Spectral properties of singular Sturm-Liouville operators with indefinite weight sgn \( x \)

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We consider a singular Sturm-Liouville expression with the indefinite weight sgn \( x \).
To this expression there is naturally a self-adjoint operator in some Krein space associated. We characterize the local definitizability of this operator in a neighbourhood of \( \infty \). Moreover, in this situation, the point \( \infty \) is a regular critical point. We construct an operator \( A = (\text{sgn} \, x)(-d^2/dx^2 + q) \) with non-real spectrum accumulating to a real point. The obtained results are applied to several classes of Sturm-Liouville operators.

1. Introduction

We consider the singular Sturm-Liouville differential expression

\[
a(y)(x) = (\text{sgn} \, x)(-y''(x) + q(x)y(x)), \quad x \in \mathbb{R},
\]

with the signum function as indefinite weight and a real potential \( q \in L^1_{\text{loc}}(\mathbb{R}) \). We assume that (1.1) is in the limit point case at both \( -\infty \) and \( +\infty \). This differential expression is naturally connected with a self-adjoint operator \( A \) in the Krein space \((L^2(\mathbb{R}), [, , ]))\) (see e.g. [12]), where the indefinite inner product \([, , ]\) is defined by

\[
[f, g] = \int_{\mathbb{R}} f(x) g(x) \text{sgn} \, x \, dx, \quad f, g \in L^2(\mathbb{R}).
\]

The operator \( J : f(x) \mapsto (\text{sgn} \, x)f(x) \) is a fundamental symmetry in the Krein space \((L^2(\mathbb{R}), [, , ]))\). Let us define the operator \( L := JA \). Then \( L = -d^2/dx^2 + q \) is a self-adjoint Sturm-Liouville operator in the Hilbert space \( L^2(\mathbb{R}) \). It was shown in [12] that if \( L \) is a non-negative operator in the Hilbert space sense then \( A \) is a definitizable operator with \( \infty \) as a regular critical point.

In general, the operator \( A \) may be not definitizable (in Section 3 we give a criterion). However, under certain assumptions, \( A \) is still locally definitizable over an appropriate subset of \( \mathbb{C} \). It seems that the first result of such type was obtained in [5] for the operator \( y \mapsto \frac{1}{2} \left((py')' + qy\right) \) with \( w \) as indefinite weight function. Note that in [5] \( w \) may have many turning points, but rather strong assumptions on the spectra of certain associated self-adjoint operators are supposed.

As a main result we show the equivalence of the semi-boundedness from below of the operator \( L \) and the local definitizability of the operator \( A \) in a neighbourhood of \( \infty \). Moreover, we give a precise description of the domain of definitizability of \( A \). If \( L \) is semi-bounded from below, we show the existence of a decomposition \( A = A_{\infty} + A_b \) such that the operator \( A_{\infty} \) is similar to a self-adjoint operator in
the Hilbert space sense and \( A \) is a bounded operator, that is, the point \( \infty \) is a regular critical point. Hence, the non-real spectrum of \( A \) remains bounded. But, in contrast to the case of a non-negative operator \( L \), now the non-real spectrum may accumulate to the real axis. We prove in Section 4 the existence of an even continuous potential \( q \) with a sequence of non-real eigenvalues of \( A \) accumulating to a real point. This potential \( q \) can be chosen in such a way that \( A \) is definitizable over \( \mathbb{C} \setminus \{0\} \).

Finally, in Section 5, we discuss the spectrum and the sets of definitizability of \( A \) for various classes of potentials \( q \).

Differential operators with indefinite weights appears in many areas of physics and applied mathematics (see [4, 21, 28, 43] and references therein). Under certain assumptions such operators are definitizable; this case was studied extensively (see [8, 12, 13, 14, 15, 18, 19, 20, 32, 35, 36, 42, 44, 47] and references therein). In [5, 6, 7, 29, 31, 33, 34] certain classes of differential operators that contain definitizable as well as not definitizable operators were considered.

**Notation:** Let \( T \) be a linear operator in a Hilbert space \( \mathcal{H} \). In what follows \( \text{dom}(T) \), \( \ker(T) \), \( \text{ran}(T) \) are the domain, kernel, range of \( T \), respectively. We denote the resolvent set by \( \rho(T) \); \( \sigma(T) := \mathbb{C} \setminus \rho(T) \) stands for the spectrum of \( T \). By \( \sigma_{\text{disc}}(T) \) the set of eigenvalues of \( T \) is indicated. The discrete spectrum \( \sigma_{\text{disc}}(T) \) is the set of isolated eigenvalues of finite algebraic multiplicity; the essential spectrum is \( \sigma_{\text{ess}}(T) := \sigma(T) \setminus \sigma_{\text{disc}}(T) \). We denote the indicator function of a set \( S \) by \( \chi_S(\cdot) \).

2. Sturm-Liouville operators with the indefinite weight \( \text{sgn} \ x \)

2.1. Differential operators

We consider the differential expression

\[
\ell(y)(x) = -y''(x) + q(x)y(x), \quad x \in \mathbb{R}
\]

with a real potential \( q \in L^1_{\text{loc}}(\mathbb{R}) \). Throughout this paper it is assumed that we have limit point case at both \( -\infty \) and \( +\infty \). We set

\[
a(y)(x) = (\text{sgn} \ x) (-y''(x) + q(x)y(x)), \quad x \in \mathbb{R}.
\]

Let \( \mathcal{D} \) be the set of all \( f \in L^2(\mathbb{R}) \) such that \( f \) and \( f' \) are absolutely continuous with \( \ell(f) \in L^2(\mathbb{R}) \). On \( \mathcal{D} \) we define the operators \( A \) and \( L \) as follows:

\[
\text{dom}(A) = \text{dom}(L) = \mathcal{D}, \quad Ay = a(y), \quad Ly = \ell(y).
\]

We equip \( L^2(\mathbb{R}) \) with the indefinite inner product

\[
[f, g] := \int_{\mathbb{R}} (\text{sgn} \ x)f(x)\overline{g(x)}dx, \quad f, g \in L^2(\mathbb{R}).
\]

Then \( (L^2(\mathbb{R}), [\cdot, \cdot]) \) is a Krein space (for the definition of a Krein space and basic notions therein we refer to [2]). A fundamental symmetry \( J \) in \( (L^2(\mathbb{R}), [\cdot, \cdot]) \) is given by

\[
(Jf)(x) = (\text{sgn} \ x)f(x), \quad f \in L^2(\mathbb{R}).
\]
Obviously, 

\[ A = JL \]

holds.

Since the differential expressions \( a(\cdot) \) and \( \ell(\cdot) \) are in the limit point case both at \(+\infty\) and \(-\infty\), the operator \( L \) is self-adjoint in the Hilbert space \( L^2(\mathbb{R}) \). As \( A = JL \), the operator \( A \) is self-adjoint in the Krein space \( L^2(\mathbb{R}, [\cdot, :]) \).

**Definition 2.1.** We shall say that \( A \) is the operator associated with the differential expression \( a(\cdot) \).

### 2.2. Titchmarsh-Weyl coefficients

In the following we denote by \( C_\pm \) the set \( \{ z \in \mathbb{C} : \pm \text{Im} z > 0 \} \). Let \( c_\lambda(x) \) and \( s_\lambda(x) \) denote the fundamental solutions of the equation

\[ -y''(x) + q(x)y(x) = \lambda y(x), \quad x \in \mathbb{R}, \quad (2.3) \]

which satisfy the following conditions

\[ c_\lambda(0) = s'_\lambda(0) = 1; \quad c'_\lambda(0) = s_\lambda(0) = 0. \]

Since the equation (2.3) is limit-point at \(+\infty\), the Titchmarsh-Weyl theory (see, for example, \([40]\)) states that there exists a unique holomorphic function \( m_+(\lambda) \), \( \lambda \in C_+ \cup C_- \), such that the function \( s_\lambda(\cdot) - m_+(\lambda)c_\lambda(\cdot) \) belongs to \( L^2(\mathbb{R}_+) \). Similarly, the limit point case at \(-\infty\) yields the fact that there exists a unique holomorphic function \( m_-(\lambda) \), \( \lambda \in C_+ \cup C_- \), such that \( s_\lambda(\cdot) + m_-(\lambda)c_\lambda(\cdot) \in L^2(\mathbb{R}-) \). The function \( m_+ (m_-) \) is called the Titchmarsh-Weyl m-coefficient for (2.3) on \( \mathbb{R}_+ \) (on \( \mathbb{R}_- \), respectively).

