NON-REAL EIGENVALUES OF SINGULAR INDEFINITE STURM-LIOUVILLE OPERATORS

JUSSI BEHRNDT, QUTAIBE KATATBEH, AND CARSTEN TRUNK

ABSTRACT. We study a Sturm-Liouville expression with indefinite weight of the form \( \text{sgn} \left( -d^2/dx^2 + V \right) \) on \( \mathbb{R} \) and the non-real eigenvalues of an associated selfadjoint operator in a Krein space. For real-valued potentials \( V \) with a certain behaviour at \( \pm \infty \) we prove that there are no real eigenvalues and the number of non-real eigenvalues (counting multiplicities) coincides with the number of negative eigenvalues of the selfadjoint operator associated to \( -d^2/dx^2 + V \) in \( L^2(\mathbb{R}) \). The general results are illustrated with examples.

1. Introduction

We consider a singular Sturm-Liouville differential expression of the form

\[
(1.1) \quad \text{sgn} \left( x \right) \left( -f''(x) + V(x)f(x) \right), \quad x \in \mathbb{R},
\]

with the signum function as indefinite weight and a real-valued locally summable function \( V \). Under the assumption that \( -d^2/dx^2 + V \) is in the limit point case at \( +\infty \) and \( -\infty \) the maximal operator \( A \) associated to (1.1) is selfadjoint in the Krein space \( (L^2(\mathbb{R}), [\cdot, \cdot]) \), where the indefinite inner product \( [\cdot, \cdot] \) is defined by

\[
(1.2) \quad [f, g] = \int_{\mathbb{R}} f(x)g(x) \text{sgn}(x) \, dx, \quad f, g \in L^2(\mathbb{R}).
\]

The spectral properties of indefinite Sturm-Liouville operators differ essentially from the spectral properties of selfadjoint Sturm-Liouville operators in the Hilbert space \( L^2(\mathbb{R}) \), e.g. the real spectrum of \( A \) necessarily accumulates to \( +\infty \) and \( -\infty \) and \( A \) may have non-real eigenvalues which possibly accumulate to the real axis, see [3, 4, 9, 15, 16, 19]. For further indefinite Sturm-Liouville problems, applications and references, see, e.g. [2, 6, 7, 11, 13, 22, 25].

The main objective of this paper is to study the number of non-real eigenvalues of the operator \( A \). For this it will be assumed that the negative spectrum of the selfadjoint definite Sturm-Liouville operator \( Bf = -f'' + Vf \) consists of \( \kappa < \infty \) eigenvalues. Then the hermitian form \( [A, \cdot, \cdot] \) has \( \kappa \) negative squares and it follows from the considerations in [9] and [20] that the spectrum \( \sigma(A) \) of \( A \) in the open upper half-plane \( \mathbb{C}^+ \) consists of at most \( \kappa \) eigenvalues (counting multiplicities). Inspired by results of I. Knowles from [17, 18] we give a sufficient condition on \( V \) such that \( \sigma(A) \cap \mathbb{C}_+ \) consists of exactly \( \kappa \) eigenvalues (counting multiplicities) and the continuous spectrum of \( A \) covers the whole real line, see Theorem 2.3 and Corollary 2.4 below. These results can be viewed as a partial answer of the open problem X. in [25, pg. 300]. We present two explicitly solvable examples illustrating our

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results. In the first example potentials of secans hyperbolicus type are considered and with the help of numerical methods we find $\kappa$ different eigenvalues in $\mathbb{C}_+$. The second example shows that in general non-real eigenvalues of $A$ may have nontrivial Jordan chains and hence the number of distinct eigenvalues in $\mathbb{C}_+$ is less than $\kappa$.

