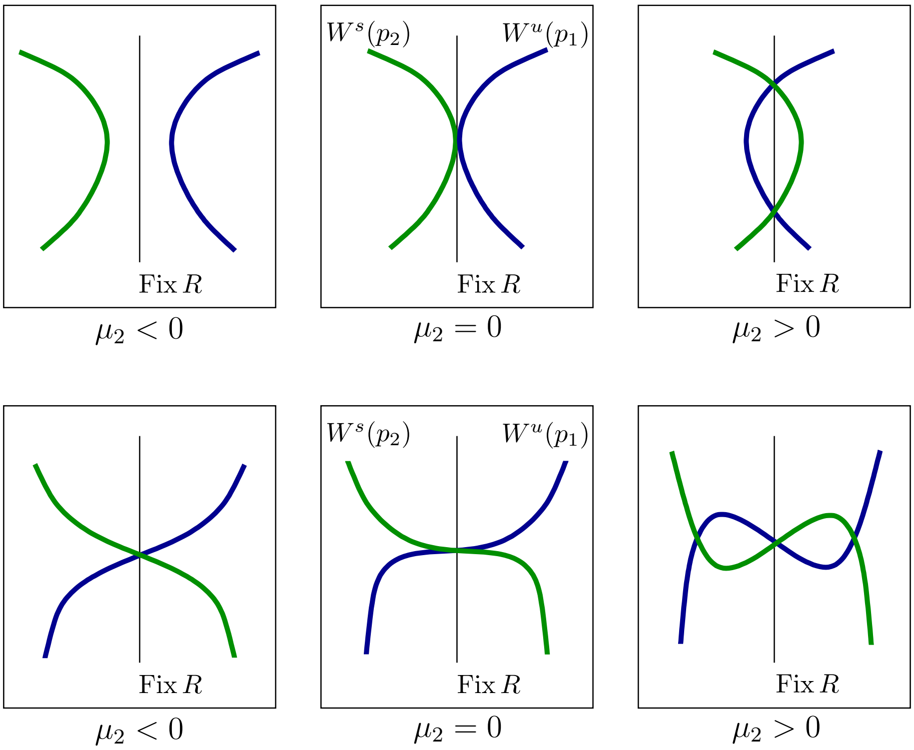

Figure 3: Upper row: unfolding of the traces of $W^u(p_1, \mu)$ and $W^s(p_2, \mu)$ in Σ_2 corresponding to a non-elementary symmetric degenerate T-point heteroclinic cycle in $\mathbb{R}^3$. At $\mu_2 = 0$ the common tangency of $W^u(p_1, \mu)$ and $W^s(p_2, \mu)$ is parallel to $U \subset \text{Fix}\, R$ (in $\mathbb{R}^3$ we have $U = \text{Fix}\, R$). Lower row: unfolding of the traces of $W^u(p_1, \mu)$ and $W^s(p_2, \mu)$ in Σ_2 corresponding to an elementary symmetric degenerate T-point heteroclinic cycle in $\mathbb{R}^3$. At $\mu_2 = 0$ the common tangency of $W^u(p_1, \mu)$ and $W^s(p_2, \mu)$ is parallel to $\text{Fix}\,(-R)$.

Furthermore, we assume that the equilibria p_1 and p_2 are real saddles. More precisely we assume:

(H 9) The leading stable and unstable eigenvalues λ^s and λ^u, respectively, of p_1 are real and simple.

The leading eigenvalues are the ones which are closest to the imaginary axis. Note that due to Hypotheses (H 1) and (H 4) the eigenvalues of p_2 arise from the eigenvalues of p_1 by multiplying with (-1). We assume

(H 10) $\quad \lambda^u < |\lambda^s|.$

In the same way as in [18] we make some further hypotheses ensuring that the T-point has codimension two:

(H 11) $\quad \Gamma_i \not\subset W^{ss}(p_i), \quad \Gamma_i \not\subset W^{uu}(p_{i+1}), \quad i = 1, 2.$

Here $W^{ss}(p)$ and $W^{uu}(p)$ denotes the strong stable manifold and the strong unstable manifold, respectively, of the equilibrium p. This is a standard non-orbit flip condition. Furthermore, we assume a slight modification of the standard non-inclination flip condition for Γ_1. To this end we introduce the local extended-unstable manifold $W^{eu}(p_2)$ of p_2 (this is an invariant manifold whose tangent space at p_2 comprises the unstable and weakest stable directions), and correspondingly the extended-stable manifold $W^{es}(p_1)$ of p_1. Note that these manifolds are not uniquely defined. However their tangent spaces along q_1 are well defined. With this notation the non-inclination flip condition reads

(H 12) $\quad W^{eu}(p_2) \pitchfork_{q_1(0)} W^{es}(p_1).$

Our main result regarding the dynamics nearby Γ is the following:

Theorem 1.1. *Assume Hypotheses (H 1)-(H 12). There is a number $N_{sd} \in \mathbb{R}^+$ and there is a constant $\delta^s > 1$ such that for $\mu = (\mu_1, \mu_2)$, $\mu_1 > 0$ and $\mu_2 > N_{sd}\mu_1^{\delta^s}$ the following applies: There is a neighbourhood of $\Gamma_2 \cap \Sigma_2$ containing a set $\mathcal{S}_\mu$ which is invariant under the first-return-map Π_μ (defined by the flow $\{\phi_\mu^t\}$), and $(\mathcal{S}_\mu, \Pi_\mu)$ is topologically conjugated to the full shift on two symbols. Moreover, $\mathcal{S}_\mu$ is R-invariant.*

The signs of μ_i, $i = 1, 2$, are due to a sign condition regarding some coefficients in the bifurcation equation, see (3.3). To prove Theorem 1.1 we proceed in the spirit of [18]. To this end we adapt Lin's method in such a way that we can handle the time-reversing symmetry and the common tangency of the manifolds $W^u(p_1)$ and $W^s(p_2)$ in Σ_2. Roughly speaking the symbols are related to the two intersection points of $W^u(p_1, \mu)$ and $W^s(p_2, \mu)$ in Σ_2, cf. right panel in the upper row in Figure 3. Note that for $\mu_1 = 0$ these intersection points correspond to non-degenerate symmetric T-points. According to results in [18] one may expect 1-periodic orbits in the unfolding of these T-points. Indeed, more specifically, the symbols for the shift dynamics are related to those periodic orbits.

Moreover we show that there is a subset $\mathcal{S}_{\mu,R}$ of $\mathcal{S}_\mu$ generating symmetric f-orbits. And we show that $(\mathcal{S}_{\mu,R}, \Pi_\mu)$ is topologically conjugated to a system which is chaotic in the sense of Devaney. In particular $\mathcal{S}_{\mu,R}$ contains all periodic orbits up to period six.

Recall that in the context of reversible vector fields an orbit $\mathcal{O}$ is called *symmetric* if $R\mathcal{O} = \mathcal{O}$. According to [31, Lemma 3] or [25, Theorem 4.1] the following holds true:

- An orbit γ is symmetric if and only if $\mathcal{O} \cap \operatorname{Fix} R \neq \emptyset$.

- A symmetric periodic orbit intersects $\operatorname{Fix} R$ in exactly two different points.
- A symmetric orbit which is not periodic intersects $\operatorname{Fix} R$ in exactly one point.

We remark that although a general assumption in [31] demands an even dimensional phase space, the arguments in the proof of [31, Lemma 3] do not rely on this assumption. We also emphasise that according to [25, Theorem 4.3(iii)], symmetric periodic orbits are generically isolated in the present context, cf. also Hypothesis (H 2).

Theorem 1.2. *Assume Hypotheses (H 1)-(H 12). Let $\delta^s > 1$ be the constant according to Theorem 1.1. There is a number $N_{nr} \in \mathbb{R}^+$, $N_{nr} < N_{sd}$ such that for $\mu = (\mu_1, \mu_2)$, $\mu_1 > 0$ and $\mu_2 < N_{nr}\mu_1^{\delta^s}$ there is no recurrent dynamics near $\Gamma_1 \cup \Gamma_2$.*

Similar to the above, the signs of μ_i, $i = 1, 2$, are due to a sign condition (3.3).

Summarising, the statements of the foregoing theorems lead to the bifurcation diagram depicted in Figure 4.

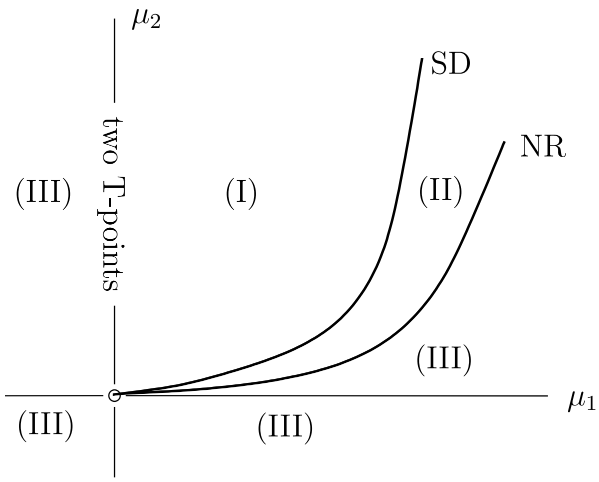

Figure 4: Bifurcation diagram for the unfolding of a non-elementary T-point. (I): μ-area for shift dynamics, (II): shift dynamics dissolves, (III): no recurrent dynamics near $\Gamma_1 \cup \Gamma_2$. The SD curve is given by $\mu_2 = N_{sd}\mu_1^{\delta^s}$ ($\mu_1 > 0$), and the NR curve is given by $\mu_2 = N_{nr}\mu_1^{\delta^s}$ ($\mu_1 > 0$).

We say that a periodic orbit $\mathcal{O}$ within a sufficiently small neighbourhood of Γ has period $N \in \mathbb{N}$, or that $\mathcal{O}$ is an N-periodic orbit, if $\mathcal{O}$ follows the T-point cycle N times before closing the loop. In a preliminary study of the disappearance of the horseshoe in the region (II), we consider N-periodic orbits with $N \leq 4$. Our study suggests that (for fixed μ_1 and increasing μ_2) all symmetric periodic orbits emerge either in the course of a saddle-center bifurcation or in the course of a subharmonic bifurcation. A more detailed bifurcation scenario is presented in Section 4. We note that with our tool to gain the corresponding bifurcation equations, Lin's method, we are unable to make stability statements. However, due to the reversing symmetry there are no symmetric asymptotically stable periodic orbits, and so this excludes saddle-node bifurcations of those orbits.

Regarding one-periodic orbits near Γ we have

Theorem 1.3. *Assume Hypotheses (H 1)-(H 12). Let (μ_1, μ_2) be within the region (II). There is a function $\kappa_{sc}(\cdot)$ such that for $\mu_2 > \kappa_{sc}(\mu_1)$ there are two one-periodic orbits. These two orbits merge at $\mu_2 = \kappa_{sc}(\mu_1)$ and disappear if μ_2 becomes smaller than $\kappa_{sc}(\mu_1)$. All one-periodic orbits are symmetric.*

Regarding two-periodic orbits near Γ we have

Theorem 1.4. *Assume Hypotheses (H 1)-(H 12). Let (μ_1, μ_2) be within the region (II). There is a function $\kappa_{pd}(\cdot)$ such that for $\mu_2 > \kappa_{pd}(\mu_1)$ there is exactly one two-periodic orbit, which emerges from a one-periodic orbit in the course of a period doubling bifurcation at $\mu_2 = \kappa_{pd}(\mu_1)$. For $\mu_2 < \kappa_{pd}(\mu_1)$ there is no two-periodic orbit.*

Generically we have $\kappa_{pd}(\mu_1) > \kappa_{sc}(\mu_1)$. We comment on this fact in Section 3.2.

Theorem 1.5. *Assume Hypotheses (H 1)-(H 12). Let (μ_1, μ_2) be within the region (II). There is a function $\kappa_{3sh}(\cdot)$ such that for $\mu_2 > \kappa_{3sh}(\mu_1)$ there is a pair of symmetric 3-periodic orbits, which emerges from a one-periodic orbit in the course of a subharmonic bifurcation at $\mu_2 = \kappa_{3sh}(\mu_1)$. For $\mu_2 < \kappa_{3sh}(\mu_1)$ there are no 3-periodic orbits.*

Theorem 1.6. *Assume Hypotheses (H 1)-(H 12). Let (μ_1, μ_2) be within the region (II).*

- *(i) There is a function $\kappa_{4sh}(\cdot)$ such that for $\mu_2 > \kappa_{4sh}(\mu_1)$ there is a pair of symmetric 4-periodic orbits, which emerges from a one-periodic orbit in the course of a subharmonic bifurcation at $\mu_2 = \kappa_{4sh}(\mu_1)$.*
- *(ii) There is a function $\kappa_{2pd}(\cdot)$ such that for $\mu_2 > \kappa_{2pd}(\mu_1)$ there is a 4-periodic orbit, which emerges from the two-periodic orbit in the course of a period doubling bifurcation at $\mu_2 = \kappa_{2pd}(\mu_1)$.*

In what follows we briefly discuss some related work. A lot of work has been done concerning the dynamics near T-points, see for example the references in [18]. Here we mention works that are related to the degeneracy of the T-point or to T-points in systems with an extra structure.

Bykov [3] claimed that in systems without any prescribed structure, such as time-reversibility, double T-points create a suspended horseshoe. Here we make a similar statement in the context of reversible systems, cf. Theorem 1.1. Furthermore, we study double T-points in the context of the unfolding of a degenerate T-point. In this unfolding we also study the disappearance of symmetric periodic orbits which are related to the shift dynamics, cf Theorems 1.3-1.6. As a result of this we present a tentative bifurcation diagram comprising symmetric orbits contained in $\mathcal{S}_{\mu,\mathcal{R}}$, see Figure 11.

Lamb et al. [26] studied symmetric T-points in reversible systems in $\mathbb{R}^3$. Besides their restriction to $\mathbb{R}^3$ the main difference to the present work is that they considered non-degenerate T-points (where Γ_2 is robust). Also they assumed in contrast to our assumption (H 9) that λ^u is complex. Their results are based on the study of an appropriate return map.

In the present paper we use Lin's method, cf. [28, 16] as the main tool for studying the nearby dynamics of Γ. This method has proved to be a powerful tool in studying dynamics near T-points, cf. [18]. Here, however we need to adapt the framework given in [18] to the context of reversible systems. Simultaneously the additional common tangent direction U has to be incorporated into the method.

Labouriau and Rodrigues studied T-points in equivariant systems in a series of papers [19, 20, 21] and found very involved dynamics, also beyond shift dynamics. However, the setting they consider is very different from ours, for example they consider complex leading eigenvalues in contrast to our assumption (H 9).

In [10] Fernández-Sánchez et al. presented a model equation for a T-point in $\mathbb{R}^3$ connecting two saddle-focus equilibria whose two-dimensional manifolds intersect non-transversely. The authors studied homoclinic orbits (to the same equilibrium) and show that these undergo a saddle-node bifurcation close to the T-point. It can be guessed that nearby periodic orbits undergo the same type of bifurcation. This effect, however can be seen as the non-reversible counterpart to the bifurcations of periodic orbits we described in Theorems 1.3-1.6. The paper [11] deals numerically with a similar object under $\mathbb{Z}_2$-symmetry.

In [9] reversible T-points are studied in $\mathbb{R}^3$. Assuming complex eigenvalues they relate the dynamics in an unfolding of the T-point to the Cocoon bifurcations; an accumulation of parameter values for which there exist heteroclinic tangencies between the equilibria.

The rest of the paper is organised as follows. In Section 2 we outline how Lin's method can be adapted to the present setting. Central to this method are the so-called *Lin orbits*. These are sequences of partial orbits where *jumps* in certain directions are allowed between two consecutive partial orbits. We show how those orbits can be constructed, and how one can derive determination equations for actual orbits. Based on these determination equations we prove our main theorems in Section 3. Finally we discuss the aforementioned tentative bifurcation diagram for symmetric periodic orbits in Section 4.

2 Lin's method

In this section we outline how Lin's method can be adapted to the reversible setting. In doing so we assume throughout that the T-point Γ is non-elementary, Hypothesis (H 8).

At the core of Lin's method are the so-called *Lin orbits*. In the context of T-points such orbits consist of pieces $X_{1,i}$ and $X_{2,i}$ of actual orbits, $X := (X_{1,i}, X_{2,i})_{i\in\mathbb{Z}}$, see also [18]. The orbit piece $X_{1,i}$ starts in Σ_1, follows Γ_1 until it reaches a neighbourhood of p_1, then follows Γ_2 until it terminates in Σ_2. Similarly the orbit piece $X_{2,i}$ starts in Σ_2, follows Γ_2 until it reaches a neighbourhood of p_2, then follows Γ_1 until it terminates in Σ_1. Between two consecutive orbit pieces $X_{2,i}$ and $X_{1,i+1}$ there may be a jump $\Xi_{1,i}$ in a particular direction $Z_1 \subset Y_1$. In addition, there may be a jump $\Xi_{2,i}$ in a particular direction $Z_2 \subset Y_2$ between the two consecutive orbit pieces $X_{1,i}$ and $X_{2,i}$. We refer to Figure 5 below for a visualisation.

Now, let $2\omega_{1,i}$ and $2\omega_{2,i}$ be (prescribed) transition times of $X_{1,i}$ and $X_{2,i}$ from Σ_1 to Σ_2 and Σ_2 to Σ_1, respectively. For sufficiently large $\omega_{j,i}$ we build sequences

$$\boldsymbol{\omega} := ((\omega_{1,i}, \omega_{2,i}))_{i\in\mathbb{Z}}.$$

Furthermore, we consider sequences built of sufficiently small $u_i \in U$

$$\mathbf{u} := (u_i)_{i\in\mathbb{Z}}, \quad u_i \in U.$$

It can be proved that for each μ which is sufficiently close to 0, and each such sequence $\boldsymbol{\omega}$ and $\mathbf{u}$, there exists a unique Lin orbit $X(\mathbf{u}, \boldsymbol{\omega}, \mu)$, see Theorem 2.9 below.

By setting the jumps $\Xi_{j,i}$ ($j = 1, 2,\ i \in \mathbb{Z}$) equal to zero one finds real orbits staying close to the heteroclinic cycle Γ for all time. Therefore the bifurcation equation for orbits staying close to Γ reads

$$\boldsymbol{\Xi} := \big(\Xi_{1,i}(\mathbf{u}, \boldsymbol{\omega}, \mu), \Xi_{2,i}(\mathbf{u}, \boldsymbol{\omega}, \mu)\big)_{i\in\mathbb{Z}} = 0.$$

The subspace Z_1 is defined as follows:

$$Z_1 := (T_{q_1(0)}W^s(p_1) + T_{q_1(0)}W^u(p_2))^\perp, \tag{2.1}$$

where again the orthogonality is with respect to the R-invariant inner-product $\langle\cdot,\cdot\rangle$. We further define

$$W_1^+ = T_{q_1(0)}W^s(p_1) \cap Y_1 \quad \text{and} \quad W_1^- = T_{q_1(0)}W^u(p_2) \cap Y_1.$$

Taking Hypothesis (H 5) into consideration we have the following direct sum decomposition of $\mathbb{R}^{2n+1}$:

$$\mathbb{R}^{2n+1} = \operatorname{span}\{f(q_1(0), 0)\} \oplus W_1^+ \oplus W_1^- \oplus Z_1.$$

Note that in accordance with Hypotheses (H 4) and (H 5) we find that

$$\dim Z_1 = 2.$$

The following lemma states that R-invariant subspaces can be decomposed into subspaces of $\operatorname{Fix} R$ and $\operatorname{Fix}(-R)$.

Lemma 2.1. *Let Z be an R-invariant subspace. Then $Z = (Z \cap \operatorname{Fix} R) \oplus (Z \cap \operatorname{Fix}(-R))$.*

Proof. Since $\operatorname{Fix} R \cap \operatorname{Fix}(-R) = \{0\}$ it suffices to show that each element of $z \in Z$ can be written as a sum of elements of $\operatorname{Fix} R$ and $\operatorname{Fix}(-R)$. The representation $z = (z+Rz)+(z-Rz)$ meets this condition. ∎

So, taking the time-reversing symmetry of the vector field and the symmetry of Γ_1 into consideration we find

Lemma 2.2. *Assume (H 1)-(H 5) and (H 7). Then the space Z_1 has a direct sum decomposition into one-dimensional subspaces of $\operatorname{Fix} R$ and $\operatorname{Fix}(-R)$.*

Proof. Due to (H 1) and $q_1(0) \in \operatorname{Fix} R$ the vector field direction $f(q_1(0), 0)$ belongs to $\operatorname{Fix}(-R)$, and the spaces W_1^+ and W_1^- are R-images of each other. So $W_1^+ \oplus W_1^-$ is R-invariant and hence, according to Lemma 2.1, it has a direct sum decomposition into $(n-1)$-dimensional subspaces of $\operatorname{Fix} R$ and $\operatorname{Fix}(-R)$, recall $\dim W_1^+ = \dim W_1^- = n - 1$. Since Z_1 is R-invariant it has a direct sum decomposition into subspaces of $\operatorname{Fix} R$ and $\operatorname{Fix}(-R)$, cf. again Lemma 2.1. Now, counting dimensions gives the lemma. ∎

To obtain corresponding statements related to Γ_2 we define in the same way as Z_1

$$Z_2 := (T_{q_2(0)} W^s(p_2) + T_{q_2(0)} W^u(p_1))^\perp.$$

Note that in accordance with Hypotheses (H 4) and (H 6) we find that

$$\dim Z_2 = 1,$$

and further

$$W_2^+ = \big(T_{q_2(0)} W^s(p_2) \cap Y_2\big) \ominus U, \quad W_2^- = \big(T_{q_2(0)} W^u(p_1) \cap Y_2\big) \ominus U.$$

Again $\ominus$ refers to a orthogonal decomposition. With that we find

$$\mathbb{R}^{2n+1} = \operatorname{span}\{f(q_2(0), 0)\} \oplus W_2^+ \oplus W_2^- \oplus U \oplus Z_2. \tag{2.2}$$

Note that by construction both U and its complement (in the direct sum decomposition (2.2)) are R-invariant.