We put

\[ M_\pm(\lambda) := \pm m_\pm(\pm\lambda). \]

**Definition 2.2.** The function \( M_+(\cdot) \) (\( M_-(\cdot) \)) is said to be the Titchmarsh-Weyl coefficient of the differential expression \( a(\cdot) \) on \( \mathbb{R}_+ \) (on \( \mathbb{R}_- \)).

It is easy to see that for \( \lambda \in C_+ \cup C_- \) the functions

\[ \psi^\pm_\lambda(x) := \begin{cases} s_\pm_\lambda(x) - M_\pm(\lambda)c_\pm_\lambda(x), & x \in \mathbb{R}_+, \\ 0, & x \in \mathbb{R}_- \end{cases} \quad (2.4) \]

belong to \( L^2(\mathbb{R}) \). Moreover, the following formula (see \([40]\)) for the norms of \( \psi^\pm_\lambda \) in \( L^2(\mathbb{R}) \) holds true

\[ \|\psi^\pm_\lambda(x)\|^2 = \frac{\text{Im} M_\pm(\lambda)}{\text{Im} \lambda}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (2.5) \]

A holomorphic function \( G : C_+ \cup C_- \rightarrow \mathbb{C} \) is called Nevanlinna function or of class \( (R) \), see e.g. \([27]\), if \( G(\lambda) = \overline{G(\lambda)} \) and \( \text{Im} \lambda \cdot \text{Im} G(\lambda) \geq 0 \) for \( \lambda \in C_+ \cup C_- \). It
follows easily from (2.5) that the functions $M_+$ and $M_-$ (as well as $m_\pm$) belong to the class ($\mathcal{R}$). Moreover, the functions $M_\pm$ have the following asymptotic behavior

$$M_\pm(\lambda) = \pm \frac{i}{\sqrt{-\lambda}} + O\left(\frac{1}{|\lambda|}\right), \quad (\lambda \to \infty, \ 0 < \delta < \arg \lambda < \pi - \delta) \quad (2.6)$$

for $\delta \in \left(0, \frac{\pi}{2}\right)$, see [17]. Here and below $\sqrt{z}$ is the branch of the multifunction on the complex plane $\mathbb{C}$ with the cut along $\mathbb{R}^+$, singled out by the condition $\sqrt{-1} = i$.

2.3. The non-real spectrum of $A$

In the following we identify functions $f \in L^2(\mathbb{R})$ with elements $(f_+ + f_-)$, where $f_\pm := f |_{\mathbb{R}_\pm} \in L^2(\mathbb{R}_\pm)$. Similarly we write $q_\pm := q |_{\mathbb{R}_\pm} \in L^1_{\text{loc}}(\mathbb{R}_\pm)$. Note that the differential expressions $-\frac{d^2}{dx^2} + q_+$ and $\frac{d^2}{dx^2} - q_-$ in $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$ are both regular at the endpoint $0$ and in the limit point case at the singular endpoint $+\infty$ and $-\infty$, respectively. Therefore the operators

$$A_{\text{min}}^+ f_+ = -f_+'' + q_+ f_+ \quad \text{and} \quad A_{\text{min}}^- f_- = f_-'' - q_- f_-$$

defined on

$$\text{dom } A_{\text{min}}^\pm = \{ f_\pm \in D_{\text{max}}^\pm : f_\pm(0) = f_\pm'(0) = 0 \},$$

with

$$D_{\text{max}}^+ = \{ f_+ \in L^2(\mathbb{R}_+) : f_+, f_+'' \text{ absolutely continuous, } -f_+'' + q_+ f_+ \in L^2(\mathbb{R}_+) \},$$

$$D_{\text{max}}^- = \{ f_- \in L^2(\mathbb{R}_-) : f_-, f_-'' \text{ absolutely continuous, } f_-'' - q_- f_- \in L^2(\mathbb{R}_-) \},$$

are closed symmetric operators in the Hilbert spaces $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$, respectively, cf. [45, 46], with deficiency indices $(1, 1)$. The adjoint operators $(A_{\text{min}}^\pm)^*$ in the Hilbert space $L^2(\mathbb{R}_\pm)$ are the usual maximal operators defined on $D_{\text{max}}^\pm$.

We introduce the operators

$$A_0^+ f_+ = -f_+'' + q_+ f_+ \quad \text{and} \quad A_0^- f_- = f_-'' - q_- f_-$$

defined on

$$\text{dom } A_0^\pm = \{ f_\pm \in D_{\text{max}}^\pm : f_\pm(0) = 0 \},$$

Evidently, $A_0^\pm$ are self-adjoint extensions of $A_{\text{min}}^\pm$ in the Hilbert spaces $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$, respectively, cf. [45, 46]. In the following we consider $\text{dom } A_{\text{min}}^\pm$ as subsets of $L^2(\mathbb{R})$. Then above considerations imply the following lemma.

Lemma 2.3. Let dom $A_{\text{min}} := \text{dom } A_{\text{min}}^+ \oplus \text{dom } A_{\text{min}}^-$ and let the operator $A_{\text{min}}$ be defined on $\text{dom } A_{\text{min}}$,

$$A_{\text{min}} := \begin{pmatrix} A_{\text{min}}^+ & 0 \\ 0 & A_{\text{min}}^- \end{pmatrix},$$
Definition 3.1. For a self-adjoint operator $A$ in $\mathcal{L}^2(\mathbb{R}) = L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_-)$. Then $A_{min}$ is a closed symmetric operator in the Hilbert space $L^2(\mathbb{R})$ with deficiency indices $(2, 2)$. Moreover, we have

$$A_{min} = A |_{\text{dom } A_{min}}, \quad A = A_{min}^* |_{\mathcal{D}},$$

where

$$\mathcal{D} = \text{dom}(A) = \{ f = (f_+^*, f_-^*) : f_+^* \in \text{dom}(A_{min}^+), f_-^* \in \text{dom}(A_{min}^-) \}.$$

In the following proposition we collect some spectral properties of $A$.

Proposition 2.4. Let $A$ be the operator associated with the differential expression $a(\cdot)$. Then:

(i) $\{ \lambda \in \mathbb{C} \setminus \mathbb{R} : M_+(\lambda) = M_-(\lambda) \} = \sigma_p(A) \setminus \mathbb{R};$

(ii) $\{ \lambda \in \mathbb{C} \setminus \mathbb{R} : M_+(\lambda) \neq M_-(\lambda) \} = \rho(A) \setminus \mathbb{R};$

(iii) $\rho(A) \neq \emptyset.$

(iv) The essential spectrum $\sigma_{ess}(A)$ of $A$ is real and

$$\sigma_{ess}(A) = \sigma_{ess}(A^+_0) \cup \sigma_{ess}(A^-_0).$$

The sets $\sigma_p(A) \cap \mathbb{C}_{\pm}$ are at most countable with possible limit points belonging to $\sigma_{ess}(A) \cup \{ \infty \}.$

For a proof of Proposition 2.4 we refer to [34, Proposition 2.5] and [30, 31]. We mention only that the statements (iii) and (iv) follow from the first and second statement and (2.6).

3. Criteria for definitizability

3.1. Definitizable and locally definitizable operators

Let $(\mathcal{H}, [\cdot, \cdot])$ be a Krein space and let $A$ be a closed operator in $\mathcal{H}$. We define the extended spectrum $\sigma_e(A)$ of $A$ by $\sigma_e(A) := \sigma(A)$ if $A$ is bounded and $\sigma_e(A) := \sigma(A) \cup \{ \infty \}$ if $A$ is unbounded. We set $\rho_e(A) := \overline{\mathbb{C}} \setminus \sigma_e(A)$. A point $\lambda_0 \in \mathbb{C}$ is said to belong to the approximative point spectrum $\sigma_{ap}(A)$ of $A$ if there exists a sequence $(x_n) \subset \text{dom}(A)$ with $\|x_n\| = 1$, $n = 1, 2, \ldots$, and $\|(A - \lambda_0)x_n\| \to 0$ if $n \to \infty$. For a self-adjoint operator $A$ in $\mathcal{H}$ all real spectral points of $A$ belong to $\sigma_{ap}(A)$ (see e.g. [9, Corollary VI.6.2]).