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2. Eigenvalues of indefinite Sturm-Liouville operators

In this section we consider the indefinite Sturm-Liouville differential expression on $\mathbb{R}$ given by (1.1), where $V : \mathbb{R} \to \mathbb{R}$ is a real function with $V \in L^1_{\text{loc}}(\mathbb{R})$. We equip the Hilbert space $(L^2(\mathbb{R}), \langle \cdot, \cdot \rangle)$ with the indefinite inner product $[\cdot, \cdot]$ defined in (1.2) and denote the corresponding Krein space $(L^2(\mathbb{R}), [\cdot, \cdot])$ by $L^2_{\text{sgn}}(\mathbb{R})$. As a corresponding fundamental symmetry we choose $J := \text{sgn}(\cdot)$, hence we have $[\cdot, \cdot] = (J \cdot, \cdot)$ and $[J \cdot, \cdot] = (\cdot, \cdot)$. For the basic properties of indefinite inner product spaces and linear operators therein, we refer to [1] and [8].

Suppose that the definite Sturm-Liouville differential expression (2.1)

$$-\frac{d^2}{dx^2} + V$$

is in the limit point case at $\pm \infty$, that is, for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there exist (up to scalar multiples) unique solutions of the differential equation $-y'' + Vy = \lambda y$ which are square integrable in a neighbourhood of $+\infty$ and $-\infty$, respectively. A sufficient criterion for (2.1) to be in the limit point case at $\pm \infty$ is, e.g.,

$$\liminf_{|x| \to \infty} \frac{V(x)}{x^2} > -\infty,$$

cf. [24, Satz 13.27] or [25, Example 7.4.1]1. Denote by $D_{\text{max}}$ the linear space of all $f \in L^2(\mathbb{R})$ such that $f$ and $f'$ are absolutely continuous and $-f'' + Vf \in L^2(\mathbb{R})$ holds. Then it is well-known that the maximal operator

(2.2)

$$Bf := -f'' + Vf, \quad \text{dom } B = D_{\text{max}},$$

associated to (2.1) is selfadjoint in the Hilbert space $L^2(\mathbb{R})$ and all eigenvalues are simple, i.e., dim ker $(B - \lambda) = 1$ for $\lambda \in \sigma_p(B)$. As a consequence we obtain the following statement for the operator $A = JB$.

Proposition 2.1. Assume that (2.1) is in the limit point case at $\pm \infty$. Then the indefinite Sturm-Liouville operator defined by

(2.3)

$$(Af)(x) = \text{sgn}(x)(-f''(x) + V(x)f(x)), \quad x \in \mathbb{R}, \quad \text{dom } A = D_{\text{max}},$$

is selfadjoint in the Krein space $L^2_{\text{sgn}}(\mathbb{R})$ and the eigenspaces $\ker (A - \lambda)$, $\lambda \in \sigma_p(A)$, have dimension one.

In the following it will be assumed that condition (I) holds.

(I) The set $\sigma(B) \cap (-\infty, 0)$ consists of $\kappa < \infty$ eigenvalues.

Hence, the selfadjoint operator $B$ in the Hilbert space $L^2(\mathbb{R})$ is semi-bounded from below and the eigenvalues do not accumulate to zero from the negative half-axis. A sufficient condition for (I) to hold is, e.g., $\int_{\mathbb{R}}(1 + x^2)|V(x)|dx < \infty$ for continuous $V$, cf. [21].

We collect some properties of the non-real spectrum of the indefinite Sturm-Liouville operator $A$ in the next proposition. Recall first, that the spectrum of a

1In the formulation of [25, Example 7.4.1] a minus sign is missing.
selfadjoint operator in a Krein space is symmetric with respect to the real axis and denote by $L_\lambda(A)$ the algebraic eigenspace of $A$ corresponding to an eigenvalue $\lambda$.

**Proposition 2.2.** The spectrum of the indefinite Sturm-Liouville operator $A$ in the open upper half-plane $\mathbb{C}_+$ consists of at most finitely many eigenvalues with

$$
\sum_{\lambda \in \sigma(A) \cap \mathbb{C}_+} \dim L_\lambda(A) \leq \kappa,
$$

where $\kappa$ is as in (I). In particular, for some $\tilde{\kappa} \leq \kappa$ we have

$$
\mathbb{C} \setminus \mathbb{R} \subset \rho(A) \cup \{\lambda_1, \lambda_2, \ldots, \lambda_{\tilde{\kappa}}, \tilde{\lambda}_{\tilde{\kappa}}\}.
$$

If $V(x) = V(-x)$, $x \in \mathbb{R}$, then $\sigma_p(A)$ is symmetric with respect to the imaginary axis.