Lemma 2.3. *Assume (H 1)-(H 8) then* $Z_2 \subset \operatorname{Fix}(-R)$.

Proof. The proof runs along the lines of the proof of Lemma 2.2. ∎

2.1 Splitting of the stable and unstable Manifolds

The first step of Lin's method is to study the splitting of the stable and unstable manifolds in Σ_i as the parameter μ is varied from zero. Neglecting the symmetry for the moment, the behaviour in Σ_1 is described by the following lemma.

Lemma 2.4 (Lemma 3.1 in [18])**.** *Assume (H 4) and (H 5). For each μ which is sufficiently close to* 0 *there is a unique pair* $(q_1^+(\mu)(\cdot), (q_1^-(\mu)(\cdot))$ *of solutions of* (1.1) *such that:*

(i) $q_1^+(\mu)(0) \in \Sigma_1 \cap W^s(p_1, \mu), \quad q_1^-(\mu)(0) \in \Sigma_1 \cap W^u(p_2, \mu)$,

(ii) $|q_1^+(\mu)(t) - q_1(t)|$ *small* $\forall t \in \mathbb{R}^+$ *and* $|q_1^-(\mu)(t) - q_1(t)|$ *small* $\forall t \in \mathbb{R}^-$,

(iii) $q_1^+(\mu)(0) - q_1^-(\mu)(0) \in Z_1$.

For the proof of this lemma we refer to [18].

In $\mathbb{R}^3$, $n = 1$ the statement in Σ_1 is obvious. In this case both the stable manifold of p_1 and the unstable manifold of p_2 are one-dimensional, and the intersection with the two-dimensional hyperplane Σ_1 consists of single points in each case. The heteroclinic connection Γ_1 generally splits up under perturbation. Let $q_{1,\mu}^+$ and $q_{1,\mu}^-$ be determined by the 'first hit' of the stable manifold of p_1 and the unstable manifold of p_2, respectively. Of course $q_{1,\mu}^+ - q_{1,\mu}^- \in Z_1$, recall that $Z_1 = Y_1$ in this case. So, in a trivial way, for each μ we find a unique pair of orbits in the stable manifold of p_1 and the unstable manifold of p_2, respectively, such that the difference of their first hits in Σ_1 is in Z_1.

Taking the time-reversing symmetry into consideration we additionally find

Lemma 2.5. *Assume (H 1)-(H 5) and (H 7), and let $q_1^{\pm}$ be the (unique) partial orbits according to Lemma 2.4. Then*

(i) $Rq_1^+(\mu)(t) = q_1^-(\mu)(-t)$,

(ii) $q_1^+(\mu)(0) - q_1^-(\mu)(0) \in \mathrm{Fix}\,(-R) \cap Z_1$.

Proof. The pair $(Rq_1^-(\mu)(\cdot), Rq_1^+(\mu)(\cdot))$ also fulfils (i)-(iii) of Lemma 2.4. Therefore the uniqueness part of Lemma 2.4 provides $Rq_1^+(\mu)(t) = q_1^-(\mu)(-t)$ and hence in particular $Rq_1^+(\mu)(0) = q_1^-(\mu)(0)$. ∎

Based on the last two lemmas we define

$$\xi_1^{\infty}(\mu) := q_1^+(\mu)(0) - q_1^-(\mu)(0). \tag{2.3}$$

In Z_1 we introduce coordinates in a similar way as in [18, Section 4.1]. Here, however, we additionally take account of the reversing symmetry: According to Hypotheses (H 1), (H 7) and (H 12) there are linearly independent ζ_1^1, $\zeta_1^2 = R\zeta_1^1$ such that

$$T_{q_1(0)}W^{eu}(p_2) \cap Y_1 = W_1^- \oplus \mathrm{span}\,\{\zeta_1^2\}, \quad T_{q_1(0)}W^{es}(p_1) \cap Y_1 = W_1^+ \oplus \mathrm{span}\,\{\zeta_1^1\}.$$

Indeed, there is an R-invariant scalar product $\langle \cdot, \cdot \rangle$ (see also Remark 2.6), which is in accordance with (1.2), and in respect of which we have

$$\zeta_1^1 \perp \zeta_1^2, \quad W_1^+ \perp W_1^- \quad \text{and} \quad \zeta_1^i \perp W_1^{\pm},\ i = 1, 2. \tag{2.4}$$

That is $Z_1 = \mathrm{span}\,\{\zeta_1^1, \zeta_1^2\}$, cf. (2.1). Furthermore, because of $\zeta_1^2 = R\zeta_1^1$ we have

$$\zeta_1^1 - \zeta_1^2 \in \mathrm{Fix}\,(-R). \tag{2.5}$$

Remark 2.6. Note that by definition Y_1 is R-invariant, cf. (1.2). Now, let $\langle\langle \cdot, \cdot \rangle\rangle_{Y_1}$ be a scalar product on Y_1 such that $\zeta_1^1 \perp \zeta_1^2$, $W_1^+ \perp W_1^-$ and $\zeta_1^i \perp W_1^{\pm}$, $i = 1, 2$. Then, due to the symmetry of the involved quantities, $\langle \cdot, \cdot \rangle_{Y_1} := \langle\langle \cdot, \cdot \rangle\rangle_{Y_1} + \langle\langle R(\cdot), R(\cdot) \rangle\rangle_{Y_1}$ is an R-invariant scalar product on Y_1 such that (2.4) is valid with respect to $\langle \cdot, \cdot \rangle_{Y_1}$. Finally we define a scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{2n+1}$ by $\langle \cdot|_{Y_1}, \cdot|_{Y_1} \rangle := \langle \cdot, \cdot \rangle_{Y_1}$ and $Y_1 \perp \mathrm{span}\,\{f(q_1(0), 0)\}$. Since $f(q_1(0), 0) \in \mathrm{Fix}(-R)$, this scalar product is R-invariant. □

Furthermore, we assume that the splitting of $W^u(p_2)$ and $W^s(p_1)$ is only caused by μ_1, recall from the introduction that μ_1 unfolds Γ_1. So we get

$$D_1\xi_1^{\infty}(0) \neq 0.$$

Together with Lemma 2.5 and (2.5) we may write

$$\xi_1^{\infty}(\mu) = \begin{pmatrix} \mu_1 \\ -\mu_1 \end{pmatrix}.$$

Next we turn to the splitting $W^u(p_1, \mu)$ and $W^s(p_2, \mu)$ in Σ_2. We find the following

Lemma 2.7. *Assume (H4) and (H6). For each pair* (u, μ) *which is sufficiently close to* $(0, 0)$ *there is a unique pair* $(q_2^+(u,\mu)(\cdot), (q_2^-(u,\mu)(\cdot))$ *of solutions of* (1.1) *such that:*

(i) $q_2^+(u,\mu)(0) \in \Sigma_2 \cap W^s(p_2,\mu)$, $\quad q_2^-(u,\mu)(0) \in \Sigma_2 \cap W^u(p_1,\mu)$,

(ii) $|q_2^+(u,\mu)(t) - q_2(t)|$ *small* $\forall t \in \mathbb{R}^+$ *and* $|q_2^-(u,\mu)(t) - q_2(t)|$ *small* $\forall t \in \mathbb{R}^-$,

(iii) $q_2^+(u,\mu)(0) - q_2^-(u,\mu)(0) \in Z_2$,

(iv) $P_U\big(q_2^\pm(u,\mu)(0)\big) = u$.

Here P_U *is the projection onto* U *related to the decomposition* (2.2).

We note that due to the R-invariance of the direct sum decomposition (2.2) (cf. the explanation following (2.2)), we have

$$RP_U = P_U R. \tag{2.6}$$

For the proof of Lemma 2.7 we refer to the proof of Lemma 1 in [15]. Similarly to (2.3) we define

$$\xi_2^\infty(u,\mu) := q_2^+(u,\mu)(0) - q_2^-(u,\mu)(0). \tag{2.7}$$

Similar to the considerations in Σ_1 we assume that the splitting of $W^u(p_1)$ and $W^s(p_2)$ is only caused by μ_2, or in other words that $\xi_2^\infty(u,\mu) = \xi_2^\infty(u,\mu_2)$. In order to determine ξ_2^∞ we have to incorporate Hypothesis (H 8).

Lemma 2.8. *Assume (H1)-(H8). Then* $Rq_2^+(u,\mu)(t) = q_2^-(u,\mu)(-t)$.

Proof. The lemma follows from the uniqueness statement of Lemma 2.7 and Lemma 2.7(iv) together with $U \subset \operatorname{Fix} R$. ∎

Furthermore, the quadratic tangency translates into the following analytical expression related with ξ_2^∞, cf.:

$$\left.\frac{\partial(\xi_2^\infty, D_1\xi_2^\infty)}{\partial(u,\mu_2)}\right|_{(0,0)} \quad \text{has rank two.}$$

With that ξ_2^∞ can be transformed into the simple form, cf. [15] or [17].

$$\xi_2^\infty(u,\mu) = \mu_2 - u^2.$$

We also refer to Figure 3. Note that in the depicted situation the jump ξ_2^∞ is measured in the direction perpendicular to $\operatorname{Fix} R$.

2.2 Construction of Lin orbits

The next step in the method is to search for orbits $X_{j,i}$, $j = 1, 2$, $i \in \mathbb{Z}$, composing the Lin orbits $X = (X_{1,i}, X_{2,i})_{i\in\mathbb{Z}}$ which we introduced at the beginning of Section 2.

We denote solutions of (1.1) by $x_{j,i}(\cdot)$, corresponding to the orbits $X_{j,i}$ with $x_{j,i}(0) \in \Sigma_j$ and $x_{j,i}(2\omega_{j,i}) \in \Sigma_{j+1}$; throughout the term "$j + 1$" is computed *modulo* 2. Actually $x_{1,i}(\cdot)$ is composed of solutions $x^+_{1,i}(\cdot)$ and $x^-_{2,i}(\cdot)$ which are defined on $[0, \omega_{1,i}]$ and $[-\omega_{1,i}, 0]$, respectively. Similarly $x_{2,i}(\cdot)$ is composed of solutions $x^+_{2,i}(\cdot)$ and $x^-_{1,i}(\cdot)$ which are defined on $[0, \omega_{2,i}]$ and $[-\omega_{2,i}, 0]$, respectively. This requires the coupling conditions

$$x^+_{j,i}(\omega_{j,i}) = x^-_{j+1,i}(-\omega_{j,i}),\ j = 1, 2, \tag{2.8}$$

and the jump conditions

$$\Xi_{1,i} := x^+_{1,i+1}(0) - x^-_{1,i}(0) \in Z_1, \quad i \in \mathbb{Z}, \tag{2.9}$$

$$\Xi_{2,i} := x^+_{2,i}(0) - x^-_{2,i}(0) \in Z_2, \quad i \in \mathbb{Z}. \tag{2.10}$$

We refer to Figure 5 for a visualisation.

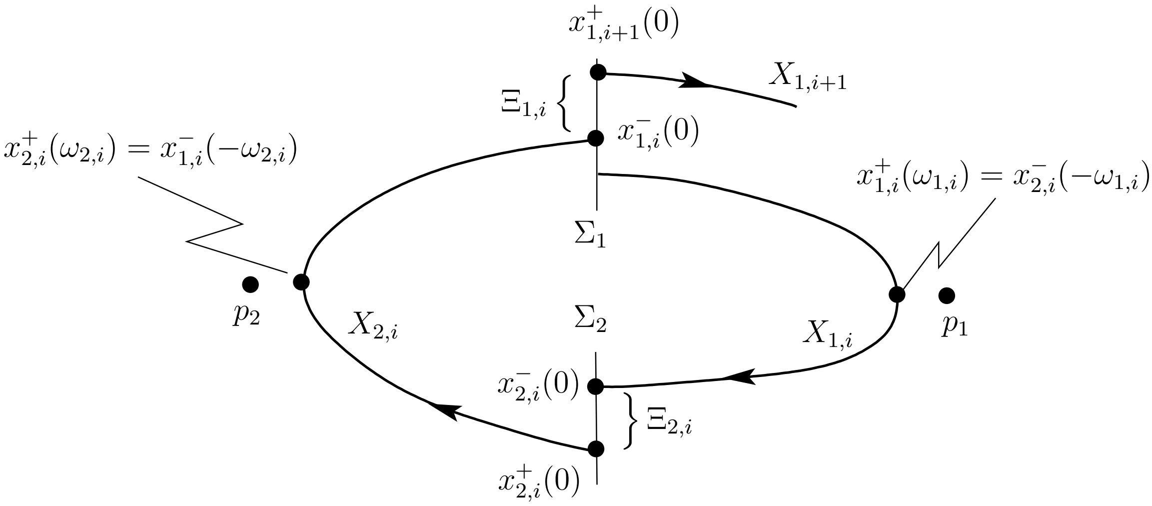

Figure 5: Ingredients of Lin orbits.

Our main existence result in this respect is the following:

Theorem 2.9. *Assume Hypotheses (H1)-(H7). Then there are constants c_μ, c_u and Ω such that for each μ with $|\mu| < c_\mu$, each $\mathbf{u}$ with $|u_i| < c_u$ and each $\boldsymbol{\omega}$ with $\omega_{j,i} > \Omega$, $j = 1, 2$, $i \in \mathbb{Z}$, there is a unique sequence of solutions $x^\pm_{j,i}(\mathbf{u}, \boldsymbol{\omega}, \mu)(\cdot)$, $j = 1, 2$, $i \in \mathbb{Z}$, of* (1.1) *satisfying the coupling condition* (2.8) *and the jump condition* (2.9), (2.10).
Moreover, these solutions satisfy

$$x^+_{2,i}(\mathbf{u}, \boldsymbol{\omega}, \mu)(0) - q^+_2(u_i, \mu)(0) \in W^+_2 \oplus W^-_2 \oplus Z_2.$$

For the proof of this theorem we refer to [16].

In other words the theorem states that for the corresponding $(\mathbf{u}, \boldsymbol{\omega}, \mu)$ there is a Lin orbit $X(\mathbf{u}, \boldsymbol{\omega}, \mu) = (X_{1,i}, X_{2,i})_{i\in\mathbb{Z}}$, where the $X_{j,i}$ are related to the functions $x^\pm_{j,i}$ as described at the beginning of this section.

Furthermore, together with (2.10) and Lemma 2.7(iv) we find

$$P_U\big(x^\pm_{2,i}(\mathbf{u}, \boldsymbol{\omega}, \mu)(0)\big) = u_i. \tag{2.11}$$

Based on Theorem 2.9 we define the following.

Definition 2.10. *Two Lin orbits* $X(\mathbf{u},\boldsymbol{\omega},\mu) = (X_{1,i}, X_{2,i})_{i\in\mathbb{Z}}$ *and* $\hat{X}(\hat{\mathbf{u}},\hat{\boldsymbol{\omega}},\mu) = (\hat{X}_{1,i}, \hat{X}_{2,i})_{i\in\mathbb{Z}}$ *are equal,* $X = \hat{X}$*, if there is an* $i_0 \in \mathbb{Z}$ *such that* $X_{j,i} = \hat{X}_{j,i+i_0}$*, for* $j = 1, 2$ *and all* $i \in \mathbb{Z}$*.*

Definition 2.11. *A Lin orbit* $X(\mathbf{u},\boldsymbol{\omega},\mu) = (X_{1,i}, X_{2,i})_{i\in\mathbb{Z}}$ *is* k*-periodic,* $k \in \mathbb{N}$*, if for all* $i \in \mathbb{Z}$ *we have* $X_{j,i} = X_{j,i+k}$*.*

Note that the latter definition implies that $x^+_{2,i}(\mathbf{u},\boldsymbol{\omega},\mu)(0) = x^+_{2,i+k}(\mathbf{u},\boldsymbol{\omega},\mu)(0)$. Hence, due to (2.11), we have $u_i = u_{i+k}$, i.e. $\mathbf{u}$ is k-periodic. Furthermore, the definition implies that the domains of $X_{j,i}$ and $X_{j,i+k}$ coincide. This again means for the sequence $\boldsymbol{\omega}$ that $\omega_{j,i} = \omega_{j,i+k}$, i.e. $\boldsymbol{\omega}$ is k-periodic. Altogether we find

Lemma 2.12. *A Lin orbit* $X(\mathbf{u},\boldsymbol{\omega},\mu) = (X_{1,i}, X_{2,i})_{i\in\mathbb{Z}}$ *is* k*-periodic if and only if the sequences* $\mathbf{u}$ *and* $\boldsymbol{\omega}$ *are* k*-periodic.* ∎

Lemma 2.13. *Let* $X(\mathbf{u},\boldsymbol{\omega},\mu)$ *be a Lin orbit. The* R*-image* RX *of* X *is a Lin orbit* $\hat{X}(\hat{\mathbf{u}},\hat{\boldsymbol{\omega}},\mu)$ *associated to the sequences* $\hat{\mathbf{u}} = (\hat{u}_i)_{i\in\mathbb{Z}}$*,* $\hat{u}_i = Ru_{-i}$ *and* $\hat{\boldsymbol{\omega}} = (\hat{\omega}_{1,i}, \hat{\omega}_{2,i})_{i\in\mathbb{Z}}$*,* $\hat{\omega}_{j,i} = \omega_{j+1,-i}$*.*

Proof. We define solutions $\hat{x}^\pm_{j,i}(\cdot)$ of (1.1) by

$$\hat{x}^+_{j,i}(t) := Rx^-_{j,-i}(-t), \quad \hat{x}^-_{j,i}(t) := Rx^+_{j,-i}(-t). \tag{2.12}$$

The domains of $\hat{x}^\pm_{j,i}$ are strongly related to the domains of $x^\mp_{j,-i}$. So, $\hat{x}^+_{1,i}$, $\hat{x}^+_{2,i}$, $\hat{x}^-_{1,i}$ and $\hat{x}^-_{2,i}$ are defined on $[0, \omega_{2,-i}]$, $[0, \omega_{1,-i}]$, $[-\omega_{1,-i}, 0]$ and $[-\omega_{2,-i}, 0]$, respectively. With $\hat{\omega}_{j,i} := \omega_{j+1,-i}$ the functions $\hat{x}^\pm_{j,i}(\cdot)$ satisfy the coupling conditions (2.8) and also the jump conditions (2.9), (2.10). Also $P_U\hat{x}^+_{2,i}(0) = P_U Rx^-_{2,-i}(0) = RP_U x^-_{2,-i}(0) = Ru_{-i} = \hat{u}_i$, cf. also (2.11) and (2.6). We denote the (partial) orbits connecting Σ_1 and Σ_2 corresponding to the solutions $\hat{x}^\pm_{j,i}$ (as constructed above - see also Figure 5) by $\hat{X}_{j,i}$, $j = 1, 2$, $i \in \mathbb{Z}$.

Altogether our construction shows that $\hat{X} = (\hat{X}_{1,i}, \hat{X}_{2,i})$ is a Lin orbit associated to $(\hat{\mathbf{u}},\hat{\boldsymbol{\omega}},\mu)$. Since $\hat{X}_{j,i} = RX_{j+1,-i}$, $j = 1, 2$, $i \in \mathbb{Z}$, we have $RX = \hat{X}$. ∎

Corollary 2.14. *Let* X *be a Lin orbit and let* $(\Xi_{1,i}, \Xi_{2,i})_{i\in\mathbb{Z}}$ *and* $(\hat{\Xi}_{1,i}, \hat{\Xi}_{2,1})_{i\in\mathbb{Z}}$ *be the sequences of jumps corresponding to* X *and* RX*, respectively. Then* $\hat{\Xi}_{1,i} = -R\Xi_{1,-i-1}$ *and* $\hat{\Xi}_{2,i} = -R\Xi_{2,-i}$*.*

Proof. According to (2.9), (2.10)) and (2.12) we find

$$\begin{aligned}
\hat{\Xi}_{1,i} &= \hat{x}^+_{1,i+1}(0) - \hat{x}^-_{1,i}(0) = -R(x^+_{1,-i}(0) - x^-_{1,-i-1}(0)) = -R\Xi_{1,-i-1},\\
\hat{\Xi}_{2,i} &= \hat{x}^+_{2,i}(0) - \hat{x}^-_{2,i}(0) = -R(x^+_{2,-i}(0) - x^-_{2,-i}(0)) = -R\Xi_{2,-i}.
\end{aligned}$$

∎

According to Lemma 2.3 we find $\Xi_{2,i} \in \mathrm{Fix}\,(-R)$ and hence $\hat{\Xi}_{2,i} = \Xi_{2,-i}$.