First we recall the notions of spectral points of positive and negative type.

The following definition was given in [37], [39] (for bounded self-adjoint operators).

Definition 3.1. For a self-adjoint operator $A$ in $\mathcal{H}$ a point $\lambda_0 \in \sigma(A)$ is called a spectral point of positive (negative) type of $A$ if $\lambda_0 \in \sigma_{ap}(A)$ and for every sequence $(x_n) \subset \text{dom}(A)$ with $\|x_n\| = 1$ and $\|(A - \lambda_0)x_n\| \to 0$ for $n \to \infty$, we have

$$\liminf_{n \to \infty} [x_n, x_n] > 0 \quad (\text{resp. } \limsup_{n \to \infty} [x_n, x_n] < 0).$$
Illya Karabash and Carsten Trunk

The point $\infty$ is said to be of positive (negative) type of $A$ if $A$ is unbounded and for every sequence $(x_n) \subset \text{dom}(A)$ with $\lim_{n \to \infty} \|x_n\| = 0$ and $\|Ax_n\| = 1$ we have

$$\liminf_{n \to \infty} [Ax_n, Ax_n] > 0 \quad (\text{resp. } \limsup_{n \to \infty} [Ax_n, Ax_n] < 0).$$

We denote the set of all points of $\sigma_e(A)$ of positive (negative) type by $\sigma_{++}(A)$ (resp. $\sigma_{--}(A)$). We shall say that an open subset $\delta$ of $\mathbb{R} (= \mathbb{R} \cup \infty)$ is of positive type (negative type) with respect to $A$ if

$$\delta \cap \sigma_e(A) \subset \sigma_{++}(A) \quad (\text{resp. } \delta \cap \sigma_e(A) \subset \sigma_{--}(A)).$$

An open set $\delta$ of $\mathbb{R}$ is called of definite type if $\delta$ is of positive or negative type with respect to $A$.

The sets $\sigma_{++}(A)$ and $\sigma_{--}(A)$ are contained in $\mathbb{R}$. The non-real spectrum of $A$ cannot accumulate at a point belonging to an open set of definite type.

Recall, that a self-adjoint operator $A$ in a Krein space $(\mathcal{H}, [\cdot, \cdot])$ is called definitizable if $\rho(A) \neq \emptyset$ and there exists a rational function $p \neq 0$ having poles only in $\rho(A)$ such that $[p(A)x, x] \geq 0$ for all $x \in \mathcal{H}$. Then the non-real part of the spectrum of $A$ consists of no more than a finite number of points. Moreover, $A$ has a spectral function $E$ defined on the ring generated by all connected subsets of $\mathbb{R}$ whose endpoints do not coincide with the points of some finite set which is contained in $\{t \in \mathbb{R} : p(t) = 0\} \cup \{\infty\}$ (see [38]).

A self-adjoint operator in a Krein space is definitizable if and only if it is definitizable over $\mathcal{U}$ in the sense of the following definition (see e.g. [24, Definition 4.4]), which localizes the notion of definitizability.

**Definition 3.2.** Let $\Omega$ be a domain in $\mathbb{C}$ such that

$$\Omega \quad \text{is symmetric with respect to } \mathbb{R}, \quad \Omega \cap \mathbb{R} \neq \emptyset,$$

and the domains $\Omega \cap \mathbb{C}^+, \Omega \cap \mathbb{C}^-$ are simply connected.

Let $A$ be a self-adjoint operator in the Krein space $(\mathcal{H}, [\cdot, \cdot])$ such that $\sigma(A) \cap (\Omega \setminus \mathbb{R})$ consists of isolated points which are poles of the resolvent of $A$, and no point of $\Omega \cap \mathbb{R}$ is an accumulation point of the non-real spectrum $\sigma(A) \setminus \mathbb{R}$ of $A$. The operator $A$ is called definitizable over $\Omega$, if the following holds.

(i) For every closed subset $\Delta$ of $\Omega \cap \mathbb{R}$ there exist an open neighbourhood $\mathcal{U}$ of $\Delta$ in $\mathbb{C}$ and numbers $m \geq 1, M > 0$ such that

$$\|(A - \lambda)^{-1}\| \leq M(|\lambda| + 1)^{2m-2}|\text{Im}\lambda|^{-m}$$

for all $\lambda \in \mathcal{U} \setminus \mathbb{R}$. (3.3)

(ii) Every point $\lambda \in \Omega \cap \mathbb{R}$ has an open connected neighbourhood $I_\lambda$ in $\mathbb{R}$ such that both components of $I_\lambda \setminus \{\lambda\}$ are of definite type (cf. Definition 3.1) with respect to $A$.

A self-adjoint operator definitizable over $\Omega$ where $\Omega$ is as in Definition 3.2 possesses a local spectral function $E$. For the construction and the properties of this
Let Lemma 3.5. The following lemma is a easy consequence of Definitions 3.1 and 3.2.

Theorem 3.3. Let $T_1$ and $T_2$ be self-adjoint operators in the Krein space $\mathcal{H}$, let $\rho(T_1) \cap \rho(T_2) \cap \Omega \neq \emptyset$ and assume that

$$(T_1 - \lambda_0 I)^{-1} - (T_2 - \lambda_0 I)^{-1}$$

is a finite rank operator for some $\lambda_0 \in \rho(T_1) \cap \rho(T_2)$. Then $T_1$ is definitizable over $\Omega$ if and only if $T_2$ is definitizable over $\Omega$.

Moreover, if $T_1$ is definitizable over $\Omega$ and $\Delta \subset \Omega \cap \mathbb{R}$ is an open interval with end point $\eta \in \Omega \cap \mathbb{R}$ and $\Delta$ is of positive type (negative type) with respect to $T_1$, then there exist open interval $\Delta'$, $\Delta' \subset \Delta$, with endpoint $\eta$ such that $\Delta'$ is of positive type (resp. negative type) with respect to $T_2$.

3.2. Definitizability of $A$

In this section we will give conditions which ensures the definitizability of the operator $A$ from Definition 2.1. The following definition is needed below.

Definition 3.4. We shall say that the sets $S_1$ and $S_2$ of real numbers are separated by a finite number of points if there exists a finite ordered set $\{\alpha_j\}_{j=1}^N \in \mathbb{N}$, $\alpha_j < \alpha_{j+1}$, such that one of the sets $S_j$, $j = 1, 2$, is a subset of $\bigcup_{k \text{ is even}} [\alpha_k, \alpha_{k+1}]$ and another one is a subset of $\bigcup_{k \text{ is odd}} [\alpha_k, \alpha_{k+1}]$.

The operator $A^+_0 \oplus A^-_0$, where $A^\pm_0$ are defined as in Section 2.3, is fundamentally reducible (cf. [22, Section 3]) in the Krein space $L^2(\mathbb{R}), [\cdot, \cdot]$) (cf. (2.2)). Hence the following lemma is a easy consequence of Definitions 3.1 and 3.2.

Lemma 3.5. Let $\lambda \in \mathbb{R}$. Then $\lambda \in \sigma_{++}(A^+_0 \oplus A^-_0)$ $\lambda \in \sigma_{--}(A^+_0 \oplus A^-_0)$ if and only if $\lambda \in \sigma(A^+_0) \setminus \sigma(A^-_0)$ $\lambda \in \sigma(A^-_0) \setminus \sigma(A^+_0)$, resp. The operator $A^+_0 \oplus A^-_0$ is definitizable if and only if the sets $\sigma(A^+_0)$ and $\sigma(A^-_0)$ are separated by a finite number of points.

It follows from Proposition 2.4 and $\sigma(A^+_0 \oplus A^-_0) \subset \mathbb{R}$ that $\rho(A) \cap \rho(A^+_0 \oplus A^-_0) \neq \emptyset$. Let $\lambda_0 \in \rho(A) \cap \rho(A^+_0 \oplus A^-_0)$. The operators $A^+_0 \oplus A^-_0$ and $A$ are extensions of $A_{\min}$
and \( \dim \left( \text{dom}(A_0^+ \oplus A_0^-)/\text{dom}(A_{\min}) \right) = \dim \left( \text{dom}(A)/\text{dom}(A_{\min}) \right) = 2 \). This implies that
\[
(A_0^+ \oplus A_0^- - \lambda_0 I)^{-1} - (A - \lambda_0 I)^{-1}
\]
is an operator of rank 2. Then [25] and Lemma 3.5 imply the following theorem.