**Proof.** Assumption (I) and the relation $[A, f] = (J Af, f) = (Bf, f)$, $f \in D_{\max}$, imply that the hermitian form $[A, \cdot]$ has exactly $\kappa$ negative squares, that is, there exists a $\kappa$-dimensional subspace $\mathcal{M}$ in $D_{\max}$ such that $[A, f] < 0$ if $f \in \mathcal{M}$, $f \neq 0$, but no $\kappa + 1$ dimensional subspace with this property. This, together with well-known properties of operators with $\kappa$ negative squares, see, e.g. [20], [9] and [5, Theorem 3.1 and § 4.2], imply (2.4) and (2.5).

Moreover, if $V$ is symmetric, then $\lambda$ is an eigenvalue of $A$ with corresponding eigenfunction $x \mapsto y(x)$ if and only if $-\lambda$ is an eigenvalue of $A$ with corresponding eigenfunction $x \mapsto y(-x)$. Therefore, as $\sigma_p(A)$ is symmetric with respect to the real axis, $\sigma_p(A)$ is also symmetric with respect to the imaginary axis. 

Under some additional assumptions on $V$ we prove the absence of eigenvalues on the real axis and, hence, improve the estimate in (2.4). By $\sigma_\kappa(A)$ we denote the continuous part of the spectrum of $A$, i.e. the set of all $\lambda \in \sigma(A) \setminus \sigma_p(A)$ such that the range of $A - \lambda I$ is dense.

**Theorem 2.3.** Assume that condition (I) holds and that there exist real functions $q$ and $r$ with $V = q + r$ such that $\lim_{|x| \to \infty} r(x) = \lim_{|x| \to \infty} q(x) = 0$, $r$ is locally of bounded variation and

$$
\lim_{t \to \infty} \frac{1}{\log t} \int_{-t}^{t} |q(x)| dx = \lim_{t \to \infty} \frac{1}{\log t} \int_{-t}^{t} |r(x)| dx = 0.
$$

Then $\sigma_\kappa(A) \setminus \{0\} = \mathbb{R} \setminus \{0\}$ and hence zero is the only possible real eigenvalue of the indefinite Sturm-Liouville operator $A$. If, in addition, $0 \notin \sigma_p(B)$, then we have $\sigma_\kappa(A) = \mathbb{R}$ and

$$
\sum_{\lambda \in \sigma_p(A) \cap \mathbb{C}_+} \dim L_\lambda(A) = \kappa.
$$

**Proof.** Let $\lambda$ be an eigenvalue of $A$ and let $y$ be a corresponding eigenfunction. Then $y$ satisfies the equations

$$
-y''(x) + V(x)y(x) = \lambda y(x), \quad x \in (0, \infty),
$$

and

$$
y''(x) - V(x)y(x) = \lambda y(x), \quad x \in (-\infty, 0).
$$

Condition (2.6) implies

$$
\lim_{t \to \infty} \frac{1}{\log t} \int_{0}^{t} |q(x)| dx = \lim_{t \to \infty} \frac{1}{\log t} \int_{0}^{t} |r(x)| dx = 0.
$$
This and [18, Theorem 3.2] applied to (2.8) yields $\lambda \not\in (0, \infty)$. Similarly, (2.6) implies
\[
\lim_{t \to \infty} \frac{1}{\log t} \int_{-t}^{0} |q(x)| dx = \lim_{t \to \infty} \frac{1}{\log t} \int_{-t}^{0} |dr(x)| = 0,
\]
and, with [18, Theorem 3.2] applied to (2.9), we find $\lambda \not\in (-\infty, 0)$. Therefore, as a selfadjoint operator in a Krein space has no real points in the residual spectrum (see, e.g., [8, Corollary VI.6.2]), we obtain
\[
(\sigma(A) \cap (\mathbb{R}\setminus\{0\})) \subset \sigma_{c}(A) \quad \text{and} \quad \sigma_{p}(A) \subset \{0\} \cup \mathbb{C}\setminus\mathbb{R}.
\]
Moreover, from $A = JB$ we get $0 \in \sigma_{p}(A)$ if and only if $0 \in \sigma_{p}(B)$. Hence, if $0 \notin \sigma_{p}(B)$ we conclude $\sigma_{p}(A) \subset \mathbb{C}\setminus\mathbb{R}$. Since the operator $A$ has exactly $\kappa$ negative squares, cf. the proof of Proposition 2.2, it follows from, e.g. [5, Theorem 3.1] that $A$ has $\kappa$ eigenvalues (counted with multiplicities) in $\mathbb{C}^{+}$ and thus (2.7) holds.

