Spectral Theory for Second Order Systems and Indefinite Sturm-Liouville Problems

According to the *Habilitationsordung der Technischen Universität Ilmenau*, the following papers are presented in this survey. They are listed in chronological order within each of the three subjects considered: Spectral and Perturbation Theory in Krein Spaces, Second Order Systems and Indefinite Sturm-Liouville Problems.

Spectral and Perturbation Theory in Krein Spaces

[H1] T. Ya. Azizov, J. Behrndt, P. Jonas and C. Trunk, *Spectral points of type π_+ and type π_- for closed linear relations in Krein spaces*, submitted.

[H2] T. Ya. Azizov, P. Jonas and C. Trunk, *Small perturbation of selfadjoint and unitary operators in Krein spaces*, to appear in J. Operator Theory.

[H3] T. Ya. Azizov, J. Behrndt, P. Jonas and C. Trunk, *Compact and finite rank perturbations of linear relations in Hilbert spaces*, Integral Equations Operator Theory **63** (2009), 151-163.

[H4] J. Behrndt, F. Philipp and C. Trunk, *Properties of the spectrum of type π_+ and type π_- of self-adjoint operators in Krein spaces*, Methods Funct. Anal. Topology **12** (2006), 326-340.

[H5] T. Ya. Azizov, P. Jonas and C. Trunk, *Spectral points of type π_+ and π_- of self-adjoint operators in Krein spaces*, J. Funct. Anal. **226** (2005), 114-137.

Second Order Systems

[H6] C. Trunk, *Analyticity of semigroups related to a class of block operator matrices*, Oper. Theory Adv. Appl. **195** (2009), 257-271.

[H7] B. Jacob and C. Trunk, *Spectrum and analyticity of semigroups arising in elasticity theory and hydromechanics*, Semigroup Forum **79** (2009), 79-100.

[H8] B. Jacob, C. Trunk and M. Winklmeier, *Analyticity and Riesz basis property of semigroups associated to damped vibrations*, Journal of Evolution Equations **8** (2008), 263-281.

[H9] C. Trunk, *Spectral theory for operator matrices related to models in mechanics*, Math. Notes **83** (2008), 843-850.

[H10] B. Jacob, K. Morris and C. Trunk, *Minimum-phase infinite-dimensional second-order systems*, IEEE Transactions on Automatic Control **52** (2007), 1654-1665.

[H11] B. Jacob and C. Trunk, *Location of the spectrum of operator matrices which are associated to second order equations*, Operators and Matrices **1** (2007), 45-60.

Indefinite Sturm-Liouville Problems

[H12] J. Behrndt, Q. Katatbeh and C. Trunk, *Non-real eigenvalues of singular indefinite Sturm-Liouville operators*, Proc. Amer. Math. Soc. **137** (2009), 3797-3806.

[H13] I. Karabash and C. Trunk, *Spectral properties of singular Sturm-Liouville operators*, Proc. R. Soc. Edinb. Sect. A **139** (2009), 483-503.

[H14] J. Behrndt, Q. Katatbeh and C. Trunk, *Accumulation of complex eigenvalues of indefinite Sturm-Liouville operators*, J. Phys. A: Math. Theor. **41** (2008), 244003.

[H15] J. Behrndt and C. Trunk, *On the negative squares of indefinite Sturm-Liouville operators*, J. Differential Equations **238** (2007), 491-519.

[H16] J. Behrndt and C. Trunk, *Sturm-Liouville operators with indefinite weight functions and eigenvalue depending boundary conditions*, J. Differential Equations **222** (2006), 297-324.

The above mentioned papers were written in close collaboration with the following colleagues:

- Prof. Dr. T. Ya. Azizov, Voronezh State University, Russia
- Priv.-Doz. Dr. Jussi Behrndt, Technische Universität Berlin, Germany
- Prof. Dr. Birgit Jacob, Universität Paderborn, Germany
- my PhD advisor, the late Dr. sci. nat. Peter Jonas, Technische Universität Berlin, Germany
- Dr. I. Karabash, University of Calgary, Calgary, Canada
- Prof. Dr. Q. Katatbeh, Jordan University of Science and Technology, Irbid, Jordan
- Prof. Dr. K. Morris, University of Waterloo, Waterloo, Ontario, Canada
- my PhD student Dipl.-Math. F. Philipp, Technische Universität Ilmenau, Germany
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In all of the above collaborations each of the authors contributed a comparable amount to the paper. Hence, in a joint paper with two, three or four authors, each of them contributed one half, one third or one quarter, respectively.

1 Preface

A classical way to treat differential equations is to associate suitable linear operators in appropriate normed spaces of functions, e.g. Sobolev spaces or spaces of square integrable functions. The behaviour of solutions of these differential equations can be described via spectral properties of the associated linear operators. Large classes of differential equations exhibit some symmetries which often imply the selfadjointness (or symmetry) of the corresponding linear operator with respect to some inner product. In many problems, this inner product is not positive definite and classical Hilbert space theory cannot be used.

In this survey we will present two types of differential equations: Ordinary differential equations of Sturm-Liouville type with a weight function changing its sign and second order systems which appear in the context of partial differential equations. In the latter application, the indefinite structure arises when one rewrites the second order system as a Cauchy problem in a larger space. The operator appearing in this Cauchy problem turns out to be selfadjoint with respect to some indefinite inner product. In the former application, the indefinite structure appears due to the sign change of the weight function: the weighted Sturm-Liouville problem is selfadjoint only within an inner product which contains the indefinite weight.

The indefinite inner products in these two problems share one common property. They are inner products of Krein spaces. This subject first appeared over 65 years ago in a paper of P. Dirac on quantum field theory [42], see also W. Pauli [135]. The first mathematical treatment was given by L.S. Pontryagin [136] and was then continued by M.G. Krein, I.S. Iohvidov, Ju.P. Ginzburg, R.S. Phillips, H. Langer, M.A. Naimark, Ju.L. Smul'jan and many others.

It is the aim of this survey to show how recent results in the spectral and perturbation theory for operators which are selfadjoint in a Krein space can be used to study the above mentioned problems. We proceed as follows: We give an introduction to the spectral and perturbation theory in Krein spaces and present recent results in Section 2. Second order systems are discussed in Section 3 and, finally, indefinite Sturm-Liouville problems in Section 4.

2 Spectral and Perturbation Theory in Krein space

In this section, we present results in the spectral and perturbation theory of (selfadjoint) operators and relations in Krein spaces contained in the following papers.

- [H1] T. Ya. Azizov, J. Behrndt, P. Jonas and C. Trunk, *Spectral points of type π_+ and type π_- for closed linear relations in Krein spaces*, submitted.
- [H2] T. Ya. Azizov, P. Jonas and C. Trunk, *Small perturbation of selfadjoint and unitary operators in Krein spaces*, to appear in J. Operator Theory.
- [H3] T. Ya. Azizov, J. Behrndt, P. Jonas and C. Trunk, *Compact and finite rank perturbations of linear relations in Hilbert spaces*, Integral Equations Operator Theory **63** (2009), 151-163.
- [H4] J. Behrndt, F. Philipp and C. Trunk, *Properties of the spectrum of type π_+ and type π_- of self-adjoint operators in Krein spaces*, Methods Funct. Anal. Topology **12** (2006), 326-340.
- [H5] T. Ya. Azizov, P. Jonas and C. Trunk, *Spectral points of type π_+ and π_- of selfadjoint operators in Krein spaces*, J. Funct. Anal. **226** (2005), 114-137.

The main contribution of these five papers is the development of the local perturbation theory for closed operators in Krein spaces. For this purpose, the new notion of spectral points of type π_+ and π_- is introduced in [H5] (for selfadjoint operators in Krein spaces) and in [H1] (for closed operators in Krein spaces). These spectral points are characterized via approximative eigensequences. This approach has the advantage that it does not make use of a local spectral function. The spectral points of type π_+ and π_- coincide, in the case of locally definitizable selfadjoint operators, with those spectral points which have a local spectral function mapping small neighbourhoods onto Pontryagin spaces. This in turn gives also a new and elementary technique for proving that a given operator is locally definitizable.

The main property of spectral points of type π_+ and π_- is that they are invariant under compact perturbations and under perturbations small in gap. This is even true for arbitrary closed operators and, more general, closed relations ([H1, H3]) in Krein spaces. Moreover, in the case of locally definitizable selfadjoint operators, they allow a much more simple proof for the fact that a spectral point of type π_+ (or π_-) becomes either an inner point of the spectrum of the perturbed operator or it becomes an eigenvalue of type π_+ (or π_-, repectively) such that the perturbed operator is still locally definitizable in a small neighbourhood.

2.1 Spectral Points of Type π_+ and π_- for Selfadjoint Operators

Let $(\mathcal{H}, [\cdot,\cdot])$ be a Krein space. We briefly recall that a complex linear space $\mathcal{H}$ with a Hermitian nondegenerate sesquilinear form $[\cdot,\cdot]$ is called a *Krein space* if there exists a so-called *fundamental decomposition*

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \tag{2.1}$$

with subspaces $\mathcal{H}_\pm$ being orthogonal to each other with respect to $[\cdot,\cdot]$ such that $(\mathcal{H}_\pm, \pm[\cdot,\cdot])$ are Hilbert spaces. To each decomposition (2.1) there corresponds a Hilbert space inner product $(\cdot,\cdot)$ and a selfadjoint operator J (the *fundamental symmetry*) with $J^2 = I, J = J^{-1}$ and $[x,y] = (Jx,y)$ for $x, y \in \mathcal{H}$.

In the following, all topological notions are understood with respect to some Hilbert space norm $\|\cdot\|$ on $\mathcal{H}$ such that $[\cdot,\cdot]$ is $\|\cdot\|$-continuous. Any two such norms are equivalent (see, e.g., [111]). If $\mathcal{H}_-$ ($\mathcal{H}_+$) is finite dimensional, then $(\mathcal{H}, [\cdot,\cdot])$ is called a *Pontryagin space with finite rank of negativity* (resp. *positivity*). For basic properties of Krein spaces we refer to [98] and to the monographs [4, 22].

Let $(\mathcal{H}, [\cdot,\cdot])$ be a Krein space and let A be a bounded or unbounded selfadjoint linear operator in the Krein space $(\mathcal{H}, [\cdot,\cdot])$, i.e., A coincides with its adjoint A^+ with respect to the indefinite inner product $[\cdot,\cdot]$. If an operator is selfadjoint with respect to some Krein space inner product, then its spectral properties differ essentially from the spectral properties of selfadjoint operators in Hilbert spaces, e.g., the spectrum $\sigma(A)$ of A is in general not real and even $\sigma(A) = \mathbb{C}$ may occur. If, besides selfadjointness, further assumptions on A are imposed, then the situation becomes more interesting and challenging from a spectral theoretic point of view.

Let, e.g., A be a $[\cdot,\cdot]$-nonnegative selfadjoint operator in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ with a nonempty resolvent set. Then $\sigma(A) \subset \mathbb{R}$ holds and the spectral points of A in $(0,\infty)$ and $(-\infty,0)$ are of *positive type* and *negative type*, respectively, i.e., each point in $\sigma(A)\cap(0,\infty)$ ($\sigma(A)\cap(-\infty,0)$) belongs to the approximate point spectrum[1] $\sigma_{ap}(A)$ of A and for every approximative eigensequence (x_n) the accumulation points of the sequence $([x_n,x_n])$ are positive (negative, respectively). These spectral points are introduced and studied by P. Lancaster, H. Langer, A. Markus and V. Matsaev in [106, 113] for bounded selfadjoint operators.

Not surprising, spectral points of positive and negative type are in general not stable under finite rank and compact perturbations. However, if the nonnegative selfadjoint operator A from above is perturbed by a finite rank operator F such that the resulting operator $B = A+F$ is selfadjoint, then the Hermitian form $[B\cdot,\cdot]$ is still nonnegative on the complement of a suitable finite dimensional subspace. Therefore, if (x_n) is an approximative eigensequence corresponding to $\lambda \in \sigma(B)\cap(0,\infty)$ ($\lambda \in \sigma(B)\cap(-\infty,0)$) and all x_n belong to a suitable linear manifold of finite codimension, then all accumulation points of the sequence $([x_n,x_n])$ are again positive (resp. negative). In [H5] the latter property of approximative eigensequence serves as a definition of so-called spectral points of *type* π_+ and *type* π_-, respectively, for an arbitrary selfadjoint operator A in a Krein space which we recall here. Note (see e.g. Corollary VI.6.2 in [22] and [45, page 242]) that all real spectral points and all boundary points of $\sigma(A)$ in $\mathbb{C}$ belong to $\sigma_{ap}(A)$.

Definition 2.1. [H5] *For a selfadjoint operator A in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ a point $\lambda_0 \in \sigma_{ap}(A)$ is called a spectral point of* type π_+ (type π_-) of A *if there exists*

[1]The *approximate point spectrum of* a closed operator A is denoted by $\sigma_{ap}(A)$ and consists of all $\lambda \in \mathbb{C}$ such that there exists a sequence (x_n) in dom A with $\|x_n\| = 1$, $n = 1, 2, \ldots$,

$$\|x_n\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \|Ax_n - \lambda x_n\| = 0.$$

The sequence (x_n) is called an *approximative eigensequence*.

a linear manifold $\mathcal{H}_0 \subset \mathcal{H}$ *with* $\operatorname{codim} \mathcal{H}_0 < \infty$ *such that for every sequence* (x_n) *in* $\mathcal{H}_0 \cap \operatorname{dom} A$ *with*

$$\|x_n\| = 1, \ n = 1, 2, \ldots, \ \text{and} \ \lim_{n\to\infty} \|(A - \lambda_0)x_n\| = 0 \tag{2.2}$$

we have

$$\liminf_{n\to\infty} \,[x_n, x_n] > 0 \quad (\text{resp.} \ \limsup_{n\to\infty} \,[x_n, x_n] < 0). \tag{2.3}$$

The point ∞ *is said to be a point of* type π_+ (type π_-) *if* A *is unbounded and if there exists a linear manifold* $\mathcal{H}_0 \subset \mathcal{H}$ *with* $\operatorname{codim} \mathcal{H}_0 < \infty$ *such that for every sequence* (x_n) *in* $\mathcal{H}_0 \cap \operatorname{dom} A$ *with*

$$\|Ax_n\| = 1, \ n = 1, 2, \ldots, \ \text{and} \ \lim_{n\to\infty} \|x_n\| = 0$$

we have

$$\liminf_{n\to\infty} \,[Ax_n, Ax_n] > 0 \quad (\text{resp.} \ \limsup_{n\to\infty} \,[Ax_n, Ax_n] < 0).$$

We denote the set of all points of type π_+ *(type* π_-*) of* A *by* $\sigma_{\pi_+}(A)$ *(resp.* $\sigma_{\pi_-}(A)$*).*

It is convenient to include the point ∞ in the set $\sigma_{\pi_+}(A)$ (resp. $\sigma_{\pi_-}(A)$) as in Definition 2.1. Hence, in the following, we will use the notion of the *extended spectrum* $\widetilde{\sigma}(A)$ of A which is defined by $\widetilde{\sigma}(A) := \sigma(A)$ if A is bounded and $\widetilde{\sigma}(A) := \sigma(A) \cup \{\infty\}$ if A is unbounded. Moreover, we set $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ and $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.

If in Definition 2.1 for all sequences (x_n) in $\mathcal{H} \cap \operatorname{dom} A$ with (2.2), property (2.3) follows (i.e. $\mathcal{H}_0 = \mathcal{H}$), then λ_0 is a spectral point of positive (resp. negative) type, cf. [106, 113]. An analogous statement holds for the point ∞. The set of all points of positive type (negative type) of A is denoted by $\sigma_{++}(A)$ (resp. $\sigma_{--}(A)$) ([H5]). Hence,

$$\sigma_{++}(A) \subset \sigma_{\pi_+}(A) \quad \text{and} \quad \sigma_{--}(A) \subset \sigma_{\pi_-}(A).$$

Moreover, spectral points of positive and negative type are real [113],

$$\sigma_{++}(A) \subset \mathbb{R} \quad \text{and} \quad \sigma_{--}(A) \subset \mathbb{R}.$$

It is shown in [113] for bounded selfadjoint operators A that if an interval I satisfies

$$I \cap \widetilde{\sigma}(A) \subset \sigma_{++}(A) \cup \sigma_{--}(A), \tag{2.4}$$

then there exists an open neighbourhood $\mathcal{U}$ in $\mathbb{C}$ of I with $\mathcal{U} \setminus \mathbb{R} \subset \rho(A)$ and a local spectral function E of A of so-called positive type, i.e. $(E(\delta)\mathcal{H}, [\cdot,\cdot])$ is a Hilbert space if $\delta \subset I$ with $\delta \cap \widetilde{\sigma}(A) \subset \sigma_{++}(A)$. Roughly speaking, the spectral properties of the operator A are locally along δ the same as of a selfadjoint operator in a Hilbert space.