**Theorem 3.6 ([30, 31]).** The operator \( A \) is definitizable if and only if the sets \( \sigma(A_0^+) \) and \( \sigma(A_0^-) \) are separated by a finite number of points.

**Example 3.7.** Let \( q \) be a constant potential, \( q(x) \equiv c, \ c \in \mathbb{R} \). It is easy to calculate that \( \sigma(A_0^+) = [c, +\infty) \) and \( \sigma(A_0^-) = (-\infty, -c] \). Thus, Corollary 3.6 implies that the operator \( (\text{sgn} \ x)(-d^2/dx^2 + c) \) is definitizable in the Krein space \( L^2(\mathbb{R}, \text{sgn} \ x \ dx) \) if and only if \( c \geq 0 \).

### 3.3. Local definitizability of \( A \)

In this subsection we consider Sturm-Liouville operators defined as in Section 2 and we prove that the operator \( A \) is a definitizable operator in a certain neighbourhood of \( \infty \) (in the sense of the Krein space \( (L^2(\mathbb{R}), [, ,]) \)) if and only if the operator \( L \) is semi-bounded from below (in the sense of the Hilbert space \( L^2(\mathbb{R}) \)).

**Remark 3.8.** Clearly, \( L \geq \eta_0 > -\infty \) whenever \( q(x) \geq \eta_0 > -\infty, \ x \in \mathbb{R} \).

The operator \( A_0^+ \oplus A_0^- \) is a self-adjoint operator both in the Hilbert space \( L^2(\mathbb{R}) \) and in the Krein space \( (L^2(\mathbb{R}), [, ,]) \), cf. (2.2).

**Lemma 3.9.** The following statements are equivalent:

(i) The operator \( L \) is semi-bounded from below.

(ii) There exists \( R > 0 \) such that the operator \( A_0^+ \oplus A_0^- \) is definitizable over the domain \( \{ \lambda \in \mathbb{C} : |\lambda| > R \} \).

**Proof.** (i) \( \Rightarrow \) (ii). Since \( A_0^+ \oplus A_0^- \) is a self-adjoint operator in the Hilbert space \( L^2(\mathbb{R}) \), we see that
\[
\sigma(A_0^+ \oplus A_0^-) \subset \mathbb{R} \quad \text{and (3.3) holds for all} \ \lambda \in \mathbb{C} \setminus \mathbb{R} \quad \text{with} \ m = 1.
\] (3.4)

Assume that \( L \geq \eta_0 \). The operator \( L \) is a self-adjoint extension of \( A_{\min}^+ \oplus (-A_{\min}^-) \), hence the operator \( A_{\min}^+ \) is semi-bounded from below, \( A_{\min}^+ \geq \eta_0 \), and \( A_{\min}^- \) is semi-bounded from above, \( A_{\min}^- \leq -\eta_0 \). The operators \( A_0^+ \) are self-adjoint extensions in \( L^2(\mathbb{R}_+) \) of the symmetric operators \( A_{\min}^+ \) with deficiency indices \((1,1)\). Hence the spectrum of \( A_0^+ \) \((A_0^-)\) lies, with the possible exception of at most one normal eigenvalue, in \([\eta_0, \infty) \) (in \((-\infty, -\eta_0]\), respectively), see e.g. [1, Section VII.85].

Choose \( R := \eta_0 \). Lemma 3.5 implies that the set \((R, +\infty)\), with the possible exception of at most one eigenvalue, is of positive type and the set \((-\infty, -R)\), with the possible exception of at most one eigenvalue, is of negative type with respect to \( A_0^+ \oplus A_0^- \). Thus, the operator \( A_0^+ \oplus A_0^- \) is definitizable over \( \{ \lambda \in \mathbb{C} : |\lambda| > R \} \).

(i) \( \Leftrightarrow \) (ii) Obviously, the Sturm-Liouville operator \( A_0^+ \) \((A_0^-)\) is not semi-bounded from above (below, resp.). That is,
\[
\sup \sigma(A_0^+) = +\infty, \quad \inf \sigma(A_0^-) = -\infty.
\] (3.5)
Assume that $L$ is not semi-bounded from below. Then $A_{\min}^+$ or $-A_{\min}$ is not semi-bounded from below. Thus, $\inf \sigma(A_0^+) = -\infty$ or $\sup \sigma(A_0^-) = +\infty$.

Consider the case

$$\inf \sigma(A_0^+) = -\infty. \quad (3.6)$$

It follows from (3.6), (3.5) and Lemma 3.5 that

$$(\infty, -r) \cap \sigma_{++}(A_0^+ \oplus A_0^-) \neq \emptyset \quad \text{and} \quad (\infty, -r) \cap \sigma_{--}(A_0^+ \oplus A_0^-) \neq \emptyset$$

for all $r > 0$. Thus, by definition, the operator $A_0^+ \oplus A_0^-$ is not definitizable over $\{ \lambda \in \mathbb{C} : |\lambda| > r \}$ for arbitrary $r > 0$. The case $\sup \sigma(A_0^-) = +\infty$ can be considered in the same way.

The following theorem is one of the main results.

**Theorem 3.10.** The following assertions are equivalent:

(i) The operator $L$ is semi-bounded from below.

(ii) There exists $R > 0$ such that the operator $A$ is definitizable over the domain $\{ \lambda \in \mathbb{C} : |\lambda| > R \}$.

**Proof.** It follows from Proposition 2.4 (iii) and $\sigma(A_0^+ \oplus A_0^-) \subset \mathbb{R}$ that $\rho(A) \cap \rho(A_0^+ \oplus A_0^-) \neq \emptyset$. Let $\lambda_0 \in \rho(A) \cap \rho(A_0^+ \oplus A_0^-)$. The operators $A_0^+ \oplus A_0^-$ and $A$ are extensions of $A_{\min}$ and $\text{dim}(\text{dom}(A_0^+ \oplus A_0^-)/\text{dom}(A_{\min})) = \text{dim}(\text{dom}(A)/\text{dom}(A_{\min})) = 2$. This implies that

$$(A_0^+ \oplus A_0^- - \lambda_0 I)^{-1} - (A - \lambda_0 I)^{-1} \quad (3.7)$$

is an operator of rank 2. Combining Lemma 3.9 and Theorem 3.3, Theorem 3.10 is proved.

By Theorem 3.10, the semi-boundedness of $L$ implies the definitizability of $A$ over some domain. Now we give a precise description of the domain of definitizability of $A$ in terms of the spectra of $A_0^+$ and $A_0^-$. Let $T$ be an operator such that $\sigma(T) \subset \mathbb{R}$. Let us introduce the sets $\sigma_{\text{left}}(T)$ and $\sigma_{\text{right}}(T)$ by the following way: a point $\lambda \in \mathbb{R} = \mathbb{R} \cup \{ \pm \infty \}$ is said to belong to $\sigma_{\text{left}}(T)$ ($\sigma_{\text{right}}(T)$) if there exists an increasing (resp. decreasing) sequence $\{ \lambda_n \}_{1}^{\infty} \subset \sigma(T)$ such that $\lim_{n \to \infty} \lambda_n = \lambda$.

Note that

$$\sigma_{\text{left}}(T) \cup \sigma_{\text{right}}(T) \subset \sigma_{\text{ess}}(T) \cup \{ \infty \}. \quad (3.8)$$

For differential operators $A_0^\pm$, equality holds in (3.8) since every point of $\sigma_{\text{ess}}(A_0^\pm)$ is an accumulation point of $\sigma(A_0^\pm)$.

We put

$$S_A := \left( \sigma_{\text{left}}(A_0^+) \cap \sigma_{\text{left}}(A_0^-) \right) \cup \left( \sigma_{\text{right}}(A_0^+) \cap \sigma_{\text{right}}(A_0^-) \right). \quad (3.9)$$

**Theorem 3.11.** Let $\Omega$ be a domain in $\mathbb{C}$ such that (3.1)-(3.2) are fulfilled. Then the operator $A = (\text{sgn} \ x)(-d^2/dx^2 + q)$ is definitizable over $\Omega$ if and only if $\Omega \subset \Omega_A$, where $\Omega_A := \mathbb{C} \setminus S_A$. 