It remains to show $\mathbb{R} \subset \sigma(A)$. For this, consider the differential expressions $\ell_{+} = -\frac{d^{2}}{dx^{2}} + V$ on $\mathbb{R}^{+}$ and $\ell_{-} = \frac{d^{2}}{dx^{2}} - V$ on $\mathbb{R}^{-}$. By assumption $\ell_{+}$ and $\ell_{-}$ are regular at zero and in the limit point case at $\infty$ and $-\infty$, respectively. Let $A_{+}$ and $A_{-}$ be selfadjoint realizations of $\ell_{+}$ and $\ell_{-}$ in the Hilbert spaces $L^{2}(\mathbb{R}^{+})$ and $L^{2}(\mathbb{R}^{-})$, respectively, e.g. corresponding to Dirichlet boundary conditions at zero. Under our assumptions
\[
\lim_{x \to \infty} V(x) = 0 \quad \text{and} \quad \lim_{x \to -\infty} V(x) = 0
\]
and it is well known that $[0, \infty) \subset \sigma(A_{+})$ and $(-\infty, 0] \subset \sigma(A_{-})$ holds. Since the rank of the operator
\[
(A - \lambda)^{-1} - ((A_{+} \times A_{-}) - \lambda)^{-1}, \quad \lambda \in \rho(A) \cap \rho(A_{+} \times A_{-}),
\]
is at most two and $\sigma(A_{+} \times A_{-}) = \mathbb{R}$ we conclude $\mathbb{R} \subset \sigma(A)$.

A sufficient condition on $V$ such that condition (I), (2.6) and $0 \notin \sigma_{p}(B)$ hold is given in the next corollary, cf. [23, Theorem 14.10], [25, §6.3] and [17, Remark after Corollary 3.3].

**Corollary 2.4.** Assume that there exists $x_{0} > 0$ with
\[
(2.10) \quad -\frac{1}{4x^{2}} \leq V(x) \leq \frac{3}{4x^{2}} \quad \text{for all} \quad x \in \mathbb{R}\setminus(-x_{0}, x_{0}).
\]
Then $\sigma_{c}(A) = \mathbb{R}$ and
\[
\sum_{\lambda \in \sigma_{p}(A) \cap \mathbb{C}^{+}} \dim \mathcal{L}_{\lambda}(A) = \kappa.
\]

**Remark 2.5.** We mention that (even under the condition (2.10)) for $\lambda \in \sigma_{p}(A)$ $\dim \mathcal{L}_{\lambda}(A) > 1$ may happen, i.e. there exists a Jordan chain of length greater than one and the non-real spectrum does not consist of $\kappa$ distinct eigenvalues. In Section 4 we give an example for an indefinite singular Sturm-Liouville operator with such a Jordan chain corresponding to a non-real eigenvalue.
3. A numerical example: Secans hyperbolicus potentials

In this section we compute the non-real eigenvalues of singular indefinite Sturm-Liouville operators with potentials given by

\[(3.1) \quad V_\kappa(x) = -\kappa(\kappa + 1) \text{sech}^2(x), \quad x \in \mathbb{R} \quad \text{and} \quad \kappa \in \mathbb{N},\]

with the help of the software package Mathematica (Wolfram Research).