In [H5] this is generalized to unbounded operators and to spectral points of type π_+/π_-. It is proved in [H5] that a real spectral point λ_0 of type π_+ of a selfadjoint operator A in a Krein space, which is not an interior point of the spectrum, has a deleted neighbourhood[2] consisting only of spectral points of positive type or of points from $\rho(A)$ and the growth of the resolvent $(A - \lambda)^{-1}$ can be estimated by some power of $|\operatorname{Im} \lambda|^{-1}$ for non-real λ in a neighbourhood of λ_0. This is precisely the definition of locally (in a neighbourhood of λ_0) definitizable operators (see, e.g., [73]). Locally definitizable operators appeared first in a paper by H. Langer in 1967 (see [110]) without having a name at that time. Later, in a series of papers, P. Jonas studied these operators and introduced the notion of locally definitizable operators, cf. [66, 67, 69, 73]. This class of operators will be of particular interest in the following, hence we recall here the definition of locally definitizable operators or, more precisely, operators definitizable over some subset of $\overline{\mathbb{C}}$.

[2] A deleted neighbourhood of a point λ_0 is the set $\mathcal{U} \setminus \{\lambda_0\}$, where $\mathcal{U}$ is a neighbourhood of λ_0.

Definition 2.2. *Let Ω be a domain in $\overline{\mathbb{C}}$ which is symmetric with respect to $\mathbb{R}$ such that $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$ and the intersection with the open upper and lower half-plane are simply connected. Let A be a selfadjoint operator in the Krein space $(\mathcal{H}, [\cdot,\cdot])$ such that $\sigma(A) \cap (\Omega \setminus \overline{\mathbb{R}})$ consists of isolated points which are poles of the resolvent of A, and no point of $\Omega \cap \overline{\mathbb{R}}$ is an accumulation point of the non-real spectrum of A. The operator A is called* definitizable over Ω *if the following holds.*

(i) *For every closed subset Δ of $\Omega \cap \overline{\mathbb{R}}$ there exist an open neighbourhood $\mathcal{U}$ of Δ in $\overline{\mathbb{C}}$ and numbers $m \geq 1$, $M > 0$ such that*

$$\|(A-\lambda)^{-1}\| \leq M(|\lambda|+1)^{2m-2}|\mathrm{Im}\,\lambda|^{-m}$$

for all $\lambda \in \mathcal{U} \setminus \overline{\mathbb{R}}$.

(ii) *Every point $\lambda \in \Omega \cap \overline{\mathbb{R}}$ has an open connected neighbourhood I_λ in $\overline{\mathbb{R}}$ such that each of the two components of $I_\lambda \setminus \{\lambda\}$ is of positive or of negative type.*

Recall that a selfadjoint operator A in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ is called *definitizable* if $\rho(A) \neq \emptyset$ and there exists a rational function $r \neq 0$ with poles only in $\rho(A_0)$ such that $r(A_0) \in \mathcal{L}(\mathcal{H})$ is a nonnegative operator in $(\mathcal{H}, [\cdot,\cdot])$, $[r(A_0)x, x] \geq 0$ for all $x \in \mathcal{H}$ (see [109, 111]). It follows from [73, Theorem 4.7]) that A is definitizable if and only if A is locally definitizable over $\overline{\mathbb{C}}$. Definitizable operators are introduced and comprehensively studied by H. Langer in [109, 111] and appear in many applications (see e.g. [15, 28, 30, 35, 40, 50, 77, 78, 108, 109, 111, 116] and Sections 3 and 4 below).

Using the notion of locally definitizable operators, the main result of [H5] reads as follows.

Theorem 2.3. **[H5]** *Let A be a selfadjoint operator in the Krein space $(\mathcal{H}, [\cdot,\cdot])$, and let I be a closed connected subset of $\overline{\mathbb{R}}$ such that*

$$I \cap \widetilde{\sigma}(A) \subset \sigma_{\pi_+}(A) \cup \sigma_{\pi_-}(A) \tag{2.5}$$

holds and that each point of I is an accumulation point of $\rho(A)$. Then there exists a domain Ω in $\overline{\mathbb{C}}$ symmetric with respect to $\mathbb{R}$ with $\Omega \cap \mathbb{C}^+$ and $\Omega \cap \mathbb{C}^-$ being simply connected such that $I \subset \Omega$ and A is definitizable over Ω.

It follows from [73, Section 3.4 and Remark 4.9] that in the situation of Theorem 2.3 the operator A has a local spectral function $E(\delta)$ defined for all finite unions δ of connected subsets of $\Omega \cap \overline{\mathbb{R}}$, $\overline{\delta} \subset \Omega \cap \overline{\mathbb{R}}$, the endpoints of which belong to $\Omega \cap \overline{\mathbb{R}}$ and are of positive or negative type with respect to A. Contrary to the case of an interval satisfying (2.4), the local spectral function is no longer of positive type. Instead, we have the following.

Theorem 2.4. **[H5]** *Let A be definitizable over Ω and let E be the spectral function of A. A real point $\lambda \in \sigma(A) \cap \Omega$ belongs to $\sigma_{\pi_+}(A)$ ($\sigma_{\pi_-}(A)$) if and only if there exists a bounded open interval $\Delta \subset \Omega$, $\lambda \in \Delta$, such that $E(\Delta)$ is defined and $(E(\Delta)\mathcal{H}, [\cdot,\cdot])$ is a Pontryagin space with finite rank of negativity (resp. positivity).*

That is, the spectral properties of the operator A in a neighbourhood of a point of type π_+ are the same as of a selfadjoint operator in a Pontryagin space. In the following theorem we illustrate some of these properties. Here we denote by $\sigma_{ess}(A)$ the *essential spectrum*[3] of A.

[3] A closed, densely defined operator A in some Banach space is called *Fredholm* if the dimension of the kernel of A and the codimension of the range of A are finite. The set

$$\sigma_{ess}(A) := \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not Fredholm}\}$$

is called the *essential spectrum* of A.

Theorem 2.5. *Let A be a selfadjoint operator in $(\mathcal{H}, [\cdot,\cdot])$ with*

$$\widetilde{\sigma}(A) = \sigma_{++}(A) \cup \sigma_{--}(A). \tag{2.6}$$

Then A is similar to a selfadjoint operator in a Hilbert space.

If A with $\rho(A) \neq \emptyset$ satisfies instead of (2.6) the following condition

$$\sigma_{ess}(A) \subset \mathbb{R} \quad \text{and} \quad \widetilde{\sigma}(A) = \sigma_{\pi_+}(A) \quad (\text{resp. } \widetilde{\sigma}(A) = \sigma_{\pi_-}(A)),$$

then $(\mathcal{H}, [\cdot,\cdot])$ is a Pontryagin space with finite rank of negativity (resp. positivity). Moreover, the non-real spectrum of A consists of at most finitely many points which belong to $\sigma_p(A) \setminus \sigma_{ess}(A)$.

2.2 Compact Perturbations of Definitizable and Locally Definitizable Operators

The notion of points of type π_+ and π_- is particularly convenient when compact perturbations are considered. Under a compact perturbation a spectral point of type π_+ remains a spectral point of type π_+ or becomes a point from the resolvent set:

Theorem 2.6. [H5] *Let A_0 and A_1 be selfadjoint operators in the Krein space $(\mathcal{H}, [\cdot,\cdot])$. Assume that $\rho(A_0) \cap \rho(A_1) \neq \emptyset$ and that for some (and hence for all) $\mu \in \rho(A_0) \cap \rho(A_1)$ the difference*

$$(A_0 - \mu)^{-1} - (A_1 - \mu)^{-1} \text{ is compact.} \tag{2.7}$$

Then

$$(\sigma_{\pi_+}(A_0) \cup \rho(A_0)) \cap \mathbb{R} = (\sigma_{\pi_+}(A_1) \cup \rho(A_1)) \cap \mathbb{R},$$
$$(\sigma_{\pi_-}(A_0) \cup \rho(A_0)) \cap \mathbb{R} = (\sigma_{\pi_-}(A_1) \cup \rho(A_1)) \cap \mathbb{R}.$$

Moreover, $\infty \in \sigma_{++}(A_0)$ ($\infty \in \sigma_{--}(A_0)$) if and only if $\infty \in \sigma_{++}(A_1)$ (resp. $\infty \in \sigma_{--}(A_0)$).

Theorem 2.6 together with the results presented in the preceding Section 2.1 give the following perturbation result for locally definitizable operators in Krein spaces from [H5] (which is presented here in a slightly different form).

Theorem 2.7. [H5] *Let A_0, A_1 be selfadjoint operators in the Krein space $(\mathcal{H}, [\cdot,\cdot])$ with $\sigma_{ess}(A_0) \subset \mathbb{R}$. Let A_0 be definitizable over a domain $\Omega \subset \overline{\mathbb{C}}$ with $\Omega \cap \overline{\mathbb{R}} = I$. Assume that $\rho(A_1) \cap \Omega \neq \emptyset$ and for some $\mu \in \rho(A_0) \cap \rho(A_1)$ (2.7) holds. If*

$$I \cap \widetilde{\sigma}(A_0) \subset \sigma_{\pi_+}(A_0) \cup \sigma_{\pi_-}(A_0),$$

then A_1 is definitizable over Ω and

$$I \cap \widetilde{\sigma}(A_1) \subset \sigma_{\pi_+}(A_1) \cup \sigma_{\pi_-}(A_1).$$

Theorem 2.7 has a long list of well-known precursers: H. Langer proved in [110] 1967 the assertion of Theorem 2.7 in the case of a bounded selfadjoint fundamentally reducible[4] operator A_0 such that the difference of the resolvents (2.7) belongs to the so-called Matsaev-class. Recall that the Matsaev-class consists of all compact operators with s-numbers (s_j) satisfying $\sum_{j=1}^{\infty}(2j-1)^{-1}s_j < \infty$. P. Jonas extended

[4]Operators which are selfadjoint in a Krein space and at the same time selfadjoint with respect to some Hilbert space inner product $(\cdot,\cdot)$ such that $[\cdot,\cdot]$ is $\|\cdot\|$-continuous are called *fundamental reducible.*

this result in [67, 68] to unbounded selfadjoint fundamentally reducible operators A_0 such that (2.7) belongs again to the Matsaev-class. In the paper [113] of H. Langer, A. Markus and V. Matsaev in 1997 these assumptions are relaxed: The assertions of Theorem 2.7 are proved for the case of a bounded selfadjoint (no more fundamentally reducible) operator such that (2.7) is compact (no more of Matsaev-class). We mention that the proof of this result from [113] is based upon the existence of maximal spectral subspaces (cf. [126]). Moreover, it is formulated in terms of the so-called eigenvalues of finite index of negativity[5] which are precisely the spectral points of type π_+ being not of positive type, cf. [H5]. Finally, J. Behrndt and P. Jonas succeeded to prove the assertions of Theorem 2.7 in 2005 (cf. [17]). We mention that the proofs given in [17] and [H5] use completely different methods. Both papers were published in 2005 but [17] was submitted more than one year earlier as [H5].

Note that Theorem 2.7 is not suitable for operators A_0 being nonnegative in a neighbourhood of ∞, or, more precisely, for operators A_0 with $\infty \in \widetilde{\sigma}(A_0) \setminus \{\sigma_{\pi_+}(A_0) \cup \sigma_{\pi_-}(A_0)\}$. However, this case is intensively studied and we refer to [17, 31, 65, 67, 69, 71, 151, 152].

Theorem 2.7 also applies to definitizable operators in Krein spaces. Based on the Theorems 2.6, 2.7 and [H6, Theorem 2.6] we obtain the following perturbation result for definitizable operators, which follows already from the results in the frequently cited paper [74] of P. Jonas and H. Langer from 1979.

Theorem 2.8. *Let A_0 and A_1 be selfadjoint operators in the Krein space $(\mathcal{H}, [\cdot,\cdot])$. Assume that $\rho(A_0) \cap \rho(A_1) \neq \emptyset$ and that for some (and hence for all) $\mu \in \rho(A_0) \cap \rho(A_1)$ the difference* (2.7) *is compact. If A_0 is a definitizable operator with*

$$\sigma_{ess}(A_0) \subset \mathbb{R} \quad \textit{and} \quad \widetilde{\sigma}(A_0) = \sigma_{\pi_+}(A_0) \cup \sigma_{\pi_-}(A_0), \tag{2.8}$$

then A_1 is a definitizable operator and (2.8) *holds for A_0 replaced by A_1.*

We mention that in [74] it is also shown that the perturbed operator A_1 remains definitizable if A_0 is definitizable and the difference (2.7) is of finite rank.

Theorem 2.8 is in the following sense optimal (cf. [74, Proposition 3]): To every bounded definitizable selfadjoint operator in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ with $\widetilde{\sigma}(A_0) \setminus \{\sigma_{\pi_+}(A_0) \cup \sigma_{\pi_-}(A_0)\} \neq \emptyset$ there exists a compact selfadjoint operator K in $(\mathcal{H}, [\cdot,\cdot])$ such that the operator $A + K$ is not definitizable.

In [H4] the investigation of spectral points of type π_+ and type π_- of selfadjoint operators started in [H5] is continued. A sharp lower bound for the codimension of the linear manifold $\mathcal{H}_0$ occurring in Definition 2.1 is given in [H4] and this number is smaller or equal to the negativity (resp. positivity) index of the spectral subspaces corresponding to small intervals containing λ_0. Moreover, in [H4], a special finite dimensional perturbation is constructed which turns a real point of type π_+ (type π_-) into a point of positive (resp. negative) type.

We note that the concept of spectral points of positive/negative type is also used as a standard tool in the analysis of selfadjoint operator functions. For further details on the sign type properties of an associated linear operator (i.e. the linearization) in a Krein space and the local spectral functions for selfadjoint operator functions we refer to [112, 114, 115].

[5] In [113] an eigenvalue of finite index of negativity of a bounded selfadjoint operator A is defined in the following way: Assume $\lambda_0 \in \sigma_p(A)$ and assume that there exists an open interval (α, β) with $\lambda_0 \in (\alpha, \beta)$ and $(\alpha, \beta) \setminus \{\lambda_0\} \subset \sigma_{++}(A) \cup \rho(A)$. Using the local spectral function we find a restriction of A to some spectral subset with resolvent set in (α, α') and (β', β), $\alpha < \alpha' < \lambda_0 < \beta' < \beta$. The interval (α', β') is a spectral set of the restriction in the sense of Dunford, hence we have a spectral projection. If this spectral projection projects onto a Pontryagin space with finite rank of negativity, then the point λ_0 is called in [113] an eigenvalue of finite index of negativity.

2.3 Compact Perturbations and Perturbations Small in Gap of Linear Relations

The notions of spectral points of positive/negative type and of type π_+/π_- extend naturally to non-selfadjoint operators and to closed linear relations. This is done in [H1].

Recall that closed linear relations in a Hilbert or Krein space $\mathcal{H}$ are closed linear subspaces of the Cartesian product $\mathcal{H} \times \mathcal{H}$. Linear operators are always identified with linear relations via their graphs. For the definitions of the usual operations with relations like the inverse, the spectrum etc. we refer to [2, 41], and to the monographs [27, 53]. Here the (extended) set of regular type $\widetilde{r}(A)$ of a closed linear relation A is defined by $\widetilde{r}(A) := \mathbb{C} \setminus \sigma_{ap}(A)$ if $0 \in \sigma_{ap}(A^{-1})$ and $\widetilde{r}(A) := \overline{\mathbb{C}} \setminus \sigma_{ap}(A)$ otherwise, where $\sigma_{ap}(A)$ is the approximate point spectrum[6] of A.

Definition 2.9. **[H1]** *Let A be a closed linear relation in a Krein space $(\mathcal{H}, [\cdot,\cdot])$. A point $\lambda_0 \in \sigma_{ap}(A)$ is said to be of* type π_+ (type π_-) *with respect to A, if there exists a linear relation $S \subset A$ with $\operatorname{codim}_A S < \infty$ such that for every sequence $\left(\binom{x_n}{\tilde{x}_n}\right)$ with $\binom{x_n}{\tilde{x}_n} \in S$, $n = 1, 2 \ldots$, $\|x_n\| = 1$ and $\lim_{n\to\infty} \|\tilde{x}_n - \lambda_0 x_n\| = 0$ we have*

$$\liminf_{n\to\infty}\, [x_n, x_n] > 0 \quad (\textit{resp.}\ \limsup_{n\to\infty}\, [x_n, x_n] < 0).$$

A similar definition is given for the point ∞, see [H1]. If for $\lambda \in \sigma_{\pi_+}(A)$ (resp. $\lambda \in \sigma_{\pi_-}(A)$) it is possible to choose in Definition 2.9 $S = A$, then we call λ a point of *positive type* (resp. *negative type*) of A. As in Section 2.1 we denote the set of all points of positive, of negative type, of type π_+ and of type π_- by $\sigma_{++}(A)$, $\sigma_{--}(A)$, $\sigma_{\pi_+}(A)$ and $\sigma_{\pi_-}(A)$, respectively.