Proof. Arguments from the proof of Theorem 3.10 show that it is enough to prove the theorem for the operator \( A^+_0 \oplus A^-_0 \).

Let \( \lambda \in \mathcal{S}_A \) and let \( I_\lambda \) be an open connected neighbourhood of \( \lambda \). Then (3.9) and Lemma 3.5 imply that one of the components of \( I_\lambda \setminus \{\lambda\} \) is not of definite type. So if \( A^+_0 \oplus A^-_0 \) is definitizable over \( \Omega \), then \( \lambda \notin \Omega \).

Conversely, if \( \mathcal{S}_A \neq \mathbb{R} \), then condition (ii) from Definition 3.2 is fulfilled for \( \Omega_A = \mathbb{T} \setminus \mathcal{S}_A \). Taking (3.4) into account, we see that \( A^+_0 \oplus A^-_0 \) is definitizable over \( \Omega_A \).

Remark 3.12. Note that \( \Omega_A \cap \mathbb{R} = \emptyset \) is equivalent to \( \sigma_{ess}(A^+_0) = \sigma_{ess}(A^-_0) = \mathbb{R} \). In the converse case, (3.1)-(3.2) are fulfilled for \( \Omega_A \) and it is the greatest domain over which the operator \( A \) is definitizable.

The following statement is a simple consequence of Theorem 3.10, Theorem 3.11, and (3.8).

Corollary 3.13. Assume that \( L \) is semi-bounded from below. Then the operator \( A \) is definitizable over the set \( \mathbb{T} \setminus (\sigma_{ess}(A^+_0) \cap \sigma_{ess}(A^-_0)) \).

3.4. Regularity of the critical point \( \infty \)

In the sequel we will use a result which follows easily from [12, Lemma 3.5 (iii)] and [12, Theorem 3.6 (i)].

Proposition 3.14. If the operator \( \tilde{L} := -d^2/dx^2 + \tilde{q}(x) \), for some real \( \tilde{q} \in L^1_{loc}(\mathbb{R}) \), defined on \( \mathcal{D} \) is nonnegative in the Hilbert space \( L^2(\mathbb{R}) \), then the operator \( \tilde{A} := (\text{sgn } x)\tilde{L} \) is definitizable and \( \infty \) is a regular critical point of \( \tilde{A} \).

The following theorem can be considered as the main result of this note.

Theorem 3.15. Assume that assertions (i), (ii) of Theorem 3.10 hold true. Then there exists a decomposition

\[
A = A_\infty + A_0 \tag{3.10}
\]

such that the operator \( A_\infty \) is similar to a self-adjoint operator in the Hilbert space sense and \( A_0 \) is a bounded operator.

Remark 3.16. The conclusion of Theorem 3.15 is equivalent to the regularity of critical point \( \infty \) of the operator \( A \).

Proof of Theorem 3.15. Assume that \( A \) is an operator definitizable over \( \{\lambda \in \mathbb{T} : |\lambda| > R\} \), \( R > 0 \). By Theorem 3.10, this is equivalent to the fact that \( L \geq \eta_0 \) for certain \( \eta_0 \in \mathbb{R} \).

Denote by \( E^A \) the spectral function of \( A \). Choose \( r > R \) such that \( \sigma(A) \setminus \mathbb{R} \subset \{\lambda \in \mathbb{C} : |\lambda| \leq r\} \) and \( E^A(\mathbb{R} \setminus (-r, r)) \) is defined. Then \( A \) decomposes,

\[
A = A_1 + A_0, \quad A_1 := A \upharpoonright \text{dom}(A) \cap (E^A(\mathbb{R} \setminus (-r, r))L^2(\mathbb{R})), \quad A_0 := A \upharpoonright \text{dom}(A) \cap ((I - E^A(\mathbb{R} \setminus (-r, r)))L^2(\mathbb{R}))
\]

and the following statements holds (cf. [22, Theorem 2.6]):
$A_1$ is a definitizable operator in the Krein space $(E^A(\mathbb{R} \setminus (-r, r))L^2(\mathbb{R}), [\cdot, \cdot])$.

$A_0$ is a bounded operator and $\sigma(A_0) \subset \{ \lambda : |\lambda| \leq r \}$. Let us show that $\infty$ is not a singular critical point of $A_1$.

Consider the operator $A_2$ defined by $A_2 = A_1 + 0$, where the direct sum is considered with respect to the decomposition

$$L^2(\mathbb{R}) = E^A(\mathbb{R} \setminus (-r, r))L^2(\mathbb{R}) \oplus (I - E^A(\mathbb{R} \setminus (-r, r)))L^2(\mathbb{R}),$$

and $0$ is the zero operator in the subspace $\text{ran}(I - E^A(\mathbb{R} \setminus (-r, r)))$. Since $A_0$ is a bounded operator, we have

$$\text{dom}(A_2) = \text{dom} A.$$ 

Moreover, $\infty$ is not a singular critical point of $A_2$ if and only if $\infty$ is not a singular critical point of $A$.

Now we prove that $\infty$ is not a singular critical point of $A_2$. Let $\eta_1 < \eta_0$. Since $L \geq \eta_0$, we see that $L - \eta_1 I$ is a uniformly positive operator in the Hilbert space $L^2(\mathbb{R})$ (i.e., $L - \eta_1 I \geq \delta > 0$). Therefore $A := J(L - \eta_1 I)$,

$$\tilde{A} y(x) = (\text{sgn} x)(-y''(x) + q(x)y(x) - \eta_1 y(x)), \quad \text{dom}(\tilde{A}) = \text{dom}(A),$$

is a definitizable nonnegative operator in the Krein space $(L^2(\mathbb{R}), [\cdot, \cdot])$. By Proposition 3.14, $\infty$ is not a singular critical point of $\tilde{A}$. The Čurgus criterion of the regularity of critical point $\infty$, see [11, Corollary 3.3], implies that $\infty$ is not a singular critical point of the operator $A_2$. So $\infty$ is not a singular critical point of $A_1$. It follows from $L \geq \eta_0$ and Lemma 3.5 that for sufficiently large $r_1 > 0$ the set $(-\infty, -r_1]$ is of negative type and the set $[r_1, +\infty)$ is of positive type with respect to $A_0^\perp \oplus A_0^\perp$. Combining this with Theorem 3.3, we obtain that there exists $r_2 \geq r_1$ such that $(-\infty, -r_2]$ is of negative type and the set $[r_2, +\infty)$ is of positive type with respect to the operator $A$. Evidently, we obtain the desired decomposition

$$A = A_\infty + A_0, \quad A_\infty := A | \text{dom}(A) \cap (E^A(\mathbb{R} \setminus (-r_2, r_2))L^2(\mathbb{R})), \quad A_0 := A | \text{dom}(A) \cap ((I - E^A(\mathbb{R} \setminus (-r_2, r_2)))L^2(\mathbb{R})),$$

where $A_0$ is a bounded operator and $A_\infty$ is similar to a self-adjoint operator in the Hilbert space sense.

4. Accumulation of non-real eigenvalues to a real point

By Proposition 2.4 (i), the non-real spectrum $\sigma(A) \setminus \mathbb{R}$ of $A$ consists of eigenvalues. Let $S_A$ be the set defined by (3.9). The following proposition is a consequence of Theorems 3.11 and 3.10.

**Proposition 4.1.** If $\lambda$ is an accumulation point of $\sigma(A) \setminus \mathbb{R}$, then $\lambda \in S_A$. In particular, if the operator $L = -d^2/dx^2 + q(x)$ is semi-bounded from below, then non-real spectrum of $A$ is a bounded set.
The goal of this subsection is to show that there exists a potential \( q \) continuous in \( \mathbb{R} \) such that the set of non-real eigenvalues of the operator \( A = (\text{sgn} \ x)(-d^2/dx^2 + q(x)) \) has a real accumulation point.

It is well known (e.g. [40]) that \( M_+ \), the Titchmarsh-Weyl m-coefficient for (2.3) (see Subsection 2.2), admits the following integral representation

\[
M_+(\lambda) = \int_{\mathbb{R}} \frac{d\Sigma_+(t)}{t - \lambda}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]

where \( \Sigma_+(\cdot) \) is a nondecreasing scalar function such that \( \int_{\mathbb{R}} (1 + |t|)^{-1} d\Sigma_+(t) < \infty \).