It is well known that the number of negative eigenvalues of the definite Sturm-Liouville operator \(Bf = -f'' + V_\kappa f\) in (2.2) is exactly \(\kappa\) and condition (I) from Section 2 holds, see, e.g. [12]. Moreover, \(V_\kappa\) satisfies (2.10) and hence by Theorem 2.3 and Corollary 2.4 the continuous spectrum of the indefinite Sturm-Liouville operator

\[(Af)(x) = \text{sgn}(x)(-f''(x) + V_\kappa(x)f(x)), \quad x \in \mathbb{R}, \quad \text{dom}\ A = D_{\text{max}},\]

in the Krein space \(L^2_{\text{sgn}}(\mathbb{R})\) coincides with \(\mathbb{R}\) and \(\sum_{\lambda \in \sigma(A) \cap C_+} \dim \mathcal{L}_\lambda(A) = \kappa\) holds. In order to determine the non-real eigenvalues of \(A\) we divide the problem into two parts,

\[(3.2) \quad -y''(x; \lambda) + V_\kappa(x)y(x; \lambda) = \lambda y(x; \lambda), \quad x \in \mathbb{R}_+, \]

\[y''(x; \lambda) - V_\kappa(x)y(x; \lambda) = \lambda y(x; \lambda), \quad x \in \mathbb{R}_-.\]

Since the potential \(V_\kappa\) in (3.1) satisfies \(V_\kappa(x) = V_\kappa(-x)\) for \(x \in \mathbb{R}\), it follows that a function \(x \mapsto h(x; \lambda), x \in \mathbb{R}_+,\) is a solution of the first differential equation if and only if \(x \mapsto h(-x; -\lambda), x \in \mathbb{R}_-,\) is a solution of the second differential equation in (3.2). Moreover, as both singular endpoints \(\infty\) and \(-\infty\) are in the limit point case, each of the equations in (3.2) has (up to scalar multiples) a unique square integrable solution. Since the functions in \(\text{dom} A\) and their derivatives are continuous at the point 0 it follows that \(\lambda \in \mathbb{C} \setminus \mathbb{R}\) is an eigenvalue of \(A\) if and only if for the square integrable solution \(x \mapsto h(x; \lambda), x \in \mathbb{R}_+,\) of the first equation in (3.2)

\[(3.3) \quad h(0; \lambda) = \gamma h(0; -\lambda) \quad \text{and} \quad h'(0; \lambda) = -\gamma h'(0; -\lambda)\]

holds for some \(\gamma \in \mathbb{C}\). For non-real \(\lambda\) we have \(h(0; \lambda) \neq 0\) and \(h(0; -\lambda) \neq 0\) and therefore (3.3) is satisfied if and only if the function

\[(3.4) \quad \mu \mapsto M(\mu) := \frac{h'(0; \mu)}{h(0; \mu)} + \frac{h'(0; -\mu)}{h(0; -\mu)}, \quad \mu \in \mathbb{C} \setminus \mathbb{R},\]

has a zero at \(\lambda\). As the equations in (3.2) are explicitly solvable in terms of Legendre functions we can determine numerically the zeros of \(M\) within the working default precision of the software package Mathematica.

The Figures 1, 2 and 3 show the non-real eigenvalues of \(A\) for the cases \(\kappa = 5, \kappa = 30\) and \(\kappa = 100\). Here we find 5, 30 and 100, respectively, distinct eigenvalues in \(\mathbb{C}^+\) and hence \(\dim \mathcal{L}_\lambda(A) = 1\) for each eigenvalue \(\lambda \in \mathbb{C} \setminus \mathbb{R},\) cf. Remark 2.5. Note also, that by the symmetry of \(V_\kappa\) there is at least one pair of eigenvalues on the imaginary axis if \(\kappa\) is odd.
Figure 2. The operator \((Ay)(x) := \text{sgn}(x)(-y''(x) + V_{30}(x)y(x)), \ x \in \mathbb{R},\) where \(V_{30}(x) = -30 \cdot 31 \text{sech}^2(x)\) has \(\kappa = 30\) pairs of non-real eigenvalues.

