Vorlesung 7 Mathe II

MB-Matr-2011 Danny Kubiak

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Satz 15+16 (zusammengefasst)

$\vec{g}: \mathbb{R}^n \to \mathbb{R}^m$, $\vec{f}: \mathbb{R}^m \to \mathbb{R}^p$ differenzierbar

Dann ist $\vec{f} \circ \vec{g}$ $\mathbb{R}^n \to \mathbb{R}^p$ differenzierbar mit:

*

$$(\vec{f} \circ \vec{g})\, \vec{x} = \vec{f}'(\vec{g}(\vec{x})) \cdot \vec{g}'(\vec{x})$$

**

$$\frac{\partial f_i \circ g}{\partial x_j}(\vec{x}) = \sum_{l=1}^{m} \frac{\partial f_i}{\partial y_l}(\vec{g}(\vec{x})) \cdot \frac{\partial g_l}{\partial x_j}(\vec{x})$$

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

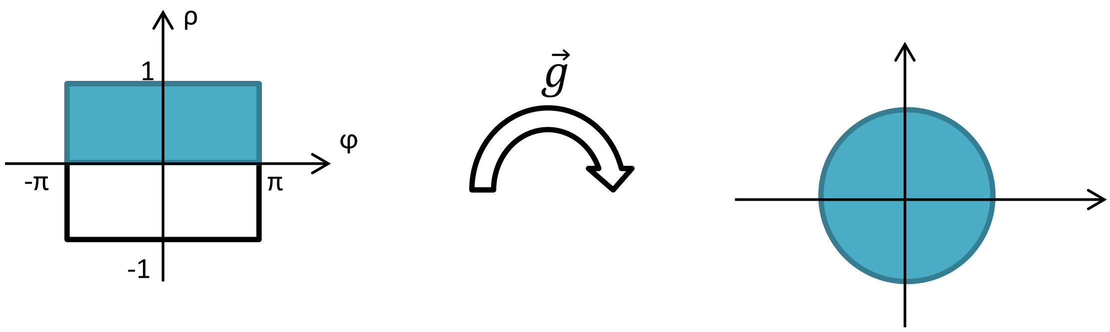
$$\vec{g}(\vec{x}) = \begin{pmatrix} g_1(\vec{x}) \\ \vdots \\ g_m(\vec{x}) \end{pmatrix} \qquad \vec{f}(\vec{y}) = \begin{pmatrix} f_1(\vec{y}) \\ \vdots \\ f_p(\vec{y}) \end{pmatrix}$$

Bsp.: $\vec{f}: \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{g}: \mathbb{R}^2 \to \mathbb{R}^2$, $n = m = p = 2$

$$\vec{f}(x,y) = \begin{pmatrix} \dfrac{-1}{\sqrt{x^2 + y^2 + 1}} \\ x + y \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$$

$$\vec{g}(\rho,\varphi) = \begin{pmatrix} \rho \cdot \cos\varphi \\ \rho \cdot \sin\varphi \end{pmatrix} = \begin{pmatrix} g_1(\rho,\varphi) \\ g_2(\rho,\varphi) \end{pmatrix}$$

„Polarkoordinaten“



$$0 \le \rho \le 1$$

$$-\pi \le \varphi \le \pi$$

Berechne * und ** nach f_1 und $x_1 = \rho$:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \rho \\ \varphi \end{pmatrix} \qquad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Ableitungen:

$$\frac{\partial g_1}{\partial \rho}(\rho, \varphi) = \cos\varphi \qquad \frac{\partial g_1}{\partial \varphi}(\rho, \varphi) = -\rho \cdot \sin\varphi$$

$$\frac{\partial g_2}{\partial \rho}(\rho, \varphi) = \sin\varphi \qquad \frac{\partial g_2}{\partial \varphi}(\rho, \varphi) = \rho \cdot \cos\varphi$$

$$\frac{\partial f_1}{\partial x}(x, y) = \frac{2x}{(\sqrt{x^2+y^2}+1)^2 \cdot 2 \cdot \sqrt{x^2+y^2}} = \frac{x}{\underbrace{(\sqrt{x^2+y^2}+1)^2}_{\rho+1} \cdot \underbrace{\sqrt{x^2+y^2}}_{\rho}}$$