The function \( \Sigma_+ \) is called a spectral function of the boundary value problem

\[
-y''(x) + q_+(x)y(x) = \lambda y(x), \quad y'(0) = 0, \quad x \in [0, +\infty).
\]  

This means that the self-adjoint operator \( A_+^0 \) introduced in Subsection 2.3 is unitary equivalent to the operator of multiplication by the independent variable in the Hilbert space \( L^2(\mathbb{R}, d\Sigma_+(t)) \). This fact obviously implies

\[
\sigma(A_+^0) = \text{supp}(d\Sigma_+), \quad (4.2)
\]

where \( \text{supp} \ d\tau \) denotes the topological support of a Borel measure \( d\Sigma_+ \) on \( \mathbb{R} \) (i.e., \( \text{supp} \ d\Sigma_+ \) is the smallest closed set \( \Omega \subset \mathbb{R} \) such that \( d\Sigma_+(\mathbb{R} \setminus \Omega) = 0 \)).

**Lemma 4.2.** Assume that \( q \) is an even potential, \( q(x) = q(-x) \), \( x \in \mathbb{R} \). If \( \varepsilon > 0 \), then \( \varepsilon i \in \sigma_p(A) \) if and only if \( \Re M_+(i\varepsilon) = 0 \).

**Proof.** Since \( q \) is even, we get \( m_+(\lambda) = m_-(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R} \). So \( M_-(i\varepsilon) = -M_+(i\varepsilon) \). Since \( M_+ \) is a Nevanlinna function, we see that \( M_+(-i\varepsilon) = \overline{M_+(i\varepsilon)} \). Thus,

\[
M_+(i\varepsilon) - M_-(i\varepsilon) = M_+(i\varepsilon) + \overline{M_+(i\varepsilon)} = 2 \Re M_+(i\varepsilon).
\]

Proposition 2.4 completes the proof. \( \square \)

The following lemma follows easily from the Gelfand–Levitan theorem (see e.g. [41, Subsection 26.5]).

**Lemma 4.3.** Let \( \Sigma(t), t \in \mathbb{R}, \) be a nondecreasing function such that

\[
\int_{-\infty}^{T_1} d\Sigma(t) = 0 \quad \text{and} \quad \int_{-\infty}^{s} d\Sigma(t) = \int_{0}^{s} \frac{1}{\pi \sqrt{t}} \frac{\pi}{\sqrt{\pi}} dt = \text{for all } s > T_2,
\]

with certain constants \( T_1, T_2 \in \mathbb{R}, T_1 < T_2 \). Then there exists a potential \( q_+ \) continuous in \( [0, +\infty) \), such that \( \Sigma(t) \) is a spectral function of the boundary value problem

\[
-y''(x) + q_+(x)y(x) = \lambda y(x), \quad y'(0) = 0, \quad x \in [0, +\infty).
\]

**Lemma 4.4.** There exist a nondecreasing function \( \Sigma(t), t \in \mathbb{R}, \) with the following properties:
(i) $\Sigma(t) = \Sigma_1(t) + \Sigma_2(t)$, where

$$\Sigma_1 \in AC_{loc}(\mathbb{R}), \quad \Sigma_1'(t) = \begin{cases} 0, & t \in (-\infty, 1), \\ \frac{1}{\pi \sqrt{t}}, & t \in (1, +\infty), \end{cases}$$

and the measure $d\Sigma_2$ has the form

$$d\Sigma_2(t) = \sum_{k=1}^{+\infty} h_k \delta(t - s_k),$$

where $h_k > 0$, $s_k \in (-1, 1)$, $k \in \mathbb{N}$; $\sum_{k=1}^{+\infty} h_k < \infty$, (4.6)

(here $\delta(t)$ is the Dirac delta-function).

(ii) Conditions (4.3)-(4.4) are valid for $\Sigma$ with $T_1 = -1$ and $T_2 = 1$.

(iii) There exists a sequence $\varepsilon_k > 0$, $k \in \mathbb{N}$, such that $\lim_{k \to \infty} \varepsilon_k = 0$ and $r(\varepsilon_k) = 0$, $k \in \mathbb{N}$, where the function $r(\varepsilon)$, $\varepsilon > 0$, is defined by

$$r(\varepsilon) := \text{Re} \int_{\mathbb{R}} \frac{1}{t - i\varepsilon} d\Sigma(t) = \int_{\mathbb{R}} \frac{t}{t^2 + \varepsilon^2} d\Sigma_1(t).$$

Proof. Let $h_k = 2^{-k+1}/\pi$. Then

$$\sum_{k=1}^{+\infty} h_k = 2/\pi.$$ (4.7)

Now, if $s_k \notin (-1, 1)$ for all $k \in \mathbb{N}$, then $\Sigma$ possesses property (ii). We should only choose $\{s_k\}_{k=1}^{+\infty} \subset (-1, 1)$ such that statements (iii) holds true.

Consider for $\varepsilon \geq 0$ the functions

$$r_0(\varepsilon) = \int_{1}^{\infty} \frac{t}{t^2 + \varepsilon^2} d\Sigma_1(t)$$

and

$$r_n(\varepsilon) := \int_{1}^{\infty} \frac{t}{t^2 + \varepsilon^2} d\Sigma_1(t) + \sum_{k=1}^{n} \frac{s_k h_k}{s_k^2 + \varepsilon^2}, \quad n \in \mathbb{N}.$$

Let $s_k \neq 0$ for all $k \in \mathbb{N}$. Then $r_n$ are well-defined and continuous on $[0, +\infty)$. Besides, $\lim_{n \to \infty} r_n(\varepsilon) = r(\varepsilon)$ for all $\varepsilon > 0$. It is easy to see that $\lim_{\varepsilon \to \infty} r_n(\varepsilon) = 0$, $n \in \mathbb{N}$. Since $r_n$ are continuous on $[0, +\infty)$, we see that

$$\text{SUP}_n := \sup_{\varepsilon \in [0, +\infty)} |r_n(\varepsilon)| < \infty, \quad n \in \mathbb{N}.$$

Now we give a procedure to choose $s_k \in (-1, 1) \setminus \{0\}$. 

Singular Sturm-Liouville operators with indefinite weight $\text{sgn} x$
Let $s_1$ be an arbitrary number in $(-1, 0)$ such that
\[
\frac{s_1 h_1}{s_1^2 + \varepsilon^2} \bigg|_{\varepsilon = |s_1|} = \frac{1}{\pi} \frac{1}{2s_1} < -\text{SUP}_0 - 1,
\]
in other words, \[ -\frac{1}{2\pi(\text{SUP}_0 + 1)} < s_1 < 0. \]
Then
\[
r_1(|s_1|) = r_0(|s_1|) + \frac{s_1 h_1}{s_1^2 + \varepsilon^2} \bigg|_{\varepsilon = |s_1|} < r_0(|s_1|) - \sup_{\varepsilon \in [0, +\infty)} |r_0(\varepsilon)| - 1 < -1. \tag{4.8}
\]
Let
\[
\{s_k\}_{k=2}^{\infty} \in (-b_1, b_1) \setminus \{0\} \quad \text{with certain} \quad b_1 \in (0, |s_1|/2). \tag{4.9}
\]
Let us show that we may choose a number $b_1$ such that (4.9) implies
\[
r(|s_1|) < 0. \tag{4.10}
\]
Indeed, (4.8) and (4.7) yield
\[
r(|s_1|) = r_1(|s_1|) + \sum_{k=2}^{\infty} \frac{s_k h_k}{s_k^2 + \varepsilon^2} \bigg|_{\varepsilon = |s_1|} < \notag
\]
\[
< -1 + \sum_{k=2}^{\infty} \frac{h_k |s_k|}{s_k^2} < -1 + \frac{b_1}{s_1^2} \sum_{k=2}^{\infty} h_k < -1 + \frac{2b_1}{\pi s_1^2}
\]
and therefore (4.10) is valid whenever $0 < b_1 < \pi s_1^2/2$.
Similarly, there exist $s_2 \in (0, b_1)$ such that
\[
\frac{s_2 h_2}{s_2^2 + \varepsilon^2} \bigg|_{\varepsilon = s_2} = \frac{1}{\pi} \frac{1}{2s_2} > \text{SUP}_1 + 1,
\]
and therefore
\[
r_2(s_2) > 1.
\]
Further, there exist $b_2 \in (0, s_2/2)$ such that \[ \{s_k\}_{k=2}^{\infty} \subset (-b_2, b_2) \setminus \{0\} \] implies that $r(s_2) > 0$.
Continuing this process, we obtain a sequence \[ \{s_k\}_{k=1}^{\infty} \subset (-1, 1) \setminus \{0\} \] with the following properties:
\[
s_k \in (-1, 0) \quad \text{if } k \text{ is odd}, \quad s_k \in (0, 1) \quad \text{if } k \text{ is even},
\]
\[
|s_1| > \frac{|s_1|}{2} > |s_2| > \frac{|s_2|}{2} > |s_3| > \ldots > |s_k| > \frac{|s_k|}{2} > |s_{k+1}| > \ldots, \tag{4.11}
\]
\[
r(|s_k|) < 0 \quad \text{if } k \text{ is odd}, \quad r(|s_k|) > 0 \quad \text{if } k \text{ is even}. \tag{4.12}
\]
It is easy to show that $r$ is continuous on $(0, +\infty)$. Combining this with (4.12), we see that there exists $\varepsilon_k \in \{|s_{k-1}|, |s_k|\}$ such that $r(\varepsilon_k) = 0$, $k \in \mathbb{N}$. Besides, (4.11) implies $\lim |s_k| = \lim \varepsilon_k = 0$. \qed
Theorem 4.5. There exist an even potential $\hat{q}$ continuous on $\mathbb{R}$ and a sequence $\{\varepsilon_k\}_1^\infty \subset \mathbb{R}_+$ such that