Definition 2.15. *A Lin orbit* X *is symmetric if* $X = RX$*.*

Lemma 2.16. *A Lin orbit $X(\mathbf{u}, \boldsymbol{\omega}, \mu)$ is symmetric if and only if there is an $i_0 \in \mathbb{Z}$ such that*

$$\omega_{j,i} = \omega_{j+1,-i-i_0}, \quad \textit{and} \quad u_i = u_{-i-i_0}, \quad i \in \mathbb{Z}, j \in \{1, 2\} \tag{2.13}$$

Proof. First we assume that X is symmetric, i.e. $X = RX =: \hat{X}$. Then, according to Definition 2.10 there is an $i_0 \in \mathbb{Z}$ such that $X_{j,i} = \hat{X}_{j,i+i_0}$. Furthermore, according to the considerations at the end of the proof of Lemma 2.13, we have $\hat{X}_{j,i+i_0} = RX_{j+1,-i-i_0}$. Hence

$$X_{j,i} = RX_{j+1,-i-i_0}. \tag{2.14}$$

This immediately gives $\omega_{j,i} = \omega_{j+1,-i-i_0}$. Also, according to (2.11) and (2.14) we find

$$u_i = P_U x^+_{2,i}(0) = P_U R x^-_{2,-i-i_0}(0) = u_{-i-i_0}.$$

Now, conversely we assume (2.13). Let $X(\mathbf{u}, \boldsymbol{\omega}, \mu)$ be the corresponding (unique) Lin orbit. The proof of Lemma 2.13 provides a representation of RX, and it turns out that $RX = X$. ■

From Corollary 2.14 and Lemma 2.16 we get

Corollary 2.17. *Let $X(\mathbf{u}, \boldsymbol{\omega}, \mu)$ be a symmetric Lin orbit, and let $i_0 \in \mathbb{Z}$ be such that (2.13) is satisfied. Then $R\,\Xi_{1,i} = -\Xi_{1,-i-i_0-1}$ and $R\,\Xi_{2,i} = -\Xi_{2,-i-i_0}$.* ■

2.3 The determination equation for orbits near Γ

A Lin orbit is a real orbit if all the jumps between two consecutive partial orbits are zero:

$$\Xi_{j,i}(\mathbf{u}, \boldsymbol{\omega}, \mu) = 0, \quad j = 1, 2, \, i \in \mathbb{Z}. \tag{2.15}$$

With Theorem 2.9 we may write these jumps as

$$\begin{aligned} \Xi_{1,i}(\mathbf{u}, \boldsymbol{\omega}, \mu) &= x^+_{1,i+1}(\mathbf{u}, \boldsymbol{\omega}, \mu)(0) - x^-_{1,i}(\mathbf{u}, \boldsymbol{\omega}, \mu)(0), \\ \Xi_{2,i}(\mathbf{u}, \boldsymbol{\omega}, \mu) &= x^+_{2,i}(\mathbf{u}, \boldsymbol{\omega}, \mu)(0) - x^-_{2,i}(\mathbf{u}, \boldsymbol{\omega}, \mu)(0). \end{aligned}$$

As in [18, Section 3.2] we write

$$x^\pm_{1,i}(\mathbf{u}, \boldsymbol{\omega}, \mu)(t) = q^\pm_1(\mu)(t) + v^\pm_{1,i}(\mathbf{u}, \boldsymbol{\omega}, \mu)(t), \tag{2.16}$$

and correspondingly, cf. also Lemma 2.7,

$$x^\pm_{2,i}(\mathbf{u}, \boldsymbol{\omega}, \mu)(t) = q^\pm_2(u_i, \mu)(t) + v^\pm_{2,i}(\mathbf{u}, \boldsymbol{\omega}, \mu)(t). \tag{2.17}$$

With that we find, cf. also (2.3) and (2.7),

$$\begin{aligned} \Xi_{1,i}(\mathbf{u}, \boldsymbol{\omega}, \mu) &= \xi^\infty_1(\mu) + \xi_{1,i}(\mathbf{u}, \boldsymbol{\omega}, \mu), \\ \Xi_{2,i}(\mathbf{u}, \boldsymbol{\omega}, \mu) &= \xi^\infty_2(u_i, \mu) + \xi_{2,i}(\mathbf{u}, \boldsymbol{\omega}, \mu), \end{aligned}$$

where

$$\begin{aligned}\xi_{1,i}(\mathbf{u},\boldsymbol{\omega},\mu) &= v^+_{1,i+1}(\mathbf{u},\boldsymbol{\omega},\mu)(0) - v^-_{1,i}(\mathbf{u},\boldsymbol{\omega},\mu)(0),\\ \xi_{2,i}(\mathbf{u},\boldsymbol{\omega},\mu) &= v^+_{2,i}(\mathbf{u},\boldsymbol{\omega},\mu)(0) - v^-_{2,i}(\mathbf{u},\boldsymbol{\omega},\mu)(0).\end{aligned} \tag{2.18}$$

In order to obtain appropriate representations of $\xi_{j,i}(\mathbf{u},\boldsymbol{\omega},\mu)$ we follow the lines of [18, Section 4.1], while taking into consideration that in the present context some quantities depend on u. The following lemma can be seen as counterpart to [18, Lemma 4.4].

Lemma 2.18. *Assume Hypotheses (H 1)-(H 12). Then the jump $\xi_{1,i}(\mathbf{u},\boldsymbol{\omega},\mu)$ can be written in the form*

$$\begin{aligned}\xi_{1,i}(\mathbf{u},\boldsymbol{\omega},\mu) &= \left(c^1_1(u_{i+1},\mu)e^{-2\lambda^u\omega_{1,i+1}} + \mathcal{R}^1_{1,i}(\mathbf{u},\boldsymbol{\omega},\mu)\right)\zeta^1_1\\ &\quad + \left(c^2_1(u_i,\mu)e^{-2\lambda^u\omega_{2,i}} + \mathcal{R}^2_{1,i}(\mathbf{u},\boldsymbol{\omega},\mu)\right)\zeta^2_1\\ \xi_{2,i}(\mathbf{u},\boldsymbol{\omega},\mu) &= \left(c_{21}(u_i,\mu)e^{2\lambda^s\omega_{1,i}} - c_{22}(u_i,\mu)e^{2\lambda^s\omega_{2,i}} + \mathcal{R}_{2,i}(\mathbf{u},\boldsymbol{\omega},\mu)\right)\zeta_2,\end{aligned}$$

where the quantities c^k_1 and c_{2k}, $k\in\{1,2\}$, are non-zero and

$$\begin{aligned}\mathcal{R}^1_{1,i}(\mathbf{u},\boldsymbol{\omega},\mu) &= o(e^{-2\lambda^u\omega_{1,i+1}}) + o(e^{-2\lambda^u\omega_{2,i}}),\\ \mathcal{R}^2_{1,i}(\mathbf{u},\boldsymbol{\omega},\mu) &= o(e^{-2\lambda^u\omega_{1,i+1}}) + o(e^{-2\lambda^u\omega_{2,i}}),\\ \mathcal{R}_{2,i}(\mathbf{u},\boldsymbol{\omega},\mu) &= o(e^{2\lambda^s\omega_{1,i}}) + o(e^{2\lambda^s\omega_{2,i}}).\end{aligned}$$

Due to the eigenvalue condition (H 10) it is enough to isolate the leading order terms containing $e^{-2\lambda^u\omega}$-terms. Furthermore, in the proof of [18, Lemma 4.4] the geometry of the T-point in Σ_2 (transversal intersection of $W^u(p_1)$ and $W^s(p_2)$ along Γ_2) was exploited at an essential point. For that reason we have to reconsider the proof in the present situation.

Preliminaries for the proof: We start with introducing some notations and with specifying the setup.

The quantities $v^\pm_{j,i}$ introduced in (2.16) and (2.17) satisfy

$$\dot v^\pm_{1,i} = A^\pm_1(t,\mu)v^\pm_{1,i} + g^\pm_1(t,v^\pm_{1,i},\mu) \quad\text{and}\quad \dot v^\pm_{2,i} = A^\pm_2(t,u_i,\mu)v^\pm_{2,i} + g^\pm_2(t,v^\pm_{2,i},u_i,\mu)$$

where

$$A^\pm_1(t,\mu) := D_xf(q^\pm_1(\mu)(t),\mu) \quad\text{and}\quad A^\pm_2(t,u,\mu) := D_xf(q^\pm_2(u,\mu)(t),\mu)$$

and

$$\begin{aligned}g^\pm_1(t,v,\mu) &:= f(q^\pm_1(\mu)(t)+v,\mu) - f(q^\pm_1(\mu)(t),\mu) - A^\pm_1(t,\mu)v,\\ g^\pm_2(t,v,u,\mu) &:= f(q^\pm_2(u,\mu)(t)+v,\mu) - f(q^\pm_2(u,\mu)(t),\mu) - A^\pm_2(t,u,\mu)v,\end{aligned}$$

respectively.

Let $\Phi^\pm_1(\mu,t,s)$ and $\Phi^\pm_2(u,\mu,t,s)$ be the transition matrices for the equations

$$\dot v = A^\pm_1(t,\mu)v \quad\text{and}\quad \dot v = A^\pm_2(t,u,\mu)v,$$

respectively. These equations have an exponential dichotomy on $\mathbb{R}^\pm$ with corresponding projections $P^\pm_1(\mu,t)$, and $P^\pm_2(u,\mu,t)$, respectively. These projections commute with the transition matrix of the corresponding variational equation. Therefore they are determined by their image and by their kernel at $t = 0$. Since the variational equations under consideration are related to solutions within a stable or an unstable manifold, respectively, the images of these projections coincides with the corresponding tangent spaces to these manifolds. We have

$$\operatorname{im} P_1^+(\mu, 0) = T_{q_1^+(\mu)(0)} W^s(p_1), \quad \operatorname{im} P_1^-(\mu, 0) = T_{q_1^-(\mu)(0)} W^u(p_2),$$

$$\operatorname{im} P_2^+(u, \mu, 0) = T_{q_2^+(u,\mu)(0)} W^s(p_2), \quad \operatorname{im} P_2^-(u, \mu, 0) = T_{q_2^-(u,\mu)(0)} W^u(p_1).$$

Regarding the kernels of the projections at $t = 0$ we stipulate

$$\ker P_1^+(\mu, 0) = W_1^- \oplus Z_1, \quad \ker P_1^-(\mu, 0) = W_1^+ \oplus Z_1,$$

$$\ker P_2^+(u, \mu, 0) = W_2^- \oplus Z_2, \quad \ker P_2^-(u, \mu, 0) = W_2^+ \oplus Z_2.$$

In this respect we also introduce quantities $a_{j,i}^{\pm}$:

$$\big(I - P_1^+(\mu, \omega_{1,i})\big) v_{1,i}^+(\omega_{1,i}) = a_{1,i}^+, \quad \big(I - P_1^-(\mu, -\omega_{2,i})\big) v_{1,i}^-(-\omega_{2,i}) = a_{2,i}^-,$$

$$\big(I - P_2^+(u_i, \mu, \omega_{2,i})\big) v_{2,i}^+(\omega_{2,i}) = a_{2,i}^+, \quad \big(I - P_2^-(u_i, \mu, -\omega_{1,i})\big) v_{2,i}^-(-\omega_{1,i}) = a_{1,i}^-,$$

Furthermore, we denote the transition matrix of the adjoint of the variational equation along $q_1^{\pm}(\mu)(\cdot)$ by $\Psi_1^{\pm}(\mu, \cdot, \cdot)$, and correspondingly we denote the transition matrix of the adjoint of the variational equation along $q_2^{\pm}(u, \mu)(\cdot)$ by $\Psi_2^{\pm}(u, \mu, \cdot, \cdot)$.

For our analysis in the proof we assume the following setup.

We start with specifying extended unstable manifolds of p_2 with respect to the vector field $f(\cdot, \mu)$: Note that $W_2^- \oplus Z_2$ is transversal to the stable manifold of p_2, cf. also (2.2). Then (due to the inclination lemma) the image of $q_2^+(0, \mu)(0) + (W_2^- \oplus Z_2)$ under the flow belongs to an extended unstable manifold of p_2. We stipulate, cf. also Figure 6,

$$W^{eu}(p_2, \mu): \quad \{\phi_\mu^t\big(q_2^+(0, \mu)(0) + (W_2^- \oplus Z_2)\big) : t \geq 0\} \subset W^{eu}(p_2, \mu).$$

Accordingly we specify an extended stable manifold of p_1 by

$$W^{es}(p_1, \mu) := R\big(W^{eu}(p_2, \mu)\big).$$

Hence

$$\{\phi_\mu^t\big(q_2^-(0, \mu)(0) + (W_2^+ \oplus Z_2)\big) : t \leq 0\} \subset W^{es}(p_1, \mu).$$

In Remark 2.19 below we show that there are transformations respecting the reversing symmetry and realising the following assumptions.

(A 1) Let $V_{q_1(0)}$ be a neighbourhood of $q_1(0)$. For small $|\mu|$ it is true:

$$W^{eu}(p_2, \mu) \cap V_{q_1(0)} \supset q_1^-(\mu)(0) + \big(W_1^- \oplus \operatorname{span}\{\zeta_1^2\}\big),$$
$$W^{es}(p_1, \mu) \cap V_{q_1(0)} \supset q_1^+(\mu)(0) + \big(W_1^+ \oplus \operatorname{span}\{\zeta_1^1\}\big).$$

(A 2) Let V_{p_2} be a neighbourhood of p_2. For small $|\mu|$ it is true:

$$W^{eu}(p_2,\mu)\cap V_{p_2}\subset T_{p_2}W^{eu}(p_2,\mu),\quad W^{s}(p_2,\mu)\cap V_{p_2}\subset T_{p_2}W^{s}(p_2,\mu).$$

Due to the reversing symmetry for small $|\mu|$ it is then also true that within a sufficiently small neighbourhood V_{p_1} of p_1

$$W^{es}(p_1,\mu)\cap V_{p_1}\subset T_{p_1}W^{es}(p_1,\mu),\quad W^{u}(p_1,\mu)\cap V_{p_1}\subset T_{p_1}W^{u}(p_1,\mu).$$

The Assumptions (A 1) and (A 2) entail the following, we refer to Figure 6,

$$\Psi_1^-(\mu,-\omega,0)\zeta_1^1\perp T_{p_2}W^{eu}(p_2,\mu) \tag{2.19}$$

and

$$\Psi_1^+(\mu,\omega,0)\zeta_1^1\in W^{es}_{loc}(p_1,\mu),\quad \Psi_1^+(\mu,\omega,0)\zeta_1^1\perp W^{s}(p_1,\mu). \tag{2.20}$$

The latter properties imply

$$\langle\Psi_1^+(\mu,\omega,0)\zeta_1^1,e^u_{p_1}\rangle\neq 0, \tag{2.21}$$

where $e^s_{p_1}$ denotes the leading stable eigendirection of p_1.

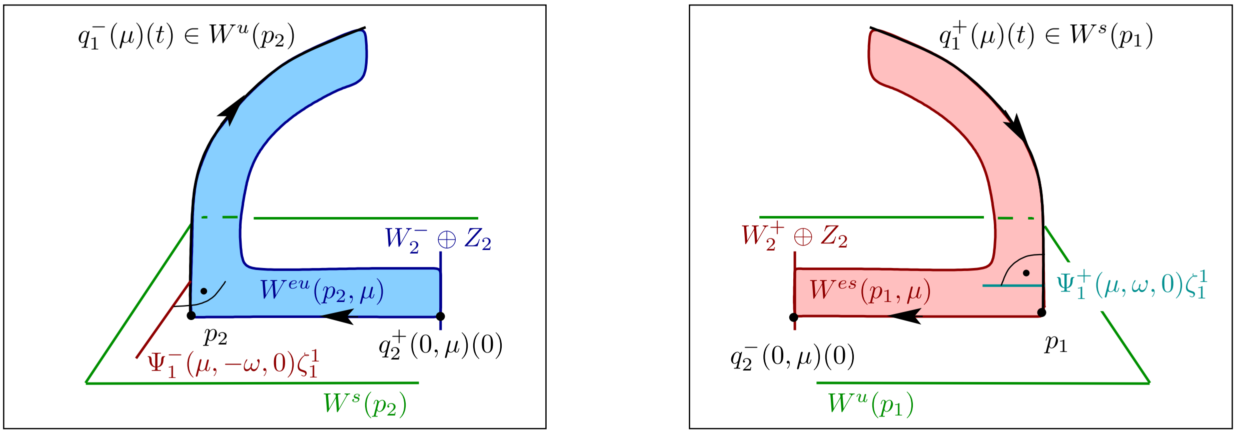

Figure 6: Sketch of the consequences (2.19), cf. left panel, and (2.21), cf. right panel, of the Assumptions (A 1) and (A 2).

Remark 2.19. In what follows we sketch transformations realising the assumptions (A 1) and (A 2). These transformations are local ones by nature, but can be globalised by means of appropriate cut-off functions. In order to preserve the reversibility of the vector field the (local) transformation near $q_1(0)$ must respect the reversing symmetry (this transformation must commute with R) and the corresponding cut-off function must be R-invariant. Let $\mathcal{T}_{p_2}$ be the transformation near p_2 realising (A 2), then $\mathcal{T}_{p_1}:=R\circ\mathcal{T}_{p_2}\circ R$ is the corresponding transformation realising the counterpart of (A 2) at p_1. Applying both $\mathcal{T}_{p_1}$ and $\mathcal{T}_{p_2}$ preserves the reversible structure.

Transformation near $q_1(0)$*:* In a first step we straighten the intersection of $W^{eu}(p_2,\mu)$ and $W^{es}(p_1,\mu)=R\big(W^{eu}(p_2,\mu)\big)$. Note that this intersection consists of a symmetric f-orbit. Hence

$W^{es}(p_1,\mu)\cap W^{eu}(p_2,\mu)$ intersects Fix R in exactly one point. $W^{es}(p_1,\mu)\cap W^{eu}(p_2,\mu)\cap V_1$ can be written as

$$W^{es}(p_1,\mu)\cap W^{eu}(p_2,\mu)\cap V_1 = q_1(0) + \operatorname{graph} h^{su},$$

$h^{su}: \operatorname{span}\{f(q_1(0),0)\} \to W_1^+ \oplus W_1^- \oplus Z_1$. Note that, due to the symmetry h^{su} commutes with R. Furthermore, write $x \in \operatorname{span}\{f(q_1(0),0)\} \oplus W_1^+ \oplus W_1^- \oplus Z_1$ in the form $x = x^f + x^r$, $x^f \in \operatorname{span}\{f(q_1(0),0)\}$ and $x^r \in W_1^+ \oplus W_1^- \oplus Z_1$. Define a transformation $\mathcal{T}^{su}$ in $\operatorname{span}\{f(q_1(0),0)\} \oplus W_1^+ \oplus W_1^- \oplus Z_1$ by $\mathcal{T}^{su}(x) := x^f + \big(x^r + h^{su}(x^f)\big)$. Note that $\mathcal{T}^{su}(x^f)$ lies on the graph of h^{su}. Hence, $W^{es}(p_1,\mu)\cap W^{eu}(p_2,\mu)\cap V_1$ can be straightened by means of the inverse of $\mathcal{T}^{su}$. Finally, since h^{su} commutes with R also $\mathcal{T}^{su}$ commutes with R. That is, $\big(\mathcal{T}^{su}\big)^{-1}$ preserves the reversible structure.

In the next step we assume that $W^{es}(p_1,\mu)\cap W^{eu}(p_2,\mu)\cap V_1$ is already straightened. For $x \in \operatorname{span}\{f(q_1(0),0)\}\oplus W_1^+ \oplus \operatorname{span}\{\zeta_1^1\} \oplus W_1^- \oplus \operatorname{span}\{\zeta_1^2\}$ we introduce coordinates $(x^f, x^+, x^1, x^-, x^2)$ and identify $q_1(0) + x \in V_1$ with $(x^f, x^+, x^1, x^-, x^2)$. Within this setting we have

$$\begin{aligned} W^{eu}(p_2)\cap V_1 &= \left\{\big(x^f, (x^+, x^1) + h^{eu}(x^f, x^-, x^2), x^-, x^2\big)\right\}, \\ W^{es}(p_1)\cap V_1 &= \left\{\big(x^f, x^+, x^1, (x^-, x^2) + h^{es}(x^f, x^+, x^1)\big)\right\}, \end{aligned}$$

where

$$\begin{aligned} h^{eu} &: \operatorname{span}\{f(q_1(0),0)\} \oplus W_1^- \oplus \operatorname{span}\{\zeta_1^2\} \to W_1^+ \oplus \operatorname{span}\{\zeta_1^1\}, \\ h^{es} &: \operatorname{span}\{f(q_1(0),0)\} \oplus W_1^+ \oplus \operatorname{span}\{\zeta_1^1\} \to W_1^- \oplus \operatorname{span}\{\zeta_1^2\}. \end{aligned}$$

Note that

$$h^{eu}(x^f,0,0) = 0 \quad \text{and} \quad h^{es}(x^f,0,0) = 0$$

and

$$R h^{eu}\big(x^f, x^-, x^2\big) = h^{es}\big(R(x^f, x^-, x^2)\big).$$

Now we define a transformation

$$\mathcal{T}: \big(x^f, x^+, x^1, x^-, x^2\big) \mapsto \big(x^f, (x^+, x^1) + h^{eu}(x^f, x^-, x^2), (x^-, x^2) + h^{es}(x^f, x^+, x^1)\big).$$

Note that $\mathcal{T}$ maps $(x^f, 0, 0, x^-, x^2)$ on the graph of h^{eu} and $(x^f, x^+, x^1, 0, 0)$ on the graph of h^{es}. Hence $\mathcal{T}^{-1}$ flattens simultaneously $W^{eu}(p_2,\mu)$ and $W^{es}(p_1,\mu)$. Finally, $\mathcal{T}$ commutes with R. Altogether $\mathcal{T}^{-1}\circ\big(\mathcal{T}^{su}\big)^{-1}$ realises (A 1).

