Technische Universität Ilmenau
Institute for Mathematics
Group for Combinatorics/Graph Theory

Master Thesis

submitted in fulfillment of the requirements for the degree of
Master of Science

GRAPH PARTITIONS

Thomas Schweser

supervised by
Prof. Dr. Michael Stiebitz

July 6, 2017

Declaration of Authorship

I hereby declare that the thesis submitted is my own unaided work. All direct or indirect sources used are acknowledged as references. This paper was not previously presented to another examination board and has not been published.

Ilmenau, July 6, 2017

Contents

1 Introduction **1**
1.1 History of Graph Partitions 1
1.2 Basic Terminology 3

2 Graph Partitions under Maximum Degree Constraints **8**
2.1 Prior Research 8
2.2 Characterizing f-partitionable Graphs 10
2.3 Applications 21

3 Graph Partitions under Minimum Degree Constraints **29**
3.1 Prior Research 29
3.2 Weighted Graph Partitions 32
3.3 Main Results 33
3.4 Preliminary Considerations 35
3.5 Proof of Theorem 3.10 36
3.6 Proof of Theorem 3.12 37
3.7 Concluding Remarks 41

4 Graph Partitions under Further Degree Constraints **43**
4.1 Graph Partitions under Average Degree Constraints 43
4.2 Graph Partitions under Color Degree Constraints 44
4.3 Graph Partitions under σ-Constraints 45

References **47**

Chapter 1

Introduction

1.1 History of Graph Partitions

Graph partitions constitute a broad field within graph theory. Lovász [23] was one of the first mathematicians to drive research forward by regarding graph partitions of simple graphs under certain degree constraints. A partition of a graph G is a sequence $(G_1, G_2, \ldots, G_p)$ of vertex disjoint induced subgraphs (that may or may not be empty) such that $V(G) = V(G_1) \cup V(G_2) \cup \ldots \cup V(G_p)$. Lovász examined partitions of simple graphs with bounded maximum degrees. In particular, he showed that, given $p \geq 2$ non-negative integers $d_1, d_2, \ldots, d_p$, each simple graph G satisfying

$$\Delta(G) < d_1 + d_2 + \ldots + d_p$$

admits a partition $(G_1, G_2, \ldots, G_p)$ such that $\Delta(G_i) < d_i$ for each $i \in \{1, 2, \ldots, p\}$. Lovász's work lead to a variety of other papers dealing with related topics. For instance, Borodin and Kostochka [6] extended his result to the case of variable functions (see chapter 2). Lovász's research probably also motivated them to deal with graph partitions with upper boundings on the degeneracy. For a graph G and a function $f : V(G) \to \mathbb{Z}_{\geq 0}$, we say that G is strictly f-degenerate if any non-empty subgraph H of G contains a vertex v such that $d_H(v) < f(v)$. Moreover, given a vector function $f = (f_1, f_2, \ldots, f_p)$, a graph G is said to be f-partitionable if G admits a partition $(G_1, G_2, \ldots, G_p)$ such that G_i is strictly f_i-degenerate for all $i \in \{1, 2, \ldots, p\}$. Borodin, Kostochka and Toft [7] analyzed, which degree-conditions are sufficient for a simple graph to be f-partitionable. They showed that for a connected simple graph G the requirement $f_1(v) + f_2(v) + \ldots + f_p(v) \geq d_G(v)$ for all $v \in V(G)$ is sufficient if and only if (G, f) is not a so called hard pair. This theorem brings along a lot of applications; for instance, it is an extension of Brooks's famous theorem, which states that a connected simple graph G does not admit a proper coloring with $\Delta(G)$ colors if and only if G is a complete graph or an odd cycle. In chapter 2, we show how to extend Borodin, Kostochka and Toft's Theorem to graphs with multiple edges and also mention some further applications.

The topic of the third chapter is graph partitions under minimum degree constraints. In 1996, Stiebitz [25] proved the following conjecture of Thomassen [26]. Given two integers $s, t \geq 1$ and a simple graph G with minimum degree at least $s + t + 1$, there always exists a partition (G_1, G_2) of G such that $\delta(G_1) \geq s$ and $\delta(G_2) \geq t$. Stiebitz even showed a generalized version of this conjecture with variable functions instead of fixed integers s and t (see chapter 3). Recently, Ban [1] managed to convey Stiebitz's result to weighted graphs. However, since his boundary was worse than Stiebitz's, this motivated us to consider the case of graphs with multiple edges. In the third chapter, our extension of Stiebitz's Theorem for graphs in general will be presented as well as extensions of theorems by Kaneko [18], Bazgan, Tuza and Vanderpooten [2] and Liu and Xu [22]. All of those regard graph partitions under minimum degree constraints with respect to constant, respectively variable functions for simple graphs with special properties.

The fourth chapter is dedicated to some further constraints under which graph partitions may be reasonable. Initially, we will present a conjecture by Csóka, Lo, Norin, Wu and Yepremyan [10] regarding graph partitions under average degree constraints. They supposed that for any real numbers $s, t > 0$ and for each graph G with average degree $av(G) \geq s+t+2$, there is a partition (G_1, G_2) of G such that $av(G_1) \geq t$ and that $av(G_2) \geq s$. Thereafter, we will go on to graph partitions of edge-colored graphs with constraints on the minimum color degree. Those were studied by Li, Fujita and Wang [21]. Finally, we will present some results on graph partitions under σ-constraints, whereby for a non-complete graph G we denote by $\sigma(G)$ the minimum degree sum of two not-adjacent vertices. Chiba and Lichiardopol [9] tried to find an analogue to Stiebitz's and Kaneko's Theorems (see chapter 2) and made a lot of progress with this in mind. However, numerous open problems still remain that will be mentioned in the last chapter. Regarding future research, the reader is encouraged to engage with those challenges.

