A model equation for the creation of shift dynamics in reversible systems

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Abstract

We consider a specific family of reversible discrete dynamical systems $(\mathbb{R}^{\mathbb{Z}}_{\lambda}, \sigma)$, where σ is the shift operator on $\mathbb{R}^{\mathbb{Z}}$. We show that for appropriate parameter values λ there is a σ-invariant set on which the system is topologically conjugated to the full shift on two symbols. Furthermore we discuss by which scenarios the symmetric periodic orbits belonging to this subset will be created.

1 Introduction

In this paper we address the transition from tame dynamics to shift dynamics in discrete reversible systems. This question is closely linked with bifurcations of periodic orbits. In this respect the keyword *period-doubling route to chaos* should be mentioned - although this phrase is mainly reserved for general systems, i.e. systems without any particular imposed structure, cf. [6, 7]. However, the principal question which arises there is the same as in the reversible context: Within the regime of shift dynamics there are infinitely many periodic orbits. The question is in which way these periodic orbits will be created. In general systems this happens typically by cascades of period-doubling bifurcations and saddle-node bifurcations. But in the context of reversible systems it is a likely supposition that this transition is mainly governed by saddle-centre bifurcations, period-doubling bifurcations and subharmonic bifurcations. The main reason for this difference derives from the fact that in the reversible context (unlike the general case) typically eigenvalues do not cross the unit circle line S^1 but will travel along the S^1. This typically leads to subharmonic bifurcations, cf. [2, 3].

In [9] we realised such a scenario in the unfolding of non-elementary T-points. There we presented a bifurcation diagram similar to the one depicted in Figure 1. It describes the first few steps in the transition to chaotic behaviour. For $\lambda < \underline{\lambda}$ there are no periodic orbits. At $\lambda = \lambda_{sc}$ a saddle-centre bifurcation of 1-periodic orbits takes place – two 1-periodic orbits will be created, a hyperbolic one and an elliptic one (indicated by the dashed line). At the bifurcation

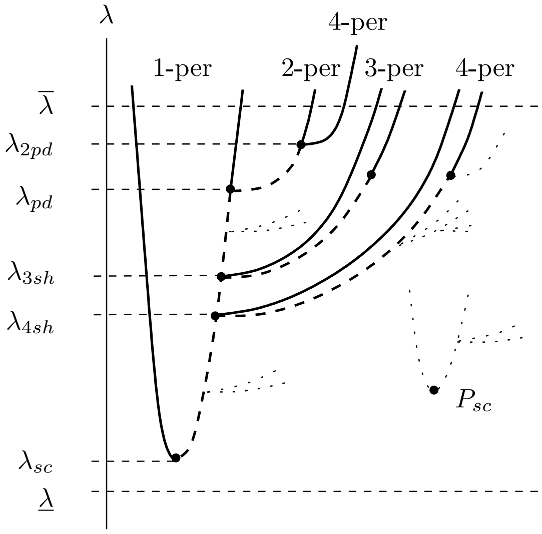

Figure 1: A tentative bifurcation diagram showing the first steps towards shift dynamics.

bifurcation point the 1-periodic orbit has two eigenvalues (equal to) 1. The eigenvalues of the elliptic periodic orbit move along the S^1 and crosses (transversely) nth roots of unity. As this happens corresponding (pairs of) n-period orbits bifurcate from the branch of 1-periodic orbits (subharmonic bifurcations at $\lambda = \lambda_{\mathrm{n}sh}$). If these eigenvalues meet again at -1 a period-doubling bifurcation takes place at $\lambda = \lambda_{pd}$.

The periodic orbits which are created in the course of subharmonic bifurcations come in pairs. One branch in each case consists of elliptic periodic orbits. Also these orbits undergo a bifurcation scenario as described for the primary elliptic branch of 1-periodic orbits. In this way we encounter a whole cascade of subharmonic bifurcations and period-doubling bifurcations leading towards shift dynamics.

However, merely by counting periodic orbits existing in the regime of shift dynamics we find that not all these orbits can be created in the course of such a cascade emanating from the branch of 1-periodic orbits. So, as indicated in Figure 1, we expect also saddle-centre bifurcations of n-periodic orbits for $n \geq 5$, which in turn give rise to further cascades of subharmonic bifurcations and period-doubling bifurcations.

So it is a natural question whether there is a 1-parameter family of reversible systems in which this scenario can be observed?

In the present paper we discuss a model system which is a promising candidate for such a family. Our considerations are motivated by our analysis in [9]. The analysis carried out there leads to a family of systems $(\mathbb{R}^{\mathbb{Z}}_{\lambda}, \sigma)$ where $\mathbb{R}^{\mathbb{Z}}_{\lambda}$ is a certain subset of the corresponding product space $\mathbb{R}^{\mathbb{Z}}$ and σ is the shift operator. Further, the systems under consideration are reversible with respect to an involution R. Here however, we consider a simplified model system which we again denote by $(\mathbb{R}^{\mathbb{Z}}_{\lambda}, \sigma)$. We refer to Section 2 for the precise setup. There we also show that there is a family of reversible systems $(\mathbb{R}^2, f_{\lambda})$ which is topologically conjugated to $(\mathbb{R}^{\mathbb{Z}}_{\lambda}, \sigma)$. We want to note that the latter is only true for the model system considered in the present paper but not for the by far more complicated system we obtained in [9]. However, here we consider alternately both systems $(\mathbb{R}^{\mathbb{Z}}_\lambda, \sigma)$ and $(\mathbb{R}^2, f_\lambda)$ depending on the purpose. The system $(\mathbb{R}^{\mathbb{Z}}_\lambda, \sigma)$ is very suitable for the detection of particular (sets of) orbits, while stability statements can be derived by studying the family $(\mathbb{R}^2, f_\lambda)$.

In Section 3 we show, cf. Theorem 3.3, that for sufficiently large λ there is a σ-invariant set Σ^2_λ on which σ is topologically conjugated to the full shift on two symbols. This implies that the restricted system $(\Sigma^2_\lambda, \sigma)$ is chaotic in the sense of Devaney. Moreover, Σ^2_λ is symmetric, i.e. it is R-invariant. In other words, for each σ-orbit $\mathcal{O}$ also its R-image $R(\mathcal{O})$ belongs to Σ^2_λ. Note that $R(\mathcal{O})$ is also a σ-orbit. Finally we show that even σ restricted the subset $\Sigma^2_{\lambda,R}$ consisting of all symmetric orbits of Σ^2_λ is chaotic in the sense of Devaney, cf. Theorem 3.6.

The rest of the paper expands on the question by which scenario the symmetric periodic orbits within $\Sigma^2_{\lambda,R}$ are created. To this end we first derive determination equations for those orbits, cf. Section 4. Based on these we determine the complete bifurcation diagram of (symmetric) periodic orbits up to period five. There we also determine, based on the considerations of $(\mathbb{R}^2, f_\lambda)$, the stability properties of those orbits up to period four. The results are compiled in the bifurcation diagram depicted in Figure 8. In the final section we study more subharmonic bifurcations from the primary branch of 1-periodic orbits. Again these considerations are related to the family $(\mathbb{R}^{\mathbb{Z}}_\lambda, \sigma)$. Our main result in this respect is Theorem 6.10 which states that for each $k \in \mathbb{N}$ a branch of $2k$-periodic orbits bifurcates from the branch of 1-periodic orbits. We want to note that this statement can, by using the same type of analysis as carried out here, be extended to the effect that the precise number of bifurcating branches can be determined, cf. Section 6.3. The same type of analysis also applies to bifurcating $(2k-1)$-periodic orbits.

Our considerations fit into general framework of discrete reversible systems as defined in [11, Section 3]: Let Ω be some set, $F : \Omega \to \Omega$ be invertible and finally let $\varrho : \Omega \to \Omega$ be an involution, i.e. $\varrho^2 = id.$, with $\varrho \circ F = F^{-1} \circ \varrho$. Then (Ω, F) is called a reversible discrete dynamical system with reversing symmetry ϱ. Both applications and theoretical studies often enforce particular structure on Ω along with corresponding properties of F and ϱ, see e.g. [5, 11, 14]. Note that, cf. Section 2, in the present situation $\Omega = \mathbb{R}^{\mathbb{Z}}$ is a topological space and $F = \sigma$ and the reversing symmetry are homeomorphisms. The state space $\Omega = \mathbb{R}^2$ of the assigned system $(\mathbb{R}^2, f_\lambda)$ is even a normed space, $F = f_\lambda$ is a diffeomorphism and the involution is linear.

An orbit $\mathcal{O}$ of the reversible system (Ω, F) is called symmetric if $\varrho(\mathcal{O}) = \mathcal{O}$. The study of symmetric periodic orbits and their bifurcations plays a major role in the theory of reversible dynamical systems, cf. [11, Section 4.1, Section 4.3]. Of particular interest are period-doubling and subharmonic bifurcations, cf. again [11] and references therein. Since bifurcating periodic orbits may have similar properties as the primary ones the bifurcating orbits may undergo again subharmonic and period-doubling bifurcations. In this way whole cascades of those bifurcations may arise, cf. [13] or [15, 16]. Although in the first paper by Vanderbauwhede the emphasis is on reversible vector fields in $\mathbb{R}^{2m}$ with a $2m$-dimensional fixed space of the involution the effect is strongly related to the corresponding behaviour of mappings - discrete dynamical systems. This becomes clear by the considerations in [16] where corresponding Poincaré maps $\mathbb{R}^{2m-1}$ are studied. The theoretical "justification" for the local subharmonic bifurcations studied in Section 5 is given by [2, Theorem 3], cf. also [3]. This theorem gives necessary conditions for subharmonic bifurcations in one-parameter families of reversible diffeomorphisms in $\mathbb{R}^{2n}$. Our system $(\mathbb{R}^2, f_\lambda)$ does fit into this regime. Basically these conditions say that a subharmonic bifurcation occurs if the eigenvalues of the linearisation of f_λ cross a root of unity transversely. We address this issue explicitly within the single subsections of Section 5, although our analysis does not rely on this theorem.

The complex dynamics addressed in [14] is due to the breaking of KAM circles while the chaotic behaviour stated in our Theorems 3.3 and 3.6 has its origin in those cascades period-doubling and subharmonic bifurcations.

In [13, 10], cf. also references therein, the aspect of self-similarity of period-doubling cascades is addressed. The question arises whether those considerations can be extended to subharmonic cascades. We do not touch this question in this paper. But maybe the presented model system is a good starting point for numerical investigations in this direction.

2 Setup of the model system

Let $\mathbb{R}^\mathbb{Z}$ be the set of bi-infinite sequences with elements in $\mathbb{R}$ equipped with the product topology. We consider the shift map on a subset $\mathbb{R}^\mathbb{Z}_\lambda$ of $\mathbb{R}^\mathbb{Z}$. For each λ we define this subset by means of the following operator

$$\mathbf{T}(\cdot,\lambda) : \mathbb{R}^\mathbb{Z} \to \mathbb{R}^\mathbb{Z}, \quad \mathbf{u} := (u_i) \mapsto (T_i(\mathbf{u},\lambda)), \quad i \in \mathbb{Z}, \tag{2.1}$$

with

$$T_i(\mathbf{u},\lambda) = \lambda + u_{i-1} + u_{i+1} - u_i^2. \tag{2.2}$$

Finally we define

$$\mathbb{R}^\mathbb{Z}_\lambda := \{\mathbf{u} \in \mathbb{R}^\mathbb{Z} : \mathbf{T}(\mathbf{u},\lambda) = 0\}. \tag{2.3}$$

Accordingly we consider $\mathbb{R}^\mathbb{Z}_\lambda$ with induced topology.

Let for the moment $\mathbf{u} \in \mathbb{R}^\mathbb{Z}$ and consider projections $p_i(\mathbf{u}) := u_i$. Recall that the product topology is the smallest topology on $\mathbb{R}^\mathbb{Z}$ such that all projections p_i, $i \in \mathbb{Z}$, are continuous. The sets of the form $p_{i_0}^{-1}(U_{i_0}) \cap \ldots \cap p_{i_N}^{-1}(U_{i_N})$, where $N \in \mathbb{N}$ and U_{i_j}, $j \in \{0,\ldots,N\}$ are open in $\mathbb{R}$, form a basis for the space $\mathbb{R}^\mathbb{Z}$, cf. [4]. Hence, $p_{i_0}^{-1}(U_{i_0}) \cap \ldots \cap p_{i_N}^{-1}(U_{i_N}) \cap \mathbb{R}^\mathbb{Z}_\lambda$ form a basis for the space $\mathbb{R}^\mathbb{Z}_\lambda$.

Next we consider the shift map σ on $\mathbb{R}^\mathbb{Z}$

$$\sigma : \mathbb{R}^\mathbb{Z} \to \mathbb{R}^\mathbb{Z}, \quad \mathbf{u} := (u_i) \mapsto \mathbf{v} := (v_i), \quad v_i = u_{i+1}, \quad i \in \mathbb{Z} \tag{2.4}$$

Lemma 2.1. *$\sigma : \mathbb{R}^\mathbb{Z} \to \mathbb{R}^\mathbb{Z}$ is a homeomorphism.*

Proof. Let V be an open neighbourhood of $\mathbf{v} = \sigma\mathbf{u}$. Then there exist $i_0,\ldots,i_N$ and open intervals $I_{i_0},\ldots,I_{i_N}$ containing $v_{i_0},\ldots,v_{i_N}$ such that

$$\mathbf{v} \in p_{i_0}^{-1}(I_{i_0}) \cap \ldots \cap p_{i_N}^{-1}(I_{i_N}) \subset V.$$

Then, from $u_{j+1} = v_j \in I_{i_j}$ it follows that $\mathbf{u} \in p_{i_0+1}^{-1}(I_{i_0}) \cap \ldots \cap p_{i_N+1}^{-1}(I_{i_N})$ and

$$\sigma\left(p_{i_0+1}^{-1}(I_{i_0}) \cap \ldots \cap p_{i_N+1}^{-1}(I_{i_N})\right) = p_{i_0}^{-1}(I_{i_0}) \cap \ldots \cap p_{i_N}^{-1}(I_{i_N}).$$

This shows the continuity of σ. In the same way it can be shown that σ^{-1} is continuous. ∎

Lemma 2.2. $\sigma\mathbf{T}(\mathbf{u},\lambda) = \mathbf{T}(\sigma\mathbf{u},\lambda)$.

Proof. According to the definition of σ, cf. (2.4), we find:

$$(\sigma\mathbf{T}(\mathbf{u},\lambda))_i = (\mathbf{T}(\mathbf{u},\lambda))_{i+1} = T_{i+1}(\mathbf{u},\lambda) = \lambda + u_i + u_{i+2} - u_{i+1}^2,$$

and

$$(\mathbf{T}(\sigma\mathbf{u},\lambda))_i = T_i(\sigma\mathbf{u},\lambda) = T_{i+1}(\mathbf{u},\lambda) = \lambda + u_i + u_{i+2} - u_{i+1}^2.$$

Hence, $(\sigma\mathbf{T}(\mathbf{u},\lambda))_i = (\mathbf{T}(\sigma\mathbf{u},\lambda))_i$. ∎

Corollary 2.3. $\mathbb{R}^{\mathbb{Z}}_{\lambda}$ *is* σ*-invariant.*

Proof. Let $\mathbf{u} \in \mathbb{R}^{\mathbb{Z}}_{\lambda}$, i.e. $\mathbf{T}(\mathbf{u},\lambda) = 0$. Then, due to Lemma 2.2 $\mathbf{T}(\sigma\mathbf{u},\lambda) = 0$, what means that $\sigma\mathbf{u} \in \mathbb{R}^{\mathbb{Z}}_{\lambda}$. ∎

Corollary 2.4. $\sigma : \mathbb{R}^{\mathbb{Z}}_{\lambda} \to \mathbb{R}^{\mathbb{Z}}_{\lambda}$ *is a homeomorphism.*

Proof. The statement follows immediately from Lemma 2.1 and Corollary 2.3. ∎

As an immediate consequence of the Corollaries 2.3 and 2.4 we get that $(\mathbb{R}^{\mathbb{Z}}_{\lambda}, \sigma)$ can be considered as a dynamical system.

Next we show that $(\mathbb{R}^{\mathbb{Z}}_{\lambda}, \sigma)$ is reversible w.r.t the involution

$$R : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}, \quad \mathbf{u} := (u_i) \mapsto \mathbf{v} := (v_i), \quad v_i = u_{-i}, \quad i \in \mathbb{Z}. \tag{2.5}$$

Lemma 2.5. $\sigma R = R\,\sigma^{-1}$.

Proof. Again we apply the definition of σ, cf. (2.4), to find $(\sigma R\mathbf{u})_i = u_{-(i+1)} = (R\sigma^{-1}\mathbf{u})_i$. ∎

Lemma 2.6. $\mathbb{R}^{\mathbb{Z}}_{\lambda}$ *is* R*-invariant.*

Proof. Let $\mathbf{u} \in \mathbb{R}^{\mathbb{Z}}_{\lambda}$, which means that $T_j(\mathbf{u},\lambda) = 0$ for all $j \in \mathbb{Z}$. We show that $R\mathbf{u} \in \mathbb{R}^{\mathbb{Z}}_{\lambda}$ by considering $(\mathbf{T}(R\mathbf{u},\lambda))_i = T_i(R\mathbf{u},\lambda) = \lambda + u_{-(i-1)} + u_{-(i+1)} - u_{-i}^2 = T_{-i}(\mathbf{u},\lambda) = 0$. ∎

Hence, $(\mathbb{R}^{\mathbb{Z}}_{\lambda}, \sigma)$ is an R-reversible dynamical system. Throughout this paper we will mainly deal with that system.

However for some considerations concerning stability of periodic orbits, cf. Section 5, it has proved useful to consider also the following associated system, which emerges from $\mathbf{T}(\mathbf{u},\lambda) = 0$. The equation $T_i(\mathbf{u},\lambda) = 0$ can be written as

$$u_{i+1} = u_i^2 - u_{i-1} - \lambda =: T_i^+(u_i, u_{i-1}, \lambda). \tag{2.6}$$

This can be seen as a second order difference equation, which defines the following dynamical system in $\mathbb{R}^2$:

$$\left.\begin{array}{rl} u_{i+1} = & u_i^2 - v_i - \lambda \\ v_{i+1} = & u_i \end{array}\right\} =: f(u_i, v_i, \lambda) =: f_\lambda(u_i, v_i). \tag{2.7}$$

Note that the space $\mathbb{R}^{\mathbb{Z}}_\lambda$ consists of the u-components of the orbits of the system (2.7).

Lemma 2.7. *The map $\mathcal{H} : \mathbb{R}^{\mathbb{Z}}_\lambda \to \mathbb{R}^2$, $\mathbf{u} = (u_i) \mapsto (u_0, u_{-1})$ is a topological conjugacy from σ to f_λ.*

Proof. By the definition of $\mathbb{R}^{\mathbb{Z}}_\lambda$, cf. (2.1) - (2.3), it is clear that any two "starting elements" u_0 and u_{-1} define the entire sequence $\mathbf{u} = (u_i)_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}_\lambda$. More precisely the u_i with $i \in \mathbb{N}$ arise out of successively application of maps T_i^+, cf. (2.6), and similarly for $i \in \mathbb{Z}^-$ the u_i arise out of successively application of maps $T_i^-(u_{i+1}, u_i, \lambda) := u_i^2 - u_{i+1} - \lambda$ (solve $T_i = 0$ for u_{i-1}, cf. (2.2)). Therefore, the map $\mathcal{H}$ is bijective.

We now prove that both $\mathcal{H}$ and $\mathcal{H}^{-1}$ are continuous. First, let U be neighbourhood of $(u_0, u_{-1}) \in \mathbb{R}^2$ then U contains a rectangle $Q = I_0 \times I_{-1}$ consisting of the Cartesian product of open intervals. Then $\mathcal{H}^{-1}(Q) = p_0^{-1}(I_0) \cap p_{-1}^{-1}(I_{-1}) \cap \mathbb{R}^{\mathbb{Z}}_\lambda$ is an open set of the induced topology on $\mathbb{R}^{\mathbb{Z}}_\lambda$. This shows that $\mathcal{H}$ is continuous.

