A43

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

$$H_0(x) = 1 \quad , \quad H_2(x) = e^{x^2}\left(-2e^{-x^2} + 4x^2 e^{-x^2}\right) = 4x^2 - 2$$

$$H_1(x) = 2x$$

Z: H_n erfüllt $y'' - 2xy' + 2ny = 0$

$$H_n'(x) = (-1)^n 2x e^{x^2} \frac{d^n}{dx^n} e^{-x^2} + (-1)^n e^{x^2} \frac{d^{n+1}}{dx^{n+1}} e^{-x^2}$$

$$= 2x H_n(x) - H_{n+1}(x)$$

$$H_n''(x) = 2H_n(x) + 2x H_n'(x) - H_{n+1}'(x) = 2H_n(x) + 2xH_n'(x) - 2x H_{n+1}(x) + H_{n+2}(x)$$

$$\overset{!}{=} 2x H_n'(x) - 2n H_n(x)$$

$$\Leftrightarrow \cancel{2H_n} \; 2(n+1) H_n(x) - 2x H_{n+1}(x) + H_{n+2}(x) = 0 \quad (*) \quad \forall n \in \mathbb{N}$$

$$\cancel{\Leftrightarrow 2(n+1) \frac{d^n}{dx^n} e^{-x^2} + 2x \frac{d^{n+1}}{dx^{n+1}} e^{-x^2} + \frac{d^{n+2}}{dx^{n+2}} e^{-x^2} = 0}$$

zeigen $(*)$ per Induktion : IA $n = 0$ ✓

Es gelte $H_{n+2}(x) = 2x H_{n+1}(x) - 2(n+1) H_n(x)$

1.) Daraus folgt: $H_{n+1}'(x) = 2x H_{n+1}(x) - H_{n+2}(x) = 2(n+1) H_n(x)$

2.) $H_{n+3}(x) = 2x H_{n+2}(x) - H_{n+2}'(x) \overset{IV}{=} 2x H_{n+2}(x) - \left(2x H_{n+1}(x) - 2(n+1) H_n(x)\right)'$

$$= 2x H_{n+2}(x) - 2H_{n+1}(x) - 2x H_{n+1}'(x) + 2(n+1) H_n'(x)$$

$$= 2x H_{n+2}(x) - 2H_{n+1}(x) - 2x H_{n+1}'(x) + 2(n+1)\left(2x H_n(x) - H_{n+1}(x)\right)$$

$$\overset{1.)}{=} 2x H_{n+2}(x) - 2(n+2) H_{n+1}(x) \qquad \square$$

Somit gilt $(*)$ $\forall n \in \mathbb{N}$ und H_n erfüllt $y'' - 2xy' + 2ny = 0$

$\boxed{A43}$

z.z.: $\int_{-\infty}^{\infty} H_n H_m e^{-x^2} dx = 0, \quad n \neq m$

$$f_n(x) = H_n(x) e^{-x^2/2}, \quad f_n'(x) = H_n'(x) e^{-x^2/2} - x H_n e^{-x^2/2}$$

$$f_n''(x) = H_n''(x) e^{-x^2/2} - 2x H_n'(x) e^{-x^2/2} - H_n e^{-x^2/2} + x^2 H_n e^{-x^2/2}$$

$$= e^{-x^2/2}\left(H_n''(x) - 2x H_n'(x) - H_n + x^2 H_n\right)$$

$$= e^{-x^2/2}\left(x^2 - 1 - 2n\right) H_n = \left(x^2 - 1 - 2n\right) f_n(x)$$

Somit: $f_n'' + (2n+1-x^2) f_n = 0, \quad f_m'' + (2m+1-x^2) f_m = 0$

$$\Rightarrow 0 = f_n'' f_m + (2n+1-x^2) f_n f_m - f_m'' f_n - (2m+1-x^2) f_m f_n$$

$$= f_n'' f_m - f_m'' f_n + 2(n-m) f_m f_n = \left(f_n' f_m - f_m' f_n\right)' + 2(n-m) f_n f_m$$

$$\Rightarrow \int_{-\infty}^{\infty} H_n H_m e^{-x^2} dx = \int_{-\infty}^{\infty} f_n f_m \, dx = -\left[f_n' f_m - f_m' f_n\right]_{-\infty}^{\infty} \cdot \frac{1}{2n-m}$$

H_n, H_n' sind Polynome $\Rightarrow$ $f_n, f_n', f_m, f_m' \to 0, \quad x \to \pm\infty$

$$\Rightarrow \int_{-\infty}^{\infty} H_n H_m e^{-x^2} dx = 0$$