Transmutation operators and efficient solution of Sturm-Liouville spectral problems

Sergii Torba

An operator T is called a transmutation operator [1] for a pair of operators A and B if it is continuous and continuously invertible on a suitable topological space and satisfy the operator equality $AT = TB$.
Transmutation operators, introduced by Delsarte, were used mostly as a theoretical tool in the solution of inverse spectral problems. We propose an efficient method for the solution of direct Sturm-Liouville spectral problems based on an approximation of transmutation operators.
In the case when $A = -\partial^2 + q(x)$ and $B = -\partial^2$, a transmutation operator T can be realized in the form of the Volterra integral operator [2]

$$Tu(x) = u(x) + \int_{-x}^{x} K(x,t)u(t)dt$$

with the integral kernel K satisfying the particular Goursat problem, whose exact solution is known only for some potentials.
We show that it is possible to approximate the integral kernel K in a form

$$K(x,t) \approx \sum_{n=0}^{N} a_n(x)t^n,$$

where the coefficients $a_n(x)$ can be easily obtained from a known (at least numerically) particular solution of the equation $Au = 0$.
Since the two linearly independent solutions of the equation $Au = \omega^2 u$ are the images of the functions $\cos\omega t$ and $\frac{\sin\omega t}{\omega}$ under the action of transmutation operator T, their approximations can be easily obtained using the proposed approximation of the integral kernel. For example,

$$T[\cos\omega t](x) \approx \cos\omega x + \sum_{n=0}^{N} a_n(x) \int_{-x}^{x} t^n \cos\omega t\, dt,$$

and the approximation is convenient because all the integrals can be evaluated exactly.
On the base of proposed approximations a new method of solution of spectral problems is developed having the following remarkable property: it allows to compute thousands of eigenvalues and eigenfunctions with uniform error bounds and with easy error control.
The talk is based on joint works with V. V. Kravchenko [3,4].

[1] B. M. Levitan: *Inverse Sturm-Liouville problems.* VSP, Zeist, 1987.

[2] V. A. Marchenko: *Sturm-Liouville operators and applications.* Birkháuser, Basel, 1986.

[3] V. V. Kravchenko and S. M. Torba: Construction of transmutation operators and hyperbolic pseudoanalytic functions, *Complex Anal. Oper. Theory* (2014), 51pp, doi:10.1007/s11785-014-0373-3.

[4] V. V. Kravchenko and S. M. Torba. Analytic approximation of transmutation operators and applications to highly accurate solution of spectral problems, submitted, available at arXiv:1306.2914, 32pp.