Next we show that also $\mathcal{H}^{-1}$ is continuous in each point $(\hat{u}, \hat{v})$. We may write $\mathcal{H}^{-1} : (\hat{u}, \hat{v}) \mapsto (s_i(\hat{u}, \hat{v}))_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}_\lambda$. The functions s_i are defined via successively application of the corresponding maps T_i^+ or T_i^- respectively, cf. in the first paragraph of this proof. Clearly $s_0(\hat{u}, \hat{v}) = \hat{u}$ and $s_{-1}(\hat{u}, \hat{v}) = \hat{v}$. Since all $T_i^\pm$ are continuous we find that all s_i are continuous mappings $\mathbb{R}^2 \to \mathbb{R}$. But note that the set $\{s_i(\cdot, \cdot)\}_{i\in\mathbb{Z}}$ is not equicontinuous with respect to i. Now, let U be an open neighbourhood of $(u_i)_{i\in\mathbb{Z}} := (s_i(\hat{u}, \hat{v}))_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}_\lambda$. Then there exist finitely many indices i, say $i_0, \dots, i_N$, $N \in \mathbb{N}$, and an $\epsilon > 0$ such that the open basic set

$$p_{i_0}^{-1}((u_{i_0} - \epsilon, u_{i_0} + \epsilon)) \cap \ldots \cap p_{i_N}^{-1}((u_{i_N} - \epsilon, u_{i_N} + \epsilon)) \cap \mathbb{R}^{\mathbb{Z}}_\lambda \subset U.$$

Due to the continuity of the s_i there is a $\delta > 0$ such that for the finitely many $i_0, \dots, i_N$

$$s_{i_j}((\hat{u} - \delta, \hat{u} + \delta) \times (\hat{v} - \delta, \hat{v} + \delta)) \subset (u_{i_j} - \epsilon, u_{i_j} + \epsilon), \quad j = 0, \dots, N.$$

Hence $\mathcal{H}^{-1}((\hat{u} - \delta, \hat{u} + \delta) \times (\hat{v} - \delta, \hat{v} + \delta)) \subset U$.

It remains to show that $\mathcal{H} \circ \sigma = f_\lambda \circ \mathcal{H}$. By using the definitions of the involved quantities it can easily be seen that for $\mathbf{u} = (u_i)$, $\mathcal{H}(\sigma(\mathbf{u})) = (u_1, u_0) = f_\lambda(\mathcal{H}(\mathbf{u}))$. ∎

Lemma 2.7 allows to choose optionally either system $(\mathbb{R}^{\mathbb{Z}}_\lambda, \sigma)$ or system $(\mathbb{R}^2, f_\lambda)$ for our considerations depending on which of the two systems is more suited for our analysis.

Remark 2.8. The existence of the topological conjugacy $\mathcal{H}$ implies in particular that an N-periodic f_λ-orbit $\{(u_1, v_1), \dots, (u_N, v_N)\}$ and its $\mathcal{H}^{-1}$-image, which is an N-periodic σ-orbit, have the same stability properties. □

On $\mathbb{R}^2$ we introduce the involution

$$\mathcal{R} : \mathbb{R}^2 \to \mathbb{R}^2, \quad (u, v) \mapsto (v, u),$$

Lemma 2.9. *The system* (2.7) *is reversible with respect to the involution* $\mathcal{R}$, *i.e.*

$$f_\lambda \circ \mathcal{R} = \mathcal{R} \circ f_\lambda^{-1}.$$

Proof. With $f^{-1}(u,v,\lambda) = (v, v^2 - u - \lambda)$ the statement can simply be verified by recalculating. ∎

Let (u,v) be a symmetric fixed point of f_λ, i.e. (u,v) generates a symmetric 1-periodic orbit of f_λ. From [5, Proposition 5.2] it follows that the eigenvalues of $Df_\lambda(u,v)$ come in quadruplets μ, $\bar{\mu}$, μ^{-1} and $\bar{\mu}^{-1}$. The following lemma shows that this property carries over to periodic orbits.

Lemma 2.10. *Let* $\mathcal{O}(u,v)$ *be a symmetric* n*-periodic orbit of* f_λ. *Then the eigenvalues* μ *of* $Df_\lambda^n(u,v)$ *come in quadruplets* μ, $\bar{\mu}$, μ^{-1} *and* $\bar{\mu}^{-1}$.

Proof. For convenience we omit the parameter λ in the notation. Since $(u,v) \in \mathcal{O}$ there is a $k \in \{0, \dots, n-1\}$ such that $\mathcal{R}(u,v) = f^k(u,v)$. Due to the reversibility of f, cf. Lemma 2.9, we find $f^n = \mathcal{R} f^{-n} \mathcal{R}$. Differentiating this equality at (u,v) yields

$$Df^n(u,v) = \mathcal{R} \circ Df^{-n}\left(\mathcal{R}(u,v)\right) \circ \mathcal{R} = \mathcal{R} \circ Df^{-n}\left(f^k(u,v)\right) \circ \mathcal{R}.$$

From that we infer that

$$\operatorname{spec} Df^n(u,v) = \operatorname{spec} Df^{-n}\left(f^k(u,v)\right).$$

Finally, differentiating $f^{-n} f^k = f^k f^{-n}$ and exploiting that (u,v) is a fixed of f^{-n} we find similarly that

$$\operatorname{spec} Df^{-n}\left(f^k(u,v)\right) = \operatorname{spec} Df^{-n}(u,v).$$

Combining these results yields the lemma. ∎

Since $Df_\lambda^n(u,v)$ is a (2,2)-matrix we find for complex eigenvalues, i.e. $\mu \neq \bar{\mu}$, that their modulus necessarily has to be equal to one, since necessarily $\mu = \mu^{-1}$.

3 Shift dynamics

In this section we consider the system $(\mathbb{R}_\lambda^{\mathbb{Z}}, \sigma)$. We show that, for sufficiently large λ, there is a σ-invariant set $\Sigma_\lambda^2 \subset \mathbb{R}_\lambda^{\mathbb{Z}}$ such that $(\Sigma_\lambda^2, \sigma)$ is topologically conjugated to the full shift on two symbols. To this end we define

$$S_\lambda^2 := \left\{ -\sqrt{\lambda}, \sqrt{\lambda} \right\}^{\mathbb{Z}}. \tag{3.1}$$

The shift operator on S_λ^2 we denote also by σ.

Further, we allocate to each $\mathbf{s} = (s_i) \in S_\lambda^2$ a sequence $\mathbf{u_s} \in \mathbb{R}_\lambda^{\mathbb{Z}}$ whose elements $u_{\mathbf{s},i}$ belong to a certain neighbourhood of s_i. The addressed set Σ_λ^2 then consists of all these sequences $\mathbf{u_s}$.

The set Σ_λ^2 can be illustrated even more intuitively accessible in terms of f_λ-orbits. For the λ under consideration there exist two 1-periodic f_λ-orbits (u^-, u^-) and (u^+, u^+) which are defined by

$$u = u^+ := 1 + \sqrt{1+\lambda} \quad \text{and} \quad u = u^- := 1 - \sqrt{1+\lambda}.$$

We refer to Section 5.1 for a more detailed discussion of 1-periodic orbits. Translated into this context the sequences of Σ_λ^2 are f_λ-orbits visiting disjoint neighbourhoods of the two 1-periodic in an order prescribed by $\mathbf{s}$.

In what follows let S_λ^2 be the set defined by (3.1). Further, in the upcoming lemma we consider $\mathbf{T}(\cdot, \lambda)$ as a mapping $l^\infty \to l^\infty$, where l^∞ as usual is equipped with the supremum norm $\|\cdot\|_\infty$. Note in this respect that $\mathbf{T}(\cdot, \lambda)$ maps indeed bounded sequences on bounded sequences. By $\mathbf{B}[\mathbf{u}, r]$ we denote a closed ball in l^∞ with radius r centred at $\mathbf{u}$.

Lemma 3.1. *For all $\lambda > 25$ the following is true: For each $\mathbf{s} \in S_\lambda^2$ there is a unique solution $\mathbf{u_s}$ of $\mathbf{T}(\mathbf{u}, \lambda) = 0$ within $\mathbf{B}[\mathbf{s}, 4]$.*

Proof. For our proof we apply the Banach fixed point theorem. To this end we rewrite $\mathbf{T}(\mathbf{u}, \lambda) = 0$ as $\mathbf{u} = \boldsymbol{\mathcal{T}}(\mathbf{u}, \lambda; \mathbf{s})$, where $\boldsymbol{\mathcal{T}}(\mathbf{u}, \lambda; \mathbf{s}) = (\mathcal{T}_i(\mathbf{u}, \lambda; \mathbf{s}))_{i\in\mathbb{Z}}$ and

$$\mathcal{T}_i(\mathbf{u}, \lambda; \mathbf{s}) := s_i - \frac{1}{2s_i}\left((u_i - s_i)^2 - (u_{i-1} + u_{i+1})\right). \tag{3.2}$$

Note that the sequence $\mathbf{s}$ depends on λ, cf. (3.1) - more precisely $s_i^2 = \lambda$, for all $i \in \mathbb{Z}$.

First we show that $\boldsymbol{\mathcal{T}}(\cdot, \lambda; \mathbf{s})\big(\mathbf{B}[\mathbf{s}, 4]\big) \subset \mathbf{B}[\mathbf{s}, 4]$: Let $\mathbf{u} \in \mathbf{B}[\mathbf{s}, 4]$. Note that $\mathbf{s} \in \mathbf{B}[\mathbf{s}, 4]$. Then

$$|\mathcal{T}_i(\mathbf{u}, \lambda; \mathbf{s}) - s_i| = \frac{1}{2\sqrt{\lambda}}|(u_i - s_i)^2 - u_{i-1} - u_{i+1}| \leq \frac{1}{2\sqrt{\lambda}}\left(4^2 + 2(\sqrt{\lambda} + 4)\right) < 4.$$

It remains to show that $\boldsymbol{\mathcal{T}}(\cdot, \lambda; \mathbf{s})$ is a contraction on $\mathbf{B}[\mathbf{s}, 4]$. To this end we show that $\boldsymbol{\mathcal{T}}(\cdot, \lambda; \mathbf{s})$ is differentiable and that the norm of its derivative is less than one. Consider

$$\begin{aligned}
\boldsymbol{\mathcal{T}}(\mathbf{u} + \mathbf{h}, \lambda; \mathbf{s}) - \boldsymbol{\mathcal{T}}(\mathbf{u}, \lambda; \mathbf{s}) &= \big(\mathcal{T}_i(\mathbf{u} + \mathbf{h}, \lambda; \mathbf{s}) - \mathcal{T}_i(\mathbf{u}, \lambda; \mathbf{s})\big)_{i\in\mathbb{Z}} \\
&= -\frac{1}{2s_i}\big((u_i + h_i - s_i)^2 - (u_i - s_i)^2 - h_{i-1} - h_{i+1}\big)_{i\in\mathbb{Z}} \\
&= -\frac{1}{2s_i}\big(2(u_i - s_i)h_i + h_i^2 - h_{i-1} - h_{i+1}\big)_{i\in\mathbb{Z}}.
\end{aligned}$$

Since $(h_i^2)_{i\in\mathbb{Z}}$ is $o(\|\mathbf{h}\|_\infty)$ as $\mathbf{h} \to 0$ we find for the derivative $D\boldsymbol{\mathcal{T}}(\mathbf{u}, \lambda; \mathbf{s})$

$$D\boldsymbol{\mathcal{T}}(\mathbf{u}, \lambda; \mathbf{s})(\mathbf{h}) = -\frac{1}{2s_i}\big(2(u_i - s_i)h_i - h_{i-1} - h_{i+1}\big)_{i\in\mathbb{Z}} \tag{3.3}$$

and hence for $\mathbf{u} \in \mathbf{B}[\mathbf{s}, 4]$

$$\left|\big(D\boldsymbol{\mathcal{T}}(\mathbf{u}, \lambda; \mathbf{s})(\mathbf{h})\big)_i\right| \leq \frac{1}{2\sqrt{\lambda}}(2 \cdot 4 + 2)\|\mathbf{h}\|_\infty.$$

Hence, since $\lambda > 25$ there is a $q < 1$ such that $\|D\boldsymbol{\mathcal{T}}(\mathbf{u}, \lambda; \mathbf{s})\| \leq q$, for all $\mathbf{u} \in \mathbf{B}[\mathbf{s}, 4]$. ∎

Let $\lambda > 25$ and let $\mathbf{u_s}$ be according to Lemma 3.1. Then, with $M = M(\lambda) := \sqrt{\lambda} + 4$ we define

$$\Sigma^2_\lambda := \{\mathbf{u_s} : \mathbf{s} \in S^2_\lambda\} \subset [-M, M]^{\mathbb{Z}} \subset \mathbb{R}^{\mathbb{Z}}_\lambda. \tag{3.4}$$

Note that by construction Σ^2_λ is σ-invariant. Note also that $S^2_\lambda \subset [-M, M]^{\mathbb{Z}}$.

On $[-M, M]^{\mathbb{Z}}$ we may introduce the following metric

$$d(\cdot, \cdot) : [-M, M]^{\mathbb{Z}} \times [-M, M]^{\mathbb{Z}} \to \mathbb{R}^+_0, \quad (\mathbf{u}, \mathbf{v}) \mapsto \sum_{i \in \mathbb{Z}} \frac{|u_i - v_i|}{2^{|i|}}. \tag{3.5}$$

Lemma 3.2. *The metric d defined in* (3.5) *induces the product topology on* $[-M, M]^{\mathbb{Z}}$.

Proof. Let $u \in [-M, M]^{\mathbb{Z}}$ and let $B_d(\mathbf{u}, r)$ be the open (with respect to the metric d) ball with radius r centred at $\mathbf{u}$.

First we show that $B_d(\mathbf{u}, r)$ is open in the product topology: To this end, let $\hat{\mathbf{u}} \in B_d(\mathbf{u}, r)$, $d(\mathbf{u}, \hat{\mathbf{u}}) = \hat{r} < r$. Now, let $N \in \mathbb{N}$ be that large and let $\epsilon > 0$ be that small such that

$$\sum_{|i| \geq N+1} \frac{1}{2^i} 2M < \frac{r - \hat{r}}{2} \quad \text{and} \quad \sum_{|i| \leq N} \frac{1}{2^{|i|}} \epsilon < \frac{r - \hat{r}}{2}.$$

Then the basic set $p^{-1}_{-N}((\hat{u}_{-N} - \epsilon, \hat{u}_{-N} + \epsilon)) \cap \ldots \cap p^{-1}_N((\hat{u}_N - \epsilon, \hat{u}_N + \epsilon))$ of the product topology containing $\hat{\mathbf{u}}$ is a subset of $B_d(\mathbf{u}, r)$. Hence $B_d(\mathbf{u}, r)$ is open in the product topology.

Next we show that the open balls $B_d(\mathbf{u}, r)$ form also a basis for the product topology: Let $U \subset [-M, M]^{\mathbb{Z}}$ be open (with respect to the product topology), and let $\mathbf{u} = (u_i) \in U$. Then there exist $i_0, \ldots, i_N$ and $\epsilon > 0$ such that

$$p^{-1}_{i_0}((u_{i_0} - \epsilon, u_{i_0} + \epsilon)) \cap \ldots \cap p^{-1}_{i_N}((u_{i_N} - \epsilon, u_{i_N} + \epsilon)) \subset U.$$

Now, let $j := \max\{|i_0|, \ldots, |i_N|\}$, and let r be that small such that $\frac{1}{2^j}\epsilon > r$. Then

$$B_d(\mathbf{u}, r) \subset p^{-1}_{i_0}((u_{i_0} - \epsilon, u_{i_0} + \epsilon)) \cap \ldots \cap p^{-1}_{i_N}((u_{i_N} - \epsilon, u_{i_N} + \epsilon)).$$

Hence the open d-balls form a basis for the product topology. ∎

Let both S^2_λ and Σ^2_λ be equipped with the topology induced by the product topology on $\mathbb{R}^{\mathbb{Z}}$. Then the metric introduced in (3.5) determines this topology.

Theorem 3.3. *For $\lambda > 5, 5^2$ the map $H : S^2_\lambda \to \Sigma^2_\lambda$, $H : \mathbf{s} \mapsto \mathbf{u_s}$ is a topological conjugacy for the systems (S^2_λ, σ) and $(\Sigma^2_\lambda, \sigma)$.*

Proof. First we show that $H \circ \sigma = \sigma \circ H$: Let $\mathbf{u_s} = (u_{\mathbf{s},i})_{i \in \mathbb{Z}}$ and $\mathbf{u}_{\sigma\mathbf{s}} = (u_{\sigma\mathbf{s},i})_{i \in \mathbb{Z}}$. In these terms we have to show that $\mathbf{u}_{\sigma\mathbf{s}} = \sigma\mathbf{u_s}$. Firstly, the uniqueness part of Lemma 3.1 yields $u_{\sigma\mathbf{s},i} = u_{\mathbf{s},i+1}$, and on the other hand, by definition of σ we have that also $(\sigma\mathbf{u_s})_i = u_{\mathbf{s},i+1}$.

It remains to show that H is a homeomorphism. In this respect we first note that by construction H is a bijection. Further, according to the Tychonoff theorem, cf. [4, Chap. XI, Theorem 1.4],

S^2_λ is compact, and according to [4, Chap. VII, Theorem 1.3] the space $\mathbb{R}^\mathbb{Z}$ is Hausdorff and hence Σ^2_λ is Hausdorff. Finally, since S^2_λ is compact and Σ^2_λ is Hausdorff it is enough to show that H is continuous, cf. [4, Chap. XI, Theorem 2.1].