As a first result we obtain (see [H1] and, for selfadjoint operators, [H5]) some sign-type properties of eigenvalues from the set $\sigma_{\pi_+}(A)\backslash\sigma_{++}(A)$. Recall that in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ a vector $x \in \mathcal{H}$ is called *positive* (*negative*) if $[x, x] > 0$ ($[x, x] < 0$, respectively).

Theorem 2.10. **[H1]** *Let A be a closed linear relation in a Krein space $(\mathcal{H}, [\cdot,\cdot])$. If $\lambda_0 \in \sigma_{\pi_+}(A)\backslash\sigma_{++}(A)$ ($\lambda_0 \in \sigma_{\pi_-}(A)\backslash\sigma_{--}(A)$), then λ_0 is an eigenvalue of A with a corresponding nonpositive* (*resp. nonnegative*) *eigenvector. If $\infty \in \sigma_{\pi_+}(A)\backslash\sigma_{++}(A)$ ($\infty \in \sigma_{\pi_-}(A)\backslash\sigma_{--}(A)$), then the multivalued part of A contains a nonpositive* (*resp. nonnegative*) *vector.*

In order to investigate the behaviour of spectral points of type π_+ and of type π_- under compact perturbations and perturbations small in norm we use the orthogonal projections P_A and P_B in $\mathcal{H}\oplus\mathcal{H}$ onto two closed subspaces A and B of $\mathcal{H}\oplus\mathcal{H}$. Recall that the quantity $\hat{\delta}(A, B) := \|P_A - P_B\|$ is called the *gap* between A and B, cf. [90]. We shall say that A is a compact (finite rank) perturbation of B if $P_A - P_B$ is a compact (resp. finite dimensional) operator. The following description of compact perturbations of closed linear relations is obtained in [H3].

Theorem 2.11. **[H3]** *Let A and B be closed linear relations. Then the following assertions are equivalent:*

(i) *$P_A - P_B$ is a compact operator,*

[6] We say that $\lambda \in \mathbb{C}$ belongs to the *approximate point spectrum* $\sigma_{ap}(A)$ of a closed linear relation A if there exists a sequence $\left(\binom{x_n}{\tilde{x}_n}\right)$ with $\binom{x_n}{\tilde{x}_n} \in A$, $n = 1, 2, \ldots$, such that

$$\|x_n\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \|\tilde{x}_n - \lambda x_n\| = 0.$$

(ii) *for every $\varepsilon > 0$ there exists a closed linear relation F such that $P_B - P_F$ is a finite rank operator and*

$$\hat{\delta}(A, F) = \|P_A - P_F\| < \varepsilon.$$

If, in addition, $\rho(A) \cap \rho(B) \neq \emptyset$, then A is a compact perturbation of B if and only if $(A - \lambda)^{-1} - (B - \lambda)^{-1}$ is a compact operator for some (and hence for all) $\lambda \in \rho(A) \cap \rho(B)$.

Moreover, it is shown in [H3] that A is a finite rank perturbation of B if and only if A and B are both finite dimensional extensions of their common part $A \cap B$.

We obtain in [H1] the following perturbation result for arbitrary non-selfadjoint operators (and relations) in Krein spaces. We mention that usually perturbation problems are only considered for special subclasses of closed operators, e.g., selfadjoint (see above), normal or dissipative operators in Krein spaces, see [H2, 5, 6, 10, 43, 117, 145].

Theorem 2.12. [H1] *Let A and B be closed linear relations in a Krein space $(\mathcal{H}, [\cdot, \cdot])$ and suppose that A is a compact perturbation of B. Then we have*

$$\sigma_{\pi_+}(A) \cup \widetilde{r}(A) = \sigma_{\pi_+}(B) \cup \widetilde{r}(B) \quad \text{and} \quad \sigma_{\pi_-}(A) \cup \widetilde{r}(A) = \sigma_{\pi_-}(B) \cup \widetilde{r}(B).$$

The main result in [H1] is devoted to perturbations which are small in the gap metric. Roughly speaking, we show that spectral points of type π_+ and type π_- type are stable under perturbations small in the gap metric. A similar result holds for spectral points of positive and negative type, see [H1].

Theorem 2.13. [H1] *Let A be a closed linear relation in a Krein space $(\mathcal{H}, [\cdot, \cdot])$ and let $\mathfrak{F} \subset \overline{\mathbb{C}}$ be a compact set with $\mathfrak{F} \subset \sigma_{\pi_+}(A) \cup \widetilde{r}(A)$ ($\mathfrak{F} \subset \sigma_{\pi_-}(A) \cup \widetilde{r}(A)$). Then there exists a constant $\gamma \in (0, 1)$ such that for all closed linear relations B with $\hat{\delta}(A, B) < \gamma$ we have*

$$\mathfrak{F} \subset \sigma_{\pi_+}(B) \cup \widetilde{r}(B) \quad (\textit{resp. } \mathfrak{F} \subset \sigma_{\pi_-}(B) \cup \widetilde{r}(B)).$$

The above introduced notions of spectral points of positive and negative type are very convenient in the study of fundamentally reducible closed linear relations under perturbations small in gap, see [H1]. A relation A is said to be *fundamentally reducible* if there exists a fundamental decomposition of the Krein space of the form (2.1) and A can be written as

$$A = A_+ \dotplus A_-, \qquad \text{direct sum}, \tag{2.9}$$

where $A_+ := A \cap \mathcal{H}_+^2$ and $A_- := A \cap \mathcal{H}_-^2$ are closed linear relations in the Hilbert spaces $(\mathcal{H}_+, [\cdot, \cdot])$ and $(\mathcal{H}_-, -[\cdot, \cdot])$, respectively. If λ belongs to $\mathbb{C} \setminus \sigma_{ap}(A_-)$ the estimate

$$\|\tilde{y}^- - \lambda y^-\| \geq k_{\lambda,-} \|y^-\| \tag{2.10}$$

holds for some $k_{\lambda,-} > 0$ and all $\binom{y^-}{\tilde{y}^-} \in A_-$.

The following result from [H1] can be viewed as a natural generalization of a result for bounded selfadjoint operators in [113, Theorem 4.1]. For simplicity we formulate it here only for spectral points of positive type (for spectral points of negative type, type π_+ or type π_- we refer to [H1]).

Theorem 2.14. [H1] *Let A be a fundamentally reducible closed linear relation in $\mathcal{H}$ as in* (2.9) *and let B be a closed linear relation in $\mathcal{H}$. If for some $\lambda \in \mathbb{C} \setminus \sigma_{ap}(A_-)$, $k_{\lambda,-} > 0$ as in* (2.10) *and $\gamma > 0$*

$$\hat{\delta}(A - \lambda, B - \lambda) < \gamma \quad \text{and} \quad \gamma^2 \left(1 + \frac{1}{k_{\lambda,-}^2}\right) < \frac{1}{4}$$

hold, then

$$\lambda \in \sigma_{++}(B) \cup \widetilde{r}(B).$$

Finally, it is shown in [H1] that for selfadjoint relations in Krein spaces analogous results as in Section 2.1 hold: A selfadjoint relation is locally definitizable (cf. [73]) in a neighbourhood of a point of type π_+ or type π_- which is in addition an accumulation point of the resolvent set of A. As a consequence, A possesses a local spectral function on subsets of $\mathbb{R}$ which are not in the interior of the spectrum of A and which consist only of spectral points of type π_+ or type π_- or of regular points.

2.4 Small Perturbation of Selfadjoint and Unitary Operators in Krein Spaces

In [H2] perturbations of selfadjoint operators and unitary operators in Krein spaces are studied. In contrast to the previous sections, the perturbations are now uniformly dissipative operators (resp. uniformly bi-expansive operators). Recall that an operator B in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ is called *uniformly dissipative* if there exists some $\alpha > 0$ such that for $x \in \operatorname{dom} B$ we have $\operatorname{Im}[Bx, x] \geq \alpha \|x\|^2$. A bounded operator V is said to be *bi-expansive* if

$$[Vx, Vx] \geq [x, x] \quad \text{and} \quad [V^+x, V^+x] \geq [x, x] \quad \text{for all} \quad x \in \mathcal{H}.$$

Moreover, V is called *uniformly bi-expansive* if the operator V is bi-expansive and there is an $\alpha_V > 0$ such that $[Vx, Vx] \geq [x, x] + \alpha_V \|x\|^2$ for $x \in \mathcal{H}$.

Proposition 2.15. [H2] *Let A be a selfadjoint operator in a Krein space $(\mathcal{H}, [\cdot,\cdot])$. Assume that λ_0, $\lambda_0 \in (a, b)$, is not an accumulation point of the non-real spectrum of A and that*

$$(a, b) \setminus \{\lambda_0\} \subset \sigma_{++}(A) \cup \rho(A)$$

holds. Let $a < a' < \lambda_0 < b' < b$. Then there exists a $\delta' > 0$ such that the strip

$$\{\lambda \in \mathbb{C} : a' \leq Re\,\lambda \leq b', -\delta' \leq Im\,\lambda < 0\}$$

belongs to the resolvent set of A. Moreover, if $\gamma_{\delta'}$ denotes the closed oriented curve in the complex plane which consists of the line segments connecting the points $b', b' - i\delta', a' - i\delta', a'$ and b', then there exists an $\varepsilon_0 > 0$ such that for all bounded uniformly dissipative operators B in $\mathcal{H}$ with $\|B\| \leq \varepsilon_0$ we have

$$\gamma_{\delta'} \subset \rho(A + B).$$

The following theorem can be considered as the main result of [H2].

Theorem 2.16. [H2] *Let A be a selfadjoint operator in a Krein space $(\mathcal{H}, [\cdot,\cdot])$. Assume that the assumptions of Proposition 2.15 are satisfied and let a', b', δ', ε_0 and $\gamma_{\delta'}$ be as in Proposition 2.15. Then the following assertions are valid.*

(i) *The point λ_0 belongs to $\sigma_{++}(A) \cup \rho(A)$ if and only if there exists an $\varepsilon_1 > 0$ such that for every uniformly dissipative operator B acting in $\mathcal{H}$ with $\|B\| < \varepsilon_1$ the operator $A + B$ has no spectrum inside the curve $\gamma_{\delta'}$.*

(ii) *The point λ_0 belongs to $\sigma_{\pi_+}(A)$ if and only if there exists an $\varepsilon_1 > 0$ such that for every uniformly dissipative operator B acting in $\mathcal{H}$ with $\|B\| < \varepsilon_1$ the spectrum of $A + B$ inside the curve $\gamma_{\delta'}$ consists of at most finitely many normal eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that*

$$\mathcal{M}_- := \operatorname{span}\{\mathcal{L}_{\lambda_j}(A + B) : 1 \leq j \leq k\}$$

is of finite dimension. Moreover, in this case, the dimension of $\mathcal{M}_-$ *is equal to the rank of negativity* $\kappa_-(E((a',b'))\mathcal{H})$ *of the Pontryagin space* $E((a',b'))\mathcal{H}$*, that is*

$$\dim \mathcal{M}_- = \kappa_-(E((a',b'))\mathcal{H}).$$

Here E *denotes the local spectral function of* A*, see Section* 2.1.

(iii) *The point* λ_0 *does not belong to* $\sigma_{\pi_+}(A) \cup \rho(A)$ *if and only if there exists an* $\varepsilon_1 > 0$ *such that for every uniformly dissipative operator* B *acting in* $\mathcal{H}$ *with* $\|B\| < \varepsilon_1$ *the range of the Riesz-Dunford projector corresponding to* $A + B$ *and* $\gamma_{\delta'}$ *is of infinite dimension.*

In addition, similar arguments hold true for uniformly bi-expansive perturbations of unitary operators, see [H2]. The perturbation results from [H2] are a different characterization of the spectral points of positive (resp. negative) type and of type π_+ (resp. π_-) of selfadjoint/unitary operators in Krein spaces. In the early work of L.S. Pontryagin such arguments were used in a similar manner, cf. [136].

3 Second Order Systems

In this section, we present results contained in the following papers on second order equations of the form $\ddot{z}(t) + A_0 z(t) + D\dot{z}(t) = 0$, where A_0 and D are operators in some Hilbert spaces with properties specified below.

[H6] C. Trunk, *Analyticity of semigroups related to a class of block operator matrices*, Oper. Theory Adv. Appl. **195** (2009), 257-271.

[H7] B. Jacob and C. Trunk, *Spectrum and analyticity of semigroups arising in elasticity theory and hydromechanics*, Semigroup Forum **79** (2009), 79-100.

[H8] B. Jacob, C. Trunk and M. Winklmeier *Analyticity and Riesz basis property of semigroups associated to damped vibrations*, Journal of Evolution Equations **8** (2008), 263-281.

[H9] C. Trunk, *Spectral theory for operator matrices related to models in mechanics*, Math. Notes **83** (2008), 843-850.

[H10] B. Jacob, K. Morris and C. Trunk, *Minimum-phase infinite-dimensional second-order systems*, IEEE Transactions on Automatic Control **52** (2007), 1654-1665.

[H11] B. Jacob and C. Trunk, *Location of the spectrum of operator matrices which are associated to second order equations*, Operators and Matrices **1** (2007), 45-60.

We write the second order equation $\ddot{z}(t) + A_0 z(t) + D\dot{z}(t) = 0$ as a first order equation with a block operator matrix $\mathcal{A}$ with entries $0, I, -A_0$ and D. The main aim of the papers [H7]-[H11] is the investigation of the block operator matrix $\mathcal{A}$. The block operator matrix $\mathcal{A}$ is well-studied in the literature, e.g. it is well-known that $\mathcal{A}$ is, under quite general assumptions, a generator of a C_0-semigroup of contractions. However, there are essentially no results concerning the location of the spectrum of $\mathcal{A}$ in terms of the coefficients. This is the main purpose of the papers [H6, H11]. There the location of the spectrum and of the essential spectrum of $\mathcal{A}$ in terms of the coefficients A_0 and D is described. Moreover, a simple sufficient condition for analyticity of the semigroup is derived in [H7, H6, H11]. These results are convenient and easy to apply for a given partial differential equation if certain bounds for the spectra of the operators A_0 and D are known a-priori. An explicit example for a partial differential equations modeling small transverse oscillations of a horizontal pipe is calculated in [H11]. The case of a selfadjoint damping D is considered in [H11], whereas the case of an accretive damping D is treated in [H6]. Moreover, it is shown in [H11] that parts of the spectrum of $\mathcal{A}$ are of type π_+ (cf. Section 2).

This fact is used in [H9, H10] to prove a surprisingly simple characterization for the definitizability of the operator matrix $\mathcal{A}$: Assume that the operator A_0^{-1} is a compact operator. Then the operator $\mathcal{A}$ is definitizable if and only if $0 \notin \sigma_{ess}(A_0^{-1}D)$. With the help of the spectral function for definitizable operators (cf. Section 2) a set of sufficient conditions for the existence of a Riesz basis consisting of eigenvectors and finitely many associated vectors is determined in [H8] and, under different assumptions on the damping, also in [H7].

In [H10] the second order equation is equipped with an input and an output (velocity or position measurement). The main achievement in [H10] is the proof that a wide class of second order infinite dimensional systems are minimum phase.

3.1 Location of the Spectrum of Operator Matrices Associated to Second Order Equations

In the following, the second order equation

$$\ddot{z}(t) + A_0 z(t) + D\dot{z}(t) = 0. \tag{3.1}$$

is considered. We make the following assumptions.

(A1) The stiffness operator $A_0 : \operatorname{dom} A_0 \subset H \to H$ is a selfadjoint uniformly positive operator on a Hilbert space H.

A scale of Hilbert spaces H_α is defined as follows: For $\alpha \geq 0$, we define $H_\alpha = \operatorname{dom} A_0^\alpha$ equipped with the norm $\|z\|_{H_\alpha} := \|A_0^\alpha z\|_H$ and $H_{-\alpha} = H_\alpha^*$. Here the duality is taken with respect to the pivot space H, that is, equivalently, $H_{-\alpha}$ is the completion of H with respect to the norm $\|z\|_{H_{-\alpha}} = \|A_0^{-\alpha} z\|_H$. Thus A_0 extends (restricts) to $A_0 : H_\alpha \to H_{\alpha-1}$ for $\alpha \in \mathbb{R}$. We use the same notation A_0 to denote this extension (restriction).