1.2 Basic Terminology

The majority of the introduced terminology is based upon the book from Diestel [11]. All **graphs** considered in this paper are finite and undirected, and may have multiple edges but no loops. Let $G = (V(G), E(G))$ be a graph with **vertex set** $V(G)$ and **edge set** $E(G)$. The **order** of G is the number of vertices in $V(G)$; we will denote it by $|G|$ or $|V(G)|$, respectively. A graph G of order 0 is also called the **empty graph**; we briefly write $G = \varnothing$. For $X, Y \subseteq V(G)$ let $E_G(X, Y)$ be the set of all edges joining a vertex of X with a vertex of Y, and let

$$E_G[X] = E_G(X, X),$$

as well as

$$E_G(X) = E_G(X, V(G) \setminus X).$$

Let G and H be graphs. H is called a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; we denote it by $H \subseteq G$. Furthermore, G is said to be H**-free** if G contains no graph isomorphic to H as a subgraph. The **subgraph** of G **induced** by X is denoted by $G[X]$, i. e. $V(G[X]) = X$ and $E(G[X]) = E_G[X]$. Moreover, we say that H is an **induced subgraph** of G if $H = G[X]$ for some $X \subseteq V(G)$. For a graph G and for $X \subseteq V(G)$, let $N_G(X) = \{v \in V(G) \setminus X \mid E_G(X, \{v\}) \neq \varnothing\}$ be the **neighborhood** of X in G. Instead of writing $N_G(\{v\})$, we will use $N_G(v)$. In particular, for the sake of readability we will often omit subscribts or brackets if the meaning is clear. For a vertex $v \in V(G)$, let

$$G - v = G[V(G) \setminus \{v\}].$$

Moreover, if H is an induced subgraph of G and if $v \in V(G)$, we define

$$H + v = G[V(H) \cup \{v\}].$$

Let $v \in V(G)$ be a vertex. Then, $d_G(v) = |E_G(v)|$ is the **degree** of v in G. If H is an induced subgraph of G and if $v \in V(G) \setminus V(H)$, we will also use $d_H(v)$ to express $d_{H+v}(v)$. A non-empty graph G is said to be k**-regular** for some $k \geq 0$ or just **regular**, respectively, if $d_G(v) = k$ for all $v \in V(G)$. As usual,

$$\delta(G) = \min_{v \in V(G)} d_G(v)$$

is the **minimum degree** and

$$\Delta(G) = \max_{v \in V(G)} d_G(v)$$

is the **maximum degree** of G. Furthermore, we define the **multiplicity** of two different vertices u and v by $\mu_G(u,v) = |E_G(u,v)|$. If $\mu_G(u,v) > 1$ for some vertices u and v, then we say that there are **multiple edges** between u and v, or **parallel edges**, equivalently. A **simple graph** is a graph without multiple edges. By $\mu(G)$ we denote the **maximum multiplicity** of a graph G, that is,

$$\mu(G) = \max_{u,v \in V(G), u \neq v} \mu(u,v).$$

The **average degree** of a non-empty graph G is defined by

$$av(G) = \frac{1}{|G|} \sum_{v \in V(G)} d_G(v) = \frac{2|E(G)|}{|G|}.$$

If $G = \varnothing$, we set $\delta(G) = av(G) = \Delta(G) = 0$. Thus, it obviously holds

$$\delta(G) \leq av(G) \leq \Delta(G)$$

for all graphs G. Given an integer $k \geq 0$, a graph G is said to be k**-degenerate** if every non-empty subgraph of G has minimum degree at most k. Thus, G is 0-degenerate if and only if G is edgeless. It is well known that G is k-degenerate if and only if there is an ordering of the vertices $\{v_1, v_2, \ldots, v_n\}$ such that $d_{G[\{v_1,v_2,\ldots,v_i\}]}(v_i) \leq k$ for all $i \in \{1, 2, \ldots, n\}$. Closely related to this is the **coloring number** $\mathrm{col}(G)$ of G (defined by Erdős and Hajnal [14]), which is 1 plus the maximum minimum degree of all non-empty subgraphs of G, or rather

$$\mathrm{col}(G) = 1 + \max_{\varnothing \neq H \subseteq G} \delta(H).$$

Thus, G has coloring number at most k if and only if G is $(k-1)$-degenerate and, equivalently, if there is a vertex order $\{v_1, v_2, \ldots, v_n\}$ of the vertices of G such that

$$d_{G[\{v_1,v_2,\ldots,v_i\}]}(v_i) + 1 \leq k$$

for all $i \in \{1, 2, \ldots, n\}$.

According to custom, a non-empty graph G is **connected** if for any two distinct vertices $u, v \in V(G)$ there exists a (u,v)**-path** P in G, that is, a non-empty subgraph of G such that $V(P) = \{v_1, \dots, v_\ell\}$, $E(P) = \{v_{i-1}v_i \mid i \in \{2, \dots, \ell\}\}$ and $v_1 = u$ as well as $v_\ell = v$ for some $\ell \geq 2$. As usual, we write $P = v_1v_2 \dots v_\ell$, i. e. describe P by the natural sequence of its vertices. The graph resulting from a path $P = v_1v_2 \dots v_\ell$ by adding the edge $v_\ell v_1$ is called a **cycle** of length ℓ. A **multi-cycle** results from a cycle by replacing each edge by one or more parallel edges. Moreover, the **girth** of a graph G is the least integer $\ell \geq 3$ such that G contains a cycle of length ℓ; we will denote it by $g(G)$. If G is a non-empty graph, then we will say that the maximal connected subgraphs of G are the **components** of G. A connected graph G is called a **tree** if G does not contain any cycles. Moreover, a graph G is a **forest** if each component of G is a tree.

For a graph G, a **separating vertex** of G is a vertex $v^* \in V(G)$ such that $G - v^*$ has more components than G. A **block** of a non-empty graph G is a maximal connected subgraph of G that does not contain a separating vertex.

In this paper, given a simple graph H and an integer $t \geq 1$, we denote by tH the graph that results from H by replacing each edge with t parallel edges. In particular, tK_n is the graph on n vertices such that any two vertices are joined by exactly t edges. For $t = 1$, $K_n = 1K_n$ denotes the **complete graph** on n vertices. Alike, the tC_n is the graph resulting from a cycle of order n by replacing each edge with t parallel edges.