In this part of the proof we invoke the metric d defined in (3.5) and write

$$\begin{aligned} d(\mathbf{u_s},\mathbf{u_t}) &= d\big(\boldsymbol{\mathcal{T}}(\mathbf{u_s},\lambda;\mathbf{s}),\boldsymbol{\mathcal{T}}(\mathbf{u_t},\lambda;\mathbf{t})\big) \\ &\le d\big(\boldsymbol{\mathcal{T}}(\mathbf{u_s},\lambda;\mathbf{s}),\boldsymbol{\mathcal{T}}(\mathbf{u_s},\lambda;\mathbf{t})\big) + d\big(\boldsymbol{\mathcal{T}}(\mathbf{u_s},\lambda;\mathbf{t}),\boldsymbol{\mathcal{T}}(\mathbf{u_t},\lambda;\mathbf{t})\big). \end{aligned} \tag{3.6}$$

We begin our analysis with the last summand in (3.6)

$$d\big(\boldsymbol{\mathcal{T}}(\mathbf{u_s},\lambda;\mathbf{t}),\boldsymbol{\mathcal{T}}(\mathbf{u_t},\lambda;\mathbf{t})\big) = \sum_{i\in\mathbb{Z}} \frac{1}{2^{|i|}} \left|\mathcal{T}_i(\mathbf{u_s},\lambda;\mathbf{t}) - \mathcal{T}_i(\mathbf{u_t},\lambda;\mathbf{t})\right|. \tag{3.7}$$

Exploiting the mean value theorem we find for an appropriate $\tau\in(0,1)$ the following estimate. Remind that the action of $D\boldsymbol{\mathcal{T}}$ is described by (3.3).

$$\begin{aligned} \frac{1}{2^{|i|}}\big|\mathcal{T}_i(\mathbf{u_s},\lambda;\mathbf{t}) - \mathcal{T}_i(\mathbf{u_t},\lambda;\mathbf{t})\big| &= \frac{1}{2^{|i|}}\big|D\mathcal{T}_i(\mathbf{u_t}+\tau(\mathbf{u_s}-\mathbf{u_t}),\lambda;\mathbf{t})(\mathbf{u_s}-\mathbf{u_t})\big| \\ = \frac{1}{2^{|i|}}\frac{1}{2\sqrt{\lambda}}\big|2(u_{t,i}+\tau(u_{s,i}-u_{t,i})-t_i)(u_{s,i}-u_{t,i}) &- (u_{s,i-1}-u_{t,i-1}) - (u_{s,i+1}-u_{t,i+1})\big| \\ < \frac{1}{2^{|i|}}\frac{1}{11}\big(2|u_{t,i}+\tau(u_{s,i}-u_{t,i})-t_i||u_{s,i}-u_{t,i}| &+ |u_{s,i-1}-u_{t,i-1}| + |u_{s,i+1}-u_{t,i+1}|\big) \end{aligned}$$

Now, let

$$s_i = t_i \text{ for } i\in[-N,N]\cap\mathbb{Z}. \tag{3.8}$$

Then for $i\in[-N,N]\cap\mathbb{Z}$ both $u_{s,i}$ and $u_{t,i}$ belong to an interval centred at t_i with radius 4. Hence

$$\begin{aligned} &\frac{1}{2^{|i|}}\frac{1}{11}\big(2|u_{t,i}+\tau(u_{s,i}-u_{t,i})-t_i||u_{s,i}-u_{t,i}| + |u_{s,i-1}-u_{t,i-1}| + |u_{s,i+1}-u_{t,i+1}|\big) \\ &\le \begin{cases} \frac{1}{2^{|i|}}\frac{1}{11}\big(2\cdot 4|u_{s,i}-u_{t,i}| + |u_{s,i-1}-u_{t,i-1}| + |u_{s,i+1}-u_{t,i+1}|\big), & |i|\le N \\ \frac{1}{2^{|i|}}\frac{1}{11}\big(2\cdot 2M|u_{s,i}-u_{t,i}| + |u_{s,i-1}-u_{t,i-1}| + |u_{s,i+1}-u_{t,i+1}|\big), & |i|> N \end{cases} \\ &= \begin{cases} \frac{1}{2^i}\frac{8}{11}|u_{s,i}-u_{t,i}| + \frac{1}{2^{i-1}}\frac{1}{2}\frac{1}{11}|u_{s,i-1}-u_{t,i-1}| + \frac{1}{2^{i+1}}2\frac{1}{11}|u_{s,i+1}-u_{t,i+1}|, & 1\le i\le N \\ \frac{8}{11}|u_{s,0}-u_{t,0}| + \frac{1}{2}2\frac{1}{11}|u_{s,-1}-u_{t,-1}| + \frac{1}{2}2\frac{1}{11}|u_{s,1}-u_{t,1}|, & i=0 \\ \frac{1}{2^{|i|}}\frac{8}{11}|u_{s,i}-u_{t,i}| + \frac{1}{2^{|i-1|}}2\frac{1}{11}|u_{s,i-1}-u_{t,i-1}| + \frac{1}{2^{|i+1|}}\frac{1}{2}\frac{1}{11}|u_{s,i+1}-u_{t,i+1}|, & -N\le i\le -1 \\ \frac{1}{2^{|i|}}\frac{4M}{11}|u_{s,i}-u_{t,i}| + \frac{1}{2^{|i|}}\frac{1}{11}|u_{s,i-1}-u_{t,i-1}| + \frac{1}{2^{|i|}}\frac{1}{11}|u_{s,i+1}-u_{t,i+1}|, & |i|>N. \end{cases} \end{aligned}$$

So, in accordance with (3.7) we find with an appropriate $Q > 1$ and $q = \frac{10{,}5}{11} < 1$

$$d\big(\boldsymbol{\mathcal{T}}(\mathbf{u_s},\lambda;\mathbf{t}),\boldsymbol{\mathcal{T}}(\mathbf{u_t},\lambda;\mathbf{t})\big) \le \frac{10}{11}|u_{s,0}-u_{t,0}| + \sum_{\substack{|i|\le N\\ i\neq 0}} \frac{10,5}{11}\frac{1}{2^{|i|}}|u_{s,i}-u_{t,i}| + \sum_{|i|>N} Q\frac{1}{2^{|i|}}|u_{s,i}-u_{t,i}|$$

$$\le \sum_{|i|\le N} q\frac{1}{2^{|i|}}|u_{s,i}-u_{t,i}| + \sum_{|i|>N} Q\frac{1}{2^{|i|}}|u_{s,i}-u_{t,i}|$$

$$< q\, d(\mathbf{u_s},\mathbf{u_t}) + \frac{8MQ}{2^{N+1}}.$$

So, together with (3.6) we find

$$(1-q)d(\mathbf{u_s},\mathbf{u_t}) \le d\big(\boldsymbol{\mathcal{T}}(\mathbf{u_s},\lambda;\mathbf{s}),\boldsymbol{\mathcal{T}}(\mathbf{u_s},\lambda;\mathbf{t})\big) + \frac{8MQ}{2^{N+1}}. \tag{3.9}$$

Due to the definition of $\mathcal{T}_i$, cf. (3.2), and the relation of $\mathbf{s}$ and $\mathbf{t}$ demanded in (3.8) we find with an appropriate constant C

$$d\big(\boldsymbol{\mathcal{T}}(\mathbf{u_s},\lambda;\mathbf{s}),\boldsymbol{\mathcal{T}}(\mathbf{u_s},\lambda;\mathbf{t})\big) \le \frac{1}{2^{N-1}}C.$$

Together with (3.9) we find for $\mathbf{t}\to\mathbf{s}$, i.e. $N\to\infty$, that $d(\mathbf{u_s},\mathbf{u_t})\to 0$, i.e. $\mathbf{u_t}\to\mathbf{u_s}$. That means that H is continuous. ∎

In other words Theorem 3.3 says that the system $(\Sigma^2_\lambda,\sigma)$ is topologically conjugated to the full shift on two symbols. That is $(\Sigma^2_\lambda,\sigma)$ is chaotic in the sense of Devaney. We recall Devaney's definition of a chaotic system, cf. [6]: Let X be a metric space and f be a homeomorphism on X. The discrete dynamical system (X,f) is chaotic if

(i) f is topologically transitive;
(ii) The periodic points of f are dense in X;
(iii) f has sensitive dependence on initial conditions.

From the construction it follows that:

Lemma 3.4. *Σ^2_λ is R-invariant.*

Proof. We adopt the notation from Lemma 3.1 and 3.4. Let $\mathbf{u_s}\in\Sigma^2_\lambda$. We show that $R\mathbf{u_s}=\mathbf{u}_{R\mathbf{s}}$ and it therefore belongs to Σ^2_λ.

First we note that due to Lemma 2.6 $\mathbf{T}(R\mathbf{u_s},\lambda)=0$. Further $R\mathbf{u_s}\in R\mathbf{B}[\mathbf{s},4]=\mathbf{B}[R\mathbf{s},4]$. Now, $R\mathbf{u_s}=\mathbf{u}_{R\mathbf{s}}$ follows from the uniqueness part of Lemma 3.1 ∎

It is true that Σ^2_λ is R-invariant but it contains also asymmetric orbits, cf. also Lemma 4.5 below. More precisely, with each asymmetric orbit also its R-image belongs to Σ^2_λ.

In what follows we restrict our considerations on R-symmetric orbits within Σ^2_λ:

$$\Sigma^2_{\lambda,R} := \{\mathbf{u}\in\Sigma^2_\lambda:\ R(\mathcal{O}(\mathbf{u}))=\mathcal{O}(\mathbf{u})\}. \tag{3.10}$$

Lemma 3.5. $\Sigma^2_{\lambda,R}$ *is* σ*-invariant.*

Proof. This follows immediately from $\mathcal{O}(\sigma\mathbf{u}) = \mathcal{O}(\mathbf{u})$. ∎

So $(\Sigma^2_{\lambda,R}, \sigma)$ is an R-reversible dynamical system - with the notable feature that all orbits of this system are R-symmetric.

Theorem 3.6. $(\Sigma^2_{\lambda,R}, \sigma)$ *is chaotic in the sense of Devaney.*

Proof. Similar to (3.10) we define $S^2_{\lambda,R} := \{\mathbf{s} \in S^2_\lambda : \mathcal{O}(\mathbf{s}) \text{ is } R\text{-symmetric}\}$. Note that $\Sigma^2_{\lambda,R} = H\big(S^2_{\lambda,R}\big)$, and what is more, H is a topological conjugacy for the systems $(S^2_{\lambda,R}, \sigma)$ and $(\Sigma^2_{\lambda,R}, \sigma)$. In [9, Section 3.1] we have shown that $(S^2_{\lambda,R}, \sigma)$ is chaotic (in the sense of Devaney), and that based on results by [1] this property will be carried forward to topologically conjugated systems. ∎

Finally we want to remark that due to Lemma 2.7 all the results of this section can be carried over to the system $(\mathbb{R}^2, f)$.

4 Symmetric periodic orbits

In this section we consider the R-reversible system $(\mathbb{R}^{\mathbb{Z}}_\lambda, \sigma)$. For $\mathbf{u} \in \mathbb{R}^{\mathbb{Z}}_\lambda$ we denote its σ-orbit by $\mathcal{O}(\mathbf{u})$. Recall that $\mathcal{O}(\mathbf{u})$ is called R-symmetric, or symmetric for short, if $R\big(\mathcal{O}(\mathbf{u})\big) = \mathcal{O}(\mathbf{u})$. Further, we want to note that the orbit $\mathcal{O}(\mathbf{u})$ is periodic if and only if the sequence $\mathbf{u}$ is periodic. If $\mathbf{u}$ is a n-periodic sequence with periodically recurring segments $u_1, \ldots, u_n$ we write $\mathbf{u} = (\overline{u_1, \ldots, u_n})$ and for its orbit we write $\mathcal{O}(\mathbf{u}) = \mathcal{O}\left(\overline{u_1, \ldots, u_n}\right)$.

We start with some characterisations of symmetric orbits.

Lemma 4.1. *Let* $\mathbf{u} \in \mathbb{R}^{\mathbb{Z}}_\lambda$ *and let* $\mathcal{O}(\mathbf{u})$ *be its* σ*-orbit.*

(i) $\mathcal{O}(\mathbf{u})$ *is R-symmetric if and only if*

$$\text{(a) } \exists \hat{\mathbf{u}} \in \mathcal{O}(\mathbf{u}) : R\hat{\mathbf{u}} = \hat{\mathbf{u}} \quad \textit{or} \quad \text{(b) } \exists \hat{\mathbf{u}} \in \mathcal{O}(\mathbf{u}) : R\hat{\mathbf{u}} = \sigma\hat{\mathbf{u}}$$

(ii) If $\mathcal{O}(\mathbf{u})$ *is an R-symmetric aperiodic orbit then exactly one of the conditions (a) and (b) is satisfied for exactly one* $\hat{\mathbf{u}}$*.*

(iii) The orbit $\mathcal{O}(\mathbf{u})$ *is a symmetric periodic orbit if and only if either both conditions (a) and (b) are satisfied or one of these conditions is satisfied for two different* $\hat{\mathbf{u}}^{\mathbf{1}}$, $\hat{\mathbf{u}}^{\mathbf{2}}$*.*

Proof. In [9] we proved corresponding statements for the shift on two symbols, cf. there Lemma 3.4, Lemma 3.5 and Corollary 3.6. The proofs given there can be adopted word for word in the present context. ∎

The statement in Lemma 4.1(iii) gives rise to the following definition:

Definition 4.2. *A symmetric periodic orbit* $(\mathbb{R}^{\mathbb{Z}}_{\lambda}, \sigma)$ *is called*

(i) symmetric periodic orbit of type (a,b), if both conditions (a) and (b) are satisfied,

(ii) symmetric periodic orbit of type (a,a) or (b,b), respectively, if one of the conditions (a) or (b) is satisfied for two different entries.

Lemma 4.3. *Let* $\mathcal{O}\left(\overline{u_1, \ldots, u_{2k-1}}\right)$ *be symmetric. Then it is of type (a,b).*

Proof. Let $u_i = u_{2k+1-i}$. Then, for $i = k$ we have $u_k = u_{k+1}$.

$$\begin{array}{ccc} \text{type (a)} & & \text{type (b)} \\ u_{k+1} \ldots u_{2k-1} \,\big|\, u_1 u_2 \ldots u_k & \Leftrightarrow & u_1 u_2 \ldots u_k \,\Big|\, u_{k+1} \ldots u_{2k-1} u_1 \end{array}$$

∎

Lemma 4.4. *Let* $\mathcal{O}\left(\overline{u_1, \ldots, u_{2k}}\right)$ *be symmetric then it is either of type (a,a) or of type (b,b).*

Proof. Let $\mathcal{O}\left(\overline{u_1, \ldots, u_{2k}}\right)$ be of type (a), and let $u_i = u_{2k+2-i}$. Then, for $i = k+1$ we have $u_{k+1} = u_{k+1}$.

$$\begin{array}{ccc} \text{type (a)} & & \text{type (a)} \\ u_{k+1} u_{k+2} \ldots u_{2k} \,\big|\, u_1 u_2 \ldots u_k u_{k+1} & \Leftrightarrow & u_1 u_2 \ldots u_k \, \overset{|}{\underset{|}{u_{k+1}}} \, u_{k+2} \ldots u_{2k} u_1 \end{array}$$

Now, let $\mathcal{O}\left(\overline{u_1, \ldots, u_{2k}}\right)$ be of type (b), and let $u_i = u_{2k+1-i}$. Then, for $i = k$ we have $u_k = u_{k+1}$.

$$\begin{array}{ccc} \text{type (b)} & & \text{type (b)} \\ u_{k+1} \ldots u_{2k} \,\Big|\, u_1 \ldots u_k & \Leftrightarrow & u_1 \ldots u_k \,\Big|\, u_{k+1} \ldots u_{2k} \end{array}$$

∎

Lemma 4.1 is true for any reversible discrete system - one just has to replace R and σ by the actual involution and the corresponding mapping, cf. also [11, Theorem 4.2]. So, also Definition 4.2 as well as the results achieved so far in this section can be adopted in the general context of reversible discrete systems.

In terms of the fixed point space $\operatorname{Fix} R$ of the involution R condition (a) in Lemma 4.1(i) says that $\hat{\mathbf{u}} \in \operatorname{Fix} R$. Hence a symmetric periodic orbit is of type (a,b) if it intersects $\operatorname{Fix} R$ in exactly one point, whereas a symmetric periodic orbit is of type (a,a) if it intersects $\operatorname{Fix} R$ in exactly two points. It is even possible that a symmetric periodic orbit has no point in common with $\operatorname{Fix} R$ - then it is of type (b,b). Let $\mathcal{O} = \{\mathbf{u}^1, \ldots, \mathbf{u}^{2k}\}$, $\mathbf{u}^{i+1} = \sigma \mathbf{u}^i$ and $\mathbf{u}^1 = \sigma \mathbf{u}^{2k}$, be a symmetric periodic orbits of type (b,b). Then $\mathbf{u}^1$ can be chosen in such a way such that $R\mathbf{u}^{2k} = \sigma \mathbf{u}^{2k} = \mathbf{u}^1$ and $R\mathbf{u}^k = \sigma \mathbf{u}^k = \mathbf{u}^{k+1}$, cf. also in the second part of the proof of Lemma 4.4. Similarly, let $\mathcal{O} = \{\mathbf{u}^1, \ldots, \mathbf{u}^{2k}\}$, $\mathbf{u}^{i+1} = \sigma \mathbf{u}^i$ and $\mathbf{u}^1 = \sigma \mathbf{u}^{2k}$, be a symmetric periodic orbits of type (a,a). Then $\mathbf{u}^1$ can be chosen in such a way such that $R\mathbf{u}^1 = \mathbf{u}^1$ and $R\mathbf{u}^{k+1} = \mathbf{u}^{k+1}$, cf. also in the first part of the proof of Lemma 4.4. Finally, let $\mathcal{O} = \{\mathbf{u}^1, \dots, \mathbf{u}^{2k-1}\}$, $\mathbf{u}^{i+1} = \sigma\mathbf{u}^i$ and $\mathbf{u}^1 = \sigma\mathbf{u}^{2k-1}$, be a symmetric periodic orbits of type (a,b). Then $\mathbf{u}^1$ can be chosen in such a way such that $R\mathbf{u}^1 = \mathbf{u}^1$ and $R\mathbf{u}^k = \mathbf{u}^{k+1}$, cf. also in the proof of Lemma 4.3.

4.1 Symmetric periodic orbits within $\Sigma^2_{\lambda,R}$

As explained in the introduction, the aim of this paper is to describe the creation of the period orbits belonging to $\Sigma^2_{\lambda,R}$ - by definition all these periodic orbits are symmetric. According to Devaney's definition of chaos and Theorem 3.6 there are infinitely many those symmetric periodic orbits. And from the proof of Theorem 3.6 we learn that these orbits are directly related to periodic orbits in $S^2_{\lambda,R}$.

Since for each $\lambda > 0$ the set S^2_λ represents sequences in two symbols in each case we drop λ from the notation and consider simply $S^2 := \{+,-\}^{\mathbb{Z}}$. On this set the shift map σ and the involution R act in the same way as defined in (2.4) and (2.5), respectively. Accordingly we denote by S^2_R the set of all those sequences whose σ-orbits are symmetric.

Let $\mathrm{Per}(N) \subset S^2$ be the set of periodic orbits with minimal period N, and correspondingly let $\mathrm{Per}_R(N) \subset S^2_R$ be the set of symmetric periodic orbits with minimal period N. In what follows we determine the cardinality of these sets. We denote the cardinality of a set M by $|M|$. Using the formula for multiset permutations we find

$$|\{\mathbf{u} \in S^2 : \mathcal{O}(\mathbf{u}) \text{ is } N\text{-periodic}\}| = \frac{N!}{k!(N-k)!}.$$

Note that such an N-periodic orbit visits k different elements if its minimal period is k (where of course k is a divisor of N). Let $D(N)$ the set of divisors of N which are different of 1 and N. With that we find for $N > 1$

$$|\mathrm{Per}(N)| = \frac{1}{N}\left(\sum_{k=1}^{N-1} \frac{N!}{k!(N-k)!} - \sum_{k\in D(N)} k|\mathrm{Per}(k)|\right). \tag{4.1}$$

For the computation of the cardinality $|\mathrm{Per}_R(N)|$ we exploit Lemma 4.3 or Lemma 4.4, respectively depending on whether N is odd or even.

First, let $N = 2k-1$. According to Lemma 4.3 a symmetric $2k-1$-periodic orbit $\mathcal{O}(\mathbf{u})$ is of type (a,b). If $2k-1$ is the minimal period then there is exactly one $\mathbf{v} \in \mathcal{O}(\mathbf{u})$ such that $R\mathbf{v} = \mathbf{v}$. So, to compute $|\mathrm{Per}_R(2k-1)|$ we just have to count the corresponding symmetric sequences $\mathbf{v}$ and taking into consideration that $2k-1$ is the minimal period. For $k > 1$ we get

$$|\mathrm{Per}_R(2k-1)| = \sum_{j=1}^{k-1} \frac{k!}{j!(k-j)!} - \sum_{j\in D(2k-1)} |\mathrm{Per}_R(j)|. \tag{4.2}$$

Note that all elements of $D(2k-1)$ are odd numbers.

In a similar way we deal with $\mathrm{Per}_R(2k)$. However, according to Lemma 4.4 we distinguish whether a symmetric $2k$-periodic orbit is of type (a,a) or of type (b,b). But note that in each case a symmetric $2k$-periodic orbit $\mathcal{O}(\mathbf{u})$ with minimal period $2k$ contains two different sequences $\mathbf{v}^1$ and $\mathbf{v}^2$ such $R\mathbf{v}^i = \sigma\mathbf{v}^i$, $i = 1, 2$. For $k \geq 1$ we get

$$|\mathrm{Per}_R(2k)| = \frac{1}{2}\left(\sum_{j=1}^{k}\frac{(k+1)!}{j!(k+1-j)!} + \sum_{j=1}^{k-1}\frac{k!}{j!(k-j)!}\right) - \sum_{j\in D(2k)}|\mathrm{Per}_R(j)|. \tag{4.3}$$

The first sum on the right-hand side refers to orbits of type (a,a) while the second sum refers to orbits of type (b,b). Note that for $k = 1$ the second sum is equal to zero - there is no 2-periodic of type (b,b).

Lemma 4.5. *Consider the system* $(\Sigma^2_\lambda, \sigma)$.

(i) For each $N \in \mathbb{N}$ *and each possible type* $(\Sigma^2_\lambda, \sigma)$ *has a symmetric periodic orbit with minimal period* N.

(ii) There are asymmetric periodic orbits in $(\Sigma^2_\lambda, \sigma)$.

(iii) All periodic orbits up to period 5 are symmetric.

Proof. According to our remarks at the beginning of this section it is enough to show the corresponding assertions for (S^2, σ).