(i) the operator $\hat{A}$ defined by the differential expression

\[
\text{sgn } x \left( -\frac{d^2}{dx^2} + \hat{q}(x) \right)
\]

on the natural domain $\mathcal{D}$ (see Subsection 2.1) is a self-adjoint operator in the Krein space $L^2(\mathbb{R}, [\cdot, \cdot])$;

(ii) $\{i\varepsilon_k\}_1^\infty \subset \sigma_p(\hat{A})$, i.e., $i\varepsilon_k$, $k \in \mathbb{N}$, are non-real eigenvalues of $\hat{A}$;

(iii) $\lim_{k \to \infty} \varepsilon_k = 0$;

(iv) the operator $\hat{A}$ is definitizable over the domain $\mathbb{C} \setminus \{0\}$.

Proof. (i) Let $\Sigma$ and $\{\varepsilon_k\}_1^\infty$ be from Lemma 4.4. Then, by Lemma 4.3, $\Sigma$ is a spectral function of the boundary value problem (4.1) with a certain potential $\hat{q}_+$. Let us consider an even continuous potential $\hat{q}(x) = \hat{q}_+(|x|)$, $x \in \mathbb{R}$, and the corresponding operator $\hat{A} = (\text{sgn } x) \left( -\frac{d^2}{dx^2} + \hat{q}(x) \right)$ defined as in Subsection 2.1.

It is well known that if equation (2.3) is in the limit-circle case at $+\infty$ then $M_+(\cdot)$ is a meromorphic function on $\mathbb{C}$ and the spectral function $\Sigma_+$ is a step function with jumps at the poles of $M_+(\cdot)$ only (see e.g. [10, Theorem 9.4.1]). As $\Sigma_+(t) = \Sigma(t)$, $t > 0$, this condition does not hold for the function $\Sigma$ since $\Sigma$ satisfies (4.4). Indeed, (4.4) means that $\Sigma'(t) = \frac{1}{\pi\sqrt{t}}$ for $t > T_2 = 1$ and therefore $\Sigma$ is not a step function. So (2.3) is limit-point at $+\infty$.

Since the potential $\hat{q}$ is even, the same is true for $-\infty$. Thus, $\hat{A}$ is a self-adjoint operator in the Krein space $L^2(\mathbb{R}, [\cdot, \cdot])$, see Subsection 2.1.

(ii) and (iii) follow from Lemma 4.2 and statement (iii) of Lemma 4.4.

(iv) Let $\hat{A}_0^\pm$ be the self-adjoint operators in the Hilbert spaces $L^2(\mathbb{R}_\pm)$ defined by the differential expression (4.13) in the same way as in Subsection 2.3 where $q$ is replaced by $\hat{q}$. By (4.2), $\sigma(\hat{A}_0^+) = \{s_k\}_1^\infty \cup [1, +\infty)$. Since $\hat{q}$ is even, one gets $\sigma(\hat{A}_0^-) = \{-s_k\}_1^\infty \cup (-1, -1]$. It follows from $\{s_k\}_1^\infty \subset (-1, 1)$ and $\lim_{k \to \infty} s_k = 0$ that

\[
\min \sigma_{ess}(\hat{A}_0^+) = \max \sigma_{ess}(\hat{A}_0^-) = 0
\]

and Theorem 3.13 concludes the proof.

5. Some classes of Sturm-Liouville operators

As an illustration of the results from the previous sections, we discuss in this section various potentials $q \in L^1_{\text{loc}}(\mathbb{R})$ such that the differential operator $A = (\text{sgn } x)(-d^2/dx^2 + q)$ is definitizable over specific subsets of $\mathbb{C}$. As before it is supposed that the differential expression (2.1) is in limit point case at $+\infty$ and at $-\infty$ (for instance, the letter holds if $\liminf_{|x| \to \infty} \frac{q(x)}{x^2} > -\infty$, see e.g., [47, Example 7.4.1]).
5.1. The case \( q(x) \to -\infty \)

In this subsection we assume that for some \( X > 0 \) the potential \( q \) has the following properties on the interval \((X, +\infty)\):

\[ q', q'' \text{ exist and are continuous on } (X, +\infty), \quad q(x) < 0, \quad q'(x) < 0, \quad (5.1) \]

\[ q''(x) \text{ is of fixed sign, i.e., } q''(x_1)q''(x_2) \geq 0 \text{ for all } x_1, x_2 > X, \quad (5.2) \]

\[ \lim_{x \to +\infty} q(x) = -\infty, \quad \int_X^{+\infty} |q(x)|^{-1/2} dx = \infty, \quad \text{and} \quad \limsup_{x \to +\infty} \frac{|q'(x)|}{|q(x)|^p} < \infty, \quad (5.3) \]

where \( p \in (0, 3/2) \) is a constant.

Then the well-known result of Titchmarsh (see e.g. [40, Theorems 3.4.1 and 3.4.2]) states that (2.1) is in the limit point case at \(+\infty\) and \( \sigma(A^+_0) = \mathbb{R} \). Hence the set \( S_A \) defined by (3.9) coincides with \( \sigma_{ess}(A^+_0) \cup \infty \). By Theorem 3.11, there are two cases:

(i) Let \( \sigma_{ess}(A^+_0) \neq \mathbb{R} \). Then the greatest domain over which \( A \) is definitizable is

\[ \Omega_A := \mathbb{C} \setminus \sigma_{ess}(A^+_0) \text{ (note that } \infty \notin \Omega_A). \]

(ii) Let \( \sigma_{ess}(A^+_0) = \mathbb{R} \). Then \( \Omega_A \cap \overline{\mathbb{R}} = \emptyset \) and there exists no domain \( \Omega \) in \( \mathbb{C} \) such that \( A \) is definitizable over \( \Omega \). In particular, the letter holds if the analogues of assumptions (5.1)-(5.3) are fulfilled for \( x \in (-\infty, 0] \).

Example 5.1. Let us consider the operator \( A = (\text{sgn } x)(-d^2/dx^2 - x) \). By [45, Theorem 6.6] the differential expression \(-d^2/dx^2 - x\) is in limit point case at \(+\infty\) and \(-\infty\). Assumptions (5.1)-(5.3) hold for \( x \in (0, +\infty) \), hence \( \sigma_{ess}(A^+_0) = \sigma(A) = \mathbb{R} \). On the other hand, \( \sigma_{ess}(A^-_0) = \emptyset \) (see Subsection 5.2 and [40, Section 3.1]). Therefore the operator \( A \) is definitizable over \( \mathbb{C} \) and there exists no domain \( \Omega \) in \( \mathbb{C} \) with \( \infty \in \Omega \) such that \( A \) is definitizable over \( \Omega \). By Proposition 4.1, the only possible accumulation point for non-real spectrum of \( A \) is the point \( \infty \).