Let $p \geq 1$ be an integer. A sequence $(A_1, A_2, \dots, A_p)$ of sets is called a **partition** of a set V if $A_1, A_2, \dots, A_p$ are pairwise disjoint non-empty subsets of V such that their union is V. Contrary to a partition, in a **weak partition** also empty sets are allowed. A graph G is **bipartite** if $V(G)$ admits a weak partition (A, B) such that $E(G) = E_G(A, B)$. In this case, we will say that A and B are the **classes** of G. Moreover, G is called a **complete bipartite graph** if any two vertices u, v with $u \in A$ and $v \in B$ are joined by exactly one edge. If $|A| = m$ and $|B| = n$, we denote the complete bipartite graph with classes A and B by $K_{m,n}$.

In this paper, we will often consider (weak) partitions of graphs. Let G be a graph. A **partition** or p**-partition** of G is a sequence $(G_1, G_2, \dots, G_p)$ of pairwise disjoint induced subgraphs of G such that $V(G_1) \cup V(G_2) \cup \ldots \cup V(G_p) = V(G)$. Note that in a p-partition also empty subgraphs are allowed, i. e. p-partitions may be weak by definition.

The concept of graph partitions is closely related to the concept of colorings of graphs. By a **coloring** of a graph G with color set C, we mean a function $\varphi : V(G) \to C$. Let φ be a coloring of a graph G with color set C. If $|C| = k$, we also say that φ is a k**-coloring** of G. Furthermore, for each color $c \in C$, the set $\varphi^{-1}(c)$ is called a **color class** of G with respect to φ. Very often, we require that the color of a vertex is chosen from a given color list. Therefore, we call a function $L : V(G) \to \mathfrak{P}(C)$ a **list-assignment** of G with color set

C. Consider a list-assignment L of G with color set C and a coloring φ of G with color set C. Then, we say that φ is an **L-coloring** of G if the color $\varphi(v)$ is taken from the list $L(v)$ for each $v \in V(G)$ (i. e. $\varphi(v) \in L(v)$ for all $v \in V(G)$).

Let G be a graph and let $C = \{1, 2, \ldots, k\}$ be a color set. If φ is a coloring of G with color set C, then $(G_1, G_2, \ldots, G_k)$ with $G_i = G[\varphi^{-1}(i)]$ for $i \in C$ is obviously a k-partition of G. Conversely, if $(G_1, G_2, \ldots, G_k)$ is a partition of G, then the function $\varphi : V(G) \to C$ satisfying $\varphi(v) = i$ if $v \in V(G_i)$ is a coloring of G.

Colorings and partition of graphs become a subject of interest only when some restrictions to the color classes, respectively to the parts of the partition, are imposed. For instance, a coloring or list-coloring of a graph G with color set C is called a **proper coloring**, respectively a **proper list-coloring** of G if each color class induces an edgeless subgraph. The **chromatic number** of a graph G, denoted by $\chi(G)$, is the least integer k such that G admits a proper k-coloring. Similar, the **list-chromatic number** of G, denoted by $\chi^\ell(G)$, is the least integer k such that G admits a proper L-coloring for each list-assignment L satisfying $|L(v)| \geq k$ for all $v \in V(G)$. Since $\chi^\ell(G) = k$ implies that G has a proper L-coloring for the list-assignment L with $L(v) = \{1, 2, \ldots, k\}$ for all $v \in V(G)$, we get $\chi(G) \leq \chi^\ell(G)$. Furthermore, a simple sequential coloring argument shows that

$$\chi(G) \leq \chi^\ell(G) \leq \mathrm{col}(G) \leq \Delta(G) + 1.$$

Note that the chromatic number and the list-chromatic number of a graph G is equal to the chromatic number, respectively list-chromatic number of its underlying simple graph H, that is, the graph resulting from G by replacing all multiple edges by a single edge.

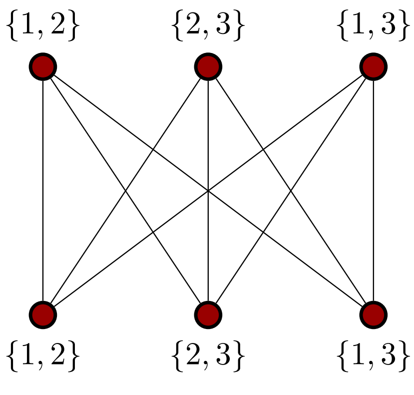

Figure 1.1: The $K_{3,3}$ admits no proper L-coloring for the displayed list-assignment L.

The list-coloring concept was introduced by Vizing [27] and, independently, by Erdős, Rubin and Taylor [13]. They furthermore showed that even for bipartite graphs G (i. e. $\chi(G) \leq 2$), the discrepancy between $\chi(G)$ and $\chi^\ell(G)$ can get arbitrarily high. To see this, consider the $K_{m,m}$ with $m = \binom{2r-1}{r}$ for some integer $r \geq 1$. Of course, it holds $\chi(K_{m,m}) = 2$ for all m. On the other hand, if we take the set $C = \{1, 2, \ldots, 2r-1\}$, then there is a list-assignment L such that each subset of C with size r is assigned to exactly one vertex in each class A and B (see Figure 1.2). Assume that there is a proper L-coloring φ of G. Then, $X = \{\varphi(a) \mid a \in A\}$ and $Y = \{\varphi(b) \mid b \in B\}$ are disjoint subsets of C and, as a consequence, one set, say X, has at most size $r-1$. However, then there is a vertex $a \in A$ such that $L(a) \cap X = \varnothing$, which is impossible. Hence, we conclude $\chi^\ell(K_{m,m}) \geq r+1$.

Chapter 2

Graph Partitions under Maximum Degree Constraints

The second chapter deals with graph partitions under maximum degree constraints. After mentioning some earlier results by Lovász, Borodin and Kostochka to clarify our motivation, we will introduce the term *hard pairs*, which will later be used to characterize some "bad" types of graphs. On this basis, given a vector function $f = (f_1, f_2, \ldots, f_p)$ we will show that there exists a partition $(G_1, G_2, \ldots, G_p)$ of a connected graph G such that for each $i \in \{1, 2, \ldots, p\}$ and for each non-empty subgraph H_i of G_i there is a vertex $v_i \in V(H_i)$ satisfying $d_{H_i}(v_i) < f_i(v_i)$ if and only if (G, f) is not a hard pair. The proof of this statement constitutes the main part of this chapter. Finally, the last section addresses applications of our main result, inter alia, a list version of Brooks's Theorem and some results on the list point-partition number of a graph.