Regarding item (i): Consider $\mathcal{O}\left(\overline{+-\ldots-}\right)$. Clearly $R\left(\overline{+-\ldots-}\right) = \left(\overline{+-\ldots-}\right)$, and therefore by Lemma 4.1(i)(a) this orbit is symmetric. Note that the given orbit is of type (a,b) or of type (a,a) depending on whether N is odd or even. Now, let N be even. Consider $\mathcal{O}\left(\overline{+-\ldots-+}\right)$. According to Lemma 4.1(i)(b) this orbit is symmetric of type (b,b).

Regarding item (ii): According to (4.1) and (4.3) we find $|\mathrm{Per}(6)| = 9$ and $|\mathrm{Per}_R(6)| = 7$. So there are two asymmetric 6-periodic orbits. By means of Lemma 4.4 it can easily be verified that these are the 6-periodic orbits $\mathcal{O}\left(\overline{++--+-}\right)$ and $\mathcal{O}\left(\overline{--++-+}\right)$.

Regarding item (iii): By means of (4.1), (4.2) and (4.3) it can easily be verified that for $N \leq 5$ $|\mathrm{Per}(N)|$ coincides with $|\mathrm{Per}_R(N)|$. We refer also to the following Remark4.6. ∎

Remark 4.6. Here we list individually the periodic orbits up to period 5:

$$\mathcal{O}\left(\overline{+}\right), \mathcal{O}\left(\overline{-}\right),\ \mathcal{O}\left(\overline{+-}\right),\ \mathcal{O}\left(\overline{++-}\right), \mathcal{O}\left(\overline{+--}\right),\ \mathcal{O}\left(\overline{+++-}\right), \mathcal{O}\left(\overline{++--}\right), \mathcal{O}\left(\overline{+---}\right),$$
$$\mathcal{O}\left(\overline{++++-}\right), \mathcal{O}\left(\overline{+++--}\right), \mathcal{O}\left(\overline{++---}\right), \mathcal{O}\left(\overline{+----}\right), \mathcal{O}\left(\overline{++-+-}\right),$$
$$\mathcal{O}\left(\overline{--+-+}\right),$$

By means of Lemma 4.3 or Lemma 4.4 it can easily be verified that all these periodic orbits are symmetric. □

According to (4.2) there are six symmetric 5-periodic orbits within $\Sigma^2_{\lambda,R}$. Further, according to our tentative bifurcation diagram, cf. Figure 1, only four of them will be created in the course of subharmonic bifurcations. Indeed, in Section 5.5.2 we will show that "the missing two 5-periodic orbits" will be created in the course of a saddle-centre bifurcation.

4.2 Determination equations for symmetric (2k-1)-periodic orbits

According to Lemma 4.3 symmetric periodic orbits with an odd minimal period are of type (a,b). So we may choose as in the proof of Lemma 4.3

$$u_2 = u_{2k-1}, \dots, u_k = u_{k+1},$$

where $u_1, \dots, u_k$ are freely selectable. Then symmetric orbits $\mathcal{O}\left(\overline{u_1, \dots, u_{2k-1}}\right)$ are determined by, cf. also (2.1)-(2.3),

$$\lambda = u_1^2 - 2u_2,\ \dots\ , \lambda = u_i^2 - (u_{i-1} + u_{i+1}),\ \dots\ , \lambda = u_k^2 - (u_{k-1} + u_k), \quad i = 2, \dots, k-1.$$

This system is equivalent to $\lambda = u_1^2 - 2u_2$ and

$$\begin{aligned} F^1_{2k-1}(\mathbf{u}) :=\ & u_1^2 - u_2^2 + ((u_1 - u_2) + (u_3 - u_2)) && = 0 \\ & && \vdots \\ F^i_{2k-1}(\mathbf{u}) :=\ & u_i^2 - u_{i+1}^2 + ((u_i - u_{i-1}) + (u_{i+2} - u_{i+1})) && = 0, \quad i = 2, \dots, k-2 \\ & && \vdots \\ & u_{k-1}^2 - u_k^2 + (u_{k-1} - u_{k-2}) && = 0. \end{aligned} \tag{4.4}$$

Obviously u_i with $u_1 = u_2 = \dots = u_k$ solve the set of equations (4.4). For each k these solutions correspond to symmetric 1-periodic orbits existing for $\lambda = u^2 - 2u$. Having this in mind and in view of our aim to consider mainly subharmonic bifurcations from the branch of 1-periodic orbits we write the set of equations (4.4) as:

$$\mathcal{M}_{2k-1} \cdot \left(u_1 - u_2, u_2 - u_3, \dots, u_{k-1} - u_k\right)^T = 0, \tag{4.5}$$

where the $(k-1, k-1)$-matrix $\mathcal{M}_{2k-1} = \mathcal{M}_{2k-1}(u_1, \dots, u_k)$ reads as follows:

$$\mathcal{M}_{2k-1} = \begin{pmatrix} u_1{+}u_2{+}1 & -1 & 0 & & & \cdots & 0 \\ -1 & u_2{+}u_3 & -1 & 0 & & & \vdots \\ 0 & -1 & u_3{+}u_4 & -1 & 0 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & 0 & -1 & u_{k\text{-}3}{+}u_{k\text{-}2} & -1 & 0 \\ \vdots & & & 0 & -1 & u_{k\text{-}2}{+}u_{k\text{-}1} & -1 \\ 0 & \cdots & & \cdots & 0 & -1 & u_{k\text{-}1}{+}u_k \end{pmatrix}. \tag{4.6}$$

Note that $\mathcal{M}_3(u_1, u_2) = u_1 + u_2 + 1$. Further, for $k = 1$ we define $\mathcal{M}_1(u) := 1$.

Solutions of (4.5) are tuples $(u_1, \dots, u_k)$:

$$\left(u_1 - u_2, u_2 - u_3, \dots, u_{k\text{-}1} - u_k\right)^T \in \ker \mathcal{M}_{2k-1}.$$

So, $(u_1, \dots, u_k)$ solves (4.5) if $u_1 = u_2 = \dots = u_k$ or, since the first $(k-2)$ lines of $\mathcal{M}_{2k-1}$ are linearly independent, if

$$F_{2k-1}\left(u_1, \dots, u_k\right) := \left(F^1_{2k-1}(\mathbf{u}), \dots, F^{k-2}_{2k-1}(\mathbf{u}), F^{k-1}_{2k-1}(\mathbf{u})\right) = 0, \tag{4.7}$$

where the F^i_{2k-1}, $i = 1, \ldots, k-2$, are defined in (4.4) and

$$F^{k-1}_{2k-1}(\mathbf{u}) := \det \mathcal{M}_{2k-1}(u_1, \ldots, u_k).$$

Therefore, cf. also (2.2),

$$M_{2k-1}(u) := \det \mathcal{M}_{2k-1}(u, \ldots, u) = 0 \quad \text{and} \quad \lambda = u^2 - 2u \tag{4.8}$$

is a necessary condition for (u, λ) being a point at which symmetric (2k-1)-periodic orbits bifurcate from the branch of 1-periodic orbits. The corresponding branches of symmetric (2k-1)-periodic orbits can be computed by solving (4.7) by means of the implicit function theorem at zeros of M_{2k-1}.

4.3 Determination equations for symmetric (2k)-periodic orbits

According to Lemma 4.3 symmetric periodic orbits with an even minimal period are either of type (a,a) or of type (b,b). In the following considerations we distinguish these both situations.

4.3.1 Symmetric (2k)-periodic orbits of type (b,b)

Let, as in the proof of Lemma 4.4,

$$u_1 = u_{2k},\ u_2 = u_{2k-1},\ \ldots,\ u_k = u_{k+1}.$$

Note that $u_1, \ldots, u_k$ are freely selectable.

Now we proceed along the lines of Section 4.2. Symmetric orbits $\mathcal{O}\left(\overline{u_1, \ldots, u_{2k}}\right)$ of type (b,b) are determined by

$$\lambda = u_1^2 - (u_1 + u_2),\ \ldots\ , \lambda = u_i^2 - (u_{i-1} + u_{i+1}),\ \ldots\ , \lambda = u_k^2 - (u_{k-1} + u_k), \quad i = 2, \ldots, k-1.$$

This system is equivalent to $\lambda = u_1^2 - (u_1 + u_2)$ and

$$\begin{array}{rlcl} F^{b,1}_{2k}(\mathbf{u}) := & u_1^2 - u_2^2 + (u_3 - u_2) & = & 0 \\ & & \vdots & \\ F^{b,i}_{2k}(\mathbf{u}) := & u_i^2 - u_{i+1}^2 + ((u_i - u_{i-1}) + (u_{i+2} - u_{i+1})) & = & 0, \quad i = 2, \ldots, k-2 \\ & & \vdots & \\ & u_{k-1}^2 - u_k^2 + (u_{k-1} - u_{k-2}) & = & 0 \end{array} \tag{4.9}$$

We write the latter set of equations as $\mathcal{M}^b_{2k} \cdot \left(u_1 - u_2, u_2 - u_3, \ldots, u_{k-1} - u_k\right)^T = 0$ where the

$(k-1,k-1)$-matrix $\mathcal{M}^b_{2k} = \mathcal{M}^b_{2k}(u_1,\dots,u_k)$ reads as follows:

$$\mathcal{M}^b_{2k} := \begin{pmatrix} u_1+u_2 & -1 & 0 & & & \dots & 0 \\ -1 & u_2+u_3 & -1 & 0 & & & \vdots \\ 0 & -1 & u_3+u_4 & -1 & 0 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & 0 & -1 & u_{k\text{-}3}+u_{k\text{-}2} & -1 & 0 \\ \vdots & & & 0 & -1 & u_{k\text{-}2}+u_{k\text{-}1} & -1 \\ 0 & \dots & & \dots & 0 & -1 & u_{k\text{-}1}+u_k \end{pmatrix}. \tag{4.10}$$

In particular we have $\mathcal{M}^b_4(u_1,u_2) = u_1+u_2$. Further, for $k=1$ we define $\mathcal{M}^b_2(u) := 1$. Finally we want to note that there is no 2-periodic orbit of type (b,b).

By analogy with (4.7) we may write the determination equations of $2k$-periodic orbits of type (b,b) in the form

$$F^b_{2k} = (u_1,\dots,u_k) := \Big(F^{b,1}_{2k}(\mathbf{u}),\dots,F^{b,k-2}_{2k}(\mathbf{u}),F^{b,k-1}_{2k}(\mathbf{u})\Big) = 0, \tag{4.11}$$

where $F^{b,i}_{2k}$, $i=1,\dots,k-2$, are defined in (4.9) and

$$F^{b,k-1}_{2k}(\mathbf{u}) := \det \mathcal{M}^b_{2k}(u_1,\dots,u_k). \tag{4.12}$$

In Section 6 we discuss bifurcating symmetric (2k)-periodic orbits of type (b,b) by solving (4.11) by means of the implicit function theorem.

4.3.2 Symmetric (2k)-periodic orbits of type (a,a)

Let, as in the proof of Lemma 4.4, the sequence $\mathbf{u}$ be defined by

$$u_1,\ u_2 = u_{2k},\ u_3 = u_{2k-1},\ \dots,\ u_k = u_{k+2},\ u_{k+1}.$$

Note that $u_1,\ \dots,\ u_{k+1}$ are freely selectable. So, symmetric orbits $\mathcal{O} = (\overline{u_1,\dots,u_{2k}})$ of type (a,a) are determined by

$$\lambda = u_1^2 - 2u_2,\ \dots\ ,\lambda = u_i^2-(u_{i-1}+u_{i+1}),\ \dots\ ,\lambda = u_{k+1}^2 - 2u_k, \quad i=2,\dots,k.$$

This system is equivalent to $\lambda = u_1^2 - 2u_2$ and

$$\begin{aligned} F^{a,1}_{2k}(\mathbf{u}) :=&\ u_1^2-u_2^2+(u_1-u_2)+(u_3-u_2) &=&\ 0 \\ && \vdots & \\ F^{a,i}_{2k}(\mathbf{u}) :=&\ u_i^2-u_{i+1}^2+((u_i-u_{i-1})+(u_{i+2}-u_{i+1})) &=&\ 0, \quad i=2,\dots,k-1 \\ && \vdots & \\ &\ u_k^2-u_{k+1}^2+(u_k-u_{k-1})+(u_k-u_{k+1}) &=&\ 0 \end{aligned} \tag{4.13}$$

As in the previous sections we write this set of equations as

$$\mathcal{M}^a_{2k} \cdot \big(u_1 - u_2, u_2 - u_3, \ldots, u_k - u_{k+1}\big)^T = 0 \tag{4.14}$$

where the (k,k)-matrices $\mathcal{M}^a_{2k} = \mathcal{M}^a_{2k}(u_1, \ldots, u_{k+1})$ is defined as follows:

$$\mathcal{M}^a_2(u_1, u_2) := u_1 + u_2 + 2, \tag{4.15}$$

and for $k \geq 2$:

$$\mathcal{M}^a_{2k} := \begin{pmatrix} u_1{+}u_2{+}\,1 & -1 & 0 & & & \cdots & 0 \\ -1 & u_2{+}u_3 & -1 & 0 & & & \vdots \\ 0 & -1 & u_3{+}u_4 & -1 & 0 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & 0 & -1 & u_{k\text{-}2}{+}u_{k\text{-}1} & -1 & 0 \\ \vdots & & & 0 & -1 & u_{k\text{-}1}{+}u_k & -1 \\ 0 & \cdots & & \cdots & 0 & -1 & u_k{+}u_{k+1}{+}\,1 \end{pmatrix}.$$

By analogy with the derivation of (4.7) or (4.11), respectively, we find the determination equation for actual symmetric (2k)-periodic orbits of type (a,a) as

$$F^a_{2k}\,(u_1, \ldots, u_{k+1}) := \Big(F^{a,1}_{2k}(\mathbf{u}), \ldots, F^{a,k-1}_{2k}(\mathbf{u}), F^{a,k}_{2k}(\mathbf{u})\Big) = 0,$$

where the $F^{a,i}_{2k}$, $i = 1, \ldots, k-1$, are defined in (4.13) and

$$F^{a,k}_{2k}(\mathbf{u}) := \det \mathcal{M}^a_{2k}(u_1, \ldots, u_{k+1}).$$

5 Selected bifurcations

In this section we study the bifurcation of periodic orbits up to period 5. To this end we consider simultaneously the systems $(\mathbb{R}^{\mathbb{Z}}_{\lambda}, \sigma)$ and $(\mathbb{R}^2, f)$. First we show that two branches of 1-periodic orbits will be created in the course of a saddle-centre bifurcation.

In Section 5.2 we show that a branch of 2-periodic orbits bifurcates in the course of a period doubling bifurcation from one of the branches of 1-period orbits. In the subsequent Sections 5.3-5.5 we consider subharmonic bifurcations of 3-, 4- and 5-periodic orbits from the branch of 1-periodic orbits and discuss the stability of the bifurcating orbits. In this course, we also address further possible bifurcations from these branches. In addition, in Section 5.4.2 we discuss the period-doubling bifurcation from 2- to 4-periodic orbits in more detail.

Finally, in Section 5.5.2 we show that apart from the subharmonic bifurcations 5-periodic orbits also will be created by a saddle-centre bifurcation.

In a concluding section we summarise our results and present a bifurcation diagram for (symmetric) periodic orbits up to period 5, cf. Figure 8.

5.1 Saddle-center bifurcation of 1-periodic orbits

Note that all 1-periodic orbits are symmetric. Consider 1-periodic orbits $\mathcal{O}(\mathbf{u}) \subset \mathbb{R}^{\mathbb{Z}}$ with

$$\mathbf{u} = (\overline{u}). \tag{5.1}$$

According to (2.1)-(2.3) those orbits belong to $\mathbb{R}^{\mathbb{Z}}_{\lambda}$ if

$$\lambda + 2u - u^2 = 0. \tag{5.2}$$

Let $\lambda_{\text{1-per}}(\cdot)$ be the solving function of (5.2):

$$\lambda_{\text{1-per}}(u) := u^2 - 2u.$$

The unique minimum of $\lambda_{\text{1-per}}(\cdot)$ we denote by λ_{sc}. Clearly

$$\lambda_{sc} = -1.$$

Hence, for $\lambda > \lambda_{sc}$ there exist two different 1-periodic orbits which are defined by

$$u = u^+ := 1 + \sqrt{1+\lambda} \quad \text{and} \quad u = u^- := 1 - \sqrt{1+\lambda}. \tag{5.3}$$

These two branches merge at λ_{sc} into one 1-periodic orbit defined by $u = u_{sc} := 1$ in the course of a saddle-center bifurcation. There are no 1-periodic orbits for $\lambda < \lambda_{sc}$. The usage of the term "saddle-center bifurcation" in this context is justified by considering the system (2.7). First we want to note that fixed points (u, u) of $f_\lambda(\cdot,\cdot)$ correspond directly to 1-periodic orbits $\mathbf{u} = (\overline{u})$ of $(\mathbb{R}^{\mathbb{Z}}_{\lambda}, \sigma)$.

Then according to the above (u, u) is a fixed point of $f_{\lambda_{\text{1-per}}(u)}(\cdot,\cdot)$. We find for the Jacobian $Df_{\lambda_{\text{1-per}}(u)}$ that

$$Df_{\lambda_{\text{1-per}}(u)}(u,u) = \begin{pmatrix} 2u & -1 \\ 1 & 0 \end{pmatrix}$$

and for the spectrum of that Jacobian we find

$$\operatorname{spec} D_{\lambda_{\text{1-per}}(u)} f(u,u) = \left\{ u + \sqrt{(u)^2 - 1}, u - \sqrt{(u)^2 - 1} \right\}. \tag{5.4}$$

Taking (5.3) into consideration, we see that $\operatorname{spec} D_{\lambda_{\text{1-per}}(u^+)} f(u^+, u^+)$ is real, while for $\lambda \in (\lambda_{sc}, 3)$ the spectrum $\operatorname{spec} D_{\lambda_{\text{1-per}}(u^-)} f(u^-, u^-)$ is complex. Due to the reversibility we know that with an eigenvalue μ of $D_{\lambda_{\text{1-per}}(u)} f(u,u)$ also μ^{-1} is an eigenvalue [5, Proposition 5.2]. Hence, for $\lambda \in (\lambda_{sc}, 3)$ all eigenvalues in question have modulus one. Altogether this shows that the fixed points (u^+, u^+) are saddles while the fixed points (u^-, u^-) are centres.

Consider for $\lambda \in [\lambda_{sc}, 3]$ the eigenvalue $\mu(\lambda)$ of $Df_\lambda(u^-, u^-)$. By (5.3) and (5.4) we find

$$\mu(\lambda) = 1 - \sqrt{1+\lambda} + \sqrt{\left(1 - \sqrt{1+\lambda}\right)^2 - 1}.$$

The real part $\Re\mu(\lambda)$ decreases monotonically from $\mu(\lambda_{sc} = -1) = 1$ to $\mu(3) = -1$. So, $\mu(\lambda)$ crosses all roots of unity transversely. Then, if a further transversality condition is satisfied, this gives rise to corresponding subharmonic bifurcations, cf. [2, Theorem 3].

In the following sections we will study these bifurcations up to period 5 in more detail.

5.2 Period-doubling bifurcation

In this section we study 2-periodic orbits. We show that the 2-periodic orbits arise in the course of a period doubling bifurcation from the u^--branch, cf. (5.3), of 1-periodic orbits. Further we consider the spectrum of Df^2 along this branch.