Figure 3. The operator \((Ay)(x) := \text{sgn}(x)(-y''(x) + V_{100}(x)y(x)), \ x \in \mathbb{R},\) where \(V_{100}(x) = -100 \cdot 101 \text{sech}^2(x)\) has \(\kappa = 100\) pairs of non-real eigenvalues.

4. A counterexample: Jordan chains of singular indefinite Sturm-Liouville operators

In this section we show that the geometric eigenspaces of a singular indefinite Sturm-Liouville operator in \(L^2_{\text{sgn}}(\mathbb{R})\) in general do not coincide with the algebraic
As in Section 3 shows that the non-real eigenvalues of the underlying differential equations (3.2) with the potentials \( V_\eta \), \( \eta \geq 0 \), of indefinite Sturm-Liouville operators in the Krein space \( L^2_{\text{sgn}}(\mathbb{R}) \), where the potentials \( V_\eta, \eta \geq 0 \), are given by

\[
V_\eta(x) = \begin{cases} 
0 & |x| \geq 1, \\
-\eta & |x| < 1, \quad \eta \geq 0.
\end{cases}
\]

The operators \( A_\eta, \eta \geq 0 \), are selfadjoint in \( L^2_{\text{sgn}}(\mathbb{R}) \) and according to Theorem 2.3 and Corollary 2.4 there are no real eigenvalues and \( \sigma_c(A_\eta) \) covers the whole real line. In the sequel we will show that the following statement holds.

**Proposition 4.1.** There exist an \( \eta_0 > 0 \) and \( \lambda_0 \in \mathbb{C}_+ \) such that

\[
2 = \dim \ker (A_{\eta_0} - \lambda_0)^2 > \dim \ker (A_{\eta_0} - \lambda_0) = 1.
\]

In order to determine the eigenvalues of the operators \( A_\eta \) we first consider the underlying differential equations (3.2) with \( V_c \) replaced by \( V_\eta \). The same reasoning as in Section 3 shows that the non-real eigenvalues of \( A_\eta \) are given by the zeros of the function

\[
M_\eta(\lambda) := \frac{h''_{\eta}(0; \lambda)}{h_{\eta}(0; \lambda)} + \frac{h''_{\eta}(0; -\lambda)}{h_{\eta}(0; -\lambda)}, \quad \lambda \in \mathbb{C}\backslash \mathbb{R},
\]

where \( h_{\eta}(\cdot, \lambda) \) is the square integrable solution of \(-y'' + V_\eta y = \lambda y \) on \( \mathbb{R}_+ \). Denote by \( \sqrt{\cdot} \) the branch of the square root with cut along \( [0, \infty) \) and \( \sqrt{x} \geq 0 \) for \( x \in [0, \infty) \). Then it is easy to check that for \( \lambda \notin [0, \infty) \) the function

\[
h_{\eta}(x; \lambda) = \begin{cases} 
\exp(i\sqrt{\lambda}x) & x > 1, \\
\alpha_\eta(\lambda) \exp(i\sqrt{\lambda + \eta}x) + \beta_\eta(\lambda) \exp(-i\sqrt{\lambda + \eta}x) & x \in [0, 1],
\end{cases}
\]

where

\[
\alpha_\eta(\lambda) = \frac{1}{2} \left( 1 + \sqrt{\lambda \beta} \right) \exp(i(\sqrt{\lambda} - \sqrt{\lambda + \eta})
\]

and

\[
\beta_\eta(\lambda) = \frac{1}{2} \left( 1 - \sqrt{\lambda \beta} \right) \exp(i(\sqrt{\lambda} + \sqrt{\lambda + \eta}),
\]

and its multiples are square integrable solutions of the first equation in (3.2) with \( V_c \) replaced by \( V_\eta \).