2.1 Prior Research

Lovász [23] was one of the first mathematicians to regard graph partitions with degree constraints. In particular, he regarded partitions with upper degree boundaries subject to constant functions. His result says that, given non-negative integers $d_1, d_2, \ldots, d_p$ with

$p \geq 2$, each simple graph G satisfying

$$\Delta(G) < d_1 + d_2 + \ldots + d_p$$

admits a p-partition $(G_1, G_2, \ldots, G_p)$ of G such that $\Delta(G_i) < d_i$ for each $i \in \{1, 2, \ldots, p\}$. Later, Borodin and Kostochka [6] conveyed this outcome to the case of variable functions. Their result trivially also holds for graphs in general.

Lemma 2.1 (Borodin and Kostochka). *Let G be a graph, and let*

$$f_1, f_2, \ldots, f_p : V(G) \to \mathbb{Z}_{\geq 1}$$

be p functions with $p \geq 2$. Assume that

$$d_G(v) < f_1(v) + f_2(v) + \cdots + f_p(v)$$

for all $v \in V(G)$. Then, there is a p-partition $(G_1, G_2, \ldots, G_p)$ of G such that $d_{G_i}(v) < f_i(v)$ for every $i \in \{1, 2, \ldots, p\}$ and every $v \in V(G_i)$.

Proof: For $p > 2$ the result easily follows from the basic case $p = 2$ by induction. For $p = 2$, we claim that a 2-partition (G_1, G_2) of G for which

$$w(G_1, G_2) = |E(G_1)| + |E(G_2)| + \sum_{v \in V(G_1)} f_2(v) + \sum_{v \in V(G_2)} f_1(v)$$

is minimum has the desired property. Otherwise, by symmetry, there is a vertex $v \in V(G_1)$ such that $d_{G_1}(v) \geq f_1(v)$. Consequently, $d_{G_2}(v) < f_2(v)$ and, therefore,

$$w(G_1 - v, G_2 + v) - w(G_1, G_2) = d_{G_2}(v) - d_{G_1}(v) + f_1(v) - f_2(v) \leq -1,$$

contradicting the choice of (G_1, G_2). ∎

In the following, we will focus on partitions of graphs with upper boundings on the degeneracy. To this end, let G be a graph, let $p \geq 1$ be an integer, and let $f : V(G) \to \mathbb{Z}_{\geq 0}$ be a function. We say that the graph G is **strictly f-degenerate** if every non-empty subgraph H of G contains a vertex $v \in V(H)$ such that $d_H(v) < f(v)$. If $f(v) = d$ for all $v \in V(G)$, then G being strictly f-degenerate is equivalent to G being $(d-1)$-degenerate and, therefore, to $\text{col}(G) \leq d$.

A function $f : V(G) \to \mathbb{Z}^p_{\geq 0}$ is called a **vector function** of G. By f_i we name the ith coordinate of f, i.e. $f = (f_1, f_2, \ldots, f_p)$. The set of all vector functions of G with p coordinates is denoted by $\mathcal{V}_p(G)$. For $f \in \mathcal{V}_p(G)$, an **f-partition** of G is a p-partition $(G_1, G_2, \ldots, G_p)$ of G such that G_i is strictly f_i-degenerate for all $i \in \{1, 2, \ldots, p\}$. If G admits an f-partition, we say that G is **f-partitionable.**

Given a vector function $f \in \mathcal{V}_p(G)$ and an arbitrary subgraph H of G, f may also be considered as a vector function of H. Because of this, we will denote the restriction of f to H by f, too. Obviously, if G is f-partitionable, then each subgraph of G is f-partitionable, as well. In the next section we will focus on the question whether or not a graph G admits an f-partition. A first assumption could be that $f_1(v) + f_2(v) + \ldots + f_p(v) \geq d_G(v)$ for all $v \in V(G)$ is sufficient for the existence of an f-partition of G. We will show that this is nearly always the case, and give a characterization of the forbidden types of graphs.

2.2 Characterizing f-partitionable Graphs

Let G be a connected graph and let $f \in \mathcal{V}_p(G)$. We say that (G, f) is a **hard pair** or, equivalently, that G is **f-hard** if one of the following conditions hold:

(H1) G is a block and there exists an index $j \in \{1, 2, \ldots, p\}$ such that

$$f_i(v) = \begin{cases} d_G(v) & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$$

for all $v \in V(G)$.

(H2) $G = tK_n$ for some $t, n \geq 1$ and there are integers $n_1, n_2, \ldots n_p \geq 0$ such that $n_1 + n_2 + \ldots + n_p = n - 1$ and that

$$f(v) = (tn_1, tn_2, \ldots, tn_p)$$

for all $v \in V(G)$.

(H3) G is a tC_n with $t \geq 1, n \geq 3$ odd and there are two integers $k \neq \ell \in \{1, 2, \ldots, p\}$ such that

$$f_i(v) = \begin{cases} t & \text{if } i \in \{k, \ell\}, \\ 0 & \text{otherwise} \end{cases}$$

for all $v \in V(G)$.

(H4) There are two hard pairs (G^1, f^1) and (G^2, f^2) with $f^1 \in \mathcal{V}_p(G^1), f^2 \in \mathcal{V}_p(G^2)$, and with $V(G^1) \cap V(G^2) = \varnothing$ such that G is obtained from $G^1 \cup G^2$ by identifying two vertices $v^1 \in V(G^1)$ and $v^2 \in V(G^2)$ to a new vertex v^*. Moreover, f satisfies

$$f(v) = \begin{cases} f^1(v) & \text{if } v \in V(G^1 - v^1), \\ f^2(v) & \text{if } v \in V(G^2 - v^2), \\ f^1(v^1) + f^2(v^2) & \text{if } v = v^* \end{cases}$$

for all $v \in V(G)$. In this case we say that (G, f) is obtained from (G^1, f^1) and (G^2, f^2) by **merging** v^1 and v^2 to v^*.

