On a class of Sturm-Liouville operators which are connected to $\mathcal{PT}$ symmetric problems

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Abstract
A prominent class of $\mathcal{PT}$-symmetric Hamiltonians is

$$ H := \frac{1}{2} p^2 + x^2 (ix)^N, \quad \text{for } x \in \Gamma $$

for some nonnegative number $N$. The associated eigenvalue problem is defined on a contour $\Gamma$ in a specific area in the complex plane (Stokes wedges), see [3, 5]. In this short note we consider the case $N = 2$ only. Here we elaborate the relationship between Stokes lines and Stokes wedges and well-known limit point/limit circle criteria from [11, 6, 10].

Keywords: non-Hermitian Hamiltonian, Stokes wedges, limit point, limit circle, $\mathcal{PT}$ symmetric operator, spectrum, eigenvalues

1 Introduction

In this paper we consider the quantum system described by the Hamiltonian

$$ H = \frac{1}{2m} p^2 - x^4, $$

(1.1)

where $g$ is real and positive, see [4] (or [3] with $N = 4$). The Hamiltonian (1.1) is of particular interest because the corresponding $-\phi^4$ quantum field theory might be a good model for describing the dynamics of the Higgs sector of the standard model as the $-\phi^4$ theory is asymptotically free and thus nontrivial, cf. [4] and the references therein. Consider the one-dimensional Schrödinger eigenvalue problem (where we assume, for simplicity, all constants equal to one)

$$ -y''(z) - z^4 y(z) = \lambda y(z), \quad z \in \Gamma, $$

(1.2)

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associated with the non-Hermitian Hamiltonian in (1.1). Here, $\lambda \in \mathbb{C}$ and the number $z$ runs along a complex contour $\Gamma$ which is within a Stokes wedge (for details we refer to Section 2). In the situation considered here, the Stokes wedge does not include the real-$x$ axis. We will not use the same complex contour that Jones and Mateo employed in their operator analysis of the Hamiltonian (1.1) in [8]. Instead we use a more simple contour which is not as smooth as the one used in [4, 8]. In this short note, we associate with (1.1) an operator in a $L^2(\mathbb{R})$ space with some boundary conditions. Moreover, we determine the cases when the expression (1.1) is in limit point or limit circle case. This classification is due to [11]; for a more recent refinement see [6, 10].

2 Limit point and limit circle classification

Recall (see, e.g., [3, 4]) that the curve $\Gamma$ is located in two Stokes wedges and tends to infinity in each of these wedges. A Stokes wedge is an open sector in the complex plane with vertex zero. In the situation considered here ($N=4$), the complex plane decomposes into six sectors, each with vertex zero, angle $\frac{\pi}{3}$, and with a boundary contained in the set of all complex numbers with

$$\arg z \in \left\{0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}.$$ 

To be more explicit: In the case considered here, we have six Stokes wedges $S_j$, $j = 1, \ldots, 6$, defined by

$$S_j = \left\{z \in \mathbb{C} : (j-1)\frac{\pi}{3} < \arg z < j\frac{\pi}{3}\right\}.$$ 

According to the rules imposed by $\mathcal{PT}$-symmetry, the contour $\Gamma$ has to satisfy some symmetry assumptions, i.e., $\Gamma$ is assumed to be located in

$$S_1 \cup S_3 = \left\{z \in \mathbb{C} : 0 < \arg z < \frac{\pi}{3} \text{ or } \frac{2\pi}{3} < \arg z < \pi\right\}. \quad (2.1)$$

However, in this note we will also consider the case when $\Gamma$ coincides with some Stokes line: $\Gamma \subset \{z \in \mathbb{C} : \arg z \in \left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}\}$. 

Let $\phi$ with $0 < \phi \leq \frac{\pi}{3}$. Here (for simplicity) we assume that $\Gamma$ is given by

$$\Gamma := \{xe^{i\phi \text{sgn} x} : x \in \mathbb{R}\}.$$ 

Note that $0 < \phi < \frac{\pi}{3}$ corresponds to the case that $\Gamma$ is contained in a Stokes wedge. This case is usually assumed, cf. [3, 4, 5, 8, 9] whereas $\phi = \frac{\pi}{3}$ corresponds to the case that $\Gamma$ coincides with some Stokes lines.