We denote the inner product on H by $(\cdot,\cdot)_H$ or $(\cdot,\cdot)$, and the duality pairing on $H_{-\alpha} \times H_\alpha$ by $(\cdot,\cdot)_{H_{-\alpha}\times H_\alpha}$. In the following we will consider $\alpha = \frac{1}{2}$. Then for $a_0 := \|A_0^{-\frac{1}{2}}\|^{-1}$ we have

$$\|z\|_{H_{\frac{1}{2}}} = \|A_0^{\frac{1}{2}} z\| \geq a_0 \|z\| \quad \text{for all } z \in H_{\frac{1}{2}}. \tag{3.2}$$

(A2) The damping operator $D : H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$ is a bounded operator such that $A_0^{-1/2} D A_0^{-1/2}$ is a bounded selfadjoint operator in H and satisfies

$$(Dz, z)_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \geq 0, \qquad z \in H_{\frac{1}{2}}.$$

System (3.1) is equivalent to the first order equation

$$\dot{x}(t) \quad = \quad \mathcal{A}x(t) \tag{3.3}$$

where $\mathcal{A} : \operatorname{dom}\mathcal{A} \subset H_{\frac{1}{2}} \times H \to H_{\frac{1}{2}} \times H$, is given by

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0 & -D \end{bmatrix}, \tag{3.4}$$

$$\operatorname{dom}\mathcal{A} = \left\{ \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right] \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \mid A_0 z + Dw \in H \right\}.$$

The operator $\mathcal{A}$ is not selfadjoint in the Hilbert space $H_{\frac{1}{2}} \times H$. We introduce an indefinite inner product $[\cdot,\cdot]$ on $H_{\frac{1}{2}} \times H$ with fundamental symmetry J given by

$$J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

That is, for $\left(\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix}\right) \in H_{\frac{1}{2}} \times H$ the inner product $[\cdot,\cdot]$ is defined via

$$[\left(\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix}\right)] := (J\left(\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix}\right)) = (x_1, x_2)_{H_{\frac{1}{2}}} - (y_1, y_2). \tag{3.5}$$

Then $(H_{\frac{1}{2}} \times H, [\cdot,\cdot])$ is a Krein space and $\mathcal{A}$ is a selfadjoint operator with respect to $[\cdot,\cdot]$, see [150, Proof of Lemma 4.5]. Moreover $\mathcal{A}$ has a bounded inverse in $H_{\frac{1}{2}} \times H$.

The block operator matrix $\mathcal{A}$ has been studied in the literature over many years. Interest in this particular model is motivated by various problems such as stabilization, see for example [18, 119, 120, 148], solvability of the Riccati equations [56] and compensator problems with partial observations [57].

It is well-known that $\mathcal{A}$ generates a C_0-semigroup of contractions in $H_{\frac{1}{2}} \times H$. This goes back to [7, 118], see also [8, 23]. Other properties of the C_0-semigroup such as analyticity have been studied in [7, 8, 23, 25, 58, 61].

In [H11] the location of the spectrum and of the essential spectrum of the generator $\mathcal{A}$ is discussed. Since $\mathcal{A}$ is a generator of a C_0-semigroup, its spectrum is contained in the closed left half-plane $\overline{\mathbb{C}_-}$. However, it is shown in [H11] that there are examples such that for each $\varepsilon > 0$ there exist operators A_0 and D satisfying **(A1)** and **(A2)** with

$$\sigma(\mathcal{A}) = \{s \in \overline{\mathbb{C}_-} \mid |s| \geq \varepsilon\}.$$

In many applications one has additional properties of the damping operator D guaranteeing that $\sigma(\mathcal{A})$ is contained in a smaller subset of $\overline{\mathbb{C}_-}$. Several authors have proved, independently of each other, that the condition

$$\beta := \inf_{z \in H_{\frac{1}{2}} \setminus \{0\}} \frac{(A_0^{-1/2} Dz, A_0^{1/2} z)_H}{\|z\|_H^2} > 0$$

is sufficient for exponential stability of the C_0-semigroup generated by $\mathcal{A}$, see for example [7, 8, 9, 23, 59, 60, 150, 153]. In [H11] two more numbers are defined,

$$\begin{aligned} \gamma &:= \sup_{z \in H_{\frac{1}{2}} \setminus \{0\}} \frac{\langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}{\|z\|_H^2}, \\ \delta &:= \inf_{z \in H_{\frac{1}{2}} \setminus \{0\}} \frac{\langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}{\|z\|_{H_{\frac{1}{2}}}^2}. \end{aligned}$$

By definition we have $\beta, \delta \in [0, \infty)$, $\gamma \in [0, \infty]$, and it is easy to see[7] that $a_0^2 \delta \leq \beta \leq \gamma$, where a_0 is as in (3.2). These numbers are now used to describe the location of the spectrum of $\mathcal{A}$.

Theorem 3.1. **[H11]** *We have*

1. *If* $\beta > 0$, *then*

$$\left\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > -\frac{\beta}{2}, \operatorname{Im} \lambda \neq 0\right\} \subset \rho(\mathcal{A}).$$

2. *If* $\gamma < \infty$, *then*

$$\left\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda < -\frac{\gamma}{2}, \operatorname{Im} \lambda \neq 0\right\} \subset \rho(\mathcal{A}).$$

3. *If* $\delta > 0$, *then*

$$\sigma(\mathcal{A}) \subset \left\{\lambda \in \mathbb{C} \mid \left|\lambda + \frac{1}{\delta}\right| \leq \frac{1}{\delta}\right\} \cup (-\infty, 0).$$

4. *If* $\langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}^2 \geq 4\|z\|_H^2 \|z\|_{H_{\frac{1}{2}}}^2$, $z \in H_{\frac{1}{2}}$, *then*

$$\sigma(\mathcal{A}) \subset (-\infty, 0).$$

[7] We mention that in [H11, page 51] the constant a_0^2 is missing.

We mention that Part 1 of Theorem 3.1 improves results from [23] and from [9].

The next theorem is one of the main results from [H11]. It describes the location of the essential spectrum of $\mathcal{A}$. Moreover, it shows that parts of the spectrum of $\mathcal{A}$ are spectral points of type π_+, cf. Section 2.1.

Theorem 3.2. **[H11]** *If $\sigma_{ess}(A_0^{-1}D) = \emptyset$, then we set $\alpha_1 := \infty$. Otherwise, let*

$$\alpha_1 := \frac{1}{2\|A_0^{-1}\|}\min\{s \in \mathbb{R} \mid s \in \sigma_{ess}(A_0^{-1}D)\}.$$

Here $\|A_0^{-1}\|$ is the operator norm of A_0^{-1} considered as an operator acting in H and $A_0^{-1}D$ is considered as an operator acting in $H_{\frac{1}{2}}$. Set $\alpha_2 := \infty$ if $\sigma_{ess}(A_0^{\frac{1}{2}}) = \emptyset$ and

$$\alpha_2 := \min\{s \in \mathbb{R} \mid s \in \sigma_{ess}(A_0^{\frac{1}{2}})\}, \text{ otherwise.}$$

Then we have:

1. *$\sigma_{ess}(\mathcal{A}) \subset (-\infty, 0) \cup \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq -\alpha_1\}$.*
2. *For $\alpha := \max\{\alpha_1, \alpha_2\}$ we have*

$$(-\alpha, 0) \cap \sigma(\mathcal{A}) \subset \sigma_{\pi_+}(\mathcal{A}).$$

In particular, no point of the interval $(-\alpha, 0)$ is an accumulation point of the non-real spectrum of $\mathcal{A}$.

The location of the essential spectrum can be described precisely under the additional assumption that A_0 has a compact resolvent.

Theorem 3.3. **[H11]** *If the operator A_0^{-1} is compact, then*

$$\sigma_{ess}(\mathcal{A}) = \left\{\lambda \in \mathbb{C}\backslash\{0\} \mid \frac{1}{\lambda} \in \sigma_{ess}(-A_0^{-1}D)\right\} \subset (-\infty, 0).$$

Here $A_0^{-1}D$ is considered as an operator acting in $H_{\frac{1}{2}}$. Moreover, no point from $\sigma_{ess}(\mathcal{A})$ is an accumulation point of the non-real spectrum of $\mathcal{A}$ and

$$(-\infty, 0) \cap \sigma(\mathcal{A}) \subset \sigma_{\pi_+}(\mathcal{A}).$$

3.2 Definitizability and Riesz basis Property of $\mathcal{A}$. Analyticity of the Generated Semigroup

Theorem 3.3 together with Theorem 2.3 show that $\mathcal{A}$ is locally definitizable. More precisely, $\mathcal{A}$ is definitizable over $\mathbb{R}$. In order to show that in this case $\mathcal{A}$ is a definitizable operator it remains to provide conditions which ensure that $\mathcal{A}$ is definitizable over a neighbourhood of ∞. This is done in [H8].

Theorem 3.4. **[H8]** *Assume that the operator A_0^{-1} is a compact operator in H and that $0 \notin \sigma_{ess}(A_0^{-1}D)$. Then*

$$\infty \in \sigma_{--}(\mathcal{A}) \quad \text{and} \quad \mathbb{R} \subset \sigma_{\pi_+}(\mathcal{A}) \cup \rho(\mathcal{A}).$$

In particular, the operator $\mathcal{A}$ is definitizable.

This result is improved considerably in [H9].

Theorem 3.5. **[H9]** *Assume that the operator A_0^{-1} is a compact operator in H. Then the operator $\mathcal{A}$ is definitizable if and only if $0 \notin \sigma_{ess}(A_0^{-1}D)$.*

Then, by standard growth properties of the resolvent of a definitizable operator (see, e.g., [111]), we obtain with [45, Chapter II, Section 4.5] the following.

Theorem 3.6. **[H8]** *Assume that A_0^{-1} is compact in H and that $0 \notin \sigma_{ess}(A_0^{-1}D)$. Then $\mathcal{A}$ generates an analytic semigroup on $H_{1/2} \times H$.*

Analyticity of the C_0-semigroup generated by $\mathcal{A}$ has been studied in many papers, see [23, 24, 25, 26, 58, 61, 62, 107]. Most of the papers require that the damping operator D is comparable with A^ρ for some $\rho \in [1/2, 1]$. In [107] the special case of a damping operator D of the form

$$D = \alpha A_0 + B, \tag{3.6}$$

is studied where $\alpha > 0$ is a constant, A_0^{-1} is compact and B is symmetric and A_0-compact. In this case the essential spectrum of the operator $A_0^{-1}D$ consists of the point $-1/\alpha$ only. If, moreover, $-1/\alpha \notin \sigma_p(\mathcal{A})$, it is shown in [107] that $\mathcal{A}$ generates an analytic semigroup on $H_{1/2} \times H$. Theorem 3.6 implies that the result from [107] holds even if $-1/\alpha \in \sigma_p(\mathcal{A})$ and for a much larger class of dampings D.

In many application it is important to have a Riesz basis[8] consisting of eigenvectors and finitely many associated vectors of $\mathcal{A}$ of the space $H_{1/2} \times H$ as it has many important implications. E.g., in this case, the semigroup is exponentially stable if and only if the spectrum of $\mathcal{A}$ is contained in the open left half-plane and uniformly bounded away from the imaginary axis.

The Riesz basis property has been shown in [107] in the situation where A_0^{-1} is a compact operator, D is of the form (3.6) for some $\alpha \geq 0$ with a symmetric operator B and $-1/\alpha \notin \sigma_p(\mathcal{A})$, if $\alpha \neq 0$ (and with some additional assumptions in the case $\alpha = 0$). Similar results are obtained [25, Appendix A] in a more special situation. All these assumptions guarantee that the essential spectrum of $\mathcal{A}$ consists of at most of one point.

In [H8] we allow more general damping operators D. In particular, the essential spectrum of $\mathcal{A}$ may contain infinitely many points.

Theorem 3.7. **[H8]** *Assume that the operator A_0^{-1} is compact in H and that*

$$0 \notin \sigma_{ess}(A_0^{-1}D), \tag{3.7}$$

where $A_0^{-1}D$ is considered as an operator acting in $H_{\frac{1}{2}}$. Assume that the set $\sigma_{ess}(A_0^{-1}D)$ is countable and has at most countably many accumulation points. Moreover, let at least one of the following conditions be satisfied.

(a) *There exists $\delta > 0$ such that for all $f \in H_{\frac{1}{2}}$ with $\|f\|_{H_{\frac{1}{2}}} = 1$ we have*

$$\left(A_0^{-1}Df, f\right)^2_{H_{\frac{1}{2}}} - 4\left(A_0^{-1}f, f\right)_{H_{\frac{1}{2}}} > \delta. \tag{3.8}$$

(b) *For all $\mu \in \sigma_{ess}(-A_0^{-1}D)$ we have either $\frac{1}{\mu} \notin \sigma_p(\mathcal{A})$ or, if $\frac{1}{\mu} \in \sigma_p(\mathcal{A})$, for every non-zero $\binom{y}{\mu^{-1}y} \in \ker(\mathcal{A} - \mu^{-1}I)$ we have*

$$\mu^2(y, w)_{H_{\frac{1}{2}}} \neq (y, w) \quad \textit{for all } \binom{w}{\mu^{-1}w} \in \ker(\mathcal{A} - \mu^{-1}I).$$

(c) $\|A_0^{-1/2}\| < \inf\{\lambda > 0 \mid \lambda \in \sigma_{ess}(A_0^{-1}D)\}$.

Then the following assertions hold.

[8] A countable set $\mathcal{M}$ of elements of a Hilbert space is said to be a *Riesz basis* if there exists an isomorphic mapping which maps $\mathcal{M}$ onto an orthonormal basis, cf. [133, Lecture VI].

1. *There exists a subspace of $H_{\frac{1}{2}} \times H$ of at most finite codimension which has a Riesz basis consisting of eigenvectors of $\mathcal{A}$.*
2. *There exists a Riesz basis of $H_{\frac{1}{2}} \times H$ consisting of eigenvectors and finitely many associated vectors of $\mathcal{A}$.*
3. *Moreover, if* (a) *holds, then $\mathcal{A}$ has no associated vectors, i.e. there are no Jordan chains of length greater than one, the spectrum of $\mathcal{A}$ is real and there exists a Riesz basis of $H_{\frac{1}{2}} \times H$ consisting of eigenvectors of $\mathcal{A}$.*

Condition (3.8) of Theorem 3.7 appears already in the celebrated works [103, 104], where the case of a bounded selfadjoint operator D and a positive compact operator A_0 is discussed. This approach is used in the proof of Theorem 3.7 in [H8].

However, the case where $A_0^{-1}D$ is a compact operator in $H_{1/2}$, and, hence, (3.7) is not satisfied, is not contained in Theorem 3.7. For results involving compact $A_0^{-1}D$ we refer to [107].

In [H6] damping operators D are considered which are in some sense large compared to A_0. In particular, in [H6] we assume instead of condition **(A2)** the following condition **(A2′)**.

(A2′) For some $\theta > 1$ the operator $D : H_{\frac{\theta}{2}} \to H_{-\frac{\theta}{2}}$ is a bounded operator such that $A_0^{-\theta/2} D A_0^{-\theta/2}$ is a bounded nonnegative selfadjoint operator in H, that is,

$$\langle Dz, z\rangle_{H_{-\frac{\theta}{2}} \times H_{\frac{\theta}{2}}} \geq 0, \qquad z \in H_{\frac{\theta}{2}}.$$

Moreover, assume, in addition, that the operator $A_0^{-\theta}D$ admits a bounded extension to an operator in H such that the bounded extension of $A_0^{-\theta}D$ to an operator in $H_{\frac{1}{2}}$ is boundedly invertible in $H_{\frac{1}{2}}$.

As above, equation (3.1) is equivalent to the first order equation $\dot{x}(t) = \mathcal{A}x(t)$, where $\mathcal{A}$ is given as in (3.4). It is shown in [H6] that the operator $\mathcal{A}$ is a closed operator in $H_{\frac{1}{2}} \times H$ and $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions on $H_{\frac{1}{2}} \times H$. Moreover, $\mathcal{A}$ is a selfadjoint operator in the Krein space $(H_{\frac{1}{2}} \times H, [\cdot,\cdot])$, where is inner product $[\cdot,\cdot]$ is given as in (3.5), cf. [H6].

Theorem 3.8. **[H6]** *Assume, in addition to our assumptions* **(A1)** *and* **(A2′)**, *that A_0^{-1} is a compact operator in H. Then the essential spectrum of $\mathcal{A}$ consists only of the point zero, $\sigma_{ess}(\mathcal{A}) = \{0\}$, and the operator $\mathcal{A}$ is definitizable with*

$$\infty \in \sigma_{--}(\mathcal{A}), \quad 0 \in \sigma_{++}(\mathcal{A}) \quad \textit{and} \quad \mathbb{R} \subset \sigma_{\pi_+}(\mathcal{A}) \cup \rho(\mathcal{A}).$$

And, in the same way as above, Theorem 3.8 implies the following.

Theorem 3.9. **[H6]** *If A_0^{-1} is a compact operator in H, then $\mathcal{A}$ generates an analytic semigroup on $H_{1/2} \times H$.*

With Theorem 3.8 it is possible to give a sufficient condition for the existence of a Riesz basis of $H_{\frac{1}{2}} \times H$ consisting of eigenvectors of $\mathcal{A}$.

Theorem 3.10. **[H6]** *If H is separable and if A_0^{-1} is a compact operator in H, then there exists a Riesz basis of $H_{\frac{1}{2}} \times H$ consisting of eigenvectors and finitely many associated vectors of $\mathcal{A}$. Moreover assume, in addition, that for all $\mu \in \sigma_{\pi_+}(\mathcal{A}) \setminus \sigma_{++}(\mathcal{A})$ and for all non-zero $\binom{y}{\mu y} \in \ker(\mathcal{A} - \mu I)$ we have*

$$(y, y)_{H_{\frac{1}{2}}} \neq \mu^2 (y, y).$$

Then there exists a Riesz basis of $H_{\frac{1}{2}} \times H$ consisting of eigenvectors of $\mathcal{A}$.