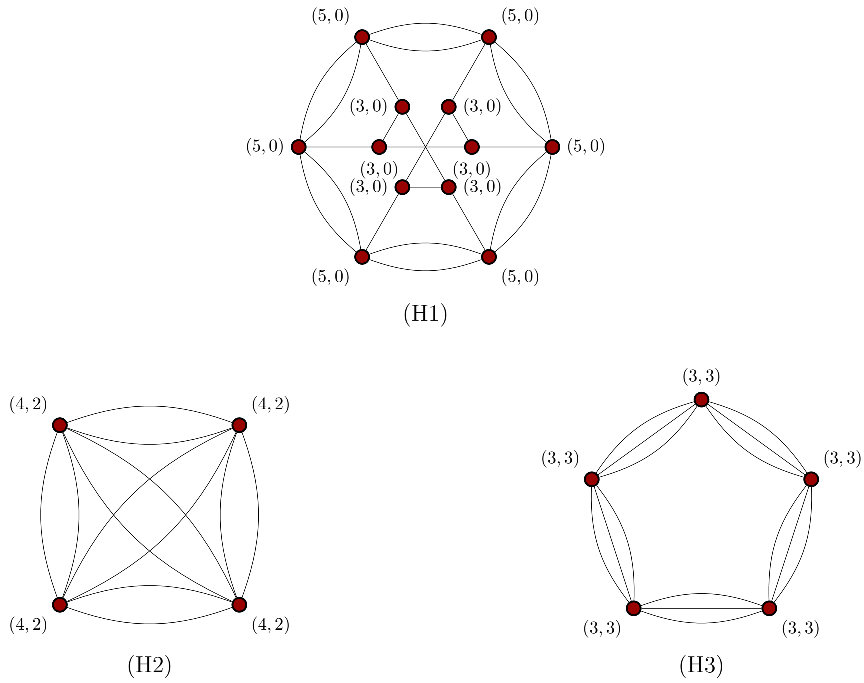

Figure 2.1: Some examples of hard pairs.

As mentioned already, if G is a graph and if $f \in \mathcal{V}_p(G)$ is a function, we will show that the condition $f_1(v) + f_2(v) + \ldots + f_p(v) \geq d_G(v)$ for all $v \in V(G)$ is not sufficient for the existence of an f-partition of G if and only if one component of G is f-hard. Note that G is f-partitionable if and only if each component of G is f-partitionable. Thus, it is adequate to consider only connected graphs. The next result was proven by Borodin, Kostochka and

Toft [7] for the class of simple graphs. Building upon their proof, we will show how to extend it to graphs in general.

Theorem 2.2 *Let G be a connected graph, and let $f \in \mathcal{V}_p(G)$ be a vector function with $p \geq 1$ such that $f_1(v) + f_2(v) + \ldots + f_p(v) \geq d_G(v)$ for all $v \in V(G)$. Then G is not f-partitionable if and only if (G, f) is a hard pair.*

Characteristics of Hard Pairs

In order to prove the "if"-direction, we first need to make some observations on hard pairs. Here, the proof for simple graphs can be copied as it stands.

Proposition 2.3 *Let G be a connected graph, and let $f \in \mathcal{V}_p(G)$ be a vector function with $p \geq 1$. If G is f-hard, then the following statements hold:*

(a) *$f_1(v) + f_2(v) + \ldots + f_p(v) = d_G(v)$ for all $v \in V(G)$.*

(b) *If u and u' are two non-separating vertices of G contained in the same block of G, then $f(u) = f(u')$ or $f_i(u) = f_i(u') = 0$ for all but one index $i \in \{1, 2, \ldots, p\}$.*

(c) *G is not f-partitionable.*

Proof: Obviously, statement (a) holds. Statement (b) follows by a simple induction on the number of blocks of G. The proof of (c) is by contradiction. To this end, we choose (G, f) as follows.

(1) (G, f) is a hard pair,

(2) there is an f-partition $(G_1, G_2, \ldots, G_p)$ of G, and

(3) $|G|$ is minimum with respect to (1) and (2).

Note that if f_i is the zero-function for some $i \in \{1, 2, \ldots, p\}$, then $G_i = \varnothing$. As a consequence, if (G, f) satisfies (H1), then there exists an index $j \in \{1, 2, \ldots, p\}$ such that $G_i = \varnothing$ for all $i \in \{1, 2, \ldots, p\} \setminus \{j\}$. Then, $G_j = G$ and $f_j(v) = d_G(v)$ for all $v \in V(G)$. Consequently, G_j is not strictly f_j-degenerate, a contradiction to (2).

If (G, f) satisfies (H2), then $G = tK_n$ and there are integers $n_1, n_2, \ldots, n_p$ such that $n_1 + n_2 + \ldots + n_p = n - 1$ and that $f(v) = (tn_1, tn_2, \ldots, tn_p)$ for all $v \in V(G)$. Hence, G_i is a tK_{m_i} for some $m_i \geq 0$ and there is a vertex $v_i \in V(G_i)$ such that $d_{G_i}(v_i) < f_i(v) = tn_i$.

Thus, $|G_i| \leq n_i$ holds for all $i \in \{1, 2, \ldots, p\}$ and, therefore, $|G| \leq n_1+n_2+\ldots+n_p = n-1$, a contradiction.

If (G, f) satisfies (H3), then $G = tC_n$ for some $t \geq 1, n \geq 3$ odd. Then, (G_k, G_ℓ) is a 2-partition of G and $f_k(v) = f_\ell(v) = t$ for all $v \in V(G)$. Since n is odd, one of the parts, say G_k, contains two vertices joined by t parallel edges. As a consequence, G_k is not strictly f_k-degenerated, a contradiction to (2).