Transformation near p_2: In principle we can proceed as for the transformation near $q_1(0)$. In this respect we note that $W^{eu}(p_2,\mu)$ and $W^{s}(p_2,\mu)$ intersect along a curve (consisting of three orbits, namely the equilibrium p_2 and two orbits within the stable manifold. In comparison with the previous case we only have to replace the f-orbit by this curve. Recall that in this case we do not have to regard the reversing symmetry. □

Proof of Lemma 2.18. To analyse the jump $\xi_{1,i}(\mathbf{u}, \boldsymbol{\omega}, \mu)$ we use the following representation, see also (2.18):

$$\begin{aligned}\xi_{1,i}(\mathbf{u},\boldsymbol{\omega},\mu) &= \sum_{j=1}^{2}\langle\zeta_1^j,\xi_{1,i}(\mathbf{u},\boldsymbol{\omega},\mu)\rangle\zeta_1^j \\ &= \sum_{j=1}^{2}\Big(\big\langle\zeta_1^j,(I-P_1^+(\mu,0))v_{1,i+1}^+(\mathbf{u},\boldsymbol{\omega},\mu)(0)-(I-P_1^-(\mu,0))v_{1,i}^-(\mathbf{u},\boldsymbol{\omega},\mu)(0)\big\rangle\Big)\zeta_1^j.\end{aligned}$$

First we consider the term $\big\langle\zeta_1^1,(I-P_1^+(\mu,0))v_{1,i+1}^+(\mathbf{u},\boldsymbol{\omega},\mu)(0)\big\rangle$. By standard theory [28, 16] we find, cf. also [18, Eqn. (4.6)] or [13, Section 3.3],

$$\begin{aligned}&\big\langle\zeta_1^1,(I-P_1^+(\mu,0))v_{1,i+1}^+(\mathbf{u},\boldsymbol{\omega},\mu)(0)\big\rangle \\ &=\big\langle\Psi_1^+(\mu,\omega_{1,i+1},0)(I-P_1^+(\mu,0))^*\zeta_1^1,\mathcal{P}_1(u_{i+1},\mu,\omega_{1,i+1})q_2^-(u_{i+1},\mu)(-\omega_{1,i+1})\big\rangle+o\big(e^{-2\lambda^u\omega_{1,i+1}}\big),\end{aligned}$$

where $\mathcal{P}_1(u,\mu,t)$ is a projection projecting on $\operatorname{im}\big(I-P_1^+(\mu,t)\big)$ along $\operatorname{im}\big(I-P_2^-(u,\mu,-t)\big)$.

Next we consider the terms in the scalar product on the right-hand side in the last equation. Due to Hypothesis (H 11) the solution $q_2^-(u,\mu)(t)$ behaves asymptotically as $t\to-\infty$ like $e^{\lambda^u t}\eta_1^u(u,\mu)$, where $\eta_1^u(u,\mu)\neq 0$ is parallel to the leading unstable eigendirection of p_1 which is spanned by $e_{p_1}^u$. The asymptotical behaviour of q_2^- becomes clear by [13, Lemma 3.3]. More precisely we find that

$$\mathcal{P}_1(u,\mu,t)q_2^-(u,\mu)(t)=e^{\lambda^u t}\eta_1^u(u,\mu)+o\big(e^{\lambda^u t}\big),\quad \text{as}\quad t\to-\infty. \tag{2.22}$$

We remark that asymptotically, as $t\to\infty$, $\mathcal{P}_1(u,\mu,t)$ tends to the projection which maps on the unstable subspace of p_1 along the stable subspace of p_1. Precise estimates of $\mathcal{P}_1(u,\mu,t)$ which finally ensure (2.22) can be found in [28] or [16], respectively. Furthermore, $\Psi_1^+(\mu,\omega_{1,i+1},0)(I-P_1^+(\mu,0))^*\zeta_1^1$ behaves asymptotically like $e^{-\lambda^u t}\hat{\eta}_1^s(\mu)$ as $t\to\infty$, where $\hat{\eta}_1^s(\mu)\neq 0$. The asymptotical behaviour becomes clear by our construction, cf. (2.20) and [13, Lemma 3.4]. More precisely we find that

$$\Psi_1^+(\mu,t,0)(I-P_1^+(\mu,0))^*\zeta_1^1=e^{-\lambda^u t}\hat{\eta}_1^s(\mu)+o\big(e^{-\lambda^u t}\big),\quad \text{as}\quad t\to\infty. \tag{2.23}$$

Altogether this yields

$$\big\langle\zeta_1^1,(I-P_1^+(\mu,0))v_{1,i+1}^+(\mathbf{u},\boldsymbol{\omega},\mu)(0)\big\rangle=\big\langle\hat{\eta}_1^s(\mu),\eta_1^u(u_{i+1},\mu)\big\rangle e^{-2\lambda^u\omega_{1,i+1}}+o\big(e^{-2\lambda^u\omega_{1,i+1}}\big).$$

According to (2.21) we find

$$c_1^1(u_{i+1},\mu):=\big\langle\hat{\eta}_1^s(\mu),\eta_1^u(u_{i+1},\mu)\big\rangle\neq 0. \tag{2.24}$$

Now we turn to the term $\big\langle\zeta_1^1,-(I-P_1^-(\mu,0))v_{1,i}^-(\mathbf{u},\boldsymbol{\omega},\mu)(0)\big\rangle$. Again by standard theory we find

$$\begin{aligned}&\big\langle\zeta_1^1,-(I-P_1^-(\mu,0))v_{1,i}^-(\mathbf{u},\boldsymbol{\omega},\mu)(0)\big\rangle \\ &\quad=\big\langle\Psi_1^-(\mu,-\omega_{2,i},0)(I-P_1^-(\mu,0))^*\zeta_1^1,\mathcal{P}_2(u_i,\mu,\omega_{2,i})q_2^+(u_i,\mu)(-\omega_{2,i})\big\rangle+o\big(e^{-2\lambda^u\omega_{2,i}}\big),\end{aligned} \tag{2.25}$$

where $\mathcal{P}_2(u,\mu,t)$ is a projection projecting on $\operatorname{im}\big(I-P_1^-(\mu,-t)\big)$ along $\operatorname{im}\big(I-P_2^+(u,\mu,t)\big)$. In quite the same way as above we find as a counterpart of (2.22)

$$\mathcal{P}_2(u,\mu,t)q_2^+(u,\mu)(t)=e^{-\lambda^u t}\eta_2^s(u,\mu)+o\big(e^{-\lambda^u t}\big),\quad \text{as}\quad t\to\infty, \tag{2.26}$$

where $\eta_2^s(u,\mu)$ is parallel to the leading stable direction of p_2. For the equivalent of (2.23) it is enough to take down

$$\Psi_1^-(\mu,t,0)(I-P_1^-(\mu,0))^*\zeta_1^1=O\big(e^{\lambda^u t}\big),\quad \text{as}\quad t\to-\infty. \tag{2.27}$$

Plugging in into (2.25) yields

$$\begin{aligned}&\big\langle\zeta_1^1,-(I-P_1^-(\mu,0))v_{1,i}^-(\mathbf{u},\boldsymbol{\omega},\mu)(0)\big\rangle\\ &\qquad=\big\langle\Psi_1^-(\mu,-\omega_{2,i},0)(I-P_1^-(\mu,0))^*\zeta_1^1,\eta_2^s(u,\mu)\big\rangle e^{-\lambda^u\omega_{2,i}}+o\big(e^{-2\lambda^u\omega_{2,i}}\big).\end{aligned}$$

Due to (2.19) and the definition of the projection P_1^- we have (see also the left panel in Figure 6),

$$\big\langle\Psi_1^-(\mu,-\omega_{2,i},0)(I-P_1^-(\mu,0))^*\zeta_1^1,\eta_2^s(u,\mu)\big\rangle=0. \tag{2.28}$$

This finally yields

$$\big\langle\zeta_1^1,-(I-P_1^-(\mu,0))v_{1,i}^-(\mathbf{u},\boldsymbol{\omega},\mu)(0)\big\rangle=o\big(e^{-2\lambda^u\omega_{2,i}}\big).$$

These considerations verify the ζ_1^1-component of the representation of $\xi_{1,i}(\mathbf{u},\boldsymbol{\omega},\mu)$. The ζ_1^2-component of $\xi_{1,i}(\mathbf{u},\boldsymbol{\omega},\mu)$ follows by applying similar arguments to the terms containing ζ_1^2. In doing so we find in particular

$$c_1^2=-\big\langle\hat\eta_2^u(\mu),\eta_2^s(u_i,\mu)\big\rangle\neq0, \tag{2.29}$$

where $\eta_2^s(u_i,\mu)$ is defined by (2.26). The vector $\hat\eta_2^u$ on the other hand is defined by

$$\Psi_1^-(\mu,t,0)(I-P_1^-(\mu,0))^*\zeta_1^1=e^{\lambda^u t}\hat\eta_2^u(\mu)+o\big(e^{\lambda^u t}\big),\quad \text{as}\quad t\to-\infty. \tag{2.30}$$

Finally, the equivalent to (2.28) reads:

$$\big\langle\Psi_1^+(\mu,-\omega_{1,i+1},0)(I-P_1^+(\mu,0))^*\zeta_1^2,\eta_1^u(u,\mu)\big\rangle=0.$$

Next we consider the jump $\xi_{2,i}$, see again (2.18):

$$\xi_{2,i}(\mathbf{u},\boldsymbol{\omega},\mu)=\big\langle\zeta_2,(I-P_2^+(u_i,\mu,0))v_{2,i}^+(\mathbf{u},\boldsymbol{\omega},\mu)(0)-(I-P_2^-(u_i,\mu,0))v_{2,i}^-(\mathbf{u},\boldsymbol{\omega},\mu)(0)\big\rangle\zeta_2.$$

Considerations such as the ones for $\xi_{1,i}$ provide:

$$c_{21}(u_i,\mu)=\big\langle\hat\eta_1^u(u_i,\mu),\eta_1^s(\mu)\big\rangle\neq0\quad\text{and}\quad c_{22}(u_i,\mu)=\big\langle\hat\eta_2^s(u_i,\mu),\eta_2^u(\mu)\big\rangle\neq0. \tag{2.31}$$

The quantities $\eta_1^s(\mu)$ and $\eta_2^u(\mu)$ are related to $q_1^+(\mu)$ and $q_1^-(\mu)$ in a way as given in (2.26) and (2.22), respectively. Alike the quantities $\hat\eta_1^u(u_i,\mu)$ and $\hat\eta_2^s(u_i,\mu)$ are related to the flows $\Psi_2^-(u_i,\mu,t,0)$ and $\Psi_2^+(u_i,\mu,t,0)$ applied to ζ_2, cf. (2.30) and (2.23), respectively, for the corresponding expressions regarding $\xi_{1,i}$. ∎

Remark 2.20. We want to note that ζ_1^1 is located in the strong stable subspace of $\Psi_1^-(\mu, t, 0)$, as $t \to -\infty$. A corresponding adaption of [13, Lemma 3.4] then allows an even better estimate as the one given in (2.27). □

We remark that $\xi_{j,i}$ depends smoothly on all variables, cf. [28] and [16]. More precisely, as counterpart to [18, Lemma 4.6] or [13, Lemma 3.8] we have

Lemma 2.21. *The derivatives* $D_k\xi_{1,i}(\mathbf{u}, \boldsymbol{\omega}, \mu)$*,* $k = 1, 2, 3$*, of the jumps* $\xi_{j,i}$ *have the following form*

$$
\begin{aligned}
D_k\xi_{1,i}(\mathbf{u}, \boldsymbol{\omega}, \mu) &= \Big(D_k\left(c_1^1(u_{i+1}, \mu)e^{-2\lambda^u \omega_{1,i+1}}\right) + \tilde{\mathcal{R}}_{1,i}^1(\mathbf{u}, \boldsymbol{\omega}, \mu)\Big)\,\zeta_1^1 \\
&\quad + \Big(D_k\left(-c_1^2(u_i, \mu)e^{-2\lambda^u \omega_{2,i}}\right) + \tilde{\mathcal{R}}_{1,i}^2(\mathbf{u}, \boldsymbol{\omega}, \mu)\Big)\,\zeta_1^2, \\
D_k\xi_{2,i}(\mathbf{u}, \boldsymbol{\omega}, \mu) &= \Big(D_k\left(c_{21}(u_i, \mu)e^{2\lambda^s \omega_{1,i}} - c_{22}(u_i, \mu)e^{2\lambda^s \omega_{2,i}}\right) + \tilde{\mathcal{R}}_{2,i}(\mathbf{u}, \boldsymbol{\omega}, \mu)\Big)\,\zeta_2,
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\mathcal{R}}_{1,i}^1(\mathbf{u}, \boldsymbol{\omega}, \mu) &= o(e^{-2\lambda^u \omega_{1,i+1}}) + o(e^{-2\lambda^u \omega_{2,i}}) \\
\tilde{\mathcal{R}}_{1,i}^2(\mathbf{u}, \boldsymbol{\omega}, \mu) &= o(e^{-2\lambda^u \omega_{1,i+1}}) + o(e^{-2\lambda^u \omega_{2,i}}) \\
\tilde{\mathcal{R}}_{2,i}(\mathbf{u}, \boldsymbol{\omega}, \mu) &= o(e^{2\lambda^s \omega_{1,i}}) + o(e^{2\lambda^s \omega_{2,i}}).
\end{aligned}
$$

■

Note that here the derivatives with respect to $\mathbf{u}$ are also considered. Statements concerning this matter are due to considerations in [16].

Finally, we consider the relations between the coefficients c_1^1 and c_1^2 or c_{21} and c_{22}, respectively, which are due to the reversing symmetry.

Lemma 2.22. *Assume (H 1)-(H 12). Then*

(i) $c_1^1(u, \mu) = c_1^2(u, \mu) =: a(u, \mu)$,

(ii) $c_{21}(u, \mu) = -c_{22}(u, \mu) =: c(u, \mu)$.

Proof. The definitions c_1^1 and c_1^2 are given in (2.24) and (2.29), respectively. Due to the reversing symmetry of the vector field, cf. (H 1), Lemma 2.7 and Lemma 2.8 we have $R\eta_1^u(u, \mu) = \eta_2^s(u, \mu)$. Similarly, taking into consideration $\zeta_1^2 = R\zeta_1^1$, we find $R\hat{\eta}_1^s(\mu) = \hat{\eta}_2^u(\mu)$. The statement (i) now follows from the R-invariance of the scalar product, cf. Section 2.1.

The statement (ii) follows by the same type of arguments but this time applied to the quantities appearing in (2.31). ■

Summarising the results of this section we find the following. Assume (H 1)-(H 12), then the determination equation (2.15) for actual orbits staying for all time close to the original T-point is equivalent to

$$
\begin{aligned}
0 =& \quad \mu_1 + a(u_{i+1}, \mu) e^{-2\lambda^u \omega_{1,i+1}} + \mathcal{R}^1_{1,i}(\mathbf{u}, \boldsymbol{\omega}, \mu) \\
0 =& - \mu_1 - a(u_i, \mu) e^{-2\lambda^u \omega_{2,i}} + \mathcal{R}^2_{1,i}(\mathbf{u}, \boldsymbol{\omega}, \mu) \\
0 =& \mu_2 - u_i^2 + c(u_i, \mu)\big(e^{2\lambda^s \omega_{1,i}} + e^{2\lambda^s \omega_{2,i}}\big) + \mathcal{R}_{2,i}(\mathbf{u}, \boldsymbol{\omega}, \mu), \\
i \in& \mathbb{Z}.
\end{aligned}
\tag{2.32}
$$

We remark that the first two equations come from setting the jump $\Xi_{1,i}$ in Σ_1 to zero while the third equation comes from setting the jump $\Xi_{2,i}$ in Σ_2 to zero.

3 Dynamics near Γ

Throughout this section we assume Hypotheses (H 1)-(H 12). That is, we assume in particular that Γ is non-elementary and that λ^u is real and is the leading eigenvalue.

3.1 Shift dynamics - Proof of Theorem 1.1

Let $\mathbf{u}$ and $\boldsymbol{\omega}$ be sequences according to Theorem 2.9. We define

$$
r_{j,i} := e^{-2\lambda^u \omega_{j,i}},\ j = 1, 2, i \in \mathbb{Z}, \qquad \mathbf{r} := ((r_{1,i}, r_{2,i}))_{i \in \mathbb{Z}}. \tag{3.1}
$$

With that (2.32) reads

$$
\begin{aligned}
0 =& \quad \mu_1 + a(u_{i+1}, \mu) r_{1,i+1} + \hat{\mathcal{R}}^1_{1,i}(\mathbf{u}, \mathbf{r}, \mu) \\
0 =& - \mu_1 - a(u_i, \mu) r_{2,i} + \hat{\mathcal{R}}^2_{1,i}(\mathbf{u}, \mathbf{r}, \mu) \\
0 =& \mu_2 - u_i^2 + c(u_i, \mu)\big(r_{1,i}^{\delta^s} + r_{2,i}^{\delta^s}\big) + \hat{\mathcal{R}}_{2,i}(\mathbf{u}, \mathbf{r}, \mu), \\
i \in& \mathbb{Z}.
\end{aligned}
\tag{3.2}
$$

The residual terms $\hat{\mathcal{R}}^j_{1,i}$ and $\hat{\mathcal{R}}_{2,i}$ arise from the corresponding $\mathcal{R}^j_{1,i}$ and $\mathcal{R}_{2,i}$ by applying (3.1). Furthermore we assume the following sign condition for the coefficients a and c:

$$
a(0,0) < 0, \quad c(0,0) < 0. \tag{3.3}
$$

This condition merely determines in which quadrant of the (μ_1, μ_2)-plane shift dynamics exists.

Assuming (3.3) we solve, for positive μ_1, the subsystem of (3.2) consisting of the first two equations (for each $i \in \mathbb{Z}$), for

$$
\mathbf{r} = \mathbf{r}(\mathbf{u}, \mu), \quad \text{where} \quad r_{j,i}(\mathbf{u}, \mu) = -\frac{1}{a(0,0)}\mu_1 + o\big(\mu_1\big),\ j = 1, 2, i \in \mathbb{Z}.
$$

It remains to solve the remaining third equations of (3.2), in which we plug in the solving function $\mathbf{r}(\mathbf{u},\mu)$:

$$0 = \mu_2 - u_i^2 + c(u_i,\mu)\big(r_{1,i}(\mathbf{u},\mu)\big)^{\delta^s} + c(u_i,\mu)\big(r_{2,i}(\mathbf{u},\mu)\big)^{\delta^s} + \hat{\mathcal{R}}_{2,i}(\mathbf{u},\mathbf{r}(\mathbf{u},\mu),\mu), \quad i\in\mathbb{Z}. \tag{3.4}$$

Equation (3.4) can be written as

$$0 = \mu_2 - u_i^2 - \frac{2c(u_i,\mu)}{a(0,0)}\mu_1^{\delta^s} + O\big(\mu_1^{\delta\delta^s}\big), \quad \delta>1, \quad i\in\mathbb{Z}. \tag{3.5}$$

Consider the truncated equation

$$0 = \xi_{sh}(u,\mu), \quad \xi_{sh}(u,\mu) := \mu_2 - u^2 - \frac{2c(u,\mu)}{a(0,0)}\mu_1^{\delta^s}.$$

This equation can be solved for μ_2:

$$\mu_2 = u^2 + \hat{C}_{sh}(u,\mu_1), \tag{3.6}$$

with $\hat{C}_{sh}(u,\mu_1) = O\big(\mu_1^{\delta^s}\big)$ and $\frac{\partial}{\partial u}\hat{C}_{sh}(u,\mu_1) = O\big(\mu_1^{\delta^s}\big)$. Hence the right-hand side of (3.6) has a unique minimum $u^*_{sh}(\mu_1)$. With that we define

$$\mu_{2}{}^*_{sh}(\mu_1) := \big(u^*_{sh}(\mu_1)\big)^2 + \hat{C}_{sh}(u^*_{sh}(\mu_1),\mu_1) = C_{sh}\mu_1^{\delta^s} + o\big(\mu_1^{\delta^s}\big),$$

where C_{sh} is a constant which is different from zero. Now, fix some

$$\mu = (\mu_1,\mu_2): \quad \mu_1 > 0 \quad \text{and} \quad \mu_2 > \mu_{2}{}^*_{sh}(\mu_1), \tag{3.7}$$

and let $u^+(\mu)$, $u^-(\mu)$ be the two solutions of (3.6), or in other words

$$\xi_{sh}(u,\mu) = 0 \quad \Leftrightarrow \quad u\in\{u^+(\mu),u^-(\mu)\}.$$

We find

$$u^\pm(\mu) = \pm\sqrt{\mu_2 - E\mu_1^{\delta^s}} + O(\mu_1^{\delta^s}), \tag{3.8}$$

where $E>0$ is some constant. To verify (3.8) just write $u = \pm\sqrt{\mu_2 - E\mu_1^{\delta^s}} + v$ and solve equation $\xi_{sh}(\pm\sqrt{\mu_2 - E\mu_1^{\delta^s}} + v,\mu) = 0$ for v.

Furthermore, according to the definition of ξ_{sh} we find

$$D_1\xi_{sh}(u^\pm,\mu) = -2\left(u^\pm + \frac{D_1c(u^\pm,\mu)}{a(0,0)}\mu_1^{\delta^s}\right). \tag{3.9}$$

With that we define sequences

$$\mathbf{u^*} := \big(u_i^*\big)_{i\in\mathbb{Z}}, \quad \text{where} \quad u_i^*\in\{u^+(\mu),u^-(\mu)\}. \tag{3.10}$$

Our goal is to show that there are μ_2 (for fixed given μ_1) and ϵ such that (3.4) can be uniquely solved in a closed ball $B(\mathbf{u^*},\epsilon)\subset l^\infty_U$ centred at $\mathbf{u^*}$ with radius

$$\epsilon < \frac{1}{2}|u^+(\mu) - u^-(\mu)|. \tag{3.11}$$

By l^∞_U we denote the set of l^∞ sequences in U equipped with the supremum norm.

Lemma 3.1. *There is a number $N_{sd} \in \mathbb{N}$ such that for $\mu = (\mu_1, \mu_2)$, $\mu_1 > 0$ and $\mu_2 > N_{sd}\mu_1^{\delta^s}$ we have the following: there is an $\epsilon < \frac{1}{2}|u^+(\mu) - u^-(\mu)|$ such that for each sequence $\mathbf{u}^*$ according to* (3.10) *there is a unique solution $(\mathbf{u}_{\mathbf{u}^*}, \mathbf{r}_{\mathbf{u}^*})$ of* (3.2) *with $\mathbf{u}_{\mathbf{u}^*} \in B(\mathbf{u}^*, \epsilon)$.*

By (3.1) the sequence $\mathbf{r}_{\mathbf{u}^*}$ defines a unique sequence $\boldsymbol{\omega}_{\mathbf{u}^*}$. Note that both $\mathbf{u}_{\mathbf{u}^*}$ and $\boldsymbol{\omega}_{\mathbf{u}^*}$ depend on μ, although we omit this dependence in our notation.