Consider 2-periodic orbits $\mathcal{O}(\mathbf{u})$ with

$$\mathbf{u} = (\overline{u_1, u_2}).$$

Note that all 2-periodic orbits are symmetric of type (a,a), cf. Definition 4.2. According to (4.14) and (4.15) those orbits are characterised by

$$(u_1 + u_2 + 2)(u_1 - u_2) = 0.$$

The solutions $u_1 = u_2$ correspond to 1-periodic orbits. So, the actual 2-periodic orbits, i.e. those whose prime period is equal to 2, are characterised by

$$u_1 = -u_2 - 2. \tag{5.5}$$

These 2-periodic orbits bifurcate from the the u^--branch of 1-periodic orbits at

$$u_1 = u_2 =: u_{pd} = -1. \tag{5.6}$$

Define $\mathbf{u}_{pd} := (\overline{u_{pd}, u_{pd}})$. The associated λ-value can be gained by solving $T_1(\mathbf{u}, \lambda) = 0$ for λ, cf. (2.2) and Section 4.3.2, respectively,

$$T_1(\mathbf{u}, \lambda) = 0 \quad \Leftrightarrow \quad \lambda = \lambda^*(\mathbf{u}) = u_1^2 - u_0 - u_2. \tag{5.7}$$

With that we find that the period-doubling bifurcation takes place at

$$\lambda_{pd} := \lambda^*(\mathbf{u}_{pd}) = 3. \tag{5.8}$$

Note that for $\lambda > \lambda_{pd}$ there exists exactly one branch of 2-periodic orbits: Let $\mathbf{u}_{\text{2-per}}$ be a 2-periodic orbit defined by (5.5), i.e. $\mathbf{u}_{\text{2-per}}(u) = (\overline{-u - 2, u})$. The associated $\lambda_{\text{2-per}}(u)$-values result from $T_1(\mathbf{u}_{\text{2-per}}, \lambda) = 0$:

$$\lambda_{\text{2-per}}(u) := \lambda^*(\mathbf{u}_{\text{2-per}}(u)) = (u + 1)^2 + 3. \tag{5.9}$$

Now, the existence of merely one branch of 2-periodic orbits follows from the facts that $\lambda_{\text{2-per}}(\cdot)$ takes its minimum at $u = u_{pd}$ and that $\mathbf{u}_{\text{2-per}}(u_{pd} + v) = \mathbf{u}_{\text{2-per}}(u_{pd} - v)$ and $\lambda_{\text{2-per}}(u_{pd} + v) = \lambda_{\text{2-per}}(u_{pd} - v)$.

We want to remark that, cf. Section 5.1, $\operatorname{spec} Df_{\lambda_{pd}}(u_{pd}, u_{pd}) = \{-1\}$. Since (u^-, u^-) is a fixed point of $f_{\lambda_{pd}}(\cdot, \cdot)$ it follows immediately that

$$\operatorname{spec} Df^2_{\lambda_{pd}}(u_{pd}, u_{pd}) = \{1\}. \tag{5.10}$$

Next we consider $\operatorname{spec} Df^2_\lambda$ along the 2-per branch: To this end we note that

$$f^2_\lambda(u, v) = \begin{pmatrix} (u^2 - v - \lambda)^2 - u - \lambda \\ u^2 - v - \lambda \end{pmatrix}, \tag{5.11}$$

and accordingly

$$Df_\lambda^2(u,v) = \begin{pmatrix} 4u(u^2 - v - \lambda) - 1 & -2(u^2 - v - \lambda) \\ 2u & -1 \end{pmatrix}.$$

For (u,v) belonging to the branch of 2-periodic orbits, that is $v = -u - 2$ and $\lambda = \lambda_{\text{2-per}}(u)$, cf. above, we therefore find

$$Df^2_{\lambda_{\text{2-per}}(u)}(u, -u-2) = \begin{pmatrix} -4u(u+2) - 1 & 2(u+2) \\ 2u & -1 \end{pmatrix}.$$

Hence the spectrum $\operatorname{spec} Df^2_{\lambda_{\text{2-per}}(u)}(u, -u-2) = \{\mu^\pm_{\text{2-per}}(u)\}$ reads

$$\mu^\pm_{\text{2-per}}(u) = -\frac{X(u)+2}{2} \pm \sqrt{\frac{(X(u)+2)^2}{4} - 1}, \quad X(u) := 4u(u+2).$$

In order to discuss whether the eigenvalues $\mu^\pm_{\text{2-per}}(u)$ are real or complex we consider the discriminant $D_2(u) := \frac{(X(u)+2)^2}{4} - 1$. The eigenvalues $\mu^\pm_{\text{2-per}}(u)$ are complex if and only if $D_2(u) < 0$. Hence, the eigenvalues $\mu^\pm_{\text{2-per}}(u)$ are complex if and only if $X \in (-4, 0)$. The latter is the case if and only if $u \in (-2, 0)$. Note that u-values in $(-2,-1]$ and in $[-1, 0)$ define the same branch of 2-periodic orbits, cf. the comments following (5.8). For that reason it is enough to consider $u \in (-2, -1]$. Associated to $u = u_{pd} = -1$ we find $X_{pd} = -4$ and $\mu^\pm_{\text{2-per}} = 1$, cf. (5.10). At $X = 0$ we have $\mu^\pm_{\text{2-per}} = -1$. In Section 5.4.2 we show that at the related

$$u =: u_{2pd} = -2 \quad \text{and} \quad \lambda_{2pd} = 4 \tag{5.12}$$

a further period doubling bifurcation takes place.

From Lemma 2.10 it follows that for $X \in (-4, 0)$ the corresponding $\mu^\pm_{\text{2-per}}$ are located on the unit circle $S^1 \subset \mathbb{C}$. Considering the corresponding real parts of the eigenvalues we see that for $u \in (-2, -1)$ the eigenvalues move monotonically along the S^1. This suggests that between λ_{pd} and λ_{2pd} subharmonic bifurcations from the branch of 2-periodic orbits will take place. However, we do not follow this line of action in the present paper.

5.3 Subharmonic bifurcations of 3-periodic orbits

Consider symmetric 3-periodic orbits $\mathcal{O}(\mathbf{u})$. According to Lemma 4.3 such an orbit is of type (a,b), and we may write

$$\mathbf{u} = (\overline{u_1, u_2, u_2}). \tag{5.13}$$

Further, according to (4.5) the determination equation for symmetric 3-periodic orbits reads

$$\mathcal{M}_3(u_1, u_2)(u_1 - u_2) = (u_1 + u_2 + 1)(u_1 - u_2) = 0.$$

So, the actual symmetric 3-periodic orbits are characterised by

$$u_2 = -u_1 - 1. \tag{5.14}$$

These 3-periodic orbits bifurcate from the u^--branch of 1-periodic orbits at

$$u_1 = u_2 =: u_{3sh} = -1/2.$$

Define $\mathbf{u}_{3sh} := (\overline{u_{3sh}, u_{3sh}, u_{3sh}})$. Using again the function λ^*, defined in (5.7), we find that 3-periodic orbits bifurcate from the branch of 1-periodic orbits at

$$\lambda_{3sh} := \lambda^*(\mathbf{u}_{3sh}) = 5/4.$$

Let $\mathbf{u}_{3\text{-per}}$ be a 3-periodic orbit defined by (5.14), i.e.

$$\mathbf{u}_{3\text{-per}}(u) = (\overline{u, -u-1, -u-1}).$$

The associated $\lambda_{3\text{-per}}(u)$-values read, cf. also (5.9),

$$\lambda_{3\text{-per}}(u) := \lambda^*(\mathbf{u}_{3\text{-per}}(u)) = (u+1)^2 + 1. \tag{5.15}$$

It is worth mentioning that $\lambda_{3\text{-per}}(\cdot)$ takes its minimum at $u_{3sc} := -1 \neq u_{3sh}$. The associated λ-value we denote by

$$\lambda_{3sc} := \lambda_{3\text{-per}}(u_{3sc}) = 1.$$

So, obviously $\lambda_{3sc} < \lambda_{3sh}$. That means that at $\lambda = \lambda_{3sh}$ the branches of 1- and 3-periodic orbits cross each other - a transcritical bifurcation takes place, see also Figure 8.

In what follows we study the stability of the 3-periodic orbits. To this end we consider the system $(\mathbb{R}^2, f_\lambda)$, and compute spec Df_λ^3 along the 3-per branch.

According to (5.11) we find $f_\lambda^3(u,v) = f_\lambda((u^2-v-\lambda)^2 - u - \lambda, u^2 - v - \lambda)$. Straightforward computations reveal that the eigenvalues $\{\mu^\pm_{3\text{-per}}(u)\}$ of the Jacobian $Df^3_{\lambda_{3\text{-per}}(u)}(u, -u-1)$ read

$$\mu^\pm_{3\text{-per}}(u) = 4u^3 + 8u^2 + 5u + 2 \pm (2u+1)\sqrt{4u^4 + 12u^3 + 13u^2 + 8u + 3}.$$

Considering the discriminant $D_3(u) := 4u^4 + 12u^3 + 13u^2 + 8u + 3$ of the eigenvalues reveals the following, cf. also left panel in Figure 2: $u = -1$ and $u = -3/2$ are the only zeros of

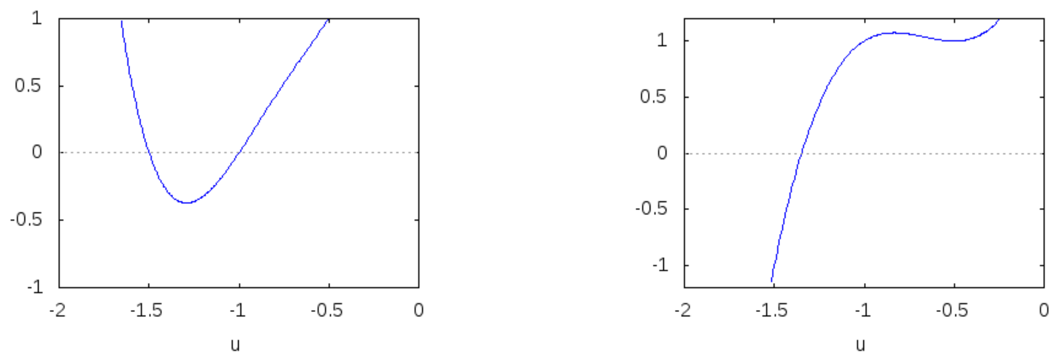

Figure 2: Left panel: The graph of the discriminant D_3; Right panel: graph of the polynomial $4u^3 + 8u^2 + 5u + 2$.

$D_3(\cdot)$. Moreover, near the subharmonic bifurcation point $u_{3sh} = -1/2$ the eigenvalues stay real. Further, while crossing the saddle-center bifurcation point $u_{3sc} = -1$ the eigenvalues become complex and further on, while crossing $u_{3pd} := -3/2$ become again real.

At the point u_{3pd} we have $\mu^{\pm}_{3\text{-per}}(u_{3pd}) = -1$. This suggests a period doubling bifurcation from the branch of 3-periodic orbits. For the associated λ-value we find $\lambda_{3pd} := \lambda_{3\text{-per}}(u_{3pd}) = 5/4$. For $u \in (u_{3pd}, u_{3sc})$ we have $\Re(\mu^{\pm}_{3\text{-per}}(u)) = 4u^3 + 8u^2 + 5u + 2$. According to the depiction in the right panel in Figure 2 the eigenvalues of Df^3_λ move (monotonically) along the unit circle S^1, cf. also Lemma 2.10. This gives rise to further subharmonic bifurcations form the branch of 3-periodic orbits. However, we are not going to study these bifurcations in this paper.

5.4 4-periodic orbits

In this section we consider symmetric 4-periodic orbits. We show that subharmonic bifurcations from the branch of 1-periodic orbits as well as period-doubling bifurcations generate 4-periodic orbits.

5.4.1 Subharmonic bifurcations of 4-periodic orbits

Let $\mathcal{O}(\mathbf{u})$ be a symmetric 4-periodic orbit. According to Lemma 4.4 those orbits are either of type (a,a) or of type (b,b).

First we assume that $\mathcal{O}(\mathbf{u})$ is of type (b,b). As explained in Section 4.3.1 we may assume that

$$\mathbf{u} = (\overline{u_1, u_2, u_2, u_1})$$

and the determination equation of those orbits reads

$$\mathcal{M}_4^b(u_1, u_2)(u_1 - u_2) = (u_1 + u_2)(u_1 - u_2) = 0.$$

So, the actual symmetric 4-periodic orbits are characterised by

$$u_1 = -u_2. \tag{5.16}$$

These 4-periodic orbits bifurcate from the u^--branch of 1-periodic orbits at $(\mathbf{u}, \lambda) = (\mathbf{u}_{4sh}, \lambda_{4sh})$, where

$$\mathbf{u}_{4sh} := (\overline{u_{4sh}})\,,\; u_{4sh} = 0 \quad \text{and} \quad \lambda_{4sh} := \lambda^*(\mathbf{u}_{4sh}) = 0. \tag{5.17}$$

Note that $\mathbf{u}_{4sh}$ satisfies (5.16) and recall that λ^* has been defined in (5.7).

Further, let $\mathbf{u}^b_{4\text{-per}}$ be a 4-periodic orbit of type (b,b) defined by (5.16), i.e.

$$\mathbf{u}^b_{4\text{-per}}(u) = (\overline{-u, u, u, -u}). \tag{5.18}$$

The associated $\lambda^b_{4\text{-per}}(u)$-values, cf also (5.9) or (5.15),

$$\lambda^b_{4\text{-per}}(u) := \lambda^*(\mathbf{u}^b_{4\text{-per}}(u)) = u^2. \tag{5.19}$$

takes its minimum at $u_{4sh} := 0$, and $\lambda^b_{4\text{-per}}(u_{4sh}) = \lambda_{4sh} = 0$.

Now, consider $\lambda^b_{4\text{-per}}(u) = c > 0$. The associated $u^\pm := \pm\sqrt{c}$ define the same 4-periodic of type (b,b):

$$\mathcal{O}(\overline{-\sqrt{c}, \sqrt{c}, \sqrt{c}, -\sqrt{c}}) = \mathcal{O}(\overline{\sqrt{c}, -\sqrt{c}, -\sqrt{c}, \sqrt{c}}).$$

That is there is exactly one branch of 4-periodic orbits of type (b,b) bifurcating.

In order to find the missing second branch bifurcating from the branch of 1-periodic orbits, cf. [2, Theorem 3], we now assume that the symmetric 4-periodic orbit $\mathcal{O}(\mathbf{u})$ is of type (a,a). As explained in Section 4.3.2 we may assume that

$$\mathbf{u} = (\overline{u_1, u_2, u_3, u_2}) \tag{5.20}$$

and according to (4.14) the determination equation of those orbits reads

$$\mathcal{M}^a_4(u_1, u_2, u_3)(u_1 - u_2, u_2 - u_3)^T = \begin{pmatrix} u_1 + u_2 + 1 & -1 \\ -1 & u_2 + u_3 + 1 \end{pmatrix} \begin{pmatrix} u_1 - u_2 \\ u_2 - u_3 \end{pmatrix} = 0. \tag{5.21}$$

According to the explanations in Section 4.3.2 this equation can be rewritten as

$$\begin{array}{rclcl} F^{a,1}_4(\mathbf{u}) & = & u_1^2 - u_2^2 + ((u_1 - u_2) + (u_3 - u_2)) & = & 0 \\ F^{a,2}_4(\mathbf{u}) & = & \det \mathcal{M}^a_4(u_1, u_2, u_3) & = & 0. \end{array} \tag{5.22}$$

In order to check whether a branch of symmetric 4-periodic orbits of type (a,a) bifurcates from the branch of 1-periodic orbits we first consider

$$M^a_4(u) := \det \mathcal{M}^a_4(u, u, u) = (2u + 2)M^b_4(u) = 4u^2 + 4u = 0. \tag{5.23}$$

The two solutions of equation (5.23) are $u_{pd} = -1$, cf. (5.6), and $u_{4sh} = 0$, cf. (5.17). Recall that u_{pd} refers to the period doubling from the branch of 1-periodic orbit. The bifurcating 2-periodic orbits, which can also be considered as symmetric 4-periodic orbits, are of type (a,a), and are exactly the ones which we find by solving (5.22) in a neighbourhood of $\mathbf{u}_{pd} = (\overline{u_{pd}})$.

Next we consider (5.22) near $\mathbf{u}_{4sh}$. To this end we employ the implicit function theorem. We find

$$F^a_4(\mathbf{u}_{4sh}) = 0 \quad \text{and} \quad \det D_{(u_1,u_2)} F^a_4(\mathbf{u}_{4sh}) = \det \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix} = 4 \neq 0.$$

So, $F^a_4(\mathbf{u}) = 0$ can be solved for $(u_1, u_2) = (u^a_1(u_3), u^a_2(u_3))$. For the solving functions we find

$$u^a_1(u_3) = -u_3 \quad \text{and} \quad u^a_2(\cdot) \text{ is even.} \tag{5.24}$$

This can easily be verified by the following observation:

$$F^{a,2}_4(-u_3, u_2, u_3) = -F^{a,1}_4(-u_3, u_2, u_3) = u_2^2 - u_3^2 + 2u_2,$$

and the equation $u_2^2 - u_3^2 + 2u_2 = 0$ can be solved for u_2 depending on u_3^2. So the corresponding branch of 4-periodic orbits is defined by

$$\mathbf{u}^a_{4\text{-per}} = \left(\overline{-u, u^a_2(u), u, u^a_2(u)}\right), \quad (u^a_2(u))^2 + 2u^a_2(u) = u^2. \tag{5.25}$$

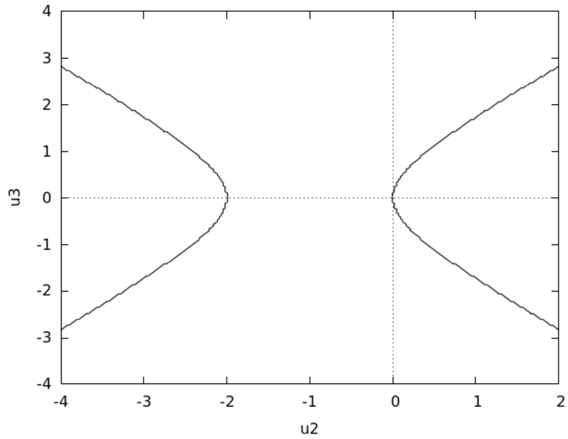

Figure 3: The hyperbola $u_2^2 - u_3^2 + 2u_2 = 0$

Note that $u_2^2 - u_3^2 + 2u_2 = 0$ defines an hyperbola, cf. Figure 3. The branch under consideration is related to the right branch displayed in this figure, namely the one to which belongs the point $(u_2, u_3) = (0, 0)$, cf. also (5.17).

As in the previous sections we find the associated $\lambda^a_{\text{4-per}}(u)$-values as

$$\lambda^a_{\text{4-per}}(u) = \lambda^*(\mathbf{u}^a_{\text{4-per}}(u)) = u^2 - 2u_2^a(u) = (u_2^a(u))^2 . \tag{5.26}$$

The latter equality follows with (5.25). Since $u_2^a(\cdot)$ is even, cf. (5.24), it follow from (5.26) that $\lambda^a_{\text{4-per}}(\cdot)$ takes its unique minimum at $u_{4sh} = 0$ and $\lambda^a_{\text{4-per}}(0) = 0$.

Note that for $\lambda^a_{\text{4-per}}(u) > \lambda_{4sh}$ there is only one branch of symmetric 4-periodic orbits of type (a,a), namely the branch $\mathbf{u}^a_{\text{4-per}}$. So, altogether for $\lambda > \lambda_{4sh}$ there exist the two different branches $\mathbf{u}^a_{\text{4-per}}$ and $\mathbf{u}^b_{\text{4-per}}$.

Next we consider spec Df^4_λ along the 4-per branches. We start with the $\mathbf{u}^b_{\text{4-per}}$-branch. According to (5.18) and (5.19) we consider the spectrum spec $Df^4_{v^2}(-v, v) = \left\{\mu^{b,\pm}_{\text{4-per}}(v)\right\}$:

$$\mu^{b,\pm}_{\text{4-per}}(v) = 8v^4 + 1 \pm 4v^2\sqrt{4v^4 + 1}.$$

This shows that the associated 4-periodic orbits are hyperbolic.

Next we consider the spectrum along the $\mathbf{u}^a_{\text{4-per}}$-branch. Set $u_2 = v$. Then from (5.24) and (5.25) it follows

$$u_1 = -u_3 = \pm\sqrt{u_2^2 + 2u_2}.$$

According to (5.25) and (5.26) the addressed spectrum spec $Df^4_{v^2}(-\sqrt{v^2 + 2v}, v) = \left\{\mu^{a,\pm}_{\text{4-per}}(v)\right\}$ reads:

$$\mu^{a,\pm}_{\text{4-per}}(v) = -8v^4 - 16v^3 + 1 \pm 4v\sqrt{4v^6 + 16v^5 + 16v^4 - v^2 - 2v}, \quad v \geq 0.$$

Recall from Figure 3 that the branch under consideration is related to $v \geq 0$. In this regard we want to remark that it makes no difference which part of the right branch in Figure 3 ($u_3 \geq 0$ or $u_3 \leq 0$) we do consider - both of them generate the same orbits. Figure 4 shows the graph of the discriminant $D_4(v) := 4v^6 + 16v^5 + 16v^4 - v^2 - 2v$. From that we infer that for $v \in (0, v_0)$, where v_0 is the positive zero of $D_4(\cdot)$, $\operatorname{spec} Df^4(-\sqrt{v^2+2v}, v, v^2) \subset S^1$. So we may expect further subharmonic bifurcations from the $\mathbf{u}^a_{4\text{-per}}$-branch. However, we shall not further pursue this line of action.