It remains to consider the case that (G, f) is obtained from two hard pairs (G^1, f^1) and (G^2, f^2) by merging v^1 and v^2 to v^*. Since G is a minimal counter-example, G^j is not f^j-partitionable for $j \in \{1, 2\}$. For $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2\}$ let $G_i^j = G_i \cap G^j$. By symmetry, we can assume that v^* belongs to G_1. Then, G_i^1 is strictly f_i^1-degenerate and G_i^2 is strictly f_i^2-degenerate for each $i \in \{2, 3, \ldots, p\}$. As a consequence, for $j \in \{1, 2\}$, the graph G_1^j is not strictly f_1^j-degenerate. Thus, for $j \in \{1, 2\}$ there is a nonempty subgraph H^j of G_1^j such that $d_{H^j}(v) \geq f_1^j(v)$ for all $v \in V(H^j)$. However, this implies that $H = H^1 \cup H^2$ is a nonempty subgraph of G_1 such that $d_H(v) \geq f_1(v)$ for all $v \in V(H)$, a contradiction. This completes the proof. ∎

Thus, the proof of the "if"-direction is complete. For the remaining direction, we will need the following term. We say that (G, f) is a **non-partitionable pair of dimension** p if G is a connected graph, $f \in \mathcal{V}_p(G)$ is a vector function satisfying

$$f_1(v) + f_2(v) + \ldots + f_p(v) \geq d_G(v)$$

for all $v \in V(G)$ and G is not f-partitionable.

Non-Partitionable Pairs

The proof of the "only if"-direction uses a reduction method which is described in the following proposition.

Proposition 2.4 *Let (G, f) be a non-partitionable pair of dimension $p \geq 1$, let z be a non-separating vertex of G, and let $j \in \{1, 2, \ldots, p\}$ be an index such that $f_j(z) \neq 0$. For the graph $G' = G - z$, define $f' \in \mathcal{V}_p(G')$ to be the vector function satisfying*

$$f'_i(v) = \begin{cases} \max\{0, f_j(v) - \mu(z, v)\} & \text{if } v \in N_G(z) \text{ and } i = j, \\ f_i(v) & \text{otherwise} \end{cases}$$

for all $v \in V(G')$ and all $i \in \{1, 2, \ldots, p\}$. Then, (G', f') is a non-partitionable pair of dimension p, and in what follows, we write $(G', f') = (G, f)/(z, j)$.

Proof: By symmetry, we may assume that $j = 1$. Then, $f_1(z) \geq 1$ and G has no f-partition. Therefore, $|G| \geq 2$ and, thus, G' is connected. If G' has an f'-partition, say $(G_1, G_2, \ldots, G_p)$, then for every $i \in \{1, 2, \ldots, p\}$ the graph G_i is strictly f_i'-degenerate and therefore strictly f_i-degenerate. Let $G_1^* = G_1 + z$. We show that G_1^* is strictly f_1-degenerate. For this to happen, choose a non-empty subgraph H^* of G_1^*. If $H = H^* \cap G_1$ is nonempty, then there is a vertex $v \in V(H)$ satisfying $d_H(v) < f_1'(v)$. But then, $d_{H^*}(v) < f_1(v)$ and there is nothing left to show. Otherwise, $V(H) = \{z\}$ and $d_H(z) = 0 < f_1(z)$. Hence, G_1^* is strictly f_1-degenerate and $(G_1^*, G_2, \ldots, G_p)$ is an f-partition of G, a contradiction. ■

By applying the above introduced reduction method, we obtain the following statements for non-partitionable pairs.

Proposition 2.5 *Let (G, f) be a non-partitionable pair of dimension $p \geq 1$. Then, the following statements hold:*

(a) $f_1(v) + f_2(v) + \ldots + f_p(v) = d_G(v)$ *for all* $v \in V(G)$.

(b) *If z is a non-separating vertex of G such that $f_j(z) \neq 0$ for some index $j \in \{1, 2, \ldots, p\}$, then $f_j(v) \geq \mu(z, v)$ for all $v \in N_G(z)$.*

(c) *If $|G| \geq 2$ and u is an arbitrary vertex of G, then $G - u$ has an f-partition and for each f-partition $(G_1, G_2, \ldots, G_p)$ of $G - u$ we get $f_i(u) = d_{G_i + u}(u)$ for every $i \in \{1, 2, \ldots, p\}$.*

Proof: We prove statement (a) by induction on the order n of G. For $n = 1$, the statement obviously holds. Let $n \geq 2$ and let v be an arbitrary vertex of G. Then, there exists a non-separating vertex $z \neq v$ in G. Due to the fact that

$$f_1(z) + f_2(z) + \ldots + f_p(z) \geq d_G(z) \geq 1,$$

it holds $f_j(z) \geq 1$ for an index $j \in \{1, 2, \ldots, p\}$. By Proposition 2.4, $(G', f') = (G, f)/(z, j)$ is a non-partitionable pair and the induction hypothesis implies that

$$f_1'(v) + f_2'(v) + \ldots + f_p'(v) = d_{G'}(v). \tag{2.1}$$

If $v \notin N_G(z)$, then the required equation clearly holds. Otherwise, $f_i'(v) = f_i(v)$ for all $i \in \{1, 2, \ldots, p\} \setminus \{j\}$ and $f_j'(v) = \max\{0, f_j(v) - \mu(z, v)\}$. Since

$$f_1(v) + f_2(v) + \ldots + f_p(v) \geq d_G(v),$$

equation (2.1) leads to $f_1(v) + f_2(v) + \ldots + f_p(v) = d_G(v)$ and the proof of (a) is complete.

The proof of (b) is by reductio ad absurdum. Then, there exist two vertices z, v and an index $j \in \{1, 2, \ldots, p\}$ such that $f_j(z) \neq 0$, $v \in N_G(z)$ and $f_j(v) < \mu(z, v)$. By symmetry, we may assume $j = 1$. Then, $(G', f') = (G, f)/(z, 1)$ is a non-partitionable pair such that $0 = f_1'(v) > f_1(v) - \mu(z, v)$, $f_i'(v) = f_i(v)$ for all $i \in \{2, 3, \ldots, p\}$ and, by (a),

$$\begin{aligned} d_G(v) - \mu(z, v) &= f_1'(v) + f_2'(v) + \ldots + f_p'(v) \\ &> f_1(v) - \mu(z, v) + f_2'(v) + \ldots + f_p'(v) \\ &= d_G(v) - \mu(z, v), \end{aligned}$$

which is impossible.