In other words the lemma says that (for the corresponding μ and $\mathbf{u}^*$) there is a unique orbit

$$X(\mathbf{u}^*, \mu) := X(\mathbf{u}_{\mathbf{u}^*}, \boldsymbol{\omega}_{\mathbf{u}^*}, \mu)$$

intersecting Σ_2 near $u^+(\mu)$ or $u^-(\mu)$, respectively, in the order prescribed by $\mathbf{u}^*$. Note that $X(\mathbf{u}_{\mathbf{u}^*}, \boldsymbol{\omega}_{\mathbf{u}^*}, \mu)$ denotes a Lin orbit which is in this case a real orbit. We also define a corresponding solution $x(\mathbf{u}^*, \mu)(\cdot)$ of (1.1) by

$$x(\mathbf{u}^*, \mu)(0) = x_{2,0}^+(\mathbf{u}_{\mathbf{u}^*}, \boldsymbol{\omega}_{\mathbf{u}^*}, \mu)(0) = x_{2,0}^-(\mathbf{u}_{\mathbf{u}^*}, \boldsymbol{\omega}_{\mathbf{u}^*}, \mu)(0) \in \Sigma_2. \tag{3.12}$$

Proof of Lemma 3.1. We employ the Banach fixed point theorem to solve (3.5).

Using Taylor expansion w.r.t. u

$$\xi_{sh}(u_i, \mu) = D_1\xi_{sh}(u_i^*, \mu)(u_i - u_i^*) + O\big((u_i - u_i^*)^2\big)$$

we rewrite equation (3.5) as follows

$$u_i = u_i^* + \big(D_1\xi_{sh}(u_i^*)\big)^{-1}\big(O\big((u_i - u_i^*)^2\big) + O\big(\mu_1^{\delta\delta^s}\big)\big) =: T_i(\mathbf{u}), \quad i \in \mathbb{Z}. \tag{3.13}$$

Note that the $O(\mu_1^{\delta\delta^s})$-term appearing in (3.13) also depends on $\mathbf{u}$ and μ_2.

In what follows we show that (3.13) can be read as a fixed point equation

$$\mathbf{u} = \mathbf{T}(\mathbf{u}) = (T_i(\mathbf{u}))_{i\in\mathbb{Z}}, \quad \mathbf{T}(\cdot) : B(\mathbf{u}^*, \epsilon) \subset l_U^\infty \to B(\mathbf{u}^*, \epsilon)$$

which satisfies the assumptions of the Banach fixed point theorem.

First we show that there are μ_2 (for fixed given μ_1) and ϵ such that $\mathbf{T}$ maps $B(\mathbf{u}^*, \epsilon)$ into itself: according to (3.13) we have

$$|T_i(\mathbf{u}) - u_i^*| \leq \left|\big(D_1\xi_{sh}(u_i^*)\big)^{-1}\right| \big(\big|O\big((u_i - u_i^*)^2\big)\big| + \big|O\big(\mu_1^{\delta\delta^s}\big)\big|\big)$$

Now we write $\mu_2 = mE\mu_1^{\delta^s}$, $m > 1$, where m has to be large enough such that (3.7) is still satisfied. Taking into consideration (3.8) and (3.9), we find

$$u^\pm(\mu) = \pm\sqrt{(m-1)E}\mu_1^{\frac{\delta^s}{2}} + O(\mu_1^{\delta^s}),$$

and

$$D_1\xi_{sh}(u^\pm, \mu) = \mp 2\sqrt{(m-1)E}\mu_1^{\frac{\delta^s}{2}} + O(\mu_1^{\delta^s}).$$

Hence, for $\kappa \in (1/2, 1)$ and sufficiently small μ_1

$$|u_i - u_i^*| \leq M\mu_1^{-\kappa\delta^s} \left(\big|O\big((u_i - u_i^*)^2\big)\big| + \big|O\big(\mu_1^{\delta\delta^s}\big)\big|\right), \quad i \in \mathbb{Z}. \tag{3.14}$$

Now, choose

$$\epsilon := \mu_1^{\lambda\delta^s}, \quad 1/2 < \kappa < \lambda, \quad \kappa + \lambda < \delta,$$

and choose μ_1 small enough such that both (3.11) and

$$\epsilon^{-\kappa/\lambda}\left(\left|O(\epsilon^2)\right| + \left|O(\epsilon^{\delta/\lambda})\right|\right) < \epsilon$$

are satisfied. Then it follows from (3.14) that for those μ_1 the operator $\mathbf{T}$ maps $B(\mathbf{u}^*, \epsilon)$ into itself.

It remains to show that $\mathbf{T}$ is contractive. This can be done by examining $\|D\mathbf{T}(\mathbf{u})\|$. We first note that $\mathbf{T}$ is indeed differentiable (as mapping $l_U^\infty \to l_U^\infty$). This follows from corresponding differentiability of $\mathbf{\Xi} = (\Xi_{1,i}(\mathbf{u}, \boldsymbol{\omega}, \mu), \Xi_{2,i}(\mathbf{u}, \boldsymbol{\omega}, \mu))_{i\in\mathbb{Z}}$, which we get from considerations in [16], cf. also Lemma 2.21.

Now, inspecting the right-hand side of (3.13), and if necessary decreasing ϵ (and so decreasing μ_1) further, we find that $\mathbf{T}$ is contractive on $B(\mathbf{u}^*, \epsilon)$.

Note, that by our construction N_{sd} can be defined by $N_{sd} := mE$. Note further that for sufficiently small μ_1, the constant m can be chosen independently of μ_1. ■

Symbolic dynamics. Let N_{sd}, μ and $\mathbf{u}^*$ be in accordance with Lemma 3.1. We define a shift operator σ on $\mathcal{S}_\mu^2 := \{u^+(\mu), u^-(\mu)\}^{\mathbb{Z}}$, equipped with the product topology, by

$$\begin{array}{rcl} \sigma: \ \{u^+(\mu), u^-(\mu)\}^{\mathbb{Z}} & \to & \{u^+(\mu), u^-(\mu)\}^{\mathbb{Z}} \\ \mathbf{u}^* = (u_i^*)_{i\in\mathbb{Z}} & \mapsto & \mathbf{v}^* = (v_i^*)_{i\in\mathbb{Z}}, \quad v_i^* = u_{i+1}^*. \end{array} \tag{3.15}$$

Let $\mathbf{u}^* = (u_i*)_{i\in\mathbb{Z}} \in \{u^+(\mu), u^-(\mu)\}^{\mathbb{Z}}$. Then the σ-orbit $\mathcal{O}_\sigma(\mathbf{u}^*)$ through $\mathbf{u}^*$ is defined by

$$\mathcal{O}_\sigma(\mathbf{u}^*) = \{\sigma^n(\mathbf{u}^*) : n \in \mathbb{Z}\}.$$

Hence, $\mathcal{O}_\sigma(\mathbf{u}^*)$ is N-periodic, or in other words, $\mathbf{u}^*$ is an N-periodic point if $u_{i+N}^* = u_i^*$, for all $i \in \mathbb{Z}$. That means that the sequence $(u_i*)_{i\in\mathbb{Z}}$ is N-periodic. In this case we write

$$(u_i^*)_{i\in\mathbb{Z}} = (\overline{u_1^* \dots u_N^*}).$$

Furthermore we define the following subset of Σ_2

$$\mathcal{S}_\mu := \left\{x(\mathbf{u}^*, \mu)(0) : \ \mathbf{u}^* \in \{u^+(\mu), u^-(\mu)\}^{\mathbb{Z}}\right\}.$$

On $\mathcal{S}_\mu$ we define a first return map Π_μ, see also Figure 5,

$$\begin{array}{rcl} \Pi_\mu: \quad \mathcal{S}_\mu & \to & \mathcal{S}_\mu \\ x(\mathbf{u}^*, \mu)(0) & \mapsto & x(\mathbf{u}^*, \mu)\big(2(\omega_{2,0} + \omega_{1,1})\big). \end{array}$$

Finally we define a one-to-one mapping $\{u^+(\mu), u^-(\mu)\}^{\mathbb{Z}} \to \mathcal{S}_\mu$ (see also (3.12))

$$\begin{array}{rcl} h_\mu: \ \{u^+(\mu), u^-(\mu)\}^{\mathbb{Z}} & \to & \mathcal{S}_\mu \\ \mathbf{u}^* & \mapsto & x(\mathbf{u}^*, \mu)(0). \end{array} \tag{3.16}$$

Lemma 3.2. *The systems* $(\mathcal{S}_\mu, \Pi_\mu)$ *and* $(\{u^+(\mu), u^-(\mu)\}^{\mathbb{Z}}, \sigma)$ *are topologically conjugated.*

Proof. We show that the mapping h_μ introduced in (3.16) is a topological conjugation. To this end we first show that h_μ is a conjugation:

$$\Pi_\mu \circ h_\mu = h_\mu \circ \sigma. \tag{3.17}$$

Equation (3.17) means that for all $\mathbf{u}^*$

$$\Pi_\mu(x(\mathbf{u}^*, \mu)(0)) = x(\sigma\mathbf{u}^*, \mu)(0).$$

The point $x(\mathbf{u}^*, \mu)(0)$ is the starting point of the orbit $X(\mathbf{u}_{\mathbf{u}^*}, \omega_{\mathbf{u}^*}, \mu)$, while $x(\sigma\mathbf{u}^*, \mu)(0)$ is the starting point of the orbit $X(\mathbf{u}_{\sigma\mathbf{u}^*}, \omega_{\sigma\mathbf{u}^*}, \mu)$, cf. (3.12). Furthermore, $\Pi_\mu(x(\mathbf{u}^*, \mu)(0)) = x(\mathbf{u}^*, \mu)\big(2(\omega_{2,0} + \omega_{1,1})\big)$ is the starting point of the orbit $X(\sigma\mathbf{u}_{\mathbf{u}^*}, \sigma\omega_{\mathbf{u}^*}, \mu)$. So, to verify (3.17) it remains to make clear that

$$X(\sigma\mathbf{u}_{\mathbf{u}^*}, \sigma\omega_{\mathbf{u}^*}, \mu) = X(\mathbf{u}_{\sigma\mathbf{u}^*}, \omega_{\sigma\mathbf{u}^*}, \mu).$$

This follows from the uniqueness statement in Lemma 3.1 which says that there is a one-to-one correspondence of $\mathbf{u}^*$ to sequences $(\mathbf{u}_{\mathbf{u}^*}, \omega_{\mathbf{u}^*})$. Clearly $\sigma\mathbf{u}^*$ is related to $(\sigma\mathbf{u}_{\mathbf{u}^*}, \sigma\omega_{\mathbf{u}^*})$. Hence, due to the uniqueness $(\sigma\mathbf{u}_{\mathbf{u}^*}, \sigma\omega_{\mathbf{u}^*}) = (\mathbf{u}_{\sigma\mathbf{u}^*}, \omega_{\sigma\mathbf{u}^*})$.

To complete the proof of the lemma it remains to show that h_μ is a homeomorphism. This part of the proof runs along the lines of [13, Section 4.2]. Here we confine ourselves to explain the major points.

Since $(\{u^+(\mu), u^-(\mu)\}^{\mathbb{Z}}$ is compact (with respect to the product topology) and $\mathcal{S}_\mu$ is Hausdorff it is enough to show that h_μ is continuous, cf. [8, Chap. XI, Theorem 2.1]. To show the continuity of h_μ we proceed as in the proof of [13, Lemma 4.7]. However, there is an extra difficulty due to the internal parameter u. To deal with that we proceed as in [17, Section 3], where a similar problem in the context of discrete systems has been considered. Now, let $\mathbf{u}^*, \mathbf{w}^* \in \{u^+(\mu), u^-(\mu)\}^{\mathbb{Z}}$, and let $\mathbf{u}_{\mathbf{u}^*} = (u_{i,\mathbf{u}^*})_{i\in\mathbb{Z}}$. Then, according to (3.16), (3.12) and (2.17) we find

$$\begin{aligned} \|h_\mu(\mathbf{u}^*) - h_\mu(\mathbf{w}^*)\| \le{}& \|q_2^+(u_{0,\mathbf{u}^*}, \mu)(0) - q_2^+(u_{0,\mathbf{w}^*}, \mu)(0)\| \\ &+ \|v_{2,0}^+(\mathbf{u}_{\mathbf{u}^*}, \boldsymbol{\omega}_{\mathbf{u}^*}, \mu)(0) - v_{2,0}^+(\mathbf{u}_{\mathbf{w}^*}, \boldsymbol{\omega}_{\mathbf{w}^*}, \mu)(0)\|. \end{aligned} \tag{3.18}$$

These terms can be estimated similarly as the related terms in the proof of [17, Lemma 3.3]. We refer in particular to equation (20) in that proof. From that we infer that both addends on the right-hand side of (3.18) are of order $O(1/k)$ if $\mathbf{u}^*$ and $\mathbf{w}^*$ coincide on a block of length k^2 centered at $i = 0$. Therefore h_μ is continuous. ∎

Reversible symbolic dynamics In what follows we identify the symbol $u^+(\mu)$ with $+$ and similarly $u^-(\mu)$ with $-$. We similarly also identify $\mathcal{S}_\mu^2 := \{u^+(\mu), u^-(\mu)\}^{\mathbb{Z}}$ with $\mathcal{S}^2 := \{-, +\}^{\mathbb{Z}}$. In this way we may consider the conjugation h_μ, cf. (3.16), as being defined on $\mathcal{S}^2$, and we may consider sequences $\mathbf{u}^*$ as elements of $\mathcal{S}^2$.

Let $\mathcal{S}^2 := \{-,+\}^{\mathbb{Z}}$ be equipped with the product topology. Consider the system $(\mathcal{S}^2, \sigma)$, where σ is the shift operator defined as in (3.15). The mapping

$$\begin{array}{rccl} \mathcal{R}: & \mathcal{S}^2 & \to & \mathcal{S}^2 \\ & \mathbf{s} & \mapsto & \mathcal{R}\mathbf{s}, \quad (\mathcal{R}\mathbf{s})_i = s_{-i} \end{array}$$

is an involution. We have

$$\operatorname{Fix}\mathcal{R} = \{\mathbf{s} \in \mathcal{S}^2 : s_i = s_{-i},\ i \in \mathbb{Z}\}.$$

Lemma 3.3. *The system $(\mathcal{S}^2, \sigma)$ is reversible w.r.t. $\mathcal{R}$, i.e. $\sigma \circ \mathcal{R} = \mathcal{R} \circ \sigma^{-1}$.*

Proof. $\sigma \circ \mathcal{R}\big((s_i)_{i\in\mathbb{Z}}\big) = (s_{-i+1})_{i\in\mathbb{Z}} = \mathcal{R} \circ \sigma^{-1}\big((s_i)_{i\in\mathbb{Z}}\big)$. ∎

For $\mathbf{s} \in \mathcal{S}^2$ the σ-orbit $\mathcal{O}_\sigma(\mathbf{s})$ is $\mathcal{R}$-symmetric if $\mathcal{R}\big(\mathcal{O}_\sigma(\mathbf{s})\big) = \mathcal{O}_\sigma(\mathbf{s})$. So,

$$\mathcal{O}_\sigma(\mathbf{s}) \text{ is symmetric} \Leftrightarrow \exists N \in \mathbb{Z} : \sigma^N \mathcal{R}\mathbf{s} = \mathbf{s}. \tag{3.19}$$

Lemma 3.4. *The σ-orbit $\mathcal{O}_\sigma(\mathbf{s})$ is $\mathcal{R}$-symmetric if and only if*

$$\text{(a)} \quad \exists\, \hat{\mathbf{s}} \in \mathcal{O}_\sigma(\mathbf{s}) : \ \mathcal{R}\hat{\mathbf{s}} = \hat{\mathbf{s}} \qquad \textit{or} \qquad \text{(b)} \quad \exists\, \hat{\mathbf{s}} \in \mathcal{O}_\sigma(\mathbf{s}) : \ \mathcal{R}\hat{\mathbf{s}} = \sigma\hat{\mathbf{s}} \tag{3.20}$$

Proof. According to (3.19) it remains to show that the symmetry of the orbit implies (3.20). Let $\mathcal{O}_\sigma(\mathbf{s})$ be $\mathcal{R}$-symmetric and let $N \in \mathbb{Z}$ such that $\mathcal{R}\mathbf{s} = \sigma^{-N}\mathbf{s}$, cf. (3.19). Therefore, for $k \in \mathbb{Z}$ we have $\mathcal{R}\sigma^{-k}\mathbf{s} = \sigma^{-N+k}\mathbf{s}$.
Now, let $|N|$ be even. For $k = N/2$ we find $\mathcal{R}\sigma^{-N/2}\mathbf{s} = \sigma^{-N/2}\mathbf{s}$. If on the contrary $|N|$ is odd we choose $k = (N+1)/2$ and find $\mathcal{R}\sigma^{-(N+1)/2}\mathbf{s} = \sigma\sigma^{-(N+1)/2}\mathbf{s}$. ∎

Condition (a) in (3.20) says that the symmetric orbit $\mathcal{O}_\sigma(\mathbf{s})$ intersects $\operatorname{Fix}\mathcal{R}$ in $\hat{\mathbf{s}}$. Note that the sequence $\hat{\mathbf{s}}$ is symmetric with respect to the reflection element s_0. By contrast, the sequence $\hat{\mathbf{s}}$ in condition (b) in (3.20) does not belong to $\operatorname{Fix}\mathcal{R}$. This sequence is symmetric with respect to the "gap between s_0 and s_1".

Lemma 3.5. *If $\mathcal{O}_\sigma(\mathbf{s})$ is a $\mathcal{R}$-symmetric aperiodic orbit then exactly one of the conditions stated in* (3.20) *is satisfied for exactly one* $\hat{\mathbf{s}}$.

Proof. We show, provided both conditions in (3.20) are satisfied or one these conditions is satisfied for two different $\hat{\mathbf{s}}^{\mathbf{1}}$, $\hat{\mathbf{s}}^{\mathbf{2}}$ then necessarily $\mathcal{O}_\sigma(\mathbf{s})$ is periodic.

Assume that there are $\hat{\mathbf{s}}^{\mathbf{1}}$, $\hat{\mathbf{s}}^{\mathbf{2}}$ such that $\mathcal{R}\hat{\mathbf{s}}^{\mathbf{i}} = \hat{\mathbf{s}}^{\mathbf{i}}$, $i = 1, 2$, and that $\hat{\mathbf{s}}^{\mathbf{2}} = \sigma^N\hat{\mathbf{s}}^{\mathbf{1}}$. Under these assumptions we find

$$\sigma\big(\sigma^N\hat{\mathbf{s}}^{\mathbf{1}}\big) = \sigma\hat{\mathbf{s}}^{\mathbf{2}} = \sigma\mathcal{R}\hat{\mathbf{s}}^{\mathbf{2}} = \mathcal{R}\sigma^{-1}\hat{\mathbf{s}}^{\mathbf{2}} = \mathcal{R}\sigma^{N-1}\hat{\mathbf{s}}^{\mathbf{1}} = \sigma^{-N+1}\mathcal{R}\hat{\mathbf{s}}^{\mathbf{1}} = \sigma^{-N+1}\hat{\mathbf{s}}^{\mathbf{1}}.$$

Next assume that there are $\hat{\mathbf{s}}^{\mathbf{1}}$, $\hat{\mathbf{s}}^{\mathbf{2}}$ such that $\mathcal{R}\hat{\mathbf{s}}^{\mathbf{i}} = \sigma\hat{\mathbf{s}}^{\mathbf{i}}$, $i = 1, 2$, and that $\hat{\mathbf{s}}^{\mathbf{2}} = \sigma^N\hat{\mathbf{s}}^{\mathbf{1}}$. Under these assumptions we find

$$\sigma\big(\sigma^N\hat{\mathbf{s}}^{\mathbf{1}}\big) = \sigma\hat{\mathbf{s}}^{\mathbf{2}} = \mathcal{R}\hat{\mathbf{s}}^{\mathbf{2}} = \mathcal{R}\sigma^N\hat{\mathbf{s}}^{\mathbf{1}} = \sigma^{-N}\mathcal{R}\hat{\mathbf{s}}^{\mathbf{1}} = \sigma^{-N}\sigma\hat{\mathbf{s}}^{\mathbf{1}} = \sigma^{-N+1}\hat{\mathbf{s}}^{\mathbf{1}}.$$

Assume finally that there are $\hat{\mathbf{s}}^{\mathbf{1}}$ such that $\mathcal{R}\hat{\mathbf{s}}^{\mathbf{1}} = \hat{\mathbf{s}}^{\mathbf{1}}$ and $\hat{\mathbf{s}}^{\mathbf{2}}$ such that $\mathcal{R}\hat{\mathbf{s}}^{\mathbf{2}} = \sigma\hat{\mathbf{s}}^{\mathbf{2}}$, and that $\hat{\mathbf{s}}^{\mathbf{2}} = \sigma^N\hat{\mathbf{s}}^{\mathbf{1}}$. Under these assumptions we find

$$\sigma\big(\sigma^N\hat{\mathbf{s}}^{\mathbf{1}}\big) = \sigma\hat{\mathbf{s}}^{\mathbf{2}} = \mathcal{R}\hat{\mathbf{s}}^{\mathbf{2}} = \mathcal{R}\sigma^N\hat{\mathbf{s}}^{\mathbf{1}} = \sigma^{-N}\mathcal{R}\hat{\mathbf{s}}^{\mathbf{1}} = \sigma^{-N}\hat{\mathbf{s}}^{\mathbf{1}}.$$

∎

The proof of Lemma 3.5 gives a characterisation of symmetric periodic orbits.

Corollary 3.6. *The orbit $\mathcal{O}_\sigma(\mathbf{s})$ is a symmetric periodic orbit if and only if either both conditions in* (3.20) *are satisfied or one of these conditions is satisfied for two different $\hat{\mathbf{s}}^{\mathbf{1}}$, $\hat{\mathbf{s}}^{\mathbf{2}}$.*

Proof. By the proof of Lemma 3.5 it remains to show that the symmetry of a periodic orbit implies that either both conditions in (3.20) are satisfied or one these conditions is satisfied for two different $\hat{\mathbf{s}}^{\mathbf{1}}$, $\hat{\mathbf{s}}^{\mathbf{2}}$.