3.3 Location of the Spectrum of Operator Matrices with Accretive Damping

In [H7] problems of the form (3.1) with an operator A_0 satisfying **(A1)** and a bounded accretive operator D are investigated. That is, instead of **(A2)** we assume the following:

(A2″) The damping operator $D : H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$ is a bounded operator such that $A_0^{-1/2} D A_0^{-1/2}$ is a bounded accretive operator in H, i.e.

$$\operatorname{Re} \langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \geq 0, \qquad z \in H_{\frac{1}{2}}.$$

Such kind of damping operator D arises in many problems in hydrodynamics, we mention here only [94, Chapters 6.4 and 6.5]. Again, as in Section 3.1, system (3.1) is equivalent to the standard first order equation $\dot{x}(t) = \mathcal{A}x(t)$, where $\mathcal{A}$ is given as in (3.4). It is well known, see e.g. [58], that the operator $\mathcal{A}$ is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ of contractions on the state space $H_{\frac{1}{2}} \times H$.

Papers dealing with accretive damping are [9, 58, 59, 63, 146]. In [146] an instability index formula is developed. Exponential stability of the semigroup generated by $\mathcal{A}$ is studied in [9, 59, 63], where in [9, 63] estimates for the growth bound of the semigroup are given under some additional assumptions on the operator D, see also [45, Theorem VI.3.18]. In [58] sufficient conditions for analyticity of the semigroup generated by $\mathcal{A}$ are derived. The first main result of [58] shows that the semigroup is analytic, if D, considered as an operator in H, is maximal sectorial satisfying some restrictions on the semiangle (cf. [58, Theorem 3]) and if the domains of the Friedrichs extension of D and its adjoint are subsets of $\operatorname{dom} A_0^{-1/2}$. The second main result of [58] proves analyticity of the semigroup if there exist constants $\rho_1, \rho_2 > 0$ with $\rho_1 A_0^\theta \leq \operatorname{Re} D \leq \rho_2 A_0^\theta$ for some $\theta \in [1/2, 1]$.

In [H7] two properties of the operator $\mathcal{A}$ are studied: Location of the spectrum and analyticity of the generated semigroup.

Similar as in [H11] we introduce various numbers,

$$\begin{aligned}
\beta_0 &:= \inf_{z \in H_{\frac{1}{2}} \setminus \{0\}} \frac{\operatorname{Re} \langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}{\|z\|_H^2}, \\
\gamma_0 &:= \sup_{z \in H_{\frac{1}{2}} \setminus \{0\}} \frac{\operatorname{Re} \langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}{\|z\|_H^2}, \\
\delta_0 &:= \inf_{z \in H_{\frac{1}{2}} \setminus \{0\}} \frac{\operatorname{Re} \langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}{\|z\|_{H_{\frac{1}{2}}}^2}, \\
\eta_0 &:= \sup_{z \in H_{\frac{1}{2}} \setminus \{0\}} \frac{\operatorname{Re} \langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}{\|z\|_{H_{\frac{1}{2}}}^2}, \\
\nu_0 &:= \sup_{z \in H_{\frac{1}{2}} \setminus \{0\}} \frac{\operatorname{Re} \langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}{\|z\|_H \|z\|_{H_{\frac{1}{2}}}}.
\end{aligned}$$

If the damping operator D satisfies not only **(A2″)** but the stronger condition **(A2)**, then β_0, γ_0 and δ_0 coincides with β, γ and δ, respectively, from Section 3.1.

By definition, we have $\beta_0, \delta_0, \eta_0 \in [0, \infty)$, and it is easy to see that $a_0^2 \delta_0 \leq \beta_0 \leq \gamma_0$ and

$$a_0^2 \delta_0 \leq a_0^2 \eta_0 \leq a_0 \nu_0 \leq \gamma_0,$$

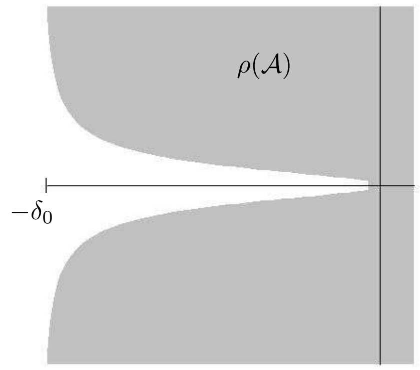

Figure 1: Theorem 3.11, Part 3, $\delta_0 > 0$ with $a_0 = 1$

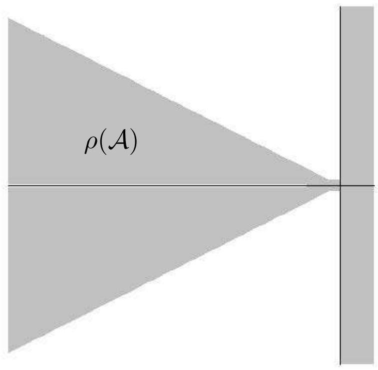

Figure 2: Theorem 3.11, Part 5, $0 < \nu_0 < 2$

where a_0 is as in (3.2). We obtain a description of the location of the spectrum, see also Figure 1 and Figure 2.

Theorem 3.11. **[H7]** *The following assertion are true.*

1. *If $\beta_0 > 0$ and if $\|D\|$ denotes the norm of the bounded operator D considered as a mapping from $H_{\frac{1}{2}}$ into $H_{-\frac{1}{2}}$, then*

$$\left\{ i\sigma \mid |\sigma| < \|D\|^{-1} \right\} \subset \rho(\mathcal{A}).$$

2. *If $\gamma_0 < \infty$, then*

$$\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leq -\gamma_0\} \subset r(\mathcal{A}).$$

In particular, if $\gamma_0 < 2a_0$, where a_0 is given by (3.2)*, then*

$$\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leq -\gamma_0\} \subset \rho(\mathcal{A}).$$

3. *If $\delta_0 > 0$, then*

$$\left\{ \lambda \in \mathbb{C} \mid \delta_0 > |\operatorname{Re}\lambda| \left(a_0^{-2} + |\lambda|^{-2} \right) \right\} \subset \rho(\mathcal{A}).$$

4. *If $0 < \eta_0 < \infty$, then*

$$\left\{ \lambda \in \mathbb{C} \mid \left| \lambda + \frac{1}{2\eta_0} \right| < \frac{1}{2\eta_0} \right\} \subset \rho(\mathcal{A}).$$

5. *If $0 < \nu_0 < 2$, then $\left(-\frac{a_0}{\nu_0} - \frac{4a_0}{\nu_0^3}, 0 \right) \subset \rho(\mathcal{A})$ and*

$$\left\{ \lambda \in \mathbb{C} \mid |\operatorname{Im}\lambda| < |\operatorname{Re}\lambda| \sqrt{\frac{4}{\nu_0^2} - 1},\ \operatorname{Im}\lambda \neq 0 \right\} \subset \rho(\mathcal{A}).$$

In [H7] further conditions guaranteeing analyticity of the semigroup generated by $\mathcal{A}$ are given. Contrary to [58], the operator D acts between $H_{1/2}$ and $H_{-1/2}$. This setup has the advantage that the operator $\mathcal{A}$ is closed. Under the additional weak assumption that there exist constants $M_0, M_1 > 0$ and $\omega_0 > 0$ such that

$$\|A_0^{1/2}(D + (\omega_0 + z)E)^{-1} A_0^{1/2} x\| \leq M_0 \|x\|, \quad x \in H, \quad \operatorname{Re} z > 0, \tag{3.9}$$

and

$$\|(\omega_0 + z)(D + (\omega_0 + z)E)^{-1}x\| \leq M_1\|x\|, \quad x \in H, \quad \operatorname{Re} z > 0, \tag{3.10}$$

where E denotes the identity on $H_{1/2}$ into $H_{-1/2}$, we obtain the following result.

Theorem 3.12. **[H7]** *If* (3.9) *and* (3.10) *are satisfied, then the operator $\mathcal{A}$ is the generator of an analytic C_0-semigroup.*

In [H7] it is shown that (3.9) and (3.10) are satisfied if

$$\delta_0 > 0.$$

Further, $\delta_0 > 0$ implies that $\omega_0 I - \mathcal{A}$ is maximal-ω-accretive[9] for some $\omega_0 > 0$, cf. [H7], which is a stronger property than analyticity of the semigroup generated by $\mathcal{A}$. Note that this result improves the second main result of [58] for $\theta = 1$ in a slightly different setup. For the location of the spectrum the following result is shown in [H7], cf. also Figure 3 and Figure 4.

Theorem 3.13. **[H7]**

1. *If $\beta_0 > 0$ and D is m-$(\arctan k)$-accretive for some $k > 0$, then*

$$M_{\beta_0} := \left\{ \mu + i\sigma \in \mathbb{C} \mid -\beta_0 < 2\mu < 0, |\sigma| > \frac{k\beta_0|\mu|}{\beta_0 - 2|\mu|} \right\} \subset \rho(\mathcal{A}).$$

2. *If $\delta_0 > 0$, then D is m-$(\arctan k)$-accretive for some $k > 0$ and*

$$M_{\beta_0} \cup \left\{ \mu + i\sigma \in \mathbb{C} \mid |\sigma| > \frac{1}{\delta_0} + k|\mu| \right\} \subset \rho(\mathcal{A}).$$

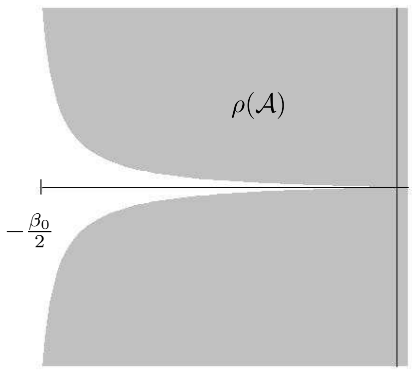

Figure 3: Theorem 3.13, Part 1, $\beta_0 > 0$

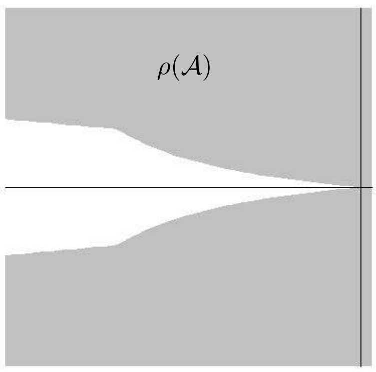

Figure 4: Theorem 3.13, Part 2, $\delta_0 > 0$

The above results are applied in [H7] to equations which model small transverse oscillations of a horizontal pipe of length 1, carrying steady-state fluid of ideal

[9]Let $\omega \in [0, \pi/2)$. A densely defined operator S in some Hilbert space is called maximal-ω-accretive or m-ω-accretive, cf., e.g., [53, page 167], if the range of $S + I$ is dense and if for all x in the domain of S we have

$$|\operatorname{Im}(Sx, x)| \leq (\tan\omega)\operatorname{Re}(Sx, x).$$

incompressible fluid. These oscillations are described by the equation (cf. e.g., [146])

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2}{\partial r^2}\left[E_0\frac{\partial^2 u}{\partial r^2}+C\frac{\partial^3 u}{\partial r^2\partial t}\right]+K\frac{\partial^2 u}{\partial t\partial r}=0, \qquad r\in(0,1), t>0. \tag{3.11}$$

Here $u(r,t)$ denote the transverse oscillations at time t and position r, and E_0, C, K are positive physical constants. The last term $K\frac{\partial^2 u}{\partial t \partial r}$ in the left hand side of (3.11) is called the gyroscopic term.

3.4 Second Order Systems. Minimum-Phase

In [H10] we study second-order systems of the form

$$\ddot{z}(t)+A_0z(t)+D\dot{z}(t)=B_0u(t), \tag{3.12}$$

equipped either with position measurements

$$y(t)=B_0^*z(t) \tag{3.13}$$

or velocity measurements

$$y(t)=B_0^*\dot{z}(t). \tag{3.14}$$

Here A_0 satisfies **(A1)**. The control operator B_0 is a bounded operator acting from $\mathbb{C}^m$ to $\mathcal{H}_{-\frac{1}{2}}$. The damping operator D satisfies **(A2)**.

These systems have been studied in the literature for more than two decades, e.g. [18, 56, 57, 119, 120, 148]. In [149, 150] these systems have been studied with the damping $D=\frac{1}{2}B_0^*B_0$ and the output $y(t)=-B_0^*\dot{z}(t)+u(t)$.

In this section we assume, in addition to **(A1)** and **(A2)**, that the following assumption is fulfilled.

(A3) For the bounded operator B_0 from $\mathbb{C}^m$ to $\mathcal{H}_{-\frac{1}{2}}$ there exists a constant $\widetilde{\beta}>0$ with

$$(Dz,z)_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}\geq \widetilde{\beta}\|B_0^*z\|^2 \quad \text{for all } z\in H_{\frac{1}{2}}.$$

Similar, as in Section 3.1, the position control system (3.12), (3.13) is equivalent to the following standard first order equation

$$\dot{x}(t) = \mathcal{A}x(t)+Bu(t) \tag{3.15}$$

$$y(t) = C_px(t), \tag{3.16}$$

where $\mathcal{A}$ is given as in (3.4) and $B:\mathbb{C}^m\to H_{\frac{1}{2}}\times H_{-\frac{1}{2}}$ and $C_p: H_{\frac{1}{2}}\times H\to\mathbb{C}^m$ are given by

$$B=\begin{bmatrix}0\\ B_0\end{bmatrix}, \quad C_p=\begin{bmatrix}B_0^* & 0\end{bmatrix}.$$

Then $\mathcal{A}$ generates a C_0-semigroup of contractions in $H_{\frac{1}{2}}\times H$, cf. Section 3.1, which we will denote by $(e^{t\mathcal{A}})_{t\geq 0}$. The velocity control system (3.12), (3.14) has the equivalent first order form (3.15) and

$$y(t)=C_vx(t), \quad t\geq 0, \tag{3.17}$$

where $C_v: H_{\frac{1}{2}}\times H_{\frac{1}{2}}\to\mathbb{C}^m$ is given by $C_v=\begin{bmatrix}0 & B_0^*\end{bmatrix}$.

In the following, we denote by $\mathbb{C}_0$ the open right-half-plane and by $\mathcal{H}^2(\mathbb{C}_0;X)$, $\mathcal{H}^\infty(\mathbb{C}_0;X)$, for some Hilbert space X, the usual Hardy spaces of X-valued functions on $\mathbb{C}_0$.

Theorem 3.14. [H10] *For $t \geq 0$ define the mappings*

$$\begin{aligned} \mathcal{B}_t : L^2(0,\infty;\mathbb{C}^m) &\rightarrow H_{\frac{1}{2}} \times H, \\ \mathcal{C}_t^p : H_{\frac{1}{2}} \times H &\rightarrow L^2(0,t;\mathbb{C}^m), \\ \mathcal{C}_t^v : H_{\frac{1}{2}} \times H_{\frac{1}{2}} &\rightarrow L^2(0,t;\mathbb{C}^m) \end{aligned}$$

via

$$\begin{aligned} \mathcal{B}_t u &= \int_0^t e^{(t-\tau)\mathcal{A}} Bu(\tau) d\tau, \\ (\mathcal{C}_t^p x_0)(s) &= C_p e^{s\mathcal{A}} x_0, \qquad \text{for } 0 \leq s \leq t \text{ and } x_0 \in H_{\frac{1}{2}} \times H, \\ (\mathcal{C}_t^v x_0)(s) &= C_v e^{s\mathcal{A}} x_0, \qquad \text{for } 0 \leq s \leq t \text{ and } x_0 \in H_{\frac{1}{2}} \times H_{\frac{1}{2}}. \end{aligned}$$

Then the mappings $\mathcal{B}_t$ from input to state and $\mathcal{C}_t^p$ from initial value to output are bounded linear operators and the mappings $\mathcal{C}_t^v$ admit extensions to bounded linear operators mapping from the state space $H_{\frac{1}{2}} \times H$ to $L^2(0,t;\mathbb{C}^m)$. For an input $u \in L^2_{\text{loc}}(0,\infty;\mathbb{C}^m)$ and zero initial condition the mappings

$$\begin{aligned} \mathcal{G}_t^p : L^2(0,t;\mathbb{C}^m) &\rightarrow L^2(0,t;\mathbb{C}^m), \\ \mathcal{G}_t^v : L^2(0,t;\mathbb{C}^m) &\rightarrow L^2(0,t;\mathbb{C}^m) \end{aligned}$$

are given by

$$\begin{aligned} (\mathcal{G}_t^p u)(s) &= C_p \mathcal{B}_s u, \qquad 0 \leq s \leq t, \\ (\mathcal{G}_t^v u)(s) &= C_v \mathcal{B}_s u, \qquad 0 \leq s \leq t. \end{aligned}$$

Then the mappings $\mathcal{G}_t^p$ and $\mathcal{G}_t^v$ from input to output are bounded linear operators.