5.2. The case \( q(x) \to +\infty \)

Let us assume that the following conditions holds with certain constants \( X, c > 0 \):

\[ q(x) \geq c \quad \text{for } x > X, \quad \text{and for any } \omega > 0, \quad \lim_{x \to +\infty} \int_x^{x+\omega} q(t) dt = +\infty. \quad (5.4) \]

Molčanov proved (see e.g., [40, Lemma 3.1.2] and [41, Subsection 24.5]) that (5.4) yields \( \sigma_{ess}(A^-_0) = \emptyset \), i.e., the spectrum of the operator \( A^-_0 \) is discrete. Besides, (5.4) implies that \( A^-_0 \) is semi-bounded from below. It follows from the results of Subsection 3.3 that the operator \( A \) is definitizable over \( \mathbb{C} \). More precisely,

(i) Let the operator \( A^-_0 \) be semi-bounded from above. Then the operator \( A \) is definitizable, \( \infty \) is a regular critical point of \( A \) (cf. [12]), and \( A \) admits decomposition (3.10).

(ii) Let \( A^-_0 \) be not semi-bounded from above. Then \( A \) is definitizable over \( \mathbb{C} \) and there exists no domain \( \Omega \) in \( \mathbb{C} \) with \( \infty \in \Omega \) such that \( A \) is definitizable over \( \Omega \). The only possible accumulation point for non-real spectrum of \( A \) is the point \( \infty \).

Note that \( A^-_0 \) is not semi-bounded from above if \( \lim_{x \to -\infty} q(x) = -\infty \).
5.3. Summable potentials

We denote by \( q_{neg}(x) := \min\{q(x), 0\} \), \( x \in \mathbb{R} \).

**Assumption 5.2.** \( \int_{t}^{t+1} |q_{neg}(x)|dx \to 0 \) as \( |t| \to \infty \).

If Assumption 5.2 is fulfilled then the differential expression \(-d^2/dx^2 + q\) is in limit point case at \( +\infty \) and \(-\infty\), cf. [46, Satz 14.21]. By [45, Theorem 15.1], \( A_{0}^{+} \) is semi-bounded from below, \( A_{0}^{-} \) is semi-bounded from above with

\[
\sigma_{\text{ess}}(A_{0}^{+}) \subset [0, +\infty) \quad \text{and} \quad \sigma_{\text{ess}}(A_{0}^{-}) \subset (-\infty, 0].
\]

This implies that the negative spectrums of the operators \( A_{0}^{+} \) and \( A_{0}^{-} \) consist of eigenvalues,

\[
\sigma(\pm A_{0}^{\pm}) \cap (-\infty, 0) = \{ \pm \lambda_{n}^\pm \}_{n=1}^{\infty} \subset \sigma_{p}(\pm A_{0}^{\pm}),
\]

where \( 0 \leq N_{\pm} \leq \infty \). Besides, \( \lim_{n \to \infty} \lambda_{n}^\pm = 0 \) if \( N_{\pm} = \infty \). Then, by Theorem 3.13, \( A \) is definitizable over \( \mathbb{C} \setminus \{0\} \). Theorems 3.11 and 3.15 imply easily the following statement.

**Theorem 5.3.** Let Assumption 5.2 be fulfilled. Then the operator \( A = (\text{sgn } x)(-d^2/dx^2 + q) \) admits the decomposition (3.10). Moreover,

(i) If \( \min \sigma_{\text{ess}}(A_{0}^{+}) > 0 \) or \( \max \sigma_{\text{ess}}(A_{0}^{-}) < 0 \), then \( A \) is a definitizable operator and \( \infty \) is a critical point of \( A \).

(ii) If \( \min \sigma_{\text{ess}}(A_{0}^{+}) = \max \sigma_{\text{ess}}(A_{0}^{-}) = 0 \) and \( N^{+} + N^{-} < \infty \), then \( A \) is a definitizable operator, \( 0 \) and \( \infty \) are critical points of \( A \).

(iii) If \( \min \sigma_{\text{ess}}(A_{0}^{+}) = \max \sigma_{\text{ess}}(A_{0}^{-}) = 0 \) and \( N^{+} + N^{-} = \infty \), then the operator \( A \) is not definitizable. It is definitizable over \( \mathbb{C} \setminus \{0\} \). In particular, \( 0 \) is the only possible accumulation point of the non-real spectrum of \( A \).

We mention (cf. [5]) that Assumption 5.2, and therefore the statements of Theorem 5.3, hold true if \( q \in L^{1}(\mathbb{R}) \).

**Remark 5.4.** By Theorem 3.15 (see also [12]) we have that if the operator \( A = (\text{sgn } x)(-d^2/dx^2 + q) \) is definitizable, then \( \infty \) is its regular critical point. In the case when \( A \) has a finite critical point, the question of the character of this critical point is difficult (see [13, 14, 18, 19, 33, 34, 32] and references therein). Let us mention one case. Assume that \( q \) is continuous in \( \mathbb{R} \) and \( \int_{\mathbb{R}} (1 + x^2)|q(x)|dx < \infty \), then \( \min \sigma_{\text{ess}}(A_{0}^{+}) = \max \sigma_{\text{ess}}(A_{0}^{-}) = 0 \) and \( N^{+} < \infty \) and \( N^{-} < \infty \) (see [40]). Therefore Theorem 5.3 (as well as [12, Proposition 1.1]) implies that \( A = (\text{sgn } x)(-d^2/dx^2 + q) \) is definitizable. It was shown (implicitly) in [18] that \( 0 \) is a regular critical point of \( A \).

In the following case, more detailed information may be obtained.

**Corollary 5.5.** Suppose \( \lim_{x \to -\infty} q(|x|) = 0 \). Then \( \min \sigma_{\text{ess}}(A_{0}^{+}) = \max \sigma_{\text{ess}}(A_{0}^{-}) = 0 \) and either the case (ii) or the case (iii) of Theorem 5.3 takes place. Moreover, the following holds.
(i) If $\liminf_{x \to \infty} x^2 q(|x|) > -1/4$, then $A$ is a definitizable operator and $0$ and $\infty$ are critical points of $A$.

(ii) If $\limsup_{x \to \infty} x^2 q(|x|) < -1/4$, then the operator $A$ is not definitizable. It is definitizable over $\mathbb{C} \setminus \{0\}$.

Proof. The statement follows directly from [16, Corollary XIII.7.57], which was proved in [16] for infinitely differentiable $q$. Actually, this proof is valid for bounded potentials $q$. Finally, note that $\lim_{x \to \infty} q(|x|) = 0$ implies that $q$ is bounded on $(-\infty, -X) \cup [X, +\infty)$ with $X$ large enough. On the other hand, $L^1$ perturbations of potential $q$ on any finite interval does not change $\sigma_{ess}(A_0^+)$, $\sigma_{ess}(A_0^-)$. Also such perturbations increase or decrease $N^+, N^-$ on finite numbers only due to Sturm Comparison Theorem (see e.g., [47, Theorem 2.6.3]). This completes the proof.

Example 5.6. Let $q(x) = -\frac{1}{1+|x|^4}$. Then Corollary 5.5 yields that the operator $A = (\text{sgn } x)(-d^2/dx^2 + q)$ is not definitizable. It is definitizable over $\mathbb{C} \setminus \{0\}$.

It was shown above that under certain assumption on the potential $q$ the operator $A = (\text{sgn } x)(-d^2/dx^2 + q)$ is not definitizable, but it is definitizable over the domain $\mathbb{C} \setminus \{\lambda_0\}$, where $\lambda_0 \in \mathbb{R}$ ($\lambda_0 = \infty$ in Example 5.1 and $\lambda_0 = 0$ in Example 5.6). In this case, unusual spectral behavior may appear near points of the set $c(A) \cup \{\lambda_0\}$ only ($c(A)$ is the set of critical points, see Subsection 3.1). Indeed, a bounded spectral projection $E^A(\Delta)$ exists for any connected set $\Delta \subset \mathbb{R} \setminus \{\lambda_0\}$ such that the endpoints of $\Delta$ do not belong to $c(A) \cup \{\lambda_0\}$. Note also that $c(A)$ is at most countable and that $\lambda_0$ is the only possible accumulation point of the non-real spectrum of $A$.

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References
Singular Sturm-Liouville operators with indefinite weight $\text{sgn} \ x$


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