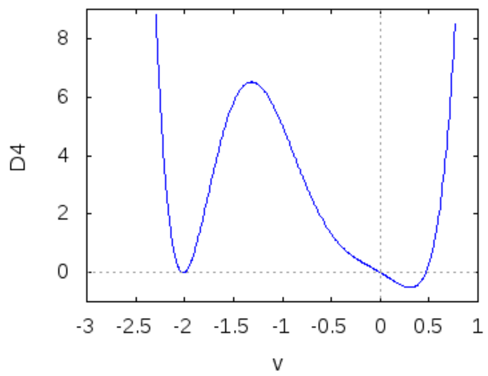

Figure 4: The discriminant D_4 of the eigenvalues $\mu^{a,\pm}_{4\text{-per}}(v)$

5.4.2 Period-doubling bifurcation from 2- to 4-periodic orbits

In this section we verify the period doubling from the branch of 2-periodic orbits announced in Section 5.2.

Note that the $\mathbf{u}^b_{4\text{-per}}$-branch does not intersect the 2-per-branch. So, to determine the period doubling bifurcation from the branch of 2-periodic orbits we consider 4-periodic orbits of type (a,a). So again we may assume that $\mathbf{u}$ has the form (5.20), and again we start with equation (5.21), which with a view to our aim we rewrite as, cf. also (5.5),

$$\mathcal{M}^{pd}_{2,4}(u_1, u_2, u_3)(u_1 + u_2 + 2, u_3 + u_2 + 2)^T = 0,$$

where

$$\mathcal{M}^{pd}_{2,4}(u_1, u_2, u_3) = \begin{pmatrix} u_1 - u_2 - 1 & 1 \\ 1 & u_3 - u_2 - 1 \end{pmatrix}.$$

In the same way as in the previous section we find the actual 4-periodic by solving $F^{pd}_{2,4}(\mathbf{u}) = 0$, where

$$\begin{aligned} F^{pd,1}_{2,4}(\mathbf{u}) &= u_1^2 - u_2^2 + ((u_1 - u_2) + (u_3 - u_2)) \\ F^{pd,2}_{2,4}(\mathbf{u}) &= \det \mathcal{M}^{pd}_{2,4}(u_1, u_2, u_3). \end{aligned}$$

Define in accordance with the explanations in Section 5.2, cf. in particular (5.12)

$$\mathbf{u}_{2pd} := \left(\overline{-u_{2pd} - 2, u_{2pd}, -u_{2pd} - 2, u_{2pd}}\right) = \left(\overline{0, -2, 0, -2}\right).$$

Recall that $\mathcal{O}(\mathbf{u}_{2pd})$ is the period-doubling point in branch of 2-periodic orbits. We find that

$$F^{pd}_{2,4}(\mathbf{u}_{2pd}) = 0 \quad \text{and} \quad \det D_{(u_1,u_2)} F^{pd}_{2,4}(\mathbf{u}_{2pd}) = \det \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} = -4 \neq 0.$$

So, near $\mathbf{u}_{2pd}$ the equation $F_{2,4}^{2pd}(\mathbf{u}) = 0$ can be solved by means of the implicit function theorem for $(u_1, u_2) = \left(u_1^{2pd}(u_3), u_2^{2pd}(u_3)\right)$. In the same way as in the previous section we find $u_1^{2pd}(u_3) = -u_3$ and $u_2^{2pd}(\cdot)$ even. The corresponding branch of 4-periodic orbits

$$\mathbf{u}_{4\text{-per}}^{a,2pd} = \left(\overline{-u, u_2^{pd}(u), u, u_2^{2pd}(u)}\right), \quad (u_2^a(u))^2 + 2u_2^a(u) = u^2.$$

is related to the left branch of the hyperbola in Figure 3.

The spectrum of Df_λ^4 along this branch is the same as for the $\mathbf{u}_{4\text{-per}}^a$-branch considered in Section 5.4.1. With the only difference that the period doubling branch under consideration is related to $v \leq -2$. So we have spec $Df_{v^2}^4(-\sqrt{v^2+2v}, v) = \left\{\mu_{4\text{-per}}^{a,2pd}(v)\right\}$:

$$\mu_{4\text{-per}}^{a,2pd}(v) = -8v^4 - 16v^3 + 1 \pm 4v\sqrt{4v^6 + 16v^5 + 16v^4 - v^2 - 2v}, \quad v \leq -2.$$

According to Figure 5 there is a $\lambda_{4pd} > \lambda_{2pd}$ such that for $\lambda \in (\lambda_{2pd}, \lambda_{4pd})$ the associated 4-periodic orbits are elliptic. This suggests further subharmonic and period-doubling bifurcations from this branch of 4-periodic orbits.

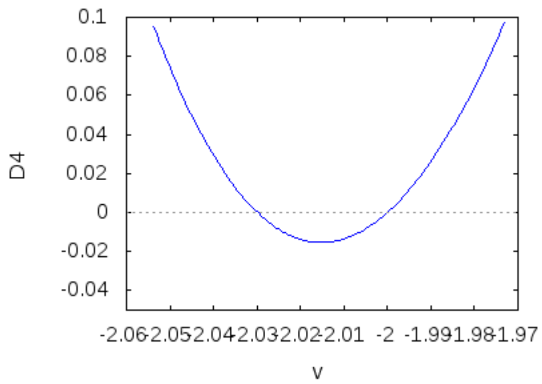

Figure 5: Enlargement of the graph of the discriminant D_4 of the eigenvalues $\mu_{4\text{-per}}^{a,pd}(v)$ near $v = -2$, cf. also Figure 4.

5.5 5-periodic orbits

Consider symmetric 5-periodic orbits $\mathcal{O}(\mathbf{u})$. According to (4.3), cf. also Remark 4.6, there are six symmetric 5-periodic orbits within $\Sigma_{\lambda,R}^2$. In this section we show that twice two branches in each case of 5-periodic orbits bifurcate from the branch of 1-periodic orbits in the course of a subharmonic bifurcation. Further we show that the remaining two 5-periodic orbits will be created in the course of a "saddle-centre" bifurcation. However, here we confine to merely discover the corresponding solution branches and drop considerations concerning stability. For that reason we placed saddle-centre within quotation marks.

According to Lemma 4.3 all 5-orbits are of type (a,b), and we may write

$$\mathbf{u} = (\overline{u_1, u_2, u_3, u_3, u_2}). \tag{5.27}$$

Further, according to (4.5) and (4.6) the determination equation for symmetric 5-periodic orbits reads

$$\mathcal{M}_5(u_1,u_2,u_3)\,(u_1-u_2,u_2-u_3)^T = \begin{pmatrix} u_1+u_2+1 & -1 \\ -1 & u_2+u_3 \end{pmatrix} \begin{pmatrix} u_1-u_2 \\ u_2-u_3 \end{pmatrix} = 0.$$

According to (4.7) the actual 5-periodic orbits are determined by $F_5(u_1,u_2,u_3)=0$:

$$\begin{array}{rcccl} F_5^1(u_1,u_2,u_3) = & (u_1+u_2+1,-1)\cdot(u_1-u_2,u_2-u_3)^T & = & 0 \\ F_5^2(u_1,u_2,u_3) = & \det\mathcal{M}_5(u_1,u_2,u_3) & = & 0. \end{array} \tag{5.28}$$

5.5.1 Subharmonic bifurcations

In order to determine the symmetric 5-periodic orbits bifurcating from the branch of 1-periodic orbits we first check the necessary condition, cf. also (4.8),

$$M_5(u) = \det\mathcal{M}_5(u,u,u) = 4u^2+2u-1 = 0.$$

This yields the potential bifurcation points, cf. again (4.8),

$$u_{5sh}^{\pm} = -\tfrac{1}{4}\pm\tfrac{\sqrt{5}}{4}, \quad \lambda_{5sh}^{\pm} = \left(u_{5sh}^{\pm}\right)^2 - 2u_{5sh}^{\pm} = \tfrac{7}{8}\mp\tfrac{\sqrt{5}^3}{8}.$$

Further

$$D_{(u_1,u_3)}F_5(u_1,u_2,u_3) = \begin{pmatrix} 2u_1+1 & 1 \\ u_2+u_3 & u_1+u_2+1 \end{pmatrix}.$$

Therefore

$$\det D_{(u_1,u_3)}F_5(u,u,u) = \begin{vmatrix} 2u+1 & 1 \\ 1 & 2u \end{vmatrix} + \begin{vmatrix} 2u+1 & 1 \\ -1+2u & 1 \end{vmatrix} = M_5(u)+2$$

and hence

$$\det D_{(u_1,u_3)}F_5(u_{5sh}^{\pm},u_{5sh}^{\pm},u_{5sh}^{\pm}) = 2.$$

So, at both points $(u_{5sh}^{\pm},u_{5sh}^{\pm},u_{5sh}^{\pm})$ the equation $F_5(u_1,u_2,u_3)=0$ can be solved for $(u_1,u_3)(u_2)$. The corresponding solving functions we denote by $u_{1,5sh}^{\pm}(\cdot)$ and $u_{3,5sh}^{\pm}(\cdot)$.

Let $\mathbf{u}_{\text{5-per}}^{\pm}$ be the 5-periodic orbits defined by means of these solving functions, i.e.

$$\mathbf{u}_{\text{5-per}}^{\pm}(u) = \overline{(u_{1,5sh}^{\pm}(u),u,u_{3,5sh}^{\pm}(u),u_{3,5sh}^{\pm}(u),u)}. \tag{5.29}$$

The associated $\lambda_{\text{5-per}}^{\pm}(u)$-values result as in the previous sections:

$$\lambda_{\text{5-per}}^{\pm}(u) := \lambda^*(\mathbf{u}_{\text{5-per}}^{\pm}(u)) = \left(u_{1,5sh}^{\pm}(u)\right)^2 - 2u. \tag{5.30}$$

Next we make clear that $\lambda_{\text{5-per}}^{\pm}(\cdot)$ takes its minimum at $u=u_{5sh}^{\pm}$: With

$$F_5(u_{1,5sh}^{\pm}(u),u,u_{3,5sh}^{\pm}(u)) \equiv 0$$

we find

$$\left(Du^{\pm}_{1,5sh}(u), Du^{\pm}_{3,5sh}(u)\right)^T = -\left(D_{(u_1,u_3)}F_5(u^{\pm}_{1,5sh}(u), u, u^{\pm}_{3,5sh}(u))\right)^{-1} D_{u_2}F_5(u^{\pm}_{1,5sh}(u), u, u^{\pm}_{3,5sh}(u)),$$

and hence

$$Du^{\pm}_{1,5sh}(u^{\pm}_{5sh}) = 2\left(2u^{\pm}_{5sh} + 1\right). \tag{5.31}$$

From (5.30) and (5.31) we finally infer that

$$D\lambda^{\pm}_{5\text{-per}}(u^{\pm}_{5sh}) = 2M_5(u^{\pm}_{5sh}) = 0.$$

Similarly we get

$$D^2\lambda^{\pm}_{5\text{-per}}(u^{\pm}_{5sh}) > 0.$$

Finally we make clear that at $\left(u^{\pm}_{5sh}, \lambda^{\pm}_{5sh}\right)$ indeed two branches of 5-periodic orbits bifurcate from the branch of 1-periodic orbits at $\mathbf{u}^{\pm}_{5sh} = (\overline{u^{\pm}_{5sh}})$. To this end we note that $Du^{\pm}_{1,5sh}(u^{\pm}_{5sh}) \neq 0$, cf. (5.31). So, three of the five entries in $\mathbf{u}^{\pm}_{5\text{-per}}(u) - \mathbf{u}^{\pm}_{5sh}$ change sign if u crosses $u^{\pm}_{5sh}$, while the remaining two (the ones related to $u^{\pm}_{3,5sh}$) have the same u-dependence, cf. also (5.29). Consequently the number of positive signs (in one period) of $\mathbf{u}^{\pm}_{5\text{-per}}(u) - \mathbf{u}^{\pm}_{5sh}$, $u < u^{\pm}_{5sh}$, does not equal the number of positive signs of $\mathbf{u}^{\pm}_{5\text{-per}}(v) - \mathbf{u}^{\pm}_{5sh}$, $v > u^{\pm}_{5sh}$. Therefore, for $u < u^{\pm}_{5sh}$ the orbit $\mathcal{O}(\mathbf{u}^{\pm}_{5\text{-per}}(u))$ cannot be equal to an orbit $\mathcal{O}(\mathbf{u}^{\pm}_{5\text{-per}}(v))$, $v > u^{\pm}_{5sh}$ (u, v close to $u^{\pm}_{5sh}$).

5.5.2 Saddle-centre bifurcation

Now we discover the remaining two (branches of) 5-periodic orbits. To this end we consider the determination equation (5.28) for 5-periodic orbits. The first equation in (5.28) can be solved for u_3. We get

$$u_3 = u_2^2 - u_1^2 - u_1 + 2u_2. \tag{5.32}$$

Plugging in into the second equation of (5.28) yields

$$u_2^3 + (u_1 + 4)u_2^2 - (u_1 - 3)(u_1 + 1)u_2 - u_1^3 - 2u_1^2 - u_1 - 1 = 0. \tag{5.33}$$

The coefficients of this polynomial we denote by

$$a(u_1) := u_1 + 4, \; b(u_1) := -(u_1 - 3)(u_1 + 1), \text{ and } c(u_1) := -u_1^3 - 2u_1^2 - u_1 - 1$$

Figure 6 displays the three solution branches of (5.33), drawn in black, together with the line $u_1 = u_2$, drawn in red, in a (u_1, u_2)-coordinate system. The upper two branches which have intersections with line $u_1 = u_2$ are associated to the subharmonic bifurcations discussed in the previous section.

The remaining lower branch is associated to the remaining 5-periodic orbits. To show that this solution branch indeed is related to saddle-centre bifurcation of 5-periodic orbits we consider an analytic representation of this branch. For that we apply Cardano's formula on the polynomial in (5.33). For the solution branch under consideration we get

$$u_2(u_1) = \sqrt{\frac{-4p(u_1)}{3}} \cos\left(\frac{1}{3}\arccos\left(-\frac{q(u_1)}{2}\sqrt{\frac{-27}{p(u_1)^3}}\right)\right) - \frac{a(u_1)}{3},$$

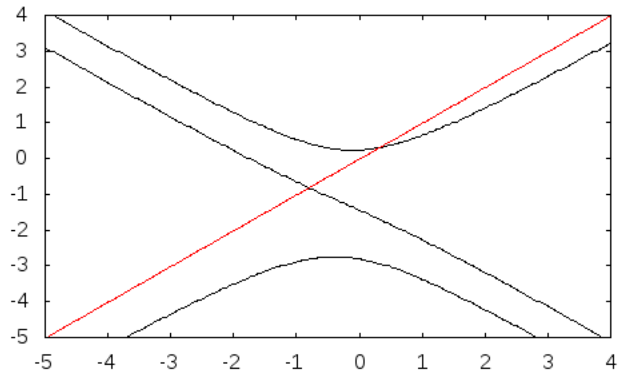

Figure 6: The solution branches of (5.33) (black) along with the line $u_1 = u_2$ (red).

where

$$p(u_1) := b(u_1) - \frac{a(u_1)^2}{3} \quad \text{and} \quad q(u_1) := \frac{2a(u_1)^3}{27} - \frac{a(u_1)b(u_1)}{3} + c(u_1).$$

The corresponding $\mathbf{u}$-sequence then reads

$$\mathbf{u}^{sc}_{5\text{-per}}(u) = \overline{(u, u_2(u), u_3(u), u_3(u), u_2(u))},$$

where $u_3(u_1)$ follows from (5.32). Again, the associated λ-values are:

$$\lambda^{sc}_{5\text{-per}}(u) := \lambda^*(\mathbf{u}^{sc}_{5\text{-per}}(u)) = u^2 - 2u_2(u).$$

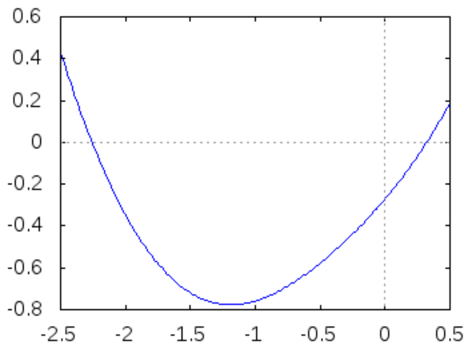

Figure 7: The graph of $\lambda^{sc}_{5\text{-per}}(\cdot)$, λ vs. u_1.

Figure 7 shows the graph of $\lambda^{sc}_{5\text{-per}}(\cdot)$. It shows that $\lambda^{sc}_{5\text{-per}}(\cdot)$ takes its minimum approximately at $u^{sc}_{1,5\text{-per}} \approx -1.19$. For the corresponding λ-value we find (approximately)

$$\hat{\lambda}^{sc}_{1,5\text{-per}} := \lambda^{sc}_{5\text{-per}}(u^{sc}_{1,5\text{-per}}) \approx -0.77.$$

Finally the same arguments as in the previous section can be applied to show that two (different) branches emanate.

5.6 Résumé

The bifurcation diagram in Figure 8 displays all bifurcations involving periodic orbits up to period 5 which we studied in the previous sections. The u-axis of the diagram in Figure 8 measures the u-value given in the representation (5.1) or the u_1-value given in the representations (5.13) and (5.27), respectively. Note that for symmetric periodic orbits of type (a,b) this value is uniquely defined - to some extent it defines the unique intersection of such an orbit with $\operatorname{Fix} R$, cf. also Section 4. But for symmetric periodic orbits with an even period such an value is not uniquely assignable. For that reason the branches of the 2- and 4-periodic orbits are only drawn schematically.

Throughout the elliptic parts of the branches are indicated by dashed lines while solid lines refer to hyperbolic orbits with the exception of the lines representing 5-periodic orbits. For those orbits we did not assess the stability behaviour. To identify this exceptions the branches for 5-periodic orbits are represented by dotted lines.

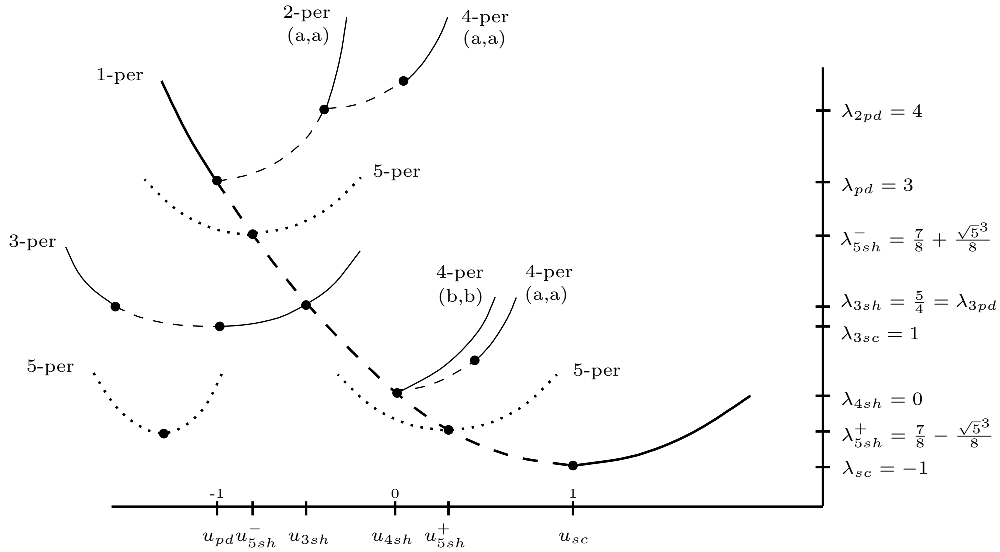

Figure 8: Bifurcation diagram for periodic orbits up to period 5. More explanations are given in the text.