The velocity measurement system (3.15), (3.17) *is well-posed and its transfer function, given by*

$$G_v(s) = sB_0^*(s^2 I + sD + A_0)^{-1} B_0,$$

satisfies $G_v \in \mathcal{H}^\infty(\mathbb{C}_0, \mathcal{L}(\mathbb{C}^m))$. The position measurement system (3.15), (3.16) *is well-posed and its transfer function, given by*

$$G_p(s) = B_0^*(s^2 I + sD + A_0)^{-1} B_0,$$

satisfies $G_p \in \mathcal{H}^\infty(\mathbb{C}_0, \mathcal{L}(\mathbb{C}^m))$.

Recall that a linear time-invariant system is called *well-posed* if and only if on any finite time interval the four maps from input and initial condition to state and output defined by $\mathcal{B}_t, e^{t\mathcal{A}}$ and $\mathcal{C}_t^p, \mathcal{G}_t^p$ ($\mathcal{C}_t^v, \mathcal{G}_t^v$, respectively) are bounded for $t > 0$. For details we refer to [149, 150] and to [155, 156]

By applying the Laplace transform to (3.15) and (3.16) (or, respectively, to (3.15) and (3.17)) one easily verifies that the transfer function maps the Laplace transform of the input on the Laplace transform of the output. Therefore, the transfer function describes the relation between input and output of the system in the frequency domain. In the case that the transfer function is a rational scalar function a system is called *minimum-phase* if its transfer function has no zeros in the right-half-plane. The minimum-phase property is useful in controller design, [64, 130, 144, e.g.].

A more general definition of minimum-phase functions exists for infinite dimensional systems.

Definition 3.15. [143, page 94] *A bounded, holomorphic function $g : \mathbb{C}_0 \rightarrow \mathbb{C}^{m\times m}$ is called* minimum-phase *if the set*

$$\{gf : f \in \mathcal{H}^2(\mathbb{C}_0;\mathbb{C}^m)\}$$

is dense in $\mathcal{H}^2(\mathbb{C}_0;\mathbb{C}^m)$.

Thus, minimum-phase functions correspond to operators given as multiplication operators in $\mathcal{H}^2(\mathbb{C}_0;\mathbb{C}^m)$ that have inverses defined on a dense subset of $\mathcal{H}^2(\mathbb{C}_0;\mathbb{C}^m)$, hence, for every $y \in \mathcal{H}^2(\mathbb{C}_0;\mathbb{C}^m)$ there is a sequence $\{u_n\}$ in $\mathcal{H}^2(\mathbb{C}_0;\mathbb{C}^m)$ with $gu_n \to y$ in $\mathcal{H}^2(\mathbb{C}_0;\mathbb{C}^m)$. This implies that a scalar minimum-phase function has no zeros in the open right-half-plane. It can be shown that a scalar rational function in $\mathcal{H}^\infty$ is minimum-phase if and only if the function has no zeros in the open right-half-plane.

To show that a transfer function of a infinite dimensional system is minimum-phase is less straightforward than for finite-dimensional systems. For example, the transfer function of a pure delay, $\exp(-s)$, has no zeros, but is not minimum-phase. Minimum-phase transfer function of infinite dimensional systems appear in, e.g., [122, 123, 124, 125, 132].

In [H10] we obtain conditions which are sufficient for the minimum-phase property of the systems under consideration. In particular, these conditions are easy to verify in applications, cf. [H10].

Theorem 3.16. [H10] *Assume that assumptions* **(A1)-(A3)** *are satisfied. If in addition one of the following conditions* (i) *or* (ii) *is satisfied,*

(i) *the resolvent of A contains the imaginary axis and the operator B_0 is injective or*

(ii) *A_0 has a compact resolvent, $(Dz, z)_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} > 0$ for any eigenvector z of A_0, and B_0 is injective,*

then the transfer functions G_p and G_v are minimum-phase functions.

In [H10] these results are applied to some well-known models with position measurements: Euler-Bernoulli beam with Kelvin-Voigt damping, and a damped plate on a bounded connected domain. It is shown in [H10] that both systems, the velocity measurement system (3.15), (3.17) and the position measurement system (3.15), (3.16), are well-posed and have minimum-phase transfer functions.

4 Indefinite Sturm-Liouville Problems

In this section we will present results for (singular) indefinite Sturm-Liouville operators which are contained in the following papers.

[H12] J. Behrndt, Q. Katatbeh and C. Trunk, *Non-real eigenvalues of singular indefinite Sturm-Liouville operators*, Proc. Amer. Math. Soc. **137** (2009), 3797-3806.

[H13] I. Karabash and C. Trunk, *Spectral properties of singular Sturm-Liouville operators*, Proc. R. Soc. Edinb. Sect. A **139** (2009), 483-503.

[H14] J. Behrndt, Q. Katatbeh and C. Trunk, *Accumulation of complex eigenvalues of indefinite Sturm-Liouville operators*, J. Phys. A: Math. Theor. **41** (2008), 244003.

[H15] J. Behrndt and C. Trunk, *On the negative squares of indefinite Sturm-Liouville operators*, J. Differential Equations **238** (2007), 491-519.

[H16] J. Behrndt and C. Trunk, *Sturm-Liouville operators with indefinite weight functions and eigenvalue depending boundary conditions*, J. Differential Equations **222** (2006), 297-324.

The main contribution of [H12, H13, H14] is the treatment of indefinite Sturm-Liouville problems of the form

$$Ay(x) := \operatorname{sgn}(x)\left(-y''(x) + q(x)y(x)\right), \quad x \in \mathbb{R},$$

where $q \in L^1_{loc}(\mathbb{R})$ is a real potential such that both $-\infty$ and $+\infty$ are in limit point case. It is easily seen that A, defined on the maximal domain, is a selfadjoint operator in the Krein space $L^2(\mathbb{R})$, where the indefinite inner product is given via the fundamental symmetry $(Jf)(x) = \operatorname{sgn}(x)f(x)$. The operator A has a nonempty resolvent set [46, 89]. The main result of [H13] is the characterization of definitizability properties of A. In particular it is shown that the operator A is a definitizable operator over a neighbourhood of ∞ if and only if the operator JA is semi-bounded from below in the Hilbert space $L^2(\mathbb{R})$. Moreover, in this case, ∞ is a regular critical point of the operator A. Via the Gelfand-Levitan theorem it is shown, that accumulation of non-real eigenvalues may occur. And in [H14] an example with such an accumulation is given.

If the operator JA is not only semi-bounded from below but its negative spectrum also consists only of κ eigenvalues, it is well-known that A is an operator with κ negative squares. In [H12] an easy criterion for the potential q is established which ensures that A has precisely κ eigenvalues in the open upper half-plane. Moreover, in [H12], an example is presented which has a Jordan chain of length greater than one in the open upper half-plane.

In [H15, H16] the number of negative squares of all selfadjoint extensions of a simple symmetric operator of defect one with finitely many negative squares in a Krein space is characterized in terms of the behaviour of an abstract Weyl function near 0 and ∞. This is done via a characterization of the negative reciprocal $-\frac{1}{\tau}$ of a function τ from D_κ, where the class D_κ is introduced and studied in [H16]. These classes are the main tool in the investigation of Weyl functions associated to symmetric operators of defect one with finitely many negative squares. The obtained results are applied to indefinite Sturm-Liouville operators.

4.1 Spectral Properties of Singular Indefinite Sturm-Liouville Operators

In [H12]-[H16] Sturm-Liouville expressions of the form

$$\ell(y)(x) := \frac{1}{r}\left(-(py')'(x) + q(x)y(x)\right), \tag{4.1}$$

on an interval $(a,b) \subset \mathbb{R}$, $r \neq 0$ a.e., $p > 0$, with locally integrable, real valued functions $r, \frac{1}{p}$ and q are considered. As usual, operators acting in weighted L^2-spaces are associated to the differential expression ℓ. By $L^2_{|r|}(a,b)$ we denote the Hilbert space of all functions f defined on (a,b) with $|f|^2|r| \in L^1(a,b)$, equipped with the usual Hilbert space scalar product

$$(f,g) := \int_a^b f(x)\overline{g(x)}\,|r(x)|\,dx, \qquad f,g \in L^2_{|r|}(a,b).$$

By $[\cdot,\cdot]$ we denote the Hermitian form

$$[f,g] := \int_a^b f(x)\overline{g(x)}\,r(x)\,dx, \qquad f,g \in L^2_{|r|}(a,b). \tag{4.2}$$

If the function r in (4.1) is positive in (a,b), then $L^2_{|r|}(a,b)$ equipped with the inner product (4.2) forms a Hilbert space. The usual selfadjoint operators associated to ℓ are obtained as restrictions (via boundary conditions) of the maximal operator $A_{\max}$ given by

$$\mathcal{D}_{\max} = \left\{f \in L^2_{|r|}(a,b) : f, pf' \in AC(a,b),\, \ell(f) \in L^2_{|r|}(a,b)\right\}, \tag{4.3}$$

$$A_{\max}f = \ell(f) = \frac{1}{r}\big(-(pf')' + qf\big). \tag{4.4}$$

Here we denote by $AC(a,b)$ the space of all locally absolute continuous functions in (a,b). The spectral theory of these operators is a classical topic, e.g. in quantum mechanics, started by the seminal papers of H. Weyl [159, 160, 161] and is covered by many monographs. We mention here only [44, 121, 139, 140, 141, 157, 158, 162].

If r is no longer assumed to be positive, then $\big(L^2_{|r|}(a,b), [\cdot,\cdot]\big)$ is no longer a Hilbert space but a Krein space with fundamental symmetry

$$(Jf)(x) = \operatorname{sgn}(r(x))f(x), \qquad f \in L^2_{|r|}(a,b). \tag{4.5}$$

Operators associated with ℓ are called *indefinite Sturm-Liouville Operators*. The minimal operator[10]

$$\begin{aligned} A_{\min}f &= \ell(f) = \frac{1}{r}\big(-(pf')' + qf\big),\\ \operatorname{dom} A_{\min} &= \left\{f \in \mathcal{D}_{\max} : f \text{ has compact support in } (a,b)\right\} \end{aligned} \tag{4.6}$$

is symmetric with respect to the indefinite inner product $[\cdot,\cdot]$ and the maximal operator $A_{\max}$ in (4.4) coincides with the adjoint with respect to $[\cdot,\cdot]$,

$$A_{\min}^{+} = A_{\max}.$$

[10]Usually, the closure of the operator given in (4.6) is called minimal. For our purposes it is convenient to call the operator given by (4.6) minimal in order to treat the case of regular and singular Sturm-Liouville expressions simultaneously.

All selfadjoint extensions of $A_{\min}$ in the Krein space $\left(L^2_{|r|}(a,b),[\cdot,\cdot]\right)$ are obtained via the multiplication of the fundamental symmetry J from (4.5) with all selfadjoint extensions of the definite Sturm-Liouville expression

$$\frac{1}{|r|}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}}{\mathrm{d}x}\right)+q\right) \tag{4.7}$$

in the Hilbert space $L^2_{|r|}(a,b)$.

The study of indefinite Sturm-Liouville problems is a classical topic dating back to the first observation made by Richardson in 1918 [142], where he showed that indefinite Sturm-Liouville problems may have non-real eigenvalues. In recent years, major contributions in [28] and in [35] to the spectral theory of selfadjoint indefinite Sturm-Liouville operators associated to ℓ were made after the basic operator theory in Krein spaces and, in particular, the theory of definitizable operators was fully developed.

In [28] it is shown that all selfadjoint extensions in the regular case (i.e. a and b are finite and $r, q, \frac{1}{p}$ are integrable in (a,b)) are definitizable operators with finitely many negative squares[11] and have a compact resolvent. A selfadjoint operator A with κ negative squares and nonempty resolvent set has a very specific spectral structure. It is a definitizable operator (cf. [111]) and the non-real spectrum of A consists of at most κ pairs $\{\mu_i,\overline{\mu}_i\}$ $i=1,\ldots,\kappa$, of eigenvalues with finite dimensional algebraic eigenspaces. If the signature of the inner product $[\cdot,\cdot]$ on the algebraic eigenspace $\mathcal{L}_\lambda(A)$ corresponding to λ is denoted by $\{\kappa_-(\lambda),\kappa_0(\lambda),\kappa_+(\lambda)\}$, then

$$\sum_{\lambda\in\sigma_p(A)\cap(-\infty,0)}(\kappa_+(\lambda)+\kappa_0(\lambda))+\sum_{\lambda\in\sigma_p(A)\cap(0,\infty)}(\kappa_-(\lambda)+\kappa_0(\lambda))+\sum_i\kappa_0(\mu_i)\leq\kappa, \tag{4.8}$$

where we have equality in the case $0\notin\sigma_p(A)$. In particular, using the notions of Section 2.1, we have

$$(0,\infty]\subset\sigma_{\pi_+}(A),\quad [-\infty,0)\subset\sigma_{\pi_-}(A). \tag{4.9}$$

By definition, all cases where the differential expression is not regular are called *singular*. In the singular case and under some additional assumptions (see [28, Propositions 2.3 and 2.4]) on the weight r and on the corresponding definite differential expression from (4.7) it is shown in [28] that all selfadjoint operators A associated to ℓ are definitizable and, as in the regular case, have finitely many negative squares, hence (4.8) and (4.9) hold. In particular in [35] the special case of a nonnegative potential q on the semi-axis $(0,\infty)$ is considered and it is shown that all selfadjoint operators A associated to ℓ have at most one negative square, hence (4.8) and (4.9) hold for $\kappa=0$ or $\kappa=1$, respectively.

Later, various results in the spectral theory of indefinite Sturm-Liouville operators were obtained. E.g., conditions for Half-range completeness and Riesz basis property of the eigenfunctions are obtained in [13, 50, 79, 134, 137, 154], eigencurve techniques are developed in, e.g., [19, 21], and [82] provides a functional model for singular indefinite Sturm-Liouville operators. Descriptions of the number of non-real eigenvalues, Sturm oscillation theorems and asymptotic estimates for real eigenvalues can be found in, e.g., [1, 3, 93, 127, 128, 129], for an overview, in particular for the regular case, we refer to the monograph [162]. Moreover, Curgus and Najman proved in [29] the similarity of the singular indefinite operator $\operatorname{sgn}(x)\frac{d^2}{dx^2}$, defined on $\mathbb{R}$, to a selfadjoint operator in a Hilbert space. In the very recent years,

[11] A closed symmetric operator S in a Krein space $(\mathcal{K},[\cdot,\cdot])$ is said to have κ *negative squares* for some $\kappa\in\mathbb{N}$ if the Hermitian form $[S\,\cdot,\cdot]$, defined on the domain of the symmetric operator S, has κ negative squares. That is, there exists a subspace of dimension κ such that $[S\,\cdot,\cdot]$ restricted to this subspace is negative definite and κ is the largest number with this property, see, e.g., [111].

and stimulated by [29], the existence of singularities, often with the special weight $r(x) = \text{sgn}(x)$ defined on $\mathbb{R}$, is studied, which describes the similarity to a selfadjoint or normal operator in a Hilbert space, see [48, 47, 81, 84, 85, 86, 87, 88, 89, 95, 96, 97]. Differential operators with indefinite weights appear in many areas of physics and applied mathematics, e.g. in the electron scattering theory [11, 12], in Forward-Backward problems in transport and scattering theory [13, 49, 52, 80], in the study of the Fokker-Planck equation [14] and in the investigation of abstract kinetic equations in [83, 138].

In the above mentioned papers, the selfadjoint operators associated to ℓ are definitizable. In general, if the differential expression is singular, this is only true under additional assumptions. In [H14] we discussed a case of a singular indefinite Sturm-Liouville operator with a so-called cut off Coulomb potential (for a study of cut off Coulomb potentials we refer e.g. to [55, 147]) and the simple weight $r(x) = \text{sgn}(x)$ on $\mathbb{R}$ which is not a definitizable operator but a locally definitizable operator being definitizable over $\overline{\mathbb{C}} \setminus \{0\}$ (for the notion of locally definitizable operators we refer to Section 2.1).

Theorem 4.1. [H14] *The operator A defined on $\mathcal{D}_{\max}$, see (4.3), given by*

$$Ay(x) := \text{sgn}(x)\left(-y''(x) - \frac{1}{1+|x|}y(x)\right), \qquad x \in \mathbb{R}, \tag{4.10}$$

is selfadjoint in the Krein space $\left(L^2_{|\text{sgn}|}(\mathbb{R}), [\cdot,\cdot]\right)$. The essential spectrum of A covers the real line,

$$\sigma_{ess}(A) = \mathbb{R}$$

and the operator A is definitizable over $\overline{\mathbb{C}} \setminus \{0\}$. Moreover, A is not definitizable over every domain in $\mathbb{C}$ which contains the point zero.