Let $\mathcal{O}_\sigma(\mathbf{s})$ be N-periodic and symmetric.

First assume that there exists a $\hat{\mathbf{s}} \in \mathcal{O}_\sigma(\mathbf{s})$ such that $\mathcal{R}\hat{\mathbf{s}} = \hat{\mathbf{s}}$, cf. condition (a) in (3.20). If $N = 2K+1$, we find that due to the symmetry we have $\sigma^{-K}\hat{\mathbf{s}} = \mathcal{R}\sigma^K\hat{\mathbf{s}}$. Furthermore, since $\mathcal{O}_\sigma(\mathbf{s})$ is $(2K+1)$-periodic we have $\sigma\big(\sigma^K\hat{\mathbf{s}}\big) = \sigma^{-K}\hat{\mathbf{s}}$. Hence condition (b) in (3.20) is satisfied with $\sigma^K\hat{\mathbf{s}}$.
If $N = 2K+2$, we find that

$$\mathcal{O}_\sigma(\mathbf{s}) = \{\sigma^{-K}\hat{\mathbf{s}}, \dots, \sigma^{-1}\hat{\mathbf{s}}, \hat{\mathbf{s}}, \sigma\hat{\mathbf{s}}, \dots, \sigma^K\hat{\mathbf{s}}, \sigma^{K+1}\hat{\mathbf{s}}\}.$$

Due to the condition $\mathcal{R}\hat{\mathbf{s}} = \hat{\mathbf{s}}$ we find that condition (a) in (3.20) is satisfied with $\sigma^{K+1}\hat{\mathbf{s}}$.

Next assume that there exists a $\hat{\mathbf{s}} \in \mathcal{O}_\sigma(\mathbf{s})$ such that $\mathcal{R}\hat{\mathbf{s}} = \sigma\hat{\mathbf{s}}$, cf. condition (b) in (3.20). If $N = 2K+2$, we find that

$$\mathcal{O}_\sigma(\mathbf{s}) = \{\sigma^{-K}\hat{\mathbf{s}}, \dots, \sigma^{-1}\hat{\mathbf{s}}, \hat{\mathbf{s}}, \sigma\hat{\mathbf{s}}, \dots, \sigma^K\hat{\mathbf{s}}, \sigma^{K+1}\hat{\mathbf{s}}\}.$$

Due to the condition $\mathcal{R}\hat{\mathbf{s}} = \sigma\hat{\mathbf{s}}$ we find that condition (b) in (3.20) is satisfied with $\sigma^{K+1}\hat{\mathbf{s}}$.
If $N = 2K+1$, we find that

$$\mathcal{O}_\sigma(\mathbf{s}) = \{\sigma^{-K+1}\hat{\mathbf{s}}, \dots, \sigma^{-1}\hat{\mathbf{s}}, \hat{\mathbf{s}}, \sigma\hat{\mathbf{s}}, \dots, \sigma^K\hat{\mathbf{s}}, \sigma^{K+1}\hat{\mathbf{s}}\}.$$

Due to the condition $\mathcal{R}\hat{\mathbf{s}} = \sigma\hat{\mathbf{s}}$ we find that condition (a) in (3.20) is satisfied with $\sigma^{K+1}\hat{\mathbf{s}}$. ∎

Indeed there are symmetric periodic orbits which intersect $\operatorname{Fix}\mathcal{R}$ only once (both conditions in (3.20) are satisfied), and there are even symmetric periodic orbits which do not intersect $\operatorname{Fix}\mathcal{R}$ at all. Examples for those orbits are $\mathcal{O}_\sigma\big((\overline{++-})\big)$ or $\mathcal{O}_\sigma\big((\overline{++--})\big)$, respectively.

Let $\mathcal{S}^2_\mathcal{R}$ denote the set of all sequences in $\mathcal{S}^2$ whose σ-orbit is symmetric

$$\mathcal{S}^2_\mathcal{R} := \{\mathbf{s} \in \mathcal{S}^2 : \mathcal{O}_\sigma(\mathbf{s}) \text{ is symmetric}\}.$$

Remark 3.7. The set $\mathcal{S}^2_{\mathcal{R}}$ is a nonempty proper subset of $\mathcal{S}^2$: With the examples given above it is clear that $\mathcal{S}^2_{\mathcal{R}}$ is nonempty. In fact all periodic orbits up to period six are contained in $\mathcal{S}^2_{\mathcal{R}}$. In order to verify that there are sequences $\mathbf{s}$ in $\mathcal{S}^2$ whose σ-orbit is asymmetric consider for instance the 7-periodic sequence $\mathbf{s} = (\overline{+++--+-})$. $\square$

By construction we find that $\mathcal{S}^2_{\mathcal{R}}$ is σ-invariant. In [16, Section 6.4.4] it has been shown that, although $\mathcal{S}^2_{\mathcal{R}}$ is not closed, the system $(\mathcal{S}^2_{\mathcal{R}}, \sigma)$ exhibits chaotic dynamics in the sense of Devaney. First we recall Devaney's definition of a chaotic system, cf. [7]: Let X be a metric space and f be a homeomorphism on X. The discrete dynamical system (X, f) is chaotic if

(i) f is topologically transitive;
(ii) The periodic points of f are dense in X;
(iii) f has sensitive dependence on initial conditions.

The result of Banks *et. al* [2], states that (i) and (ii) imply sensitive dependence on initial conditions. So it remains to verify that $(\mathcal{S}^2_{\mathcal{R}}, \sigma)$ is topologically transitive and that the periodic points are dense.

Lemma 3.8 ([16], Lemma 6.4.17)**.** *Consider $(\mathcal{S}^2_{\mathcal{R}}, \sigma)$. The set of periodic points is dense in $\mathcal{S}^2_{\mathcal{R}}$.*

Proof. We show that the set of those periodic points is even dense in $\mathcal{S}^2$. Let $\mathbf{s} := (s_i)_{i\in\mathbb{Z}} \in \mathcal{S}^2$. Then $(\overline{s_n \dots s_0 \dots s_{-n} + s_{-n} \dots \centerdot s_0 \dots s_n})$ is a symmetric periodic point coinciding with $\mathbf{s}$ on segment of length 2n+1 around "$\centerdot$". ∎

Lemma 3.9 ([16], Lemma 6.4.18)**.** *There is a dense orbit in $(\mathcal{S}^2_{\mathcal{R}}, \sigma)$.*

Proof. The proof is very similar to the proof that there is a dense orbit in $(\mathcal{S}^2, \sigma)$. Let $\mathbf{s}^k := \{\mathbf{s}^k_i = s^k_{i,1}, \dots, s^k_{i,k} : \quad s^k_{i,j} \in \{-,+\}, i \in \{1, \dots, 2^k\}\}$ be the set of all finite segments of length k. Introduce an order in $\mathbf{s}^k$ in the following way:

$$\mathbf{s}^k_i < \mathbf{s}^l_j \Leftrightarrow k < l \quad \text{or} \quad k = l, i < j.$$

Finally let $\bar{\mathbf{s}}^k_i = s^k_{i,k}, \dots, s^k_{i,1}$. The orbit through $\mathbf{s} := (\dots \bar{\mathbf{s}}^2_1 \bar{\mathbf{s}}^1_2 \bar{\mathbf{s}}^1_1 \centerdot + \mathbf{s}^1_1 \mathbf{s}^1_2 \mathbf{s}^2_1 \dots \mathbf{s}^2_4 \dots)$ is dense in $\mathcal{S}^2_{\mathcal{R}}$. ∎

An immediate consequence of this lemma is

Corollary 3.10 ([16], Corollary 6.4.19)**.** *The system $(\mathcal{S}^2_{\mathcal{R}}, \sigma)$ is topologically transitive.* ∎

Indeed, $\mathcal{S}^2_{\mathcal{R}}$ is not closed because the orbit constructed in Lemma 3.9 is even dense in $\mathcal{S}^2$. Moreover the addressed result of Banks *et. al* implies that systems which are topologically conjugated to $(\mathcal{S}^2_{\mathcal{R}}, \sigma)$ are chaotic as well.

Lemma 3.11. *For the map $h_\mu : \mathcal{S}^2 \to \mathcal{S}_\mu \subset \Sigma_2$ we have $h_\mu \circ \mathcal{R} = R \circ h_\mu$.*

Proof. The statement of the lemma immediately translates into $x(\mathcal{R}\mathbf{u}^*, \mu)(0) = R(x(\mathbf{u}^*, \mu)(0))$. This equality follows with Lemma 2.13 and the uniqueness statement in Lemma 3.1. ∎

This yields the following:

Corollary 3.12. *The statement of Lemma 3.11 implies*

(i) $R(\mathcal{S}_\mu) = \mathcal{S}_\mu$.

(ii) The system $(\mathcal{S}_\mu, \Pi_\mu)$ *is reversible w.r.t. to the involution* R*, i.e.* $\Pi_\mu \circ R = R \circ \Pi_\mu^{-1}$.

(iii) The σ*-orbit through* $\mathbf{u}^*$ *is* $\mathcal{R}$*-symmetric if and only if the* Π_μ*-orbit through* $h_\mu(\mathbf{u}^*)$ *is* R*-symmetric. Then according to Lemma 3.4 we have the following possibilities:*

(a) $\mathcal{R}\hat{\mathbf{u}}^* = \hat{\mathbf{u}}^* \Leftrightarrow h_\mu(\hat{\mathbf{u}}^*) = x(\hat{\mathbf{u}}^*, \mu)(0) \in \operatorname{Fix} R \cap \Sigma_2$*, for* $\hat{\mathbf{u}}^* \in \mathcal{O}_\sigma(\mathbf{u}^*)$,

(b) $\mathcal{R}\hat{\mathbf{u}}^* = \sigma\hat{\mathbf{u}}^* \Leftrightarrow x(\hat{\mathbf{u}}^*, \mu)(2\omega_{2,0}) \in \operatorname{Fix} R \cap \Sigma_1$*, for* $\hat{\mathbf{u}}^* \in \mathcal{O}_\sigma(\mathbf{u}^*)$.

Proof. (i): According to Lemma 3.11 we have $h_\mu\big(\mathcal{R}(\mathcal{S}^2)\big) = R\big(h_\mu(\mathcal{S}^2)\big)$. With $\mathcal{R}(\mathcal{S}^2) = \mathcal{S}^2$ and $h_\mu(\mathcal{S}^2) = \mathcal{S}_\mu$ we prove the statement.

(ii): According to Lemma 3.3 we have $\sigma\mathcal{R} = \mathcal{R}\sigma^{-1}$, where we substitute $\sigma = h_\mu^{-1}\Pi_\mu h_\mu$, cf. (3.17). Taking the statement of Lemma 3.11 into consideration we have $h_\mu^{-1}\Pi_\mu R h_\mu = h_\mu^{-1} R \Pi_\mu^{-1} h_\mu$. This gives the statement (ii).

(iii): With (3.17) and Lemma 3.11 we find $h_\mu \sigma^N \mathcal{R}(\mathbf{u}^*) = \Pi_\mu^N R h_\mu(\mathbf{u}^*)$. Now, the statement follows with (3.19) and the analogous statement for Π_μ-orbits.

(a): Let $\hat{\mathbf{u}}^* \in \mathcal{O}_\sigma(\mathbf{u}^*) \cap \operatorname{Fix} \mathcal{R}$. Of course, cf. Lemma 3.2, $h_\mu(\hat{\mathbf{u}}^*) \in \mathcal{O}_{\Pi_\mu}(h_\mu(\mathbf{u}^*))$. According to Lemma 3.11 we find $R h_\mu(\hat{\mathbf{u}}^*) = h_\mu(\mathcal{R}\hat{\mathbf{u}}^*) = h_\mu(\hat{\mathbf{u}}^*)$.

(b): By the conjugacy h_μ it is clear that the $\mathbf{u}^*$-symmetry implies the R-symmetry of the corresponding Π_μ-orbit and hence also of the corresponding f-orbit. Due to the symmetry property $\mathcal{R}\hat{\mathbf{u}}^* = \sigma\hat{\mathbf{u}}^*$ we invoke Lemma 2.16 with $i_0 = -1$ to find that $\omega_{j,i} = \omega_{j+1,-i+1}$. Finally this implies the statement. ∎

Remark 3.13. Recall that Π_μ-orbits correspond to f-orbits, i.e. to orbits of (1.1) and that symmetric periodic f-orbits intersect $\operatorname{Fix} R$ exactly twice and a symmetric aperiodic f-orbit intersects $\operatorname{Fix} R$ exactly once. Those f-orbits may intersect $\operatorname{Fix} R$ within Σ_1 or Σ_2. Which case is at hand can be read from "the symmetry of $\mathbf{u}^*$" according to (iii)(a) or (iii)(b) in the foregoing corollary. Case (iii)(a) implies an intersection of the corresponding f-orbit with $\operatorname{Fix} R \cap \Sigma_2$, while case (iii)(b) implies an intersection of the corresponding f-orbit with $\operatorname{Fix} R \cap \Sigma_1$. Both cases are possible for one orbit $\mathcal{O}(\mathbf{u}^*)$, recall Lemma 3.5 and Corollary 3.6. □

Finally we define

$$\mathcal{S}_{\mu,\mathcal{R}} := h_\mu\big(\mathcal{S}^2_{\mathcal{R}}\big).$$

From (3.17) and the σ-invariance of $\mathcal{S}^2_{\mathcal{R}}$, the set $\mathcal{S}_{\mu,\mathcal{R}}$ is Π_μ-invariant, and the systems $(\mathcal{S}^2_{\mathcal{R}}, \sigma)$ and $(\mathcal{S}_{\mu,\mathcal{R}}, \Pi_\mu)$ are topologically conjugated. Since topological transitivity and denseness of periodic point are topological properties we find by the aforementioned result by Banks *et. al* [2] that the system $(\mathcal{S}_{\mu,\mathcal{R}}, \Pi_\mu)$ is chaotic in the sense of Devaney.

In the following section we consider how this chaotic dynamics dissolves within the region (II) displayed in Figure 4.

3.2 Dissolution of shift dynamics – local bifurcations

In this section we exclusively consider N-periodic orbits $X_N(\mu)$, $N \in \{1,2,3,4\}$. Of course for μ within the region (I) the intersections of those orbits with Σ_2 belong to $\mathcal{S}_\mu$. The aim of this section is to discuss how those orbits disappear within the region (II) by decreasing μ_2.

3.2.1 Proof of Theorem 1.2

We proceed as in the Section 3.1 up to (3.5) which we repeat here

$$0 = \mu_2 - u_i^2 - \frac{2c(u_i,\mu)}{a(0,0)}\mu_1^{\delta^s} + O\big(\mu_1^{\delta\delta^s}\big), \quad \delta > 1, \quad i \in \mathbb{Z}.$$

Due to the sign condition (3.3), $a(0,0)$ and $c(0,0)$ are negative and since $c(u_i,\mu)$ is bounded (for sufficiently small u_i and μ), we have for sufficiently small μ_1

$$-\frac{2c(u_i,\mu)}{a(0,0)}\mu_1^{\delta^s} + O\big(\mu_1^{\delta\delta^s}\big) < 0.$$

This observation leads immediately to the statement of Theorem 1.2.

3.2.2 One-periodic orbits – proof of Theorem 1.3

Similar to [18], one-periodic orbits are characterised by sequences $\boldsymbol{\omega} = (\omega_{1,i}, \omega_{2,i})_{i\in\mathbb{Z}}$ and $\mathbf{u} = (u_i)_{i\in\mathbb{Z}}$ with

$$(\omega_{1,i}, \omega_{2,i}) =: (\omega_1, \omega_2), \quad u_i =: u, \quad i \in \mathbb{Z}.$$

Therefore, according to (2.32) the bifurcation equations for 1-periodic orbits reads:

$$\begin{aligned} 0 &= \quad \mu_1 + a(u,\mu)e^{-2\lambda^u\omega_1} + \mathcal{R}_1^1(\mathbf{u},\boldsymbol{\omega},\mu) \\ 0 &= -\,\mu_1 - a(u,\mu)e^{-2\lambda^u\omega_2} + \mathcal{R}_1^2(\mathbf{u},\boldsymbol{\omega},\mu) \\ 0 &= \mu_2 - u^2 + c(u,\mu)e^{2\lambda^s\omega_1} + c(u,\mu)e^{2\lambda^s\omega_2} + \mathcal{R}_2(\mathbf{u},\boldsymbol{\omega},\mu). \end{aligned} \tag{3.21}$$

Define in the same manner as in (3.1)

$$r_i := e^{-2\lambda^u\omega_i}, \quad i = 1,2.$$

Then there exists a $\delta^s > 1$ such that the bifurcation equations in the new variables read

$$\begin{aligned} 0 &= \mu_1 + a(u,\mu)r_1 + \hat{\mathcal{R}}_1^1(u,r_1,r_2,\mu) \\ 0 &= -\mu_1 - a(u,\mu)r_2 + \hat{\mathcal{R}}_1^2(u,r_1,r_2,\mu) \\ 0 &= \mu_2 - u^2 + c(u,\mu)r_1^{\delta^s} + c(u,\mu)r_2^{\delta^s} + \hat{\mathcal{R}}_2(u,r_1,r_2,\mu). \end{aligned} \tag{3.22}$$

In what follows we assume as for the proof of Theorem 1.1 the sign condition (3.3).

The first two equations in (3.22) can be solved for $(r_1, r_2)(u, \mu)$; because of (3.3) μ_1 has to be positive. After plugging in into the third equation in (3.22), this one can be solved for μ_2:

$$\mu_2 = u^2 + \hat{C}_1(u, \mu_1). \tag{3.23}$$

The right hand side of (3.23) has a unique minimum $u^*(\mu_1)$. Moreover, $u^*(\mu_1) \to 0$, as $\mu_1 \to 0$. The above arguments prove Theorem 1.3 with

$$\kappa_{sc}(\mu_1) := \big(u^*(\mu_1)\big)^2 + \hat{C}_1(u^*(\mu_1), \mu_1) = C\mu_1^{\delta^s} + o\big(\mu_1^{\delta^s}\big),$$

where, due to (3.3), C is a positive constant.

Lemma 3.14. *Let $\mu_1 > 0$, let $\mu_2 \geq \kappa_{sc}(\mu_1)$, and finally let u and (ω_1, ω_2) be values for which $X(u, (\omega_1, \omega_2), \mu)$ is a 1-periodic orbit according to Theorem 1.3. Then $RX = X$, i.e. X is symmetric.*

Proof. The statement is an immediate consequence of Hypothesis (H 8), Lemma 2.13 and the fact that, according to the analysis of (3.21) or (3.22), respectively, (ω_1, ω_2) is uniquely determined by u and μ. ∎

3.2.3 Two-periodic orbits – proof of Theorem 1.4

Two-periodic orbits are characterised by sequences $\boldsymbol{\omega} = (\omega_{1,i}, \omega_{2,i})_{i\in\mathbb{Z}}$ and $\mathbf{u} = (u_i)_{i\in\mathbb{Z}}$ with

$$(\omega_{1,i}, \omega_{2,i}) = \begin{cases} (\omega_{1,1}, \omega_{2,1}), & i \text{ odd} \\ (\omega_{1,2}, \omega_{2,2}), & i \text{ even} \end{cases}, \qquad u_i = \begin{cases} u_1, & i \text{ odd} \\ u_2, & i \text{ even} \end{cases}. \tag{3.24}$$

We refer to Figure 7 for visualisation.

With the setting of (3.1) there exists a $\delta^s > 1$ such that the bifurcation equations for two-periodic orbits reads.

$$\begin{aligned} 0 &= \mu_1 + a(u_2,\mu)r_{1,2} + \hat{\mathcal{R}}_{1,1}^1(\mathbf{u},\mathbf{r},\mu) \\ 0 &= -\mu_1 - a(u_1,\mu)r_{2,1} + \hat{\mathcal{R}}_{1,1}^2(\mathbf{u},\mathbf{r},\mu) \\ 0 &= \mu_2 - u_1^2 + c(u_1,\mu)r_{1,1}^{\delta^s} + c(u_1,\mu)r_{2,1}^{\delta^s} + \hat{\mathcal{R}}_{2,1}(\mathbf{u},\mathbf{r},\mu) \\ 0 &= \mu_1 + a(u_1,\mu)r_{1,1} + \hat{\mathcal{R}}_{1,2}^1(\mathbf{u},\mathbf{r},\mu) \\ 0 &= -\mu_1 - a(u_2,\mu)r_{2,2} + \hat{\mathcal{R}}_{1,2}^2(\mathbf{u},\mathbf{r},\mu) \\ 0 &= \mu_2 - u_2^2 + c(u_2,\mu)r_{1,2}^{\delta^s} + c(u_2,\mu)r_{2,2}^{\delta^s} + \hat{\mathcal{R}}_{2,2}(\mathbf{u},\mathbf{r},\mu). \end{aligned} \tag{3.25}$$

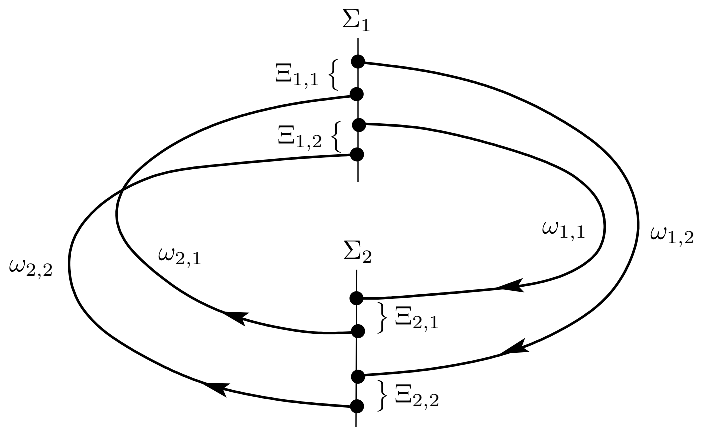

Figure 7: A two-periodic Lin orbit.