Our approach starts with the idea of Mostafazadeh in [9] to map the problem (1.2) back onto the real axis using a real parametrization. Here (contrary to [9]) we use the following parametrization $z : \mathbb{R} \to \mathbb{C}$,

$$z(x) := xe^{i\phi \text{sgn} x}.$$
Then \( y \) solves (1.2) for \( z \neq 0 \) if and only if \( w, w(x) := y(z(x)), \) solves
\[
\begin{align*}
-e^{2i\phi} w''(x) - e^{4i\phi} x^4 w(x) &= \lambda w(x) &\text{if } x > 0, \\
-e^{2i\phi} w''(x) - e^{-4i\phi} x^4 w(x) &= \lambda w(x) &\text{if } x < 0.
\end{align*}
\]

We define for a complex number \( \alpha \) the operator \( A_\alpha \) with domain \( \text{dom} A_\alpha \) in \( L^2(\mathbb{R}) \). The domain \( \text{dom} A_\alpha \) consists of all \( w \in L^2(\mathbb{R}) \) which are locally absolutely continuous on \( \mathbb{R} \) such that \( w' \) is locally absolutely continuous on \( \mathbb{R} \setminus \{0\} \) with
\[
A_\alpha w \in L^2(\mathbb{R}) \quad \text{and} \quad w'(0+) = \alpha w'(0-).
\]

For \( w \in \text{dom} A_\alpha \) we define \( A_\alpha w \) in the following way:
\[
A_\alpha w := \begin{cases} 
-e^{2i\phi} w''(x) - e^{4i\phi} x^4 w(x) &\text{if } x > 0, \\
-e^{2i\phi} w''(x) - e^{-4i\phi} x^4 w(x) &\text{if } x < 0.
\end{cases}
\]

The two (linearly independent) solutions \( y^\pm \) of (2.2) satisfy as \( x \to \infty \) (see, e.g., [7, pg. 58])
\[
y^\pm(x) \sim [e^{-4i\phi}s(x)]^{-1/4} \exp \left( \pm \int_0^\infty \text{Re } s(t)^{1/2} dt \right)
\]
with \( s(x) := -e^{6i\phi} x^4 - e^{2i\phi} \). We use the notation \( f(x) \sim g(x) \) to mean that \( f(x)/g(x) \to 1 \) as \( x \to \infty \) The same holds for the two solutions of (2.3) (as \( x \to -\infty \)) which is easily seen by replacing \( x \) by \( -x \). We have
\[
\text{Re } s(t)^{1/2} \sim -t^2 \sin 3\phi.
\]

The following theorem is the main result of this note. It is a consequence of the above observations and follows from the classification given in [11] (see also [6, 10]).

**Theorem 2.1.**

(i) If \( 0 < \phi < \frac{\pi}{2} \), then (2.2) and (2.3) are in limit point case. In particular this implies that one solution of (2.2) is not in \( L^2(\mathbb{R}^+) \) and that one solution of (2.3) is not in \( L^2(\mathbb{R}^-) \).

(ii) If \( \phi = \frac{\pi}{2} \), then (2.2) and (2.3) in limit circle case. In particular this implies that both solutions of (2.2) are in \( L^2(\mathbb{R}^+) \) and that both solution of (2.3) are in \( L^2(\mathbb{R}^-) \).

Theorem 2.1 allows the following mathematical interpretation: If \( \Gamma \) coincides with a Stokes line, then (2.2) and (2.3) are in limit circle case. If \( \Gamma \) is contained in a Stokes wedge, then (2.2) and (2.3) are in limit circle case.

### 3 Point spectrum of \( A_\alpha \) in the limit circle case

In the case \( \Gamma \) coincides with a Stokes line, both solutions of (2.2) are in \( L^2(\mathbb{R}^+) \) and that both solution of (2.3) are in \( L^2(\mathbb{R}^-) \). It is easily seen, that there exist a linear combination of these solutions which is in \( \text{dom} A_\alpha \) and the following theorem follows.
Theorem 3.1. Assume that \( \Gamma \) coincides with a Stokes line. Then the point spectrum \( \sigma_p(A_\alpha) \) of \( A_\alpha \) coincides with the complex plane,

\[
\sigma_p(A_\alpha) = \mathbb{C}.
\]

In the situation of Theorem 3.1 a boundary condition is missing. In order to avoid the situation in Theorem 3.1, one has to impose so-called boundary conditions at \( \pm \infty \), see e.g., [1, 2].

References


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