The operator considered in Theorem 4.1 arises from completely solvable models in quantum mechanics. The corresponding definite Sturm-Liouville operator JA,

$$(JAy)(x) = -y''(x) - \frac{1}{1+|x|}y(x), \qquad x \in \mathbb{R},$$

with the fundamental symmetry J,

$$Jy(x) = \text{sgn}(x)y(x), \quad \text{for } y \in L^2(\mathbb{R}). \tag{4.11}$$

is selfadjoint in the Hilbert space $L^2(\mathbb{R})$, semi-bounded from below and one verifies numerically that -0.429911 is a lower bound of the spectrum of JA. The negative eigenvalues of JA accumulate to zero (red points in Figure 5) and the half axis $[0,\infty)$ is the essential spectrum of JA.

Similarly, one can compute eigenvalues and eigenfunctions of the singular indefinite Sturm-Liouville operator A from (4.10) by using standard software packages, e.g. Mathematica (WolframResearch). The numerical calculations in [H14] suggest that the non-real eigenvalues of A accumulate to zero (blue points in Figure 5).

In [H13] we obtain conditions which guarantee that selfadjoint operators associated to singular indefinite Sturm-Liouville expressions are still locally definitizable over an appropriate subset of $\mathbb{C}$. This is done for a special choice of the weight, i.e. $r(x) = \text{sgn}(x)$. Then the operator A associated to ℓ defined on $\mathcal{D}_{\max}$ (see (4.3)) is given by

$$Ay(x) := \text{sgn}(x)\left(-y''(x) + q(x)y(x)\right), \quad x \in \mathbb{R}, \tag{4.12}$$

where $q \in L^1_{loc}(\mathbb{R})$ is a real potential. Throughout [H13] it is assumed that we have limit point case at both $-\infty$ and $+\infty$, that is, the homogeneous equation

$$-f'' + qf = \lambda f, \qquad \lambda \in \mathbb{C}\backslash\mathbb{R},$$

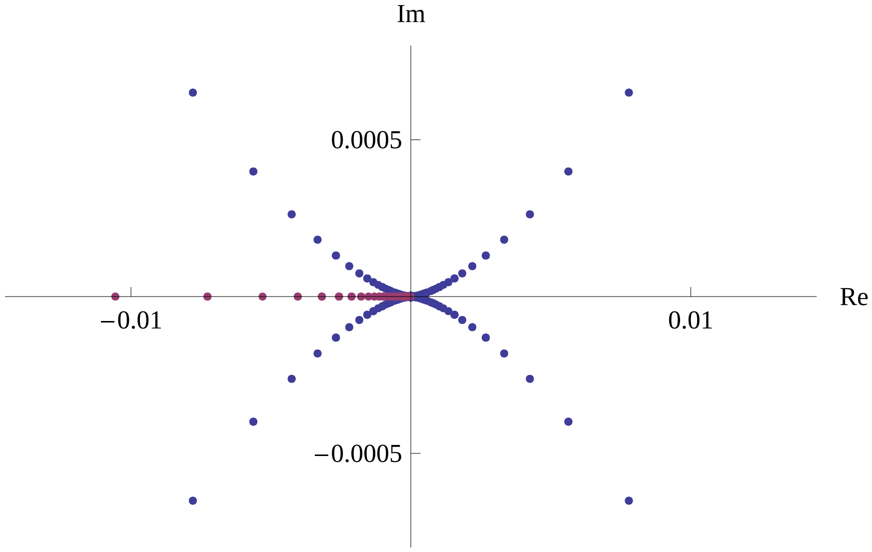

Figure 5: Nonreal eigenvalues of $(Ay)(x) := \operatorname{sgn}(x)\big(-y''(x) - \frac{y(x)}{1+|x|}\big)$ (blue dots) and negative eigenvalues of JA, where J is as in (4.11) (red dots).

has a unique solution φ_λ (up to scalar multiples) in $L^2(0,\infty)$ and a unique solution ψ_λ in $L^2(-\infty,0)$, respectively (cf. [158, 162]). Hence JA is a selfadjoint operator in the Hilbert space $L^2(\mathbb{R})$ and A is a selfadjoint operator in the Krein space $\big(L^2_{|\operatorname{sgn}|}(\mathbb{R}), [\cdot,\cdot]\big)$, where the inner product is given by

$$[f,g] = \int_{\mathbb{R}} f(x)\overline{g(x)}\operatorname{sgn}(x)\,dx = (Jf,g)_{L^2(\mathbb{R})}, \qquad f,g \in L^2(\mathbb{R}).$$

In the following, we identify functions $f \in L^2(\mathbb{R})$ with elements $\binom{f_+}{f_-}$, where $f_\pm := f\restriction_{\mathbb{R}_\pm} \in L^2(\mathbb{R}_\pm)$. Similarly we write $q_\pm := q\restriction_{\mathbb{R}_\pm} \in L^1_{loc}(\mathbb{R}_\pm)$. Note that the differential expressions

$$-\frac{d^2}{dx^2} + q_+ \qquad \text{and} \qquad \frac{d^2}{dx^2} - q_- \tag{4.13}$$

in $L^2(\mathbb{R}_+)$ and in $L^2(\mathbb{R}_-)$, respectively, are both regular at the endpoint 0 and in the limit point case at the singular endpoint $+\infty$ and $-\infty$, respectively. Therefore the operators

$$A_0^+ f_+ = -f_+'' + q_+ f_+ \quad \text{and} \quad A_0^- f_- = f_-'' - q_- f_-$$

defined on

$$\operatorname{dom} A_0^\pm = \big\{f_\pm \in \mathcal{D}^\pm_{\max} : f_\pm'(0) = 0\big\},$$

with

$$\mathcal{D}^+_{\max} = \big\{f_+ \in L^2(\mathbb{R}_+) : f_+, f_+' \text{ absolutely continuous, } -f_+'' + q_+ f_+ \in L^2(\mathbb{R}_+)\big\},$$
$$\mathcal{D}^-_{\max} = \big\{f_- \in L^2(\mathbb{R}_-) : f_-, f_-' \text{ absolutely continuous, } f_-'' - q_- f_- \in L^2(\mathbb{R}_-)\big\},$$

are selfadjoint operators in the Hilbert spaces $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$, respectively, cf. [157, 158]. In the following theorem we collect some important spectral properties of A, see [89, Proposition 2.5] and [82].

Theorem 4.2. *The operator A from* (4.12) *is a selfadjoint operator in the Krein space* $\big(L^2_{|\mathrm{sgn}|}(\mathbb{R}), [\cdot,\cdot]\big)$. *The resolvent set of A is nonempty,*

$$\rho(A) \neq \emptyset$$

and the essential spectrum $\sigma_{ess}(A)$ of A is real and

$$\sigma_{ess}(A) = \sigma_{ess}(A_0^+) \cup \sigma_{ess}(A_0^-).$$

We mention that $\rho(A) \neq \emptyset$ follows from the asymptotic behavior of the Titchmarsh-Weyl-coefficients corresponding to the differential expressions in (4.13), see [46]. This and the fact that the difference of the resolvents of A and $A_0^+ \oplus A_0^-$ is one dimensional, imply the statement on the essential spectrum of the operator A.

The operator $A_0^+ \oplus A_0^-$ is selfadjoint in the Hilbert space $L^2(\mathbb{R})$ and in the Krein space $\big(L^2_{|\mathrm{sgn}|}(\mathbb{R}), [\cdot,\cdot]\big)$. Such operators are called fundamental reducible, cf. Section 2.2. It is natural to deduce the spectral properties of A from the operator $A_0^+ \oplus A_0^-$. Obviously, we have

$$\sigma_{++}(A_0^+ \oplus A_0^-) = \sigma(A_0^+) \setminus \sigma(A_0^-) \quad \text{and} \quad \sigma_{--}(A_0^+ \oplus A_0^-) = \sigma(A_0^-) \setminus \sigma(A_0^+).$$

Moreover, a real point λ belongs to $\sigma_{\pi_+}(A_0^+ \oplus A_0^-)$ if and only it belongs to $\sigma(A_0^+) \setminus \sigma_{ess}(A_0^-)$ and it belongs to $\sigma_{\pi_-}(A_0^+ \oplus A_0^-)$ if and only it belongs to $\sigma(A_0^-) \setminus \sigma_{ess}(A_0^+)$. By Theorem 2.6 we obtain

$$\sigma_{\pi_+}(A) = \sigma_{\pi_+}(A_0^+ \oplus A_0^-) \quad \text{and} \quad \sigma_{\pi_-}(A) = \sigma_{\pi_-}(A_0^+ \oplus A_0^-).$$

Using [74] and [16] (in the case of locally definitizable operators) and arguments from [28, Lemma 3.5 (iii)] and [28, Theorem 3.6 (i)] the following theorem is shown in [H13].

Theorem 4.3. [H13] *The operator A is definitizable if and only if the operator $A_0^+ \oplus A_0^-$ is definitizable.*

The operator A is a definitizable operator over a neighbourhood of ∞ if and only if the operator JA is semi-bounded from below in the Hilbert space $L^2(\mathbb{R})$. Moreover, in this case, ∞ is a regular critical point[12] *of the operator A.*

By a standard application of the spectral function for locally definitizable operators (see Section 2.1), the fact that ∞ is a regular critical point of the operator A, cf. Theorem 4.3, implies the existence of a decomposition of the underlying space and, accordingly, a decomposition of the operator A, $A = \mathcal{A}_\infty \dot{+} \mathcal{A}_b$ such that the operator $\mathcal{A}_\infty$ is similar to a selfadjoint operator in the Hilbert space sense and $\mathcal{A}_b$ is a bounded operator.

We mention that similar results as in the Theorems 4.2 and 4.3 are obtained in [15] by a different technique for the operator associated to $y \mapsto \frac{1}{w}[-(py')' + qy]$ with w as indefinite weight function. Note that in [15] w may have many turning points, but additional assumptions on the spectra of certain associated selfadjoint operators are needed.

Via the Gelfand-Levitan theorem (see e.g. [131, Subsection 26.5]) the following existence result is obtained in [H13]. It shows, that accumulation of non-real eigenvalues (as in Figure 5) may occur.

[12] For a selfadjoint operator A definitizable over Ω a point $t \in \overline{\mathbb{R}} \cap \Omega$ is called a *critical point* of A if there is no open subset $\Delta \subset \Omega$, $t \in \Delta$, such that $\Delta \subset \sigma_{++}(A) \cup \sigma_{--}(A)$ holds. A critical point t is called *regular* if there exists an open deleted neighbourhood $\delta_0 \subset \Omega$ of t such that the set of the projections $E(\delta)$ (see Section 2.1) where δ runs through all intervals δ with $\overline{\delta} \subset \delta_0$ is bounded.

Theorem 4.4. **[H13]** *There exist an even potential $\widehat{q}$ continuous on $\mathbb{R}$ and a sequence (ϵ_n) of positive numbers converging to zero such that the operator A defined by* (4.12) *with q replaced by $\widehat{q}$ is definitizable over $\overline{\mathbb{C}} \setminus \{0\}$ and*

$$i\epsilon_n \in \sigma_p(A) \quad \textit{for } n \in \mathbb{N},$$

i.e., $i\epsilon_n$, $n \in \mathbb{N}$, are non-real eigenvalues of A.

In [H12] a special case of an operator defined as in (4.12) is considered. Additional to the general assumptions made in [H13] (that is, $q \in L^1_{loc}(\mathbb{R})$ is a real potential and we have limit point case at both $-\infty$ and $+\infty$) we assume the following.

(I) The set $\sigma(JA) \cap (-\infty, 0)$ consists of $\kappa < \infty$ eigenvalues.

Hence, the selfadjoint operator JA in the Hilbert space $L^2(\mathbb{R})$ is semi-bounded from below and the eigenvalues do not accumulate to zero from the negative half axis. A sufficient condition for **(I)** to hold is, e.g., $\int_{\mathbb{R}}(1 + x^2)|q(x)|dx < \infty$ for a continuous function q, cf. [121].

Then the Hermitian form $[A\cdot, \cdot]$ has exactly κ negative squares and it follows from the considerations in [28] and [111] (see also Theorem 4.3 and Relation (4.8) above) that the spectrum $\sigma(A)$ in the open upper half-plane consists of at most κ eigenvalues (counting multiplicities).

Under some additional assumptions on q the absence of eigenvalues on the real axis is shown in [H12]. This is done with the help of some results from I. Knowles (cf. [91, 92]). We mention that in [87, Section 4] a similar result is proved if q satisfies $\int_{\mathbb{R}}(1 + |x|)|q(x)|dx < \infty$. By $\sigma_c(A)$ we denote the continuous part of the spectrum of A, i.e. the set of all $\lambda \in \sigma(A) \setminus \sigma_p(A)$ such that the range of $A - \lambda$ is dense.

Theorem 4.5. **[H12]** *Assume that condition* **(I)** *holds and that there exist real functions v and r with $q = v + r$ such that $\lim_{|x|\to\infty} r(x) = \lim_{|x|\to\infty} v(x) = 0$, r is locally of bounded variation and*

$$\lim_{t\to\infty} \frac{1}{\log t} \int_{-t}^{t} |v(x)|dx = \lim_{t\to\infty} \frac{1}{\log t} \int_{-t}^{t} |dr(x)| = 0.$$

Then $\sigma_c(A)\backslash\{0\} = \mathbb{R}\backslash\{0\}$ and hence zero is the only possible real eigenvalue of the indefinite Sturm-Liouville operator A. If, in addition, there exists $x_0 > 0$ with

$$-\frac{1}{4x^2} \le q(x) \le \frac{3}{4x^2} \qquad \textit{for all} \quad x \in \mathbb{R} \setminus (-x_0, x_0). \tag{4.14}$$

Then $\sigma_c(A) = \mathbb{R}$ and

$$\sum_{\lambda\in\sigma_p(A),\, \mathrm{Im}\,\lambda>0} \dim \mathcal{L}_\lambda(A) = \kappa. \tag{4.15}$$

Theorem 4.5 can be viewed as a partial answer of the open problem X. in [162, pg. 300].

We mention that (even under the condition (4.14)) for $\lambda \in \sigma_p(A)$ $\dim \mathcal{L}_\lambda(A) > 1$ may happen, i.e. there exists a Jordan chain of length greater than one and the non-real spectrum consists of less than κ distinct eigenvalues. In [H12] we present an explicitly solvable example for this case.

Moreover, [H12] provides examples of potentials of hyperbolic secant type given by

$$q_\kappa(x) = -\kappa(\kappa + 1)\mathrm{sech}^2(x), \qquad x \in \mathbb{R} \;\text{ and }\; \kappa \in \mathbb{N}.$$

It is well known, see, e.g. [54], that the number of negative eigenvalues of the definite Sturm-Liouville operator $JAf = -f'' + q_\kappa f$ is exactly κ and condition **(I)** holds. Moreover, q_κ satisfies (4.14) and hence, by Theorem 4.5, the continuous spectrum of the indefinite Sturm-Liouville operator A coincides with $\mathbb{R}$ and (4.15) holds.

With the help of the software package Mathematica κ different eigenvalues in the open upper half-plane are computed numerically in [H12]. The Figures 6 and 7 below from [H12] show the non-real eigenvalues of A for the cases $\kappa = 30$ and $\kappa = 100$. Here we find 30 and 100, respectively, distinct pairs of non-real eigenvalues and hence $\dim \mathcal{L}_\lambda(A) = 1$ for each eigenvalue $\lambda \in \mathbb{C}\backslash\mathbb{R}$.

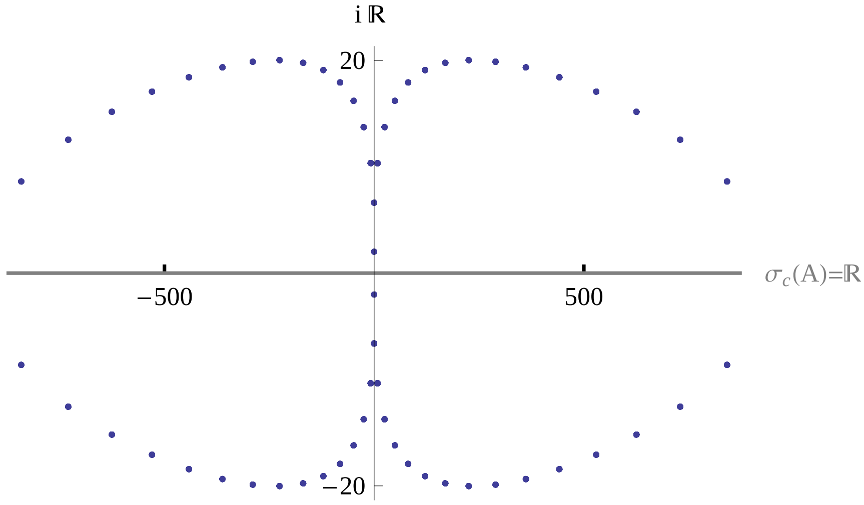

Figure 6: The operator $(Ay)(x) := \operatorname{sgn}(x)(-y''(x) + q_{30}(x)y(x))$, $x \in \mathbb{R}$, where $q_{30}(x) = -30 \cdot 31 \operatorname{sech}^2(x)$ has $\kappa = 30$ pairs of non-real eigenvalues.