Now we proceed in principle as in the previous sections. We solve the subsystem consisting of the first two equations of each block in (3.25) for

$$\mathbf{r} = (r_{1,2}, r_{2,1}, r_{1,1}, r_{2,2})(\mathbf{u}, \mu).$$

Plugging into the third equations of each block in (3.25) and solving these equations for μ_2 in each case yields two functions μ_2^k, $k = 1, 2$:

$$\mu_2^1 = u_1^2 + \hat{C}_{2,1}(\mathbf{u}, \mu_1), \qquad \mu_2^2 = u_2^2 + \hat{C}_{2,2}(\mathbf{u}, \mu_1). \tag{3.26}$$

Thus, the resulting determination equation for two-periodic orbits reads:

$$\mu_2^1 = \mu_2^2 \quad \Leftrightarrow \quad 0 = u_1^2 - u_2^2 + \hat{C}_{2,1}(\mathbf{u}, \mu_1) - \hat{C}_{2,2}(\mathbf{u}, \mu_1) =: \hat{C}(\mathbf{u}, \mu_1). \tag{3.27}$$

The right-hand side of the latter equation can be seen as a perturbation of $u_1^2 - u_2^2$. The zero set of $u_1^2 - u_2^2$ consists of two intersecting straight lines. In what follows we show that, for fixed μ_1, the solution set of the perturbation given in (3.27) still consists of two transversely intersecting curves. In the style of the shift operator σ introduced in Section 3.1 we define the mapping ζ on pairs (a, b):

$$\zeta(a, b) = (b, a).$$

Note that ζ is related to the shift σ on two-periodic sequences $(\overline{a\, b})$.

Lemma 3.15. $\zeta(\hat{C}_{2,1}(\mathbf{u}, \mu_1), \hat{C}_{2,2}(\mathbf{u}, \mu_1)) = (\hat{C}_{2,1}(\zeta\mathbf{u}, \mu_1), \hat{C}_{2,2}(\zeta\mathbf{u}, \mu_1))$.

Proof. This statement follows from the uniqueness of Lin orbits (for given $\mathbf{u}$ and $\boldsymbol{\omega}$). We also refer to the arguments used in the proof of Lemma 3.2. ∎

An immediate consequence of Lemma 3.15 is $\hat{C}_{2,1}(\mathbf{u}, \mu_1) = \hat{C}_{2,2}(\zeta\mathbf{u}, \mu_1)$, or in other words

$$\hat{C}_{2,1}(u_1, u_2, \mu_1) = \hat{C}_{2,2}(u_2, u_1, \mu_1), \tag{3.28}$$

and hence

$$\hat{C}_{2,1}(u, u, \mu_1) = \hat{C}_{2,2}(u, u, \mu_1). \tag{3.29}$$

Therefore there is a function $\hat{C}_r$ such that we may write

$$\hat{C}(\mathbf{u}, \mu_1) = (u_1 - u_2)\big(u_1 + u_2 + \hat{C}_r(\mathbf{u}, \mu_1)\big).$$

That means that one branch of two-periodic solutions exists for $u_1 = u_2$. Note that these solutions correspond to one-periodic orbits (which are passed through twice). So, the real two-periodic orbits (the ones with minimal period two) are related to solutions $u_1 \neq u_2$ of

$$0 = u_1 + u_2 + \hat{C}_r(\mathbf{u}, \mu_1). \tag{3.30}$$

It remains to show that the solutions of (3.30) form a curve (in (u_1, u_2)-space) intersecting the straight line $\{u_1 = u_2\}$ transversely.

To this end we first note that $C_i(\mathbf{u}, \mu_1)$, $i = 1, 2$, and their $\mathbf{u}$-derivatives are of order $O(\mu_1^{\delta^s})$. Having that in mind we see that

$$0 = 2u + \hat{C}_r(u, u, \mu_1)$$

has a solution $u = \hat{u}(\mu_1)$, where $\hat{u}(\mu_1) = O(\mu_1^{\delta^s})$, and hence (3.30) has a solution $(u_1, u_2) = (\hat{u}, \hat{u})$. Now we can apply the implicit function theorem to show that near $(\hat{u}, \hat{u})$ equation (3.30) can be solved for $u_2 = u_2^*(u_1, \mu_1)$. It turns out that $D_{u_1} u_2^*(\hat{u}, \mu_1) < 0$. This finally shows that the solution curve of (3.30) indeed intersects the straight line $\{u_1 = u_2\}$ transversely.

According to (3.26) and (3.27) we find that the function $\kappa_{pd}(\mu_1)$ (stated in Theorem 1.4) is defined by

$$\kappa_{pd}(\mu_1) := \hat{u}^2(\mu_1) + \hat{C}_{2,1}(\hat{u}(\mu_1), \hat{u}(\mu_1), \mu_1).$$

Similar to Lemma 3.14 we get

Lemma 3.16. *Let $\mu_1 > 0$, let $\mu_2 \geq \kappa_{pd}(\mu_1)$, and let $\mathbf{u}$ and $\boldsymbol{\omega}$, in accordance with* (3.24) *be values for which $X(\mathbf{u}, \boldsymbol{\omega}, \mu)$ is a 2-periodic orbit according to Theorem 1.3. Then $RX = X$, i.e. X is symmetric.*

Proof. The statement is, as the corresponding one of Lemma 3.14, an immediate consequence of the uniqueness: let μ be fixed and let (u_1, u_2), $u_1 \neq u_2$ such that $X(\mathbf{u}, \boldsymbol{\omega}, \mu)$ is the corresponding two periodic orbit. Note that, in accordance with the above explanations $\boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{u}, \mu)$, i.e. $\boldsymbol{\omega}$ is uniquely determined by $\mathbf{u}$ and μ. Furthermore, according to the uniqueness statements in Lemma 2.7(iv) and Theorem 2.9 we find that RX belongs to the same $\mathbf{u}$ - and hence to the same $\boldsymbol{\omega}$. Hence we have $RX = X$. ∎

Finally we comment on the size comparison of $\kappa_{pd}(\mu_1)$ and $\kappa_{sc}(\mu_1)$. According to (3.28) we may write

$$\hat{C}_{2,1}(u_1, u_2, \mu_1) = a_0 + a_1 u_1 + a_2 u_2 + O\big((u_1, u_2)^2\big)$$

$$\hat{C}_{2,2}(u_1, u_2, \mu_1) = a_0 + a_2 u_1 + a_1 u_2 + O\big((u_1, u_2)^2\big).$$

Note that $a_i = a_i(\mu_1)$, $a_i(\mu_1) = O(\mu_1^{\delta^s})$. With that

$$\hat{C}_r(u, u, \mu_1) = a_1 - a_2 + O(u^2),$$

and (3.30) can written as

$$u = \frac{a_2 - a_1}{2} + O(u^2).$$

Hence

$$\hat{u}(\mu_1) = \frac{a_2 - a_1}{2} + O(\mu_1^{2\delta^s}), \quad a_i(\mu_1) = O(\mu_1^{\delta^s}).$$

Next we compute the minimum $u^*(\mu_1)$ of the right-hand side of (3.23) in terms of a_i. By construction we have

$$\hat{C}_1(u, \mu_1) = \hat{C}_{2,1}(u, u, \mu_1) = (a_1 + a_2)u + O(u^2).$$

With that we find

$$u^*(\mu_1) = -\frac{a_2 + a_1}{2} + O(\mu_1^{2\delta^s}), \quad a_i(\mu_1) = O(\mu_1^{\delta^s}).$$

Therefore

$$a_2 \neq 0 \quad \Rightarrow \quad \hat{u}(\mu_1) \neq u^*(\mu_1). \tag{3.31}$$

Now, in accordance with the definitions of κ_{pd} and κ_{sc} we find that (3.31) implies

$$\kappa_{pd}(\mu_1) \neq \kappa_{sc}(\mu_1).$$

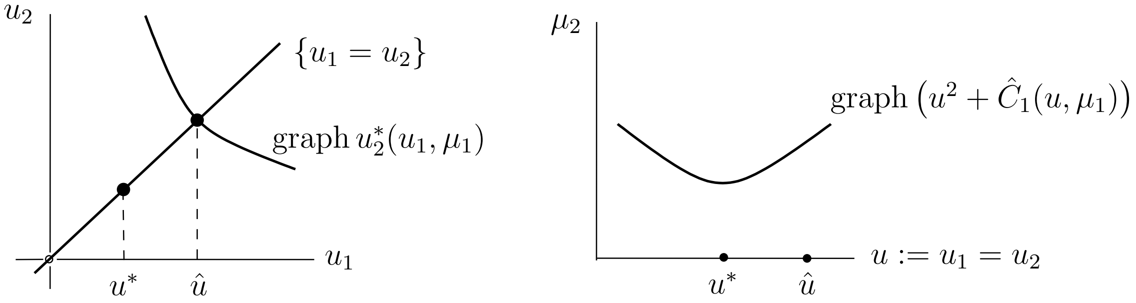

Figure 8: Let μ_1 be fixed. Left: solution branches related to two-periodic orbits - the solutions of (3.27). The straight line $\{u_1 = u_2\}$ is related to one-periodic orbits which are passed through twice. The graph of u_2^* is related to actual two-periodic orbits. Right: the graph of the function $u^2 + \hat{C}_1(u, \mu_1)$ explains the saddle center bifurcation of one-periodic orbits.

Figure 8 explains that $a_2 \neq 0$ more precisely implies

$$\kappa_{pd}(\mu_1) > \kappa_{sc}(\mu_1).$$

3.2.4 3-periodic orbits – the proof of Theorem 1.5

In this section we study symmetric 3-periodic orbits. Similar to the considerations in the previous subsections 3-periodic orbits are characterised by sequences $\boldsymbol{\omega} = (\omega_{1,i}, \omega_{2,i})_{i \in \mathbb{Z}}$ and $\mathbf{u} = (u_i)_{i \in \mathbb{Z}}$ with

$$(\omega_{1,i+3}, \omega_{2,i+3}) = (\omega_{1,i}, \omega_{2,i}), \qquad u_{i+3} = u_i, \qquad i \in \mathbb{Z}.$$

Due to Lemma 2.16, symmetric 3-periodic orbits are characterised, up to permutations, by the following restrictions (see also Figure 9):

$$\begin{aligned} \omega_{1,1} &= \omega_{2,1} \\ \omega_{1,2} &= \omega_{2,3} \qquad\qquad u_2 = u_3. \\ \omega_{1,3} &= \omega_{2,2} \end{aligned} \tag{3.32}$$

Remark 3.17. Note that the second lower indices $1, 2, 3$ of ω or the indices of u, respectively, correspond to $i = 0, 1, 2$ in the notation of Lemma 2.16. This implies that i_0 in the present setting is equal to zero. □

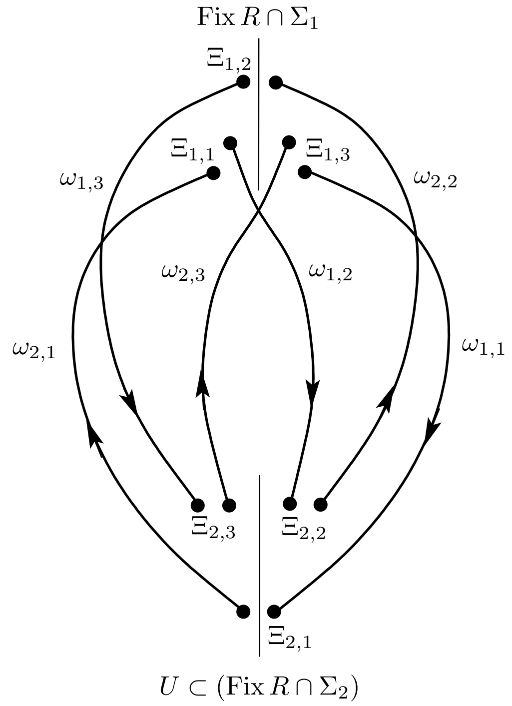

Figure 9: A symmetric 3-periodic Lin orbit.

The uniqueness statement in Theorem 2.9 implies (see again Figure 9 for a visualisation):

Lemma 3.18.

$$\begin{array}{rcl} \Xi_{2,2} = 0 & \Leftrightarrow & \Xi_{2,3} = 0 \\ \Xi_{1,1} = 0 & \Leftrightarrow & \Xi_{1,3} = 0 \\ \langle \Xi_{1,2}, \zeta_1^1 \rangle = 0 & \Leftrightarrow & \langle \Xi_{1,2}, \zeta_1^2 \rangle = 0. \end{array} \tag{3.33}$$

Proof. The statement follows mainly by means of Corollary 2.17. This immediately yields the first two equivalence statements in (3.33), recall Remark 3.17 in this respect. Also it gives

$$R\Xi_{1,2} = -\Xi_{1,2}. \tag{3.34}$$

Furthermore, write $\Xi_{1,2} = \langle \Xi_{1,2}, \zeta_1^1 \rangle \zeta_1^1 + \langle \Xi_{1,2}, \zeta_1^2 \rangle \zeta_1^2$. Exploiting $R\zeta_1^1 = \zeta_1^2$ we see that (3.34) implies the third equivalence in (3.33). ∎

According to (3.32) and (3.33) the set of equations determining symmetric 3-periodic orbit reduces to

$$\Xi_{1,1} = 0, \quad \langle \Xi_{1,2}, \zeta_1^1 \rangle = 0, \quad \Xi_{2,1} = 0, \quad \Xi_{2,2} = 0.$$

Using the notation introduced in (3.1) this set of equations can be written:

$$\begin{aligned}
0 &= \quad \mu_1 + a(u_2, \mu) r_{1,2} + \hat{\mathcal{R}}^1_{1,1}(\mathbf{u}, \mathbf{r}, \mu) \\
0 &= -\mu_1 - a(u_1, \mu) r_{1,1} + \hat{\mathcal{R}}^2_{1,1}(\mathbf{u}, \mathbf{r}, \mu) \\
0 &= \quad \mu_1 + a(u_2, \mu) r_{1,3} + \hat{\mathcal{R}}^1_{1,2}(\mathbf{u}, \mathbf{r}, \mu) \\
0 &= \mu_2 - u_1^2 + 2c(u_1, \mu) r_{1,1}^{\delta^s} + \hat{\mathcal{R}}_{2,1}(\mathbf{u}, \mathbf{r}, \mu) \\
0 &= \mu_2 - u_2^2 + c(u_2, \mu) r_{1,2}^{\delta^s} + c(u_2, \mu) r_{1,3}^{\delta^s} + \hat{\mathcal{R}}_{2,2}(\mathbf{u}, \mathbf{r}, \mu).
\end{aligned} \tag{3.35}$$

The first three equations in (3.35) can be solved for

$$\mathbf{r} := (r_{1,1}, r_{1,2}, r_{1,3})(u_1, u_2, \mu).$$

For the next few steps we proceed as in Section 3.2.3. The last two equations in (3.35) can be solved for μ_2 in each case, which yields two functions μ_2^k, $k = 1, 2$

$$\mu_2^1 = u_1^2 + \tilde{C}_{3,1}(u_1, u_2, \mu_1), \qquad \mu_2^2 = u_2^2 + \tilde{C}_{3,2}(u_1, u_2, \mu_1). \tag{3.36}$$

Thus, the resulting determination equation for symmetric 3-periodic orbits reads:

$$\mu_2^1 = \mu_2^2 \quad \Leftrightarrow \quad 0 = u_1^2 - u_2^2 + \tilde{C}_{3,1}(u_1, u_2, \mu_1) - \tilde{C}_{3,2}(u_1, u_2, \mu_1) =: \tilde{C}(u_1, u_2, \mu_1). \tag{3.37}$$

Lemma 3.19. $\tilde{C}_{3,1}(u, u, \mu_1) = \tilde{C}_{3,2}(u, u, \mu_1)$

Proof. This statement can be seen as an equivalent of (3.29). Indeed, it can be proved in a very similar way. To this end, for a start we disregard the symmetry condition (3.32) and consider the full bifurcation equation for 3-periodic orbits. If we proceed as in Section 3.2.3 we get corresponding functions $\hat{C}_{3,i}(u_1, u_2, u_3, \mu_1)$, $i = 1, 2, 3$. For these function an equivalent to Lemma 3.15 holds true. From that we find, by taking the symmetry condition (3.32) into account, the statement of the lemma. Note in this respect that $\tilde{C}_{3,i}(u_1, u_2, \mu_1) = \hat{C}_{3,i}(u_1, u_2, u_2, \mu_1)$, $i = 1, 2$. ∎

Indeed, Lemma 3.19 is a key point in our argumentation. Namely with that lemma we find

$$\tilde{C}(\mathbf{u}, \mu_1) = (u_1 - u_2)\big(u_1 + u_2 + \tilde{C}_r(\mathbf{u}, \mu_1)\big).$$

With that we can proceed as in Section 3.2.3. Eventually we get a function $\tilde{u}(\mu_1)$ which solves $0 = 2u + \tilde{C}_r(u, u, \mu_1)$. Finally, near $(u_1, u_2) = (\tilde{u}, \tilde{u})$, we can solve $0 = u_1 + u_2 + \tilde{C}_r(\mathbf{u}, \mu_1)$ for $u_2 = \tilde{u}_2^*(u_1, \mu_1)$. The function κ_{3sh} stated in Theorem 1.5 is defined by

$$\kappa_{3sh}(\mu_1) := \tilde{u}^2(\mu_1) + \tilde{C}_{3,1}(\tilde{u}(\mu_1), \tilde{u}(\mu_1), \mu_1).$$

3.2.5 4-periodic orbits – the proof of Theorem 1.6

We may expect that symmetric periodic orbits which are related to chaotic dynamics, or in other words which are related to $\mathcal{S}^2_{\mathcal{R}}$, can be continued into region (II) (cf. Figure 4). Consider the periodic orbits in $\mathcal{S}^2_{\mathcal{R}}$ with minimal period four. These correspond to the 4-periodic sequences $(\overline{+---})$, $(\overline{++--})$ and $(\overline{+++-})$. We denote the corresponding f-orbits by $\mathcal{O}_{1+}$, $\mathcal{O}_{2+}$ and $\mathcal{O}_{3+}$ respectively. In accordance with Remark 3.13 we find $\mathcal{O}_{1+}$ and $\mathcal{O}_{3+}$ have two intersections with $\operatorname{Fix} R \cap \Sigma_2$ in each case while $\mathcal{O}_{2+}$ has no intersection with $\operatorname{Fix} R \cap \Sigma_2$. Similarly we find that the 1-periodic f-orbits corresponding to $(\overline{+})$ or $(\overline{-})$ have exactly one intersection with $\operatorname{Fix} R \cap \Sigma_2$ in each case and the 2-periodic f-orbit corresponding to $(\overline{+-})$ has two intersections with $\operatorname{Fix} R \cap \Sigma_2$. We refer to Figure 10 for a visualisation.

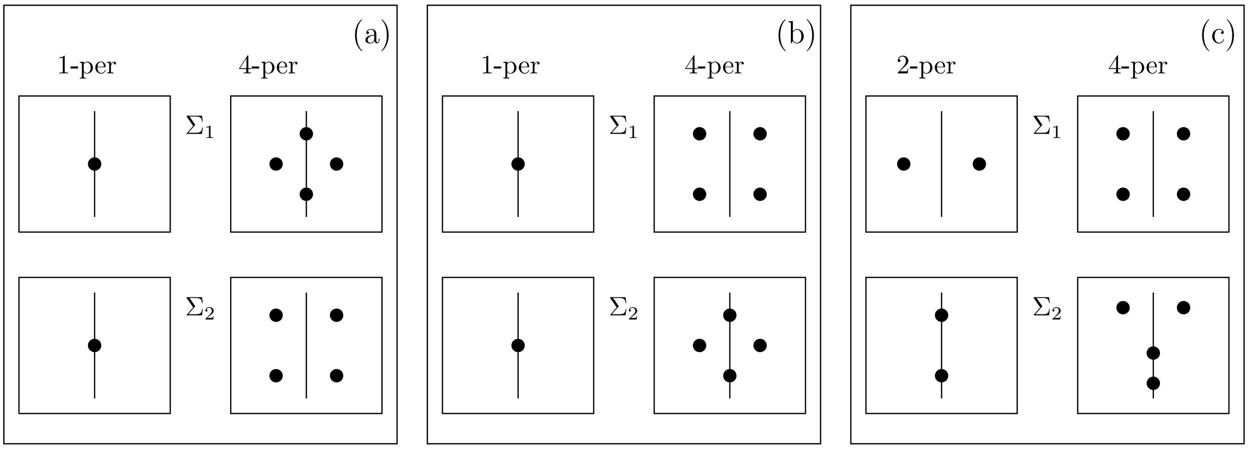

Figure 10: Possible subharmonic bifurcations from 1- or 2-periodic orbits to 4-periodic orbits. Displayed are the intersections of symmetric periodic orbits (black dots) with $\operatorname{Fix} R$ (vertical lines).

Now, consider 4-periodic f-orbits as depicted in Figure 10 which exist in region (I), cf. Figure 4. If we continue those orbits into region (II) then we find, similar to Section 3.2.4, the following: the continuation of $\mathcal{O}_{2+}$ (depicted in panel (a) of Figure 10) is related to a 4-periodic $\mathbf{u}$-sequence with $u_1 = u_2$ and $u_3 = u_4$. Similarly, the continuations of both $\mathcal{O}_{1+}$ and $\mathcal{O}_{3+}$ (depicted in the panels (b) and (c) of Figure 10), are related to a 4-periodic $\mathbf{u}$-sequence with $u_2 = u_4$. We emphasise that, without further examinations or assumptions, it is impossible to decide which of the orbits $\mathcal{O}_{1+}$ and $\mathcal{O}_{3+}$ bifurcates from a 1-periodic orbit and which one bifurcates from the branch of 2-periodic orbits.