In order to prove (c), let u be an arbitrary vertex of G and let $G' = G - u$. By construction, each component H of G' contains a vertex $u' \in N_G(u)$ and it holds

$$f_1(u') + f_2(u') + \ldots + f_p(u') > d_H(u').$$

Then, statement (a) implies that each component of G' is f-partitionable and, thus, G' is f-partitionable. Let $(G_1, G_2, \ldots, G_p)$ be an arbitrary f-partition of G', and let $i \in \{1, 2, \ldots, p\}$. The graph G is not f-partitionable, therefore, $G_i' = G_i + u$ is not strictly f_i-degenerate, i. e. there is a non-empty subgraph H of G_i' such that $d_H(v) \geq f_i(v)$ for all $v \in V(H)$. Clearly, u belongs to H and thus $f_i(u) \leq d_H(u) \leq d_{G_i'}(u)$. Since, by (a), $f_1(u) + f_2(u) + \ldots + f_p(u) = d_G(u)$ and $d_G(u) = d_{G_1'}(u) + d_{G_2'}(u) + \ldots + d_{G_p'}(u)$, we conclude $f_i(u) = d_{G_i'}(u)$ and the proof is complete. ∎

Main Result

Now we got everything necessary in order to prove the "only if"-direction. Case 1 in the proof is pretty similar to the simple case, however, for case 2 some new ideas are needed.

Theorem 2.6 *If (G, f) is a non-partitionable pair, then G is f-hard.*

Proof: Let (G, f) be a non-partitionable pair, say, of dimension p. We prove the theorem by induction on the order of G. By Proposition 2.5,

$$f_1(v) + f_2(v) + \ldots + f_p(v) = d_G(v) \quad \text{for all } v \in V(G) \tag{2.2}$$

holds.

Case 1: *G contains a separating vertex v^*.* Then, G is the union of two connected induced subgraphs G^1 and G^2 with $V(G^1) \cap V(G^2) = \{v^*\}$ and it holds $|G^j| < |G|$ for $j \in \{1, 2\}$. By applying Proposition 2.5, we obtain that $G - v^*$ has an f-partition $(G_1, G_2, \dots, G_p)$ and $f_i(v^*) = d_{G_i+v^*}(v^*)$ for all $i \in \{1, 2, \dots, p\}$. For $i \in \{1, 2, \dots, p\}$ we set $G_i^1 = G_i \cap G^1$ and $G_i^2 = G_i \cap G^2$. Then, G_i is the disjoint union of G_i^1 and G_i^2, and

$$f_i(v^*) = d_{G_i+v^*}(v^*) = d_{G_i^1+v^*}(v^*) + d_{G_i^2+v^*}(v^*). \tag{2.3}$$

For $j \in \{1, 2\}$, let $f^j \in \mathcal{V}_p(G^j)$ be the vector function satisfying

$$f_i^j(v) = \begin{cases} f_i(v) & \text{if } v \in V(G^j - v^*), \\ d_{G_i^j+v^*}(v^*) & \text{if } v = v^* \end{cases}$$

for all $v \in V(G^j)$ and all $i \in \{1, 2, \dots, p\}$. By combining (2.2) and (2.3), it follows that

$$f_1^j(v) + f_2^j(v) + \ldots + f_p^j(v) = d_{G^j}(v)$$

for $j \in \{1, 2\}$ and for each $v \in V(G^j)$. If G^j is not f^j-partitionable for some $j \in \{1, 2\}$, it follows from the induction hypothesis that G^j is f^j-hard. Thus, if G^j is not f^j-partitionable for each $j \in \{1, 2\}$, G is f-hard (by (H4)) and there is nothing left to prove. If this is not the case, we may assume, that G^1 has an f^1-partition $(G_1', G_2', \dots, G_p')$ (by symmetry). Again by symmetry, we may assume that $v^* \in V(G_1')$. Consider the p-partition $(H_1, H_2, \dots, H_p)$ of G, whereby $H_1 = G_1' \cup G_1^2 + v^*$ and $H_i = G_i' \cup G_i^2$ for $i \in \{2, 3, \dots, p\}$. By construction, H_i is strictly f_i-degenerate for each $i \in \{2, 3, \dots, p\}$. We show that H_1 is strictly f_1-degenerate. To this end, let H be a non-empty subgraph of H_1. If $H \subseteq G_1^2$, then $d_H(v) < f(v)$ holds for some vertex $v \in V(H)$, since G_1^2 is strictly f_1-degenerate. Otherwise, $H' = H \cap G_1'$ is a non-empty subgraph of G_1' and, since G_1' is strictly f_1^1-degenerate, there exists a vertex $v \in V(H')$ such that $d_{H'}(v) < f_1^1(v)$. If $v \neq v^*$, then $d_H(v) = d_{H'}(v) < f_1^1(v) = f_1(v)$ and we are done. Else, $v = v^*$ and it follows from (2.3) and from the definition of f_1^j that

$$d_H(v^*) \leq d_{H'}(v^*) + d_{G_1^2+v^*}(v^*) < f_1^1(v^*) + f_1^2(v^*) = f_1(v^*),$$

so we are done, too. Thus, H_1 is strictly f_1-degenerate and, hence, G has an f-partition, which contradicts the premise. Therefore, the first case is complete.

Case 2: *G is a block.* Assume that $G = tK_n$ for integers $t, n \geq 1$. Then, by (2.2),

$$f_1(v) + f_2(v) + \dots f_p(v) = t(n-1)$$

holds for all $v \in V(G)$. If $n = 1$, there is nothing left to prove. Hence, let $u, v \in V(G)$. Then, by Proposition 2.5(c), $G - u$ has an f-partition $(G_1, G_2, \dots, G_p)$ and $f_i(u) = d_{G_i+u}(u) = t|G_i|$ for every $i \in \{1, 2, \dots, p\}$. By symmetry, we can assume that $v \in V(G_1)$. Since G_1 is strictly f_1-degenerate and, since $f_1(u) = d_{G_1+u}(u) > d_{(G_1-v)+u}(u)$, $G_1' = (G_1 - v) + u$ is also strictly f_1-degenerate and, therefore, $(G_1', G_2, \dots, G_p)$ is an f-partition of $G - v$ satisfying $|G_1'| = |G_1|$. As a consequence, $f_i(v) = t|G_i|$ for every $i \in \{1, 2, \dots, p\}$ (by Proposition 2.5(c)). So we have $f_i(u) = f_i(v) = t|G_i|$ for each $i \in \{1, 2, \dots, p\}$ and hence, G is f-hard by (H2). In the following assume that G is not a tK_n for some $t, n \geq 1$. This implies, in particular, that $|G| \geq 3$.