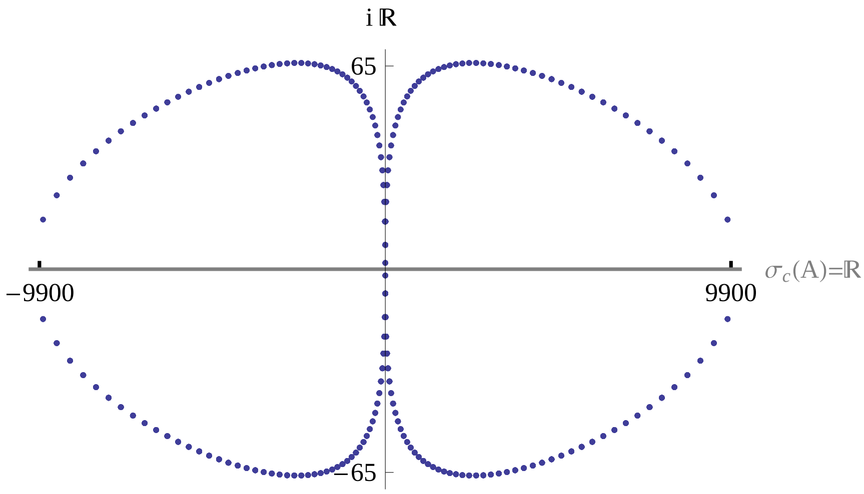

Figure 7: The operator $(Ay)(x) := \operatorname{sgn}(x)(-y''(x) + q_{100}(x)y(x))$, $x \in \mathbb{R}$, where $q_{100}(x) = -100 \cdot 101 \operatorname{sech}^2(x)$ has $\kappa = 100$ pairs of non-real eigenvalues.

4.2 The Number of Negative Squares of Extensions of Indefinite Sturm-Liouville Operators

In [H15] the number of negative squares of all selfadjoint extensions of a simple[13] symmetric operator of defect one with finitely many negative squares in a Krein space is characterized in terms of the behaviour of an abstract Weyl function near 0 and ∞. We recall the definition of the class D_κ from [H16], see Definition 4.6 below, which is the main tool in the investigation of Weyl functions associated to symmetric operators of defect one with finitely many negative squares

The class of all functions τ which are piecewise meromorphic in $\mathbb{C}\backslash\mathbb{R}$ and symmetric with respect to the real axis is denoted by $M(\mathbb{C}\backslash\mathbb{R})$. For a function $\tau \in M(\mathbb{C}\backslash\mathbb{R})$ the union of all points of holomorphy of τ in $\mathbb{C}\backslash\mathbb{R}$ and all points $\lambda \in \overline{\mathbb{R}}$ such that τ can be analytically continued to λ and the continuations from the open upper and the lower half-plane coincide is denoted by $\mathfrak{h}(\tau)$. Recall that a function $G \in M(\mathbb{C}\backslash\mathbb{R})$ belongs to the class N_κ if the kernel N_G,

$$N_G(\lambda, \mu) := \frac{G(\lambda) - G(\overline{\mu})}{\lambda - \overline{\mu}},$$

has κ negative squares[14] (see [101, 105]). The class N_0 coincides with the class of Nevanlinna functions. This class consists of functions which are holomorphic in $\mathbb{C} \setminus \mathbb{R}$ and have a nonnegative imaginary part in the open upper half-plane.

Definition 4.6. [H16] *A function $\tau \in M(\mathbb{C}\backslash\mathbb{R})$ belongs to the class D_κ, $\kappa \in \mathbb{N}_0$, if there exists a point $\lambda_0 \in \mathfrak{h}(\tau)\backslash\{\infty\}$, a function $G \in N_\kappa$ holomorphic in λ_0 and a rational function g holomorphic in $\overline{\mathbb{C}}\backslash\{\lambda_0, \overline{\lambda}_0\}$ such that*

$$\frac{\lambda}{(\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)}\tau(\lambda) = G(\lambda) + g(\lambda)$$

holds for all points λ where τ, G and g are holomorphic.

It is shown in [H16] that the number κ in Definition 4.6 does not depend on the choice of $\lambda_0 \in \mathfrak{h}(\tau)\backslash\{\infty\}$. We note that the classes D_κ, $\kappa \in \mathbb{N}_0$, are subclasses of the class of definitizable functions, see [70, 72].

One of the main results in [H15] is the classification of the class $D_{\tilde{\kappa}}$ to which the negative reciprocal $-\frac{1}{\tau}$ of a function τ from D_κ belongs.

Theorem 4.7. [H15] *Let $\tau \in \mathcal{D}_\kappa$, $\kappa \geq 1$, be not identically equal to zero. Then*

$$-\frac{1}{\tau} \in \mathcal{D}_{\widehat{\kappa}}, \quad \textit{where} \quad \widehat{\kappa} = \kappa + \Delta_0 + \Delta_\infty,$$

$$\Delta_0 = \begin{cases} -1 & \text{if } \lim\limits_{\lambda \hat{\to} 0} \tau(\lambda) \leq 0, \\ 0 & \textit{otherwise} \end{cases} \quad \textit{and} \quad \Delta_\infty = \begin{cases} 0 & \text{if } \lim\limits_{\lambda \hat{\to} \infty} \tau(\lambda) \geq 0, \\ 1 & \textit{otherwise.} \end{cases}$$

[13] A densely defined closed symmetric operator S of defect one with finitely many negative squares is said to be *simple* if there exists a selfadjoint extension A' of S with a nonempty resolvent set such that the condition

$$\mathcal{K} = \operatorname{clsp}\{\ker(S^+ - \lambda) : \lambda \in \rho(A')\}$$

holds.

[14] The kernel N_G has κ negative squares if for any $N \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_N \in \mathfrak{h}(G)$ from the open upper half-plane the Hermitian matrix

$$(N_G(\lambda_i, \lambda_j))_{i,j=1}^N$$

has at most κ negative eigenvalues, and κ is minimal with this property.

We present Theorem 4.7 above in a slightly different form as in [H15]. A similar result holds for the case $\kappa = 0$, see [H15]. Here $\lambda \hat{\to} \lambda_0$ denotes the nontangential limit from the open upper half-plane and if for some $\lambda_0 \in \overline{\mathbb{R}}$ the limit $\lim_{\lambda \hat{\to} \lambda_0} \tau(\lambda)$ exists and if it is real, then we set $\tau(\lambda_0) := \lim_{\lambda \hat{\to} \lambda_0} \tau(\lambda)$.

We assume in the following that S is a densely defined closed symmetric operator in the Krein space $\mathcal{K}$ of *defect one*, that is, there exists a selfadjoint extension A' in the Krein space $\mathcal{K}$ such that $\dim(\operatorname{dom} A'/\operatorname{dom} S) = 1$. If, in addition S has κ negative squares, then the number of negative squares of the selfadjoint extensions of S can be described in terms of an abstract Weyl function. For this we first briefly recall the notions of boundary triplets and associated Weyl functions.

Definition 4.8. *Let S be a densely defined closed symmetric operator of defect one in the Krein space $(\mathcal{K}, [\cdot,\cdot])$. We say that $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is a* boundary triplet *for S^+ if there exist linear mappings $\Gamma_0, \Gamma_1 : \operatorname{dom} S^+ \to \mathbb{C}$ such that*

$$[S^+ f, g] - [f, S^+ g] = \Gamma_1 f \, \overline{\Gamma_0 g} - \Gamma_0 f \, \overline{\Gamma_1 g}$$

holds for all $f, g \in \operatorname{dom} S^+$ and the mapping $(\Gamma_0, \Gamma_1)^\top : \operatorname{dom} S^+ \to \mathbb{C}^2$ is surjective.

For basic facts on boundary triplets and further references, see, e.g., [36, 37, 38, 39]. We recall only a few important facts. All selfadjoint extensions A_τ of S in $\mathcal{K}$ can be characterized by $A_\tau := S^+ \upharpoonright \ker(\Gamma_1 - \tau\Gamma_0)$, if $\tau \in \mathbb{R}$, and $A_\infty := S^+ \upharpoonright \ker \Gamma_0$, if $\tau = \infty$. We will briefly write A instead of A_∞.

For a point λ of regular type of S we set $\mathcal{N}_\lambda := \ker(S^+ - \lambda)$. In the following we will assume that the selfadjoint operator $A = S^+ \upharpoonright \ker \Gamma_0$ has a nonempty resolvent set. Then the functions

$$\lambda \mapsto \gamma(\lambda) := \big(\Gamma_0 \upharpoonright \mathcal{N}_\lambda\big)^{-1} \quad \text{and} \quad \lambda \mapsto M(\lambda) := \Gamma_1 \big(\Gamma_0 \upharpoonright \mathcal{N}_\lambda\big)^{-1}, \quad \lambda \in \rho(A),$$

are well defined and holomorphic on $\rho(A)$. They are called the *γ-field* and the *Weyl function* corresponding to S and the boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$, see [36]. We mention that every Weyl function is a Q-function (see e.g. [38]) in the sense of M.G. Krein and H. Langer, cf. [99, 100, 101, 102]. The γ-field and Weyl function satisfy

$$M(\lambda) = \operatorname{Re} M(\lambda_0) + \gamma(\lambda_0)^+ \big(\lambda - \operatorname{Re}\lambda_0) + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(A - \lambda)^{-1}\big)\gamma(\lambda_0) \quad (4.16)$$

for any fixed $\lambda_0 \in \rho(A)$ and all $\lambda \in \rho(A)$. Equation (4.16) establishes a connection between the resolvent of A and the Weyl function M.

Proposition 4.9. **[H16]** *Let S be a densely defined closed symmetric operator of defect one in $\mathcal{K}$ and let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^+ with Weyl function M. Assume that $A = S^+ \upharpoonright \ker \Gamma_0$ has finitely many negative squares and a nonempty resolvent set and that S is simple. Then the following holds.*

(i) *A has κ negative squares if and only if M belongs to the class D_κ,*

(ii) *$\rho(A) = \mathfrak{h}(M) \backslash \{\infty\}$.*

Now we use Theorem 4.7 in order to give a characterization of the number of negative squares of the selfadjoint extensions of a simple symmetric operator of defect one with finitely many negative squares. We note that the following can also be deduced from the abstract results obtained by V. Derkach in [36].

Theorem 4.10. **[H15]** *Let S be a densely defined closed simple symmetric operator of defect one in the Krein space $\mathcal{K}$ and let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^+. Assume that $A = S^+ \upharpoonright \ker \Gamma_0$ has $\kappa \geq 1$ negative squares and a nonempty resolvent set, let*

$$A_\tau = S^+ \upharpoonright \ker\big(\Gamma_1 - \tau\Gamma_0\big), \qquad \tau \in \mathbb{R},$$

and denote the Weyl function corresponding to $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ *by* M. *If* $M(0)$ *or* $M(\infty)$ *does not exist, we set* $M(0) := \infty$ *and* $M(\infty) := -\infty$*, respectively. Then*

$$A_\tau \text{ has } \widetilde{\kappa} = \kappa + \Delta_0 + \Delta_\infty \text{ negative squares,}$$

where

$$\Delta_0 = \begin{cases} 0, & \text{if } \tau < M(0), \\ -1, & \text{otherwise,} \end{cases} \qquad \text{and} \quad \Delta_\infty = \begin{cases} 1, & \text{if } M(\infty) < \tau, \\ 0, & \text{otherwise.} \end{cases}$$

For the special case that A is nonnegative (i.e. $\kappa = 0$) a similar statement holds, see [H15]. In this case P. Jonas and H. Langer characterized the negative squares of the canonical selfadjoint extensions of A in [75, 76].

In the following we will apply Theorem 4.10 to indefinite Sturm-Liouville differential expressions of the form (4.1) in $L^2_r(a,b)$ which are regular at the left endpoint a (that is, $-\infty < a < \infty$ and $\frac{1}{p}, q, r \in L^1(a,c)$ for some $c \in (a,b)$) and either singular (and in the limit point case) or regular at the right endpoint b. The first case that b is singular means that $b = \infty$, or at least one of the functions $\frac{1}{p}, q, r$ does not belong to $L^1(c', b)$ for some (and hence for every) $c' \in (a,b)$. In addition, it is assumed that in the singular case the differential expression $|r|^{-1}(-\frac{d}{dx}(p\frac{d}{dx}) + q)$ is limit point at b. The case regular at b means $\frac{1}{p}, q, r \in L^1(a,b)$ and $b < \infty$. For brevity we will treat both cases simultaneously.

Define a symmetric operator S in the Krein space $\big(L^2_{|r|}(a,b), [\cdot,\cdot]\big)$ by

$$\begin{aligned} Sf &:= \frac{1}{r}\big(-(pf')' + qf\big), \\ \operatorname{dom} S &:= \big\{f \in \mathcal{D}_{\max} : f(a) = (pf')(a) = 0,\ [\alpha f(b) = (pf')(b)]_{\text{reg}}\big\}, \end{aligned} \tag{4.17}$$

where $\alpha \in \overline{\mathbb{R}}$ is fixed and $[\cdot]_{\text{reg}}$ indicates that the boundary condition $\alpha f(b) = (pf')(b)$ is imposed in the regular case only. In this case $\alpha = \infty$ in (4.17) means $f(b) = 0$. The next proposition collects some properties of the operator S and its selfadjoint extensions A_τ in the Krein space $\big(L^2_{|r|}(a,b), [\cdot,\cdot]\big)$. It follows from [28, Propositions 2.2-2.5] and the statement about the simplicity of S is shown in [H16].

Proposition 4.11. [28, H15] *Let* S *be the indefinite Sturm-Liouville operator from* (4.17)*. In the case that* b *is singular it is assumed that there exists* $b' \in (a,b)$ *and* r *is of constant sign a.e. on* (b', b)*. Then* S *is a densely defined closed simple symmetric operator of defect one in the Krein space* $L^2_r(a,b)$*. The adjoint operator* S^+ *is given by*

$$\begin{aligned} S^+ f &= \frac{1}{r}\big(-(pf')' + qf\big), \\ \operatorname{dom} S^+ &= \big\{f \in \mathcal{D}_{\max} : [\alpha f(b) = (pf')(b)]_{\text{reg}}\big\}, \end{aligned}$$

and $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$, $\Gamma_0 f = f(a)$, $\Gamma_1 f = (pf')(a)$, $f \in \operatorname{dom} S^+$, *is a boundary triplet for* S^+*. All selfadjoint extensions* A_τ, $\tau \in \overline{\mathbb{R}}$, *are given by*

$$\begin{aligned} A_\tau f &= \frac{1}{r}\big(-(pf')' + qf\big), \\ \operatorname{dom} A_\tau &= \begin{cases} \{f \in \operatorname{dom} S^+ : \tau f(a) = (pf')(a)\}, & \text{if } \tau \in \mathbb{R}, \\ \{f \in \operatorname{dom} S^+ : f(a) = 0\}, & \text{if } \tau = \infty, \end{cases} \end{aligned}$$

and each A_τ, $\tau \in \overline{\mathbb{R}}$, *has a nonempty resolvent set and a finite number of negative squares.*

If $f_\lambda \in L^2_r(a,b)$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$, spans the defect subspace of S, $\ker(S^+ - \lambda) = \operatorname{sp}\{f_\lambda\}$, then the Weyl function M corresponding to the boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ from Proposition 4.11 is given by

$$M(\lambda) = \frac{\Gamma_1 f_\lambda}{\Gamma_0 f_\lambda} = \frac{(pf'_\lambda)(a)}{f_\lambda(a)}, \qquad \lambda \in \rho(A),$$

and belongs to the class D_κ, where the number κ coincides with the number of negative squares of the selfadjoint extension $A = S^+ \upharpoonright \ker\Gamma_0$ of S, see Proposition 4.9. As a consequence of Theorem 4.10 and Proposition 4.11 we obtain the following theorem.

Theorem 4.12. [H15] *Let $A = S^+ \upharpoonright \ker\Gamma_0$ be as above and assume that b is regular or that b is singular and there exists $b' \in (a,b)$ and r is of constant sign a.e. on (b',b). Then A has a nonempty resolvent set and κ negative squares, $\kappa \in \mathbb{N}_0$. Let M be the Weyl function corresponding to the boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ from Proposition 4.11. If $M(0)$ or $M(\infty)$ does not exist, we set $M(0) := \infty$ and $M(\infty) := -\infty$, respectively. If $\kappa \geq 1$, then for $\tau \in \mathbb{R}$ the operator*

$$A_\tau = S^+ \upharpoonright \ker\big(\Gamma_1 - \tau\Gamma_0\big) \text{ has } \widetilde{\kappa} = \kappa + \Delta_0 + \Delta_\infty \text{ negative squares,}$$

where

$$\Delta_0 := \begin{cases} 0, & \text{if } \tau < M(0), \\ -1, & \text{otherwise,} \end{cases} \qquad \text{and} \quad \Delta_\infty := \begin{cases} 1, & \text{if } M(\infty) < \tau, \\ 0, & \text{otherwise.} \end{cases}$$

The case $\kappa = 0$ can be treated analogously, cf. [H15]. Finally, we mention that in [20] a different description of the number of negative squares in the regular case in terms of the Titchmarsh-Weyl function is given.

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