First we consider the continuation of $\mathcal{O}_{2+}$ and show that this, as suggested by Figure 10 panel (a), bifurcates from a branch of 1-periodic orbits. This proves part of Theorem 1.6(i), namely that one branch of 4-periodic orbits bifurcates from the branch of 1-periodic orbits. However from general theory it is known that in the course of a subharmonic bifurcation two branches bifurcate, cf. the discussion below. In the second part of this section we will also discuss how it can be shown that the continuation of one of the orbits $\mathcal{O}_{1+}$ and $\mathcal{O}_{3+}$ will bifurcate from the branch of 1-periodic orbits.

By a sketch similar to the one in Figure 9 it becomes clear that the continuation of a symmetric periodic orbit $\mathcal{O}_{2+}$ is related to sequences $\boldsymbol{\omega}$ and $\mathbf{u}$ with

$$\omega_{1,1} = \omega_{2,2}, \quad \omega_{1,2} = \omega_{2,1}, \quad \omega_{1,3} = \omega_{2,4}, \quad \omega_{1,4} = \omega_{2,3},$$
$$u_1 = u_2, \quad u_3 = u_4.$$

Furthermore, as the counterpart of Lemma 3.18 we find

$$\begin{array}{rcl} \Xi_{1,2} = 0 & \Leftrightarrow & \Xi_{1,4} = 0, \\ \langle \Xi_{1,1}, \zeta_1^1 \rangle = 0 & \Leftrightarrow & \langle \Xi_{1,1}, \zeta_1^2 \rangle = 0, \\ \langle \Xi_{1,3}, \zeta_1^1 \rangle = 0 & \Leftrightarrow & \langle \Xi_{1,3}, \zeta_1^2 \rangle = 0, \end{array} \qquad \begin{array}{rcl} \Xi_{2,1} = 0 & \Leftrightarrow & \Xi_{2,2} = 0, \\ \Xi_{2,3} = 0 & \Leftrightarrow & \Xi_{2,4} = 0. \end{array}$$

With that we proceed as in Section 3.2.4. In doing so we arrive at the counterpart of (3.36)

$$\mu_2^1 = u_1^2 + \tilde{C}_{4,1}(u_1, u_3, \mu_1), \qquad \mu_2^3 = u_3^2 + \tilde{C}_{4,3}(u_1, u_3, \mu_1).$$

Thus, the resulting determination equation for symmetric 3-periodic orbits reads:

$$\mu_2^1 = \mu_2^3 \quad \Leftrightarrow \quad 0 = u_1^2 - u_3^2 + \tilde{C}_{4,1}(u_1, u_3, \mu_1) - \tilde{C}_{4,3}(u_1, u_3, \mu_1) =: \tilde{C}(u_1, u_3, \mu_1).$$

Parallel to Lemma 3.19 we find

$$\tilde{C}_{4,1}(u, u, \mu_1) = \tilde{C}_{4,3}(u, u, \mu_1).$$

This enables us to proceed along the lines of the remaining part of Section 3.2.4 and we finally obtain

$$\kappa_{4sh}(\mu_1) := \tilde{u}^2(\mu_1) + \tilde{C}_{4,1}(\tilde{u}(\mu_1), \tilde{u}(\mu_1), \mu_1),$$

where $\tilde{u}(\cdot)$ is defined in the same way as in Section 3.2.4. However we note that the function $\tilde{u}(\cdot)$ used here does not coincide with one used in Section 3.2.4.

Next we consider continuations of the orbits $\mathcal{O}_{1+}$ and $\mathcal{O}_{3+}$. For those orbits we find that they are related to sequences $\boldsymbol{\omega}$ and $\mathbf{u}$ with

$$\omega_{1,1} = \omega_{2,1}, \quad \omega_{1,2} = \omega_{2,4}, \quad \omega_{1,3} = \omega_{2,3}, \quad \omega_{1,4} = \omega_{2,2}, \quad \text{and} \quad u_2 = u_4.$$

For the jumps $\Xi_{i,j}$ we find

$$\begin{array}{rcl} \Xi_{1,1} = 0 & \Leftrightarrow & \Xi_{1,4} = 0, \\ \Xi_{1,2} = 0 & \Leftrightarrow & \Xi_{1,3} = 0, \end{array} \qquad \Xi_{2,2} = 0 \Leftrightarrow \Xi_{2,4} = 0.$$

We again proceed as in Section 3.2.4. The counterpart of (3.36) reads

$$\mu_2^1 = u_1^2 + \tilde{C}_{4,1}(u_1, u_2, u_3, \mu_1), \quad \mu_2^2 = u_2^2 + \tilde{C}_{4,2}(u_1, u_2, u_3, \mu_1), \quad \mu_2^3 = u_3^2 + \tilde{C}_{4,3}(u_1, u_2, u_3, \mu_1),$$

whereas the counterpart to (3.37) is given by

$$\begin{array}{lcl} \mu_2^1 = \mu_2^2 & \Leftrightarrow & 0 = u_1^2 - u_2^2 + \tilde{C}_{4,1}(u_1, u_2, u_3, \mu_1) - \tilde{C}_{4,2}(u_1, u_2, u_3, \mu_1) \\ \mu_2^2 = \mu_2^3 & \Leftrightarrow & 0 = u_3^2 - u_2^2 + \tilde{C}_{4,3}(u_1, u_2, u_3, \mu_1) - \tilde{C}_{4,2}(u_1, u_2, u_3, \mu_1). \end{array} \tag{3.38}$$

Define

$$\tilde{C}_{4,1} - \tilde{C}_{4,2} =: \tilde{C}_{412} \quad \text{and} \quad \tilde{C}_{4,3} - \tilde{C}_{4,2} =: \tilde{C}_{432} \tag{3.39}$$

Lemma 3.20. $\tilde{C}_{412}(u_1, u_2, u_3) = \tilde{C}_{432}(u_3, u_2, u_1)$.

Proof. Using arguments similar to that given in the proof of Lemma 3.19 we find:

$$\begin{aligned}\tilde{C}_{412}(u_1, u_2, u_3) &= \tilde{C}_{4,1}(u_1, u_2, u_3) - \tilde{C}_{4,2}(u_1, u_2, u_3) = \hat{C}_{4,1}(u_1, u_2, u_3, u_2) - \hat{C}_{4,2}(u_1, u_2, u_3, u_2)\\ &= \hat{C}_{4,3}(u_3, u_2, u_1, u_2) - \hat{C}_{4,2}(u_3, u_2, u_1, u_2)\\ &= \tilde{C}_{432}(u_3, u_2, u_1).\end{aligned}$$

∎

Using the notation introduced in (3.39), system (3.38) can be written as

$$F(u_1, u_2, u_3, \mu_1) = 0, \quad F(u_1, u_2, u_3, \mu_1) := \begin{pmatrix} u_1^2 - u_2^2 + \tilde{C}_{412}(u_1, u_2, u_3, \mu_1) \\ u_3^2 - u_2^2 + \tilde{C}_{432}(u_1, u_2, u_3, \mu_1) \end{pmatrix}. \tag{3.40}$$

Since 1-periodic solution also are covered by the equation $F(u_1, u_2, u_3, \mu_1) = 0$ we find

$$F(u, u, u, \mu_1) = 0. \tag{3.41}$$

Hence, by means of the Mean Value Theorem we find that F can be written as

$$F(u_1, u_2, u_3, \mu_1) = \begin{pmatrix} F_1^1(u_1, u_2, u_3, \mu_1) & F_3^1(u_1, u_2, u_3, \mu_1) \\ F_1^2(u_1, u_2, u_3, \mu_1) & F_3^2(u_1, u_2, u_3, \mu_1) \end{pmatrix} \begin{pmatrix} u_1 - u_2 \\ u_3 - u_2 \end{pmatrix}. \tag{3.42}$$

Furthermore, the functions F_i^j are related to the partial derivative of F with respect to (u_1, u_3). More precisely

$$\begin{aligned}\begin{pmatrix} F_1^1(u, u, u, \mu_1) & F_3^1(u, u, u, \mu_1) \\ F_1^2(u, u, u, \mu_1) & F_3^2(u, u, u, \mu_1) \end{pmatrix} &= \begin{pmatrix} 2u + D_1\tilde{C}_{412}(u, u, u, \mu_1) & D_3\tilde{C}_{412}(u, u, u, \mu_1) \\ D_1\tilde{C}_{432}(u, u, u, \mu_1) & 2u + D_3\tilde{C}_{432}(u, u, u, \mu_1) \end{pmatrix}\\ &= \begin{pmatrix} 2u + D_1\tilde{C}_{412}(u, u, u, \mu_1) & D_3\tilde{C}_{412}(u, u, u, \mu_1) \\ D_3\tilde{C}_{412}(u, u, u, \mu_1) & 2u + D_1\tilde{C}_{412}(u, u, u, \mu_1) \end{pmatrix},\end{aligned}$$

where the last equality follows from Lemma 3.20. Applying the implicit function theorem we obtain the following:

Lemma 3.21. *The equation* $2u + D_1\tilde{C}_{412}(u, u, u, \mu_1) - D_3\tilde{C}_{412}(u, u, u, \mu_1) = 0$ *has a unique solution* $\hat{u}$. ∎

Now we reconsider the equation $F(u_1, u_2, u_3, \mu_1) = 0$. To find solutions which are related to proper 4-periodic solutions we consider the system

$$\begin{aligned}\mathcal{F}^1(u_1, u_2, u_3, \mu_1) &:= \det \begin{pmatrix} F_1^1(u_1, u_2, u_3, \mu_1) & F_3^1(u_1, u_2, u_3, \mu_1) \\ F_1^2(u_1, u_2, u_3, \mu_1) & F_3^2(u_1, u_2, u_3, \mu_1) \end{pmatrix} = 0\\ \mathcal{F}^2(u_1, u_2, u_3, \mu_1) &:= \langle (F_1^1(u_1, u_2, u_3, \mu_1), F_3^1(u_1, u_2, u_3, \mu_1)), (u_1 - u_2, u_3 - u_2) \rangle = 0.\end{aligned} \tag{3.43}$$

We show that near $(u_1, u_2, u_3) = (\hat{u}, \hat{u}, \hat{u})$ the system (3.43) can be solved for $(u_1, u_2)(u_3)$, and since $\hat{u}$ is an isolated zero of $\mathcal{F}^1(\hat{u}, \hat{u}, \hat{u})$ we find that $(u_1, u_2)(u_3) \neq (u_3, u_3)$.

Due to Lemma 3.21 it is clear that $\mathcal{F}^1(\hat{u}, \hat{u}, \hat{u}) = 0$, and by its definition it is immediately clear that $\mathcal{F}^2(\hat{u}, \hat{u}, \hat{u}) = 0$. It can easily be verified that $\left.\frac{\partial(\mathcal{F}^1,\mathcal{F}^2)}{\partial(u_1,u_2)}\right|_{(\hat{u},\hat{u},\hat{u})}$ is non-singular. With the implicit function theorem this gives the solution $(u_1, u_2)(u_3)$.

In a similar way we can handle the period doubling from 2- to 4-periodic orbits. According to Section 3.2.3 all 2-periodic orbits are symmetric. These orbits intersect $\mathrm{Fix}\, R \cap \Sigma_2$ twice (see also Figure 8 panel (c)). Let $u_{2p,1}$ and $u_{2p,2}$ be the u-components of the intersection points. We have $u_{2p,1} \neq u_{2p,2}$. Finally we know that, for fixed μ_1, there exists a function u_2^* such that the graph $(u_{2p,1}, u_2^*(u_{2p,1}))$ describes the branch of 2-periodic orbits $(u_{2p,1}(\mu_2), u_{2p,2}(\mu_2))$ in the (u_1, u_2)-plane, see also Figure 8.

Equation (3.40) also covers the branch of 2-periodic solutions. So we find as the counterpart of (3.41)

$$F(u_2^*(u_2), u_2, u_2^*(u_2)) \equiv 0.$$

In the same way as explained above this leads to the representation of F

$$F(u_1, u_2, u_3, \mu_1) = \begin{pmatrix} F_1^1(u_1, u_2, u_3, \mu_1) & F_3^1(u_1, u_2, u_3, \mu_1) \\ F_1^2(u_1, u_2, u_3, \mu_1) & F_3^2(u_1, u_2, u_3, \mu_1) \end{pmatrix} \begin{pmatrix} u_1 - u_2^*(u_2) \\ u_3 - u_2^*(u_2) \end{pmatrix}.$$

Note that F_i^j are not the same as the ones in (3.42).

Further, similar to Lemma 3.21 we can show that the equation

$$F_1^1(u_2^*(u_2), u_2, u_2^*(u_2)) + F_1^3(u_2^*(u_2), u_2, u_2^*(u_2)) = 0$$

has a unique solution $\hat{u}$ (which is also different from the one stated in Lemma 3.21). By symmetry arguments, see Lemma 3.20, we find that $F(u_2^*(u_2), u_2, u_2^*(u_2))$ has rank one. This allows to proceed in exactly the same way as above from equation (3.43) on.

4 Discussion

Based on our considerations in the previous section or Theorems 1.3-1.6, respectively, it seems a likely supposition that the transition from no recurrent dynamics to shift dynamics is mainly governed by saddle-centre bifurcations, period doubling bifurcations and subharmonic bifurcations. In Figure 11 we present a part of a possible bifurcation diagram to explain the bifurcations of symmetric periodic orbits in the region (II) (of Figure 4) while taking our results into account. We emphasise that our focus here is just to describe a plausible mechanism by which the shift dynamics dissolves in the region (II), and we make no claim that our description of the dynamics in this region is complete. We expect that a general description of the dynamics in this region is considerably more complicated.

Note that in Theorems 1.3-1.6 we only made statements concerning bifurcations of symmetric periodic orbits with period less than five. Since our proofs are based on Lin's method we are not able to make any statements concerning the stability of the involved orbits. In particular, we

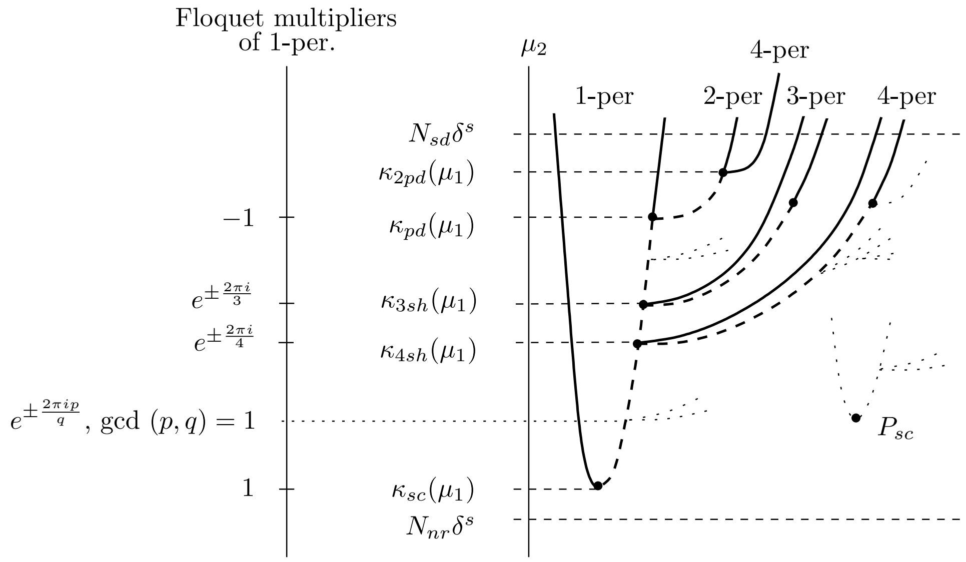

Figure 11: Possible bifurcation diagram for symmetric periodic orbits.

cannot say anything about the Floquet multipliers (of those orbits) at the bifurcation points. However, the scenario depicted in Figure 11 is supported by studies of local bifurcations in generic (codimension one) families of vector fields. First note that the $2n$-dimensional Poincaré map of a symmetric periodic orbit will inherit the reversing symmetry R, and therefore the Floquet multipliers will come in pairs that multiply to one. Generic codimension one bifurcations can therefore be reduced to a 2-dimensional centre manifold, where the dynamics on the centre manifold again inherits the reversing symmetry.

Consider first the cases where a pair of Floquet multipliers are equal to either 1 or -1. Denote the Poincaré map of the symmetric periodic orbit on the centre manifold by $L(\cdot,\mu)$, and suppose that $L(0,0)=0$, and $DL(0,0)=A_s+A_n$ is the canonical splitting of the Frechet derivative of L at $(0,0)$ into its semisimple and nilpotent part. Then it can be shown (see e.g. [23, Theorem 1.1]) that there exists an R-equivariant coordinate transformation T_μ such that

$$T_\mu^{-1}\circ L\circ T_\mu \qquad \text{and} \qquad A_s\circ\chi_\tau(\mu)$$

have the same Taylor expansion up to an arbitrarily high order at $(0,0)$, where $\chi_\tau(\mu)$ is the time-one map of the flow of a vector field $\chi(\mu)$ that commutes with A_s and has R as a reversing symmetry.

When the Floquet multipliers are equal to 1, then $A_s=I$ and generically the vector field is determined by its 2-jet [24]:

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= \mu+ax^2, \quad a\neq 0.\end{aligned}$$

The above normal form results in a saddle-centre bifurcation (see e.g. [24, Figure 1]), and the map L is equal (up to the R-equivariant coordinate transformation T_μ) to the time-one map $\chi_\tau(\mu)$ up to arbitrarily high order. Therefore, the periodic orbit generically undergoes a saddle-centre bifurcation. We remark that addition of the higher order terms to the map L will generically break the invariant tori surrounding the elliptic periodic orbit resulting in chaotic dynamics.

When the Floquet multipliers are equal to -1, then $A_s = -I$, and in this case the vector field has the generic unfolding (see again [24]):

$$\begin{aligned}\dot{x} &= \mu y + a y^3, \quad a \neq 0,\\ \dot{y} &= x.\end{aligned}$$

The above normal form gives rise to a pitchfork bifurcation (see e.g. [24, Figure 3(b,c)]). However this time the map L is equivalent (up to arbitrarily high order) to $A_s \circ \chi_\tau(\mu) = -\chi_\tau(\mu)$, and so the periodic orbit undergoes a period-doubling bifurcation. We remark that the bifurcating (period-doubled) orbit will be R-symmetric.

Furthermore, studies by Vanderbauwhede and Ciocci [29, 30, 5, 4] describe the generic situation as Floquet multipliers travel around the unit circle, as depicted in the elliptic regions in Figure 11. These results are given in the following theorem.

Theorem 4.1 ([4], Theorem 3)**.** *Let $\Phi_\lambda(\cdot)$ be a one-parameter family of reversible diffeomorphisms on $\mathbb{R}^{2n}$, and let 0 be a symmetric fixed point of Φ_0. Assume*

(i) $D\Phi_0(0)$ has a pair of simple eigenvalues $e^{\pm\frac{2\pi i p}{q}}$, where $p, q \in \mathbb{N}$, $\gcd(p,q) = 1$,

(ii) $D\Phi_0(0)$ has no other eigenvalues which are root of unity,

(iii) the continuation of the eigenvalue $e^{\frac{2\pi i p}{q}}$ crosses the root of unity transversely (on the unit circle S^1) by moving λ off zero.

Assume additionally that a further non-degeneracy condition (ND) is satisfied. Then, if $q \geq 3$, exactly two branches of symmetric q-periodic orbits bifurcate from the fixed point $x = 0$ at $\lambda = 0$. Also, for $q \geq 5$ the orbits in one branch are stable, while those in the other branch are unstable.

The non-degeneracy condition (ND) mentioned in the theorem refers to a certain coefficient in the bifurcation equation. For more details, as well as for the proof we refer to [4]. But we note that due to the symmetry of the fixed point, there is a "symmetry" for the set of eigenvalues of $D\Phi_0(0)$. This finally implies that under the assumption of the theorem $\dim \mathrm{Fix}\, R = \dim \mathrm{Fix}\,(-R)$ (where R is the involved involution). Finally we note that, although the theorem only refers to subharmonic bifurcations ($q \geq 3$), by the same methods (as used in the proof of the theorem) the continuation/bifurcation of one- and two-period orbits can also be studied. This has been done for instance in [5] in a somewhat different context.

To explain the diagram in Figure 11 we perform the following thought experiment. Consider the two branches of one-periodic orbits which merges at $\mu_2 = \kappa_{sc}(\mu_1)$. We assume that this happens in the course of a saddle-center bifurcation (note that we only proved the coalescence of the periodic orbits). Let us also assume that along one branch the periodic orbits have a pair of simple Floquet multipliers on the unit circle S^1. Increasing μ_2 causes a motion of these multipliers along S^1 until they meet again at -1 and the period doubling occurs. Along its way on S^1 these multipliers have to pass roots of unity. If this passage happens in accordance with assumptions of Theorem 4.1 then subharmonic bifurcations as suggested in Figure 11 will occur. If $q \geq 3$ then (again in accordance with Theorem 4.1) the periodic orbits in one branch are stable. In the reversible context that means that they have Floquet multipliers on S^1. They will also move on S^1 as μ_1 changes and can cross roots of unity. If at those points the assumptions of Theorem 4.1 are met again corresponding subharmonic bifurcations from this q-periodic orbit will occur. This leads to a cascade of subharmonic bifurcations. We note that a simultaneous cascade of period doubling bifurcations takes place as has been described in [27, Section 5.2] for two-dimensional reversible maps. We also refer to [22, Section 3.2].

However, we cannot explain the disappearance (for decreasing μ_2) of all the periodic orbits which are involved in the shift dynamics only by means of such cascades of subharmonic and period doubling bifurcations. Namely, simple enumerative combinatorics reveals that there are six 5-periodic orbits. All of these are symmetric. But only four of them can be created (or disappear) in the course of subharmonic bifurcations from corresponding branch of one-periodic orbits. So we conjecture that the remaining two branches undergo a saddle-center bifurcation - cf. point P_{sc} in Figure 11.

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