At $\lambda = \lambda_{sc} = -1$ the 1-periodic orbits undergo a saddle-centre bifurcation. More precisely, for $\lambda > \lambda_{sc}$ there are two 1-periodic orbits. One branch is hyperbolic and the other one is at the beginning elliptic and becomes hyperbolic at $\lambda = \lambda_{pd} = 3$.

At λ_{pd} a period-doubling bifurcation takes place - a branch of 2-periodic orbits bifurcates from the corresponding branch of 1-periodic orbits. The bifurcating 2-periodic orbits are elliptic up to $\lambda = \lambda_{2pd} = 4$ and then become hyperbolic or $\lambda > \lambda_{2pd}$. At λ_{2pd} another period-doubling bifurcation takes place - a branch of 4-periodic orbits bifurcates from the branch of 2-periodic orbits. The bifurcating 4-periodic orbits are elliptic up to $\lambda = \lambda_{4pd}$ and then become hyperbolic. We conjecture that $\lambda = \lambda_{2pd}$ is the starting point for period-doubling cascade which can be observed in the example under consideration. All 2^n-periodic orbits appearing in this cascade will be of type (a,a). In this respect we refer also to the more general Lemma 5.1 below.

Further, branches of 3-periodic, 4-periodic and 5-periodic orbits bifurcate from the branch of 1-periodic orbits. Remarkable is the behaviour of 3-periodic orbits - cf. also Section 5.3. The change of the stability behaviour along the branch of 3-periodic orbits does not coincide with the bifurcation from the branch of 1-periodic orbits. However, we want to note that the bifurcating 3-periodic orbits hold a special position also in [2, Theorem 3]. According to that theorem for all bifurcating n-periodic orbits, $n \geq 5$ we may expect a behaviour as displayed in Figure 8 for the 4-periodic orbits.

Lemma 5.1. *Let $\mathcal{O}^2_\lambda$ be a branch of symmetric 4N-periodic orbits bifurcating from a branch $\mathcal{O}_\lambda$ of symmetric 2N-periodic orbits. Then the orbits within the two branches are of the same type.*

Proof. We give the proof in terms of the system $(\mathbb{R}^{\mathbb{Z}}_\lambda, \sigma)$ for orbits of type (a,a).

Let for the critical λ-value λ_0 $\mathcal{O}_{\lambda_0} = \{\mathbf{u}^1, \ldots, \mathbf{u}^{2N}\}$, where $R\mathbf{u}^1 = \mathbf{u}^1$ and $R\mathbf{u}^{N+1} = \mathbf{u}^{N+1}$. Then, for λ close to λ_0 we have $\mathcal{O}^2_\lambda = \{\mathbf{v}^1, \ldots, \mathbf{v}^{4N}\}$. Let the $\mathbf{v}$ be numbered in such a way such that $\mathbf{v}^i$ and $\mathbf{v}^{i+2N}$ are close to $\mathbf{u}^i$. Then due to the reversing symmetry either

$$R\mathbf{v}^{N+1} = \mathbf{v}^{3N+1} \quad \text{or} \quad R\mathbf{v}^{N+1} = \mathbf{v}^{N+1}, \quad R\mathbf{v}^{3N+1} = \mathbf{v}^{3N+1}.$$

In the latter case the orbit $\mathcal{O}^2_\lambda$ is of type (a,a) as we have assumed for the primary branch $\mathcal{O}_\lambda$.

To complete the proof we now assume $R\mathbf{v}^{N+1} = \mathbf{v}^{3N+1}$. Then because of $R\,\sigma^{-1} = \sigma R$, cf. Lemma 2.5 we find

$$R\mathbf{v}^{N+1} = \mathbf{v}^{3N+1} \quad \Rightarrow \quad R\mathbf{v}^{N} = \mathbf{v}^{3N+2} \quad \Rightarrow \ldots \Rightarrow \quad R\mathbf{v}^{1} = \mathbf{v}^{4N+1} = R\mathbf{v}^{1}.$$

Similarly, exploiting that $R\mathbf{v}^{N+1} = \mathbf{v}^{3N+1}$ is equivalent to $\mathbf{v}^{N+1} = R\mathbf{v}^{3N+1}$ we find that also $R\mathbf{v}^{2N+1} = \mathbf{v}^{2N+1}$. Hence, also under this assumption the orbit $\mathcal{O}^2_\lambda$ is of type (a,a).

The same type of arguments applies also for orbits of type (b,b). ∎

Remark 5.2. Let $\mathcal{O}^2_\lambda$ be a branch of symmetric (4N-2)-periodic orbits bifurcating from a branch $\mathcal{O}_\lambda$ of symmetric (2N-1)-periodic orbits. Recall that $\mathcal{O}_\lambda$ is of type (a,b). Then, by only taking the reversing symmetry into consideration as done in the proof of Lemma 5.1, it is not possible to make an assertion concerning the type of $\mathcal{O}^2_\lambda$. □

6 Subharmonic bifurcations

In our analysis we confine to show that for each $k \in \mathbb{N}$ symmetric $2k$-periodic orbits of type (b,b) bifurcate from the branch of 1-periodic orbits, cf. Theorem 6.10 below. The corresponding determination equations were provided in Section 4.3, cf. (4.11). Similar to (4.8) we find that

$$M_{2k}(u) := \det \mathcal{M}^b_{2k}(u, \ldots, u) = 0 \quad \text{and} \quad \lambda = u^2 - 2u \tag{6.1}$$

is a necessary condition for (u, λ) being a point at which symmetric (2k)-periodic orbits of type (b,b) bifurcate from the branch of 1-periodic orbits. Further, we compute the corresponding branches of symmetric (2k)-periodic orbits by means of the implicit function theorem.

In a preliminary section we provide the necessary properties of $M_{2k}(u)$ which play an essential role in studying the necessary condition (6.1). The subsequent section is reserved for Theorem 6.10 and its proof. As a last point we discuss generalisations of that theorem on the basis of numerical computations.

6.1 Preliminaries

In this section we mainly consider properties of $M_{2k}(u)$. In this respect we first note that $\hat{u}$ is a zero of $M_{2k}(u)$ if and only if $-2\hat{u}$ is an eigenvalue of $\mathcal{M}^b_{2k}(0,\dots,0)$, cf. (4.10). This gives:

Lemma 6.1. *All zeros of $M_{2k}(u)$ are real and simple.*

Proof. Since the matrix $\mathcal{M}^b_{2k}(0,\dots,0)$ is symmetric, all its eigenvalues are real and semisimple. Further, for all μ the last $k-2$ columns of $\mathcal{M}^b_{2k}(0,\dots,0) - \mu\,\mathrm{I}$, where I is the identity matrix, are linearly independent. Hence, all eigenvalues are simple. ∎

The next lemma reveals a corresponding statement related to symmetric $2k$-periodic orbits of type (a,a).

Lemma 6.2. $M^a_{2k}(u) := \det \mathcal{M}^a_{2k}(u,\dots,u) = (2u+2)M_{2k}(u)$.

Proof. We exploit that the determinant is a multilinear mapping of the columns of the underlying matrix. To this end we write $M^a_{2k}(u)$ as the sum of four determinants arising by a sum decomposition of the first and the k^{th} column of $M^a_{2k}(u)$ as

$$(2u+1,-1,0,\dots,0)^T = (2u,-1,0,\dots,0)^T + (1,0,0,\dots,0)^T \quad \text{and}$$

$$(0,\dots,0,-1,2u+1)^T = (0,\dots,0,-1,2u)^T + (0,\dots,0,0,1)^T,$$

respectively. The calculation of these determinants immediately yields the statement. ∎

Consider the representation of $M^a_{2k}(u)$ given in the lemma. The zero of the factor $(2u+2)$ corresponds to the period doubling bifurcation from 1-periodic to 2-periodic orbits. Note that a 2-periodic orbit may also be seen as a $2k$-periodic orbit (whose minimal period is two). Concerning this matter we refer also to the discussion of $M^a_4(u) = 0$ in Section 5.4.1.

Note that $M_{2k}(u)$ is the determinant of a tridiaogonal Toeplitz matrix. For those matrices there are even explicit representations of the eigenvalues, cf. [12, Section 2]. In our case we find for the spectrum spec $(\mathcal{M}_{2k}(0,\dots,0))$ of $\mathcal{M}_{2k}(0,\dots,0)$

$$\text{spec}\ (\mathcal{M}_{2k}(0,\dots,0)) = \left\{2\cos\left(\frac{n\pi}{k}\right)\ :\ n = 1,\dots,k-1\right\}.$$

According to the remarks at the beginning of this section we find

$$M_{2k}(u) = 0 \quad \Leftrightarrow \quad u = -\cos\left(\frac{n\pi}{k}\right), \quad n = 1,\dots,k-1. \tag{6.2}$$

Further, according to [8, Theorem 2.1] determinants of tridiogonal matrices allow recursion formulas, which in our situation read as follows:

Lemma 6.3. $M_{2k}(u) = 2uM_{2k-2}(u) - M_{2k-4}(u), \quad k \geq 2;\ M_0(u) := 0, M_2(u) = 1.$

Proof. Expansion of $M_{2k}(u)$ along the first column yields the statement. ∎

Definition 6.4. *Let P be a polynomial of the form $P(u) = \sum_{n=0}^{N} a_n u^n$, $a_N \neq 0$. Then we define*

$$\deg P := N, \quad \textit{and} \quad \operatorname{coeff}(P, n) := a_n,\ n \in \{0, \ldots, N\}.$$

From the recursion formula given in the previous lemma together with $M_0(u) := 0$ and $M_2(u) = 1$ it follows immediately by induction that for all $m \in \mathbb{N}$

$$\deg(M_{2m}) = m - 1 \quad \text{and} \quad \operatorname{coeff}(M_{2m}, m-1) = 2^{m-1}. \tag{6.3}$$

Further we find

Corollary 6.5. $M_{2k}(u) = M_{2i}(u)M_{2k-2i+2}(u) - M_{2i-2}(u)M_{2k-2i}(u),\ i = 2, \ldots, k-1.$

Proof. Repeated application of Lemma 6.3 yields that there are polynomials P_i and Q_i in u such that

$$\begin{aligned} M_{2k}(u) =& P_i M_{2k-2i+2} + Q_i M_{2k-2i} \\ =& P_i \left(2uM_{2k-2i} - M_{2k-2i-2)}\right) + Q_i M_{2k-2i} \\ =& \left(2uP_i + Q_i\right) M_{2k-2i} - P_i M_{2k-2i-2}. \end{aligned}$$

Hence

$$P_{i+1} = 2uP_i + Q_i, \quad Q_{i+1} = -P_i$$

and therefore

$$P_{i+1} = 2uP_i - P_{i-1}, \quad P_2(u) = 2u = M_4(u),\ P_1(u) = 1 = M_2(u).$$

With Lemma 6.3 we infer $P_i = M_{2i}$, and with that we find $Q_i = -M_{2i-2}$. ∎

From Corollary 6.5 we conclude that

$$M_{2m}M_{2n} = M_{2m+2n-2} + M_{2m-2}M_{2n-2}, \quad m, n \in \mathbb{N}. \tag{6.4}$$

Lemma 6.6. *Let $m, n, k \in \mathbb{N}$ with $m, n < k$ and $m + n > k$. Then*

$$M_{2m}M_{2n} = M_{2k}M_{2m+2n-2k} + M_{2k-2m}M_{2k-2n} \tag{6.5}$$

and

$$M_{2k-2m}M_{2k-2n} = M_{2m}M_{2n} \mod M_{2k}. \tag{6.6}$$

Proof. According to (6.4) we find

$$\begin{aligned} M_{2m}M_{2n} &= M_{2m+2n-2} + M_{2m-2}M_{2n-2} \\ &= M_{2k}M_{2m+2n-2k} - M_{2k-2}M_{2m+2n-2k-2} + M_{2m-2}M_{2n-2}. \end{aligned} \tag{6.7}$$

Applying (6.4) repeatedly we find for $i = 1, \dots, m+n-k$:

$$\begin{aligned} M_{2m-2}M_{2n-2} - M_{2k-2}M_{2m+2n-2k-2} =& M_{2m+2n-6} + M_{2m-4}M_{2n-4} \\ & - M_{2m+2n-6} - M_{2k-4}M_{2m+2n-2k-4} \\ =& M_{2m-4}M_{2n-4} - M_{2k-4}M_{2m+2n-2k-4} \\ & \vdots \\ =& M_{2m-2i}M_{2n-2i} - M_{2k-2i}M_{2m+2n-2k-2i}, \end{aligned}$$

So, in particular for $i = m+n-k$ we find

$$M_{2m-2}M_{2n-2} - M_{2k-2}M_{2m+2n-2k-2} = M_{2k-2m}M_{2k-2n}.$$

Together with (6.7) this proves (6.5).

Further, cf. also (6.3),

$$\deg(M_{2k-2m}M_{2k-2n}) = 2k - m - n - 2 \le k - 2 < k - 1 = \deg(M_{2k}).$$

This proves (6.6). ∎

Corollary 6.7. *Let $m, n, k \in \mathbb{N}$ with $m + n \le k$. Then*

$$M_{2m}M_{2n} \mod M_{2k} = M_{2m}M_{2n}. \tag{6.8}$$

Proof. According to (6.3) we have

$$\deg(M_{2m}M_{2n}) = m + n - 2 \le k - 2 < k - 1 = \deg(M_{2k}).$$

This proves (6.8). ∎

6.2 Symmetric (2k)-periodic orbits of type (b,b)

In this section we study subharmonic bifurcations from the branch of 1-periodic orbits. In doing so we confine to bifurcating periodic orbits having an even period and more in particular to those which are of type (b,b). The corresponding determination equations $F^b_{2k} = 0$ we derived in Section 4.3. We will solve this equation near the zeros $\hat{\mathbf{u}}$ of $M_{2k}(u)$ by means of the implicit function theorem. Note that $F^b_{2k}(\hat{\mathbf{u}}) = 0$. So it remains to show that the Jacobian $DF^b_{2k}(\hat{\mathbf{u}}) = \left(D_j F^{b,i}_{2k}(\hat{\mathbf{u}})\right)_{\substack{i=1,\dots,k-1 \\ j=1,\dots,k}}$ has full rank.

According to (4.11) and (4.9) we find easily for $i < k-1$ that:

$$
\begin{array}{rcll}
D_1 F_{2k}^{b,1}(\hat{\mathbf{u}}) & = & 2\hat{u} & \\
D_i F_{2k}^{b,i}(\hat{\mathbf{u}}) & = & 2\hat{u}+1, & i = 2, \dots, k-2, \\
D_{i-1} F_{2k}^{b,i}(\hat{\mathbf{u}}) & = & -1, & i = 2, \dots, k-2, \\
D_{i+1} F_{2k}^{b,i}(\hat{\mathbf{u}}) & = & -2\hat{u}-1, & i = 2, \dots, k-2, \\
D_{i+2} F_{2k}^{b,i}(\hat{\mathbf{u}}) & = & 1, & i = 1, \dots, k-2, \\
D_j F_{2k}^{b,i}(\hat{\mathbf{u}}) & = & 0, & \text{else.}
\end{array} \tag{6.9}
$$

For $i = k-1$ we find:

Lemma 6.8.

$$
\begin{array}{rcll}
D_1 F_{2k}^{b,k-1}(\hat{\mathbf{u}}) & = & M_{2k-2}(\hat{u}) & \\
D_i F_{2k}^{b,k-1}(\hat{\mathbf{u}}) & = & M_{2i-2}(\hat{u}) M_{2k+2-2i}(\hat{u}) + M_{2i}(\hat{u}) M_{2k-2i}(\hat{u}), & i = 2, \dots, k-1, \\
D_k F_{2k}^{b,k-1}(\hat{\mathbf{u}}) & = & M_{2k-2}(\hat{u}). &
\end{array}
$$

Proof. Recall from (4.12) that $F_{2k}^{b,k-1}(\mathbf{u}) = \det \mathcal{M}_{2k}^b(u_1, \dots, u_k)$. The lemma follows from the fact that the determinant is a multilinear mapping with respect to the lines of the underlying matrix. ∎

Lemma 6.9. *Let* $\mathbf{u} = (u, \dots, u)$*. With the definition* $\sum_{n=1}^{0} =: 0$ *we find*

$$
\det\left(D_{(u_1,\dots,u_{k-1})} F_{2k}^b(\mathbf{u})\right) = \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \sum_{j=i}^{k-i} (2j-2i+1) M_{2j}(u) M_{2k-4i+2}(u) + \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor - 1} \sum_{j=1}^{i} (2k-4i-2) M_{2k-2j}(u) M_{2k-4i-2}(u). \tag{6.10}
$$

Proof. According to (6.9) we find

$$
\det\left(D_{(u_1,\dots,u_{k-1})} F_{2k}^b(\mathbf{u})\right) = \left|
\begin{array}{cccccccc}
2u & -2u-1 & 1 & 0 & & & \dots & 0 \\
-1 & 2u+1 & -2u-1 & 1 & 0 & & \dots & \vdots \\
0 & -1 & 2u+1 & -2u-1 & 1 & 0 & \dots & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
 & & 0 & -1 & 2u+1 & -2u-1 & 1 & 0 \\
 & & & 0 & -1 & 2u+1 & -2u-1 & 1 \\
0 & & & & 0 & -1 & 2u+1 & -2u-1 \\
D_1 F_{2k}^{b,k-1}(\mathbf{u}) & & & \dots & & & & D_{k-1} F_{2k}^{b,k-1}(\mathbf{u})
\end{array}
\right| .
$$

And therefore

$$\det\big(D_{(u_1,\dots,u_{k-1})}F^b_{2k}(\mathbf{u})\big) = \begin{vmatrix} 2u & -1 & 0 & & & \dots & 0 \\ -1 & 2u & -1 & 0 & & & \vdots \\ 0 & -1 & 2u & -1 & 0 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & 0 & -1 & 2u & -1 & 0 \\ 0 & & \dots & 0 & -1 & 2u & -1 \\ \mathcal{D}_1 & & \dots & \mathcal{D}_i & \dots & & \mathcal{D}_{k-1} \end{vmatrix} = \begin{vmatrix} & \mathcal{M}^b_{2k-2}(\mathbf{u}) & & \vline & 0 \\ & & & \vline & \vdots \\ & & & \vline & 0 \\ \_\_ & \_\_ & \_\_ & \vline & -1 \\ \mathcal{D}_1 & \dots & \mathcal{D}_{k-2} & & \mathcal{D}_{k-1} \end{vmatrix},$$

where

$$\mathcal{D}_j := \sum_{i=1}^{j} D_i F^{b,k-1}_{2k}(\mathbf{u}). \tag{6.11}$$

Expanding this determinant along the last line we find

$$\det\big(D_{(u_1,\dots,u_{k-1})}F^b_{2k}(\mathbf{u})\big) = \sum_{j=1}^{k-1} \mathcal{D}_{k-j} M_{2k-2j}(u) = \sum_{j=1}^{k-1} M_{2j}(u)\mathcal{D}_j.$$

Further, by the definition of the $\mathcal{D}_j$, cf. (6.11), and Lemma 6.8 we find

$$\sum_{j=1}^{k-1} M_{2j}\mathcal{D}_j = \sum_{j=1}^{k-1} M_{2j}\left[\left(\sum_{i=1}^{j-1} 2M_{2i}M_{2k-2i}\right) + M_{2j}M_{2k-2j}\right]. \tag{6.12}$$

According to (6.4) we write

$$M_{2i}M_{2k-2i} = M_{2k-2} + M_{2i-2}M_{2k-2i-2}.$$

Plugging in into (6.12) yields

$$\begin{aligned} \sum_{j=1}^{k-1} M_{2j}\mathcal{D}_j &= \sum_{j=1}^{k-1} M_{2j}\left[\left(\sum_{i=1}^{j-1} 2\big(M_{2k-2} + M_{2i-2}M_{2k-2i-2}\big)\right) + M_{2k-2} + M_{2j-2}M_{2k-2j-2}\right] \\ &= \sum_{j=1}^{k-1}\big(2j-1\big)M_{2j}M_{2k-2} + \sum_{j=1}^{k-1} M_{2j}\left[\left(\sum_{i=1}^{j-1} 2M_{2i-2}M_{2k-2i-2}\right) + M_{2j-2}M_{2k-2j-2}\right]. \end{aligned}$$