The function \( M_\eta \) in (4.1) can be expressed in terms of \( \alpha_\eta \) and \( \beta_\eta \) in the following form:

\[
M_\eta(\lambda) = i \sqrt{\lambda + \eta} \frac{\alpha_\eta(\lambda) - \beta_\eta(\lambda)}{\alpha_\eta(\lambda) + \beta_\eta(\lambda)} + i \sqrt{\eta - \lambda} \frac{\alpha_\eta(-\lambda) - \beta_\eta(-\lambda)}{\alpha_\eta(-\lambda) + \beta_\eta(-\lambda)}.
\]

We note that the values \( M_\eta(\mu), \mu \in \mathbb{R}\backslash\{0\} \), are real since the solutions fulfil \( h_{\eta}(x, \lambda) = \overline{h_{\eta}(x, \lambda)} \) for \( \lambda \in \mathbb{C}\backslash \mathbb{R} \). Let us summarize some observations in the following lemma.
Figure 4. In the first row the function $\mu \mapsto M_\eta(i\mu)$ is plotted for $\mu > 0$ and $\eta = 3.1$, $\eta = 3.2$ and $\eta = 3.3$, respectively. In the second row the corresponding non-real eigenvalues of the operators $A_{3.1}$, $A_{3.2}$ and $A_{3.3}$ are shown.

**Lemma 4.2.** A non-real number $\lambda$ is an eigenvalue of the indefinite Sturm-Liouville operator $A_\eta$ if and only if $M_\eta(\lambda) = 0$. The restriction of $M_\eta$ onto the imaginary axis is a real-valued function and the non-real eigenvalues of $A_\eta$ are symmetric with respect to the real and imaginary axis.

One can check numerically that the selfadjoint operator $B_\eta = -\frac{d^2}{dx^2} + V_\eta$, dom $B_\eta = D_{\text{max}}$, in the Hilbert space $L^2(\mathbb{R})$ has exactly two negative eigenvalues for $\eta = 3.1$, $\eta = 3.2$ and $\eta = 3.3$, cf. [12]. By Corollary 2.4 for these $\eta$ the spectral subspace of $A_\eta$ corresponding to the eigenvalues in the upper half-plane $\mathbb{C}^+$ has dimension two.

The plots in the first row of Figure 4 show the function $\mu \mapsto M_\eta(i\mu)$, $\mu \in \mathbb{R}_+$, for $\eta = 3.1$, $\eta = 3.2$ and $\eta = 3.3$, respectively. For $\eta = 3.1$ and $\eta = 3.2$ the two zeros are the eigenvalues of $A_{3.1}$ and $A_{3.2}$ in the upper half-plane $\mathbb{C}^+$ which lie on the positive imaginary axis. These eigenvalues and their counterparts in $\mathbb{C}^-$ are plotted in the second row of Figure 4. For $\eta = 3.3$ the function $\mu \mapsto M_\eta(i\mu)$ has no zeros on the positive imaginary axis. Recall that a finite system of eigenvalues is continuous under perturbations small in norm, see [14, IV.3.5]. Hence the continuity and symmetry of the eigenvalues of $A_\eta$ implies that the eigenvalues of $A_{3.3}$ are located as in the right lower plot in Figure 4. This can also be checked numerically by computing the non-real roots of $M_{3.3}$. Again by continuity properties of the point spectrum there exists an $\eta_0 \in (3.2, 3.3)$ such that the spectrum of $A_{\eta_0}$ in $\mathbb{C}^+$ (and hence also in $\mathbb{C}^-$) consists only of one eigenvalue $\lambda_0$ on the imaginary axis with corresponding algebraic eigenspace of dimension two. Recall that the dimension of the geometric eigenspaces of $A_{\eta_0}$ is at most one since $\infty$ and $-\infty$ are in the limit
point case. Hence there exists a Jordan chain of length two of \( A_{\eta_0} \) at the eigenvalue \( \lambda_0 \) (and \( \bar{\lambda}_0 \)). We remark that for the function (4.1) we have \( M_{\eta_0}(\lambda_0) = M'_{\eta_0}(\lambda_0) = 0 \).

Table 1. In bold face is the (approximative) value of \( \eta \) where the eigenvalues \( \lambda_{1,\eta} \) and \( \lambda_{2,\eta} \) of \( A_\eta \) in \( \mathbb{C} \) coincide and we have a Jordan chain of length two. With further increasing \( \eta \) the eigenvalues \( \lambda_{1,\eta} \) and \( \lambda_{2,\eta} \) move away from the imaginary axis.

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References

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