$$\frac{\partial f_1}{\partial y}(x, y) = \frac{y}{\underbrace{(\sqrt{x^2+y^2}+1)^2}_{\rho+1} \cdot \underbrace{\sqrt{x^2+y^2}}_{\rho}}$$

$$\frac{\partial f_2}{\partial x}(x, y) = 1 = \frac{\partial f_2}{\partial y}(x, y) = 1$$

$$(\vec{f} \circ \vec{g})'(\rho, \varphi) = \vec{f}'\begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \end{pmatrix} \cdot \vec{g}'\begin{pmatrix} \rho \\ \varphi \end{pmatrix}$$

$$(\vec{f} \circ \vec{g})(\rho, \varphi) = \vec{f}(\vec{g}\begin{pmatrix} \rho \\ \varphi \end{pmatrix})$$

$$=$$

$$\vec{f}\begin{pmatrix} \overbrace{\rho \cdot \cos\varphi}^{x} \\ \underbrace{\rho \cdot \sin\varphi}_{y} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{-1}{\sqrt{\rho^2 \cdot \cos^2\varphi + \rho^2 \cdot \sin^2\varphi} + 1} \\ \rho \cdot \cos\varphi + \rho \cdot \sin\varphi \end{pmatrix} \overset{\rho > 0}{=} \begin{pmatrix} \dfrac{1}{\rho+1} \\ \rho \cdot (\cos\varphi + \sin\varphi) \end{pmatrix}$$

$$\begin{bmatrix} \dfrac{1}{(\rho+1)^2} & 0 \\ \cos\varphi + \sin\varphi & \rho \cdot (-\sin\varphi + \cos\varphi) \end{bmatrix} = \begin{bmatrix} \dfrac{\rho \cdot \cos\varphi}{(\rho+1)^2 \cdot \rho} & \dfrac{\rho \cdot \sin\varphi}{(\rho+1)^2 \cdot \rho} \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\varphi & -\rho \cdot \sin\varphi \\ \sin\varphi & \rho \cdot \cos\varphi \end{bmatrix}$$

$$f'\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2+1}\cdot\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2+1}\cdot\sqrt{x^2+y^2}} \\ 1 & 1 \end{pmatrix}$$

** $\quad i = 1 = j$

$$\frac{\partial f_1 \circ \vec{g}}{\partial x_1}(\vec{x}) = \sum_{l=1}^{2} \frac{\partial f_1}{\partial y_l}\left(\vec{g}(\vec{x})\right)\frac{\partial g_l}{\partial x_1}(\vec{x}) =$$

$$\frac{\partial f_1}{\partial y_1}\left(\vec{g}(\vec{x})\right)\frac{\partial g_1}{\partial x_1}(\vec{x}) + \frac{\partial f_1}{\partial y_2}\left(\vec{g}(\vec{x})\right)\frac{\partial g_2}{\partial x_1}(\vec{x})$$

$y_1 \to x$, $\vec{g}(\vec{x}) \to \begin{pmatrix} \rho\cdot\cos\varphi \\ \rho\cdot\sin\varphi \end{pmatrix}$, $x_1 \to \rho$, $\vec{x} \to (\rho,\varphi)$, $y_2 \to y$, $x_1 \to \rho$

$$= \frac{\partial f_1}{\partial x}\begin{pmatrix} \rho\cdot\cos\varphi \\ \rho\cdot\sin\varphi \end{pmatrix}\frac{\partial g_1}{\partial \rho}\begin{pmatrix} \rho \\ \varphi \end{pmatrix} + \frac{\partial f_1}{\partial y}\begin{pmatrix} \rho\cdot\cos\varphi \\ \rho\cdot\sin\varphi \end{pmatrix}\frac{\partial g_2}{\partial \rho}\begin{pmatrix} \rho \\ \varphi \end{pmatrix}$$

$$= \frac{\rho\cdot\cos\varphi}{(\rho+1)^2\cdot\rho}\cos\varphi + \frac{\rho\cdot\sin\varphi}{(\rho+1)^2\cdot\rho}\sin\varphi = \frac{1}{(\rho+1)^2}$$