Note that $M_0 = 0$. So the latter representation reduces to

$$\begin{aligned} \sum_{j=1}^{k-1} M_{2j}\mathcal{D}_j = \sum_{j=1}^{k-1}\big(2j-1\big)M_{2j}M_{2k-2} + \sum_{j=2}^{k-2} M_{2j}\left[\left(\sum_{i=2}^{j-1} 2M_{2i-2}M_{2k-2i-2}\right) + M_{2j-2}M_{2k-2j-2}\right] \\ + M_{2k-2}\sum_{i=2}^{k-2} 2M_{2i-2}M_{2k-2i-2}. \end{aligned} \tag{6.13}$$

Again applying (6.4) we write

$$M_{2i-2}M_{2k-2i-2} = M_{2k-6} + M_{2i-4}M_{2k-2i-4}.$$

Plugging in into (6.13) yields

$$\begin{aligned}
\sum_{j=1}^{k-1} M_{2j}D_j =& \sum_{j=1}^{k-1} \big(2j-1\big)M_{2j}M_{2k-2} \\
&+ \sum_{j=2}^{k-2} M_{2j}\left[\left(\sum_{i=2}^{j-1} 2\big(M_{2k-6} + M_{2i-4}M_{2k-2i-4}\big)\right) + M_{2k-6} + M_{2j-4}M_{2k-2j-4}\right] \\
&+ M_{2k-2}\sum_{i=2}^{k-2} 2(M_{2k-6} + M_{2i-4}M_{2k-2i-4}) \\
=& \sum_{j=1}^{k-1} \big(2j-1\big)M_{2j}M_{2k-2} + \sum_{j=2}^{k-2} \big(2j-3\big)M_{2j}M_{2k-6} + (2k-6)M_{2k-2}M_{2k-6} \\
&+ \sum_{j=2}^{k-2} M_{2j}\left[\left(\sum_{i=2}^{j-1} 2M_{2i-4}M_{2k-2i-4}\right) + M_{2j-4}M_{2k-2j-4}\right] \\
&+ M_{2k-2}\sum_{i=2}^{k-2} 2(M_{2i-4}M_{2k-2i-4}).
\end{aligned}$$

And now taking again into consideration that $M_0 = 0$ yields

$$\begin{aligned}
\sum_{j=1}^{k-1} M_{2j}D_j =& \sum_{j=1}^{k-1} \big(2j-1\big)M_{2j}M_{2k-2} + \sum_{j=2}^{k-2} \big(2j-3\big)M_{2j}M_{2k-6} + (2k-6)M_{2k-2}M_{2k-6} \\
&+ \sum_{j=3}^{k-3} M_{2j}\left[\left(\sum_{i=3}^{j-1} 2M_{2i-4}M_{2k-2i-4}\right) + M_{2j-4}M_{2k-2j-4}\right] \\
&+ (M_{2k-2} + M_{2k-4})\sum_{i=3}^{k-3} 2(M_{2i-4}M_{2k-2i-4}).
\end{aligned}$$

Again applying (6.4) we write

$$M_{2i-4}M_{2k-2i-4} = M_{2k-10} + M_{2i-6}M_{2k-2i-6},$$

and get

$$\sum_{j=1}^{k-1} M_{2j}D_j = \sum_{j=1}^{k-1}\big(2j-1\big)M_{2j}M_{2k-2} + \sum_{j=2}^{k-2}\big(2j-3\big)M_{2j}M_{2k-6} + (2k-6)M_{2k-2}M_{2k-6}$$
$$+ \sum_{j=3}^{k-3} M_{2j}\left[\left(\sum_{i=3}^{j-1} 2(M_{2k-10} + M_{2i-6}M_{2k-2i-6})\right) + (M_{2k-10} + M_{2j-6}M_{2k-2j-6})\right]$$
$$+ (M_{2k-2} + M_{2k-4})\sum_{i=3}^{k-3} 2(M_{2k-10} + M_{2i-6}M_{2k-2i-6})$$
$$= \sum_{j=1}^{k-1}\big(2j-1\big)M_{2j}M_{2k-2} + \sum_{j=2}^{k-2}\big(2j-3\big)M_{2j}M_{2k-6} + \sum_{j=3}^{k-3}\big(2j-5\big)M_{2j}M_{2k-10}$$
$$+ (2k-6)M_{2k-2}M_{2k-6} + (2k-10)(M_{2k-2} + M_{2k-4})M_{2k-10}$$
$$+ \sum_{j=3}^{k-3} M_{2j}\left[\left(\sum_{i=3}^{j-1} 2(M_{2i-6}M_{2k-2i-6})\right) + (M_{2j-6}M_{2k-2j-6})\right]$$
$$+ (M_{2k-2} + M_{2k-4})\sum_{i=3}^{k-3} 2(M_{2i-6}M_{2k-2i-6}).$$

Finally, repeated application of the demonstrated argumentation yields the statement. ∎

Theorem 6.10. *For each $k \in \mathbb{N}$ there exists a branch of $2k$-periodic orbits of type (b,b) bifurcating from the branch of 1-periodic orbits.*

Proof. As we explained at the beginning of this section it remains show that there is a zero $\hat{\mathbf{u}}$ of $M_{2k}(u)$ for which $DF^b_{2k}(\hat{\mathbf{u}})$ has full rank. To this end it is enough to show that $M_{2k}(u)$ is not a divisor of $\det\big(D_{(u_1,\dots,u_{k-1})}F^b_{2k}(\mathbf{u})\big)$, or in other words

$$\det\big(D_{(u_1,\dots,u_{k-1})}F^b_{2k}(\mathbf{u})\big) \bmod M_{2k}(u) \neq 0.$$

To this end we show that, cf. Definition 6.4,

$$\operatorname{coeff}\Big(\det\big(D_{(u_1,\dots,u_{k-1})}F^b_{2k}(\mathbf{u})\big) \bmod M_{2k}(u), k-2\Big) \neq 0. \tag{6.14}$$

In our analysis we use the representation (6.10) of $\det\big(D_{(u_1,\dots,u_{k-1})}F^b_{2k}(\mathbf{u})\big)$. For evaluating the remainders under the division by $M_{2k}(u)$ of the single terms of the form $M_{2m}M_{2n}$ in (6.10) we use Lemma 6.6 and Corollary 6.7.

The representation (6.10) consists of two summands. We start with considering the term

$$\sum_{i=1}^{\lfloor \frac{k}{2} \rfloor}\sum_{j=i}^{k-i}(2j-2i+1)M_{2j}(u)M_{2k-4i+2}(u). \tag{6.15}$$

First we consider those terms $M_{2j}(u)M_{2k-4i+2}(u)$ in (6.15) for which Lemma 6.6 applies, i.e.:

$$j + k - 2i + 1 > k \quad \Leftrightarrow \quad j > 2i - 1. \tag{6.16}$$

Further, according to Lemma 6.6 we find

$$M_{2j}M_{2k-4i+2} \mod M_{2k} = M_{2k-2j}M_{4i-2}.$$

For the degree of the remainder terms we find, cf. (6.3), (6.4) and (6.16),

$$\deg(M_{2k-2j}M_{4i-2}) = k - j + 2i - 3 < k - 2.$$

Hence, since we want to show (6.14), we do not need to consider these terms furthermore.

Next we consider those terms $M_{2j}(u)M_{2k-4i+2}(u)$ in (6.15) for which Corollary 6.7 applies, i.e.:

$$j + k - 2i + 1 \le k \quad \Leftrightarrow \quad j \le 2i - 1.$$

We note that, cf. (6.4) and (6.3),

$$\deg(M_{2j}M_{2k-4i+2}) = k - 2 \quad \Leftrightarrow \quad j = 2i - 1.$$

Since we want to show (6.14), it is enough to consider terms $M_{2j}M_{2k-4i+2}$ with $j = 2i-1$. This leads, cf. (6.15), to the inequality

$$i \le 2i - 1 \le k - i.$$

While the first inequality is satisfied for all $i \ge 1$, the second inequality yields the following:

$$2i - 1 \le k - i \quad \Leftrightarrow \quad i \le \left\lfloor \frac{k+1}{3} \right\rfloor.$$

Since for $k \ge 2$ always $\lfloor \frac{k+1}{3} \rfloor \le \lfloor \frac{k}{2} \rfloor$, the terms in (6.15) having degree $k - 2$ are

$$M_{4i-2}M_{2k-4i+2}, \quad i = 1, \ldots, \left\lfloor \frac{k+1}{3} \right\rfloor.$$

Further, taking also into consideration (6.3) and (6.4), we find

$$\begin{aligned} \operatorname{coeff}\left(\sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \sum_{j=i}^{k-i} (2j - 2i + 1) M_{2j}(u) M_{2k-4i+2}(u) \bmod M_{2k}, k - 2 \right) &= 2^{k-2} \sum_{i=1}^{\lfloor \frac{k+1}{3} \rfloor} (2i - 1) \\ &= 2^{k-2} \left(\left\lfloor \frac{k+1}{3} \right\rfloor \right)^2. \end{aligned} \tag{6.17}$$

Next we consider the remaining term in (6.10)

$$\sum_{i=1}^{\lfloor \frac{k}{2} \rfloor - 1} \sum_{j=1}^{i} (2k - 4i - 2) M_{2k-2j}(u) M_{2k-4i-2}(u). \tag{6.18}$$

To this end we proceed in the same way as above.
First we consider those terms $M_{2k-2j}(u)M_{2k-4i-2}(u)$ in (6.18) for which Lemma 6.6 applies, i.e.:

$$k - j + k - 2i - 1 > k \quad \Leftrightarrow \quad k - 2i - 1 > j. \tag{6.19}$$

According to Lemma 6.6 we find for the terms under consideration

$$M_{2k-2j}M_{2k-4i-2} \mod M_{2k} = M_{2j}M_{4i+2}.$$

Exploiting (6.19) we find for the degree of the residual term

$$\deg\left(M_{2j}M_{4i+2}\right) = j + 2i - 1 < k - 2.$$

Hence, cf. again (6.14), we do not anymore consider these terms.
Next we consider those terms $M_{2k-2j}(u)M_{2k-4i-2}(u)$ in (6.18) for which Corollary 6.7 applies, i.e.:

$$k - j + k - 2i - 1 \leq k \quad \Leftrightarrow \quad k - 2i - 1 \leq j.$$

Using similar arguments as above we find for those terms

$$\deg(M_{2k-2j}M_{2k-4i-2}) = k - 2 \quad \Leftrightarrow \quad k - 2i - 1 = j.$$

And again as above this leads to the inequality

$$1 \leq k - 2i - 1 \leq i \quad \Leftrightarrow \quad \frac{k-1}{3} \leq i \leq \frac{k-2}{2}.$$

Note that $\lceil \frac{k-1}{3} \rceil \leq \lfloor \frac{k-2}{2} \rfloor$ for all $k \geq 6$. So, the terms in (6.18) having degree $k - 2$ are

$$M_{4i+2}M_{2k-4i-2}, \qquad \left\lceil \frac{k-1}{3} \right\rceil \leq i \leq \left\lfloor \frac{k-2}{2} \right\rfloor.$$

There is no such term for $k = 2, 3, 5$.
This finally implies

$$\operatorname{coeff}\left(\sum_{i=1}^{\lfloor \frac{k}{2} \rfloor - 1} \sum_{j=1}^{i} (2k - 4i - 2) M_{2k-2j}(u) M_{2k-4i-2}(u) \operatorname{mod} M_{2k}, k - 2 \right) = 2^{k-2} \sum_{i=\lceil \frac{k-1}{3} \rceil}^{\lfloor \frac{k-2}{2} \rfloor} (2k-4i-2), \tag{6.20}$$

where the sum on the right-hand side is equal to zero for $k = 2, 3, 5$.
Combining (6.17) and (6.20) we find

$$\operatorname{coeff}\left(\det\left(D_{(u_1,\dots,u_{k-1})}F^b_{2k}(\mathbf{u})\right) \operatorname{mod} M_{2k}, k-2\right) = 2^{k-2} \left(\left(\left\lfloor \frac{k+1}{3} \right\rfloor \right)^2 + \sum_{i=\lceil \frac{k-1}{3} \rceil}^{\lfloor \frac{k-2}{2} \rfloor} (2k - 4i - 2) \right)$$

$$\neq 0.$$

∎

6.3 Discussion

Theorem 6.10 is a first step towards a complete analysis of subharmonic bifurcations from the branch of 1-periodic orbits. In the following we will confine ourselves exclusively on this type of bifurcation.

Indeed, computations based upon the recursion formula for M_{2k} given in Lemma 6.3 and the representation formula (6.10) of $\det\left(D_{(u_1,\dots,u_{k-1})}F^b_{2k}(\mathbf{u})\right)$ suggest that even

$$\gcd\left(\det\left(D_{(u_1,\dots,u_{k-1})}F^b_{2k}(\mathbf{u})\right), M_{2k}(u)\right) \in \mathbb{R}, \quad \forall k \in \mathbb{N}. \tag{6.21}$$

To underpin the conjecture (6.21) we add a list displaying $M_{2k}(u)$, $\det\left(D_{(u_1,\dots,u_{k-1})}F^b_{2k}(\mathbf{u})\right)$ and its greatest common divisor:

k	$M_{2k}(u)$	$\det\left(D_{(u_1,\dots,u_{k-1})}F^b_{2k}(\mathbf{u})\right)$	gcd
2	$2u$	1	1
3	$4u^2-1$	$12u^2+2u$	1
4	$8u^3-4u$	$80u^4+24u^3-28u^2-4u+2$	2
5	$16u^4-12u^2+1$	$448u^6+160u^5-336u^4-88u^3+60u^2+10u$	1
6	$32u^5-32u^3+6u$	$2304u^8+896u^7-2752u^6-864u^5+976u^4$ $+224u^3-96u^2-12u+3$	1
7	$64u^6-80u^4+24u^2-1$	$11264u^{10}+4608u^9-18688u^8-6528u^7$ $+10496u^6+2944u^5-2288u^4-456u^3$ $+168u^2+20u$	1
8	$128u^7-192u^5+80u^3-8u$	$53248u^{12}+22528u^{11}-113664u^{10}-42496u^9$ $+88832u^8+28288u^7-31040u^6-7936u^5$ $+4736u^4+880u^3-248u^2-24u+4$	4
9	$256u^8-448u^6+240u^4-40u^2+1$	$245760u^{14}+106496u^{13}-643072u^{12}$ $-251904u^{11}+649216u^{10}+224256u^9$ $-317952u^8-93440u^7+77824u^6+18432u^5$ $-8752u^4-1512u^3+360u^2+36u$	1
10	$512u^9-1024u^7+672u^5-160u^3$ $+10u$	$1114112u^{16}+491520u^{15}-3457024u^{14}$ $-1400832u^{13}+4296704u^{12}+1570816u^{11}$ $-2735104u^{10}-880128u^9+945664u^8$ $+258432u^7-173248u^6-37984u^5$ $+15120u^4+2400u^3-480u^2-40u+5$	1

However, expending more effort than done in the proof of Theorem 6.10 one should be able to prove (6.21). In this respect we want to mention that the computation of the remainder of the polynomial division of $\det\left(D_{(u_1,\dots,u_{k-1})}F^b_{2k}(\mathbf{u})\right)$ and $M_{2k}(u)$ is just the first step in the algorithm for the computation of the greatest common divisor of these polynomials. It is also worth to mention that the recursion formulas provided in Section 6.1 will enable us to process this algorithm.

If (6.21) would have been proved for any $k \in \mathbb{N}$, then this would imply that for each zero $\hat{\mathbf{u}}$

of $M_{2k}(u)$ the Jacobian $DF^b_{2k}(\hat{\mathbf{u}})$ has full rank. And this again means that for each zero $\hat{\mathbf{u}}$ of $M_{2k}(u)$ there is assigned a branch of $2k$-periodic orbits of type (b,b) bifurcating from the branch of 1-periodic orbits. However, we want to remark that k might not be the minimal period. If m is a factor of $2k$ then an m-periodic orbit can also be seen as a $2k$-periodic orbit, cf. also the discussion subsequent to the proof of Lemma 6.2. This becomes also clear by considering the explicit representations of the zeros of M_{2k} given in (6.2): Let $l \cdot p = k$. Then according to (6.2) we find that for n equal to multiples of p the value $-\cos\left(\frac{n\pi}{k}\right)$ is a zero of M_{2k} as well as of M_{2l}. In the same way also the bifurcating symmetric orbits of type (a,a) can be analysed.

Furthermore, our considerations concerning subharmonic bifurcations from the branch of 1-periodic orbits can be extended to symmetric (2k-1)-periodic orbits in a straightforward manner. As in Lemma 6.1 we find that all zeros of $M_{2k-1}(u)$ are real and simple, and as in Lemma 6.3 we find

$$M_{2k-1}(u) = 2uM_{2k-3}(u) - M_{2k-5}(u), \quad k \geq 3.$$

Further, in completely the same way as Corollary 6.5 we prove:

$$M_{2k-1}(u) = M_{2i}(u)M_{2k-(2i-1)}(u) - M_{2i-2}(u)M_{2k-(2i+1)}(u), \quad i = 2, \dots, k-1.$$

And finally, an analysis based on the same idea as used in the proof of Lemma 6.2 yields:

$$M_{2k-1}(u) = M_{2k}(u) + M_{2k-2}(u).$$

These properties of $M_{2k-1}(u)$ allow to study $F_{2k-1}(\mathbf{u}) = 0$ near zeros $\hat{\mathbf{u}}$ of $\det \mathcal{M}_{2k-1}(u, \dots, u)$ in the same way as done in Section 6.2. For the Jacobian $DF_{2k-1}(\hat{\mathbf{u}}) = \left(D_j F^i_{2k-1}(\hat{\mathbf{u}})\right)_{\substack{i=1,\dots,k-1 \\ j=1,\dots,k}}$ we find:

$$\begin{array}{lcll}
D_{i-1}F^i_{2k-1}(\hat{\mathbf{u}}) & = & -1, & i = 2, \dots, k-2, \\
D_i F^i_{2k-1}(\hat{\mathbf{u}}) & = & 2\hat{u} + 1, & i = 1, \dots, k-2, \\
D_2 F^1_{2k-1}(\hat{\mathbf{u}}) & = & -2(\hat{u} + 1) & \\
D_{i+1}F^i_{2k-1}(\hat{\mathbf{u}}) & = & -2\hat{u} - 1, & i = 2, \dots, k-2, \\
D_{i+2}F^i_{2k-1}(\hat{\mathbf{u}}) & = & 1, & i = 1, \dots, k-2, \\
D_j F^i_{2k-1}(\hat{\mathbf{u}}) & = & 0, & \text{else}, \\
D_1 F^{k-1}_{2k-1}(\hat{\mathbf{u}}) & = & M_{2k-2}(\hat{u}) & \\
D_j F^{k-1}_{2k-1}(\hat{\mathbf{u}}) & = & M_{2j-3}(\hat{u})M_{2k+2-2j}(\hat{u}) + M_{2j-1}(\hat{u})M_{2k-2j}(\hat{u}), & j = 2, \dots, k-1, \\
D_k F^{k-1}_{2k-1}(\hat{\mathbf{u}}) & = & M_{2k-3}(\hat{u}). &
\end{array}$$

With that it can be shown (for instance by means of a computer algebra system) that for certain $k \in \mathbb{N}$ also $\gcd\left(\det\left(D_{(u_1,\dots,u_{k-1})}F^b_{2k-1}(\mathbf{u})\right), M_{2k-1}(u)\right) \in \mathbb{R}$ is true. To obtain a more general result analytically the counterparts of Lemma 6.9, Theorem 6.10 or even (6.21) need to be proved.

To round out the bifurcation analysis also examinations as performed in Section 5.4.1 and Section 5.5.1 for 4- and 5-periodic orbits, respectively, are necessary. That means that it must be shown that at each bifurcation point exactly two branches emerge. In the case of $2k$-periodic orbits we have reasonable grounds to believe that in each case one of these branches consists of orbits of type (a,a) and the other one consists of orbits of type (b,b).

In order to justify that the whole bifurcation scenario can be seen as the starting point of a cascade of similar bifurcations (ideally) also a stability analysis should be performed.

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