Claim 2.6.1 *If there is a vertex $z \in V(G)$ and an index $j \in \{1, 2, \dots, p\}$ such that $f_j(z) \neq 0$, then $(G', f') = (G, f)/(z, j)$ is a non-partitionable pair. Furthermore, the following statements hold:*

(a) *G' is f'-hard.*

(b) *$f_j(v) \geq \mu(z, v)$ for all $v \in V(G)$.*

Proof : Since G is a block and, thus, z is a non-separating vertex, (G', f') is a non-partitionable pair by Proposition 2.4. Applying the induction hypothesis then leads to G' being f'-hard. This proves (a). Due to the fact that G has no separating vertices, statement (b) follows from Proposition 2.5(b) and from the fact that $f_j(v) \geq 0 = \mu(z, v)$ if $v \notin N_G(z)$. □

Note that the above claim implies that each coordinate f_i is either constant zero or nowhere zero. If there is at most one nowhere-zero coordinate, then

$$f(v) = (0, \dots, 0, d_G(v), 0, \dots, 0)$$

for all $v \in V(G)$ and G is f-hard by (H1).

It only remains to consider the case when f has at least two nowhere-zero coordinates. By symmetry we may assume that both f_1 and f_2 are nowhere zero.

Claim 2.6.2 *Let $z \in V(G)$, let $G' = G - z$ and let u and u' be non-separating vertices contained in the same block of G'. Then, $\mu(u, z) = \mu(u', z)$.*

Proof : Consider $(G', f') = (G, f)/(z, 1)$ and $(G', f'') = (G, f)/(z, 2)$. Then, by Claim 2.6.1, G' is both f'-hard, and f''-hard and, by Proposition 2.3, we have

$$f_1(u') - \mu(u', z) = f_1'(u') = f_1'(u) = f_1(u) - \mu(u, z),$$

and also

$$f_2(u') = f_2'(u') = f_2'(u) = f_2(u).$$

By regarding f'', we obtain

$$f_1(u') = f_1''(u') = f_1''(u) = f_1(u)$$

as well as

$$f_2(u') - \mu(u', z) = f_2''(u') = f_2''(u) = f_2(u) - \mu(u, z).$$

By merging those equalities we easily obtain the statement. □

Finally, let z be an arbitrary vertex of G. We claim, that $G - z$ is not a block. Otherwise, by Claim 2.6.2, there is an integer $t \geq 1$ such that $\mu(z, u) = t$ for all $u \in V(G - z)$. As a consequence, $G - u$ is also a block for all $u \in V(G - z)$ and, since $\mu(z, u) = t$, Claim 2.6.2 implies that $\mu(u, u') = t$ for all $u, u' \in V(G)$. Hence, G is a tK_n for some $n \geq 1$, a contradiction. Therefore, $G' = G - z$ has at least two endblocks for all $z \in V(G)$.

Case 2.1: *There is a vertex $z \in V(G)$ and an end-block B of $G' = G - z$ with $|B| = 2$.* Let $V(B) = \{u, v\}$. Then, B has exactly one non-separating vertex, say u. Let $s = \mu(z, u) \geq 1$, and let $(G', f') = (G, f)/(z, 1)$ and $(G', f'') = (G, f)/(z, 2)$. Since G' has less vertices than G and since f_1 and f_2 are nowhere-zero, the induction hypothesis implies that $f'(u) = (0, t, 0, \ldots, 0)$ and $f''(u) = (t, 0, \ldots, 0)$ for some $t \geq 1$. Thus, it holds $f(u) = (t, t, 0 \ldots, 0)$ and we conclude $s = t$ and $s = \mu(u, v)$. Since $N_G(u) = \{v, z\}$ and since $G - u$ is not a block, $G - u$ has exactly two end-blocks both of cardinality 2 (by Claim 2.6.2). Due to the fact that G is connected, this implies that $G - w$ has exactly two end-blocks (both of cardinality 2) for all $w \in V(G)$, and hence, G is a multi-cycle. Furthermore, reapplying the above argumentation shows that each pair of adjacent vertices is joined by exactly s edges and that $f(w) = (s, s, 0, \ldots, 0)$ for all $w \in V(G)$. Since G is not f-partitionable, $|G|$ must obviously be odd. Hence, G is f-hard (by (H3)).

Case 2.2: *Each endblock of $G' = G - z$ has at least 3 vertices for all $z \in V(G)$.* We choose a vertex $z \in V(G)$ of minimum degree. Let again $(G', f') = (G, f)/(z, 1)$, $(G', f'') = (G, f)/(z, 2)$ and let $\mathfrak{B}$ be the set of endblocks of $G' = G - z$. For $B \in \mathfrak{B}$, we denote by v_B the only separating vertex of G' in $V(B)$.

Claim 2.6.3 *For each $B \in \mathfrak{B}$ there is an $f^* \in \{f', f''\}$, which has at least two nowhere-zero coordinates restricted to $V(B) \setminus \{v_B\}$.*

Proof : Otherwise, there exists a $B \in \mathfrak{B}$ such that f', f'' both have at most one nowhere-zero coordinate in $V(B) \setminus \{v_B\}$. Then, the induction hypothesis implies that $f'(u) = (0, d_{G'}(u), 0, \ldots, 0)$, and that $f''(u) = (d_{G'}(u), 0, \ldots, 0)$ for all $u \in V(B) \setminus \{v_B\}$ (since f_1 and f_2 are nowhere zero). Consequently, we obtain $f(u) = (d_{G'}(u), d_{G'}(u), 0, \ldots, 0)$ and $\mu(z, u) = d_{G'}(u)$ for all $u \in V(B) \setminus \{v_B\}$. In particular, Claim 2.6.2 implies that there exists an integer $t \geq 1$ such that $t = \mu(z, u) = d_{G'}(u)$ for all $u \in V(B) \setminus \{v_B\}$. Since $|B| \geq 3$ and since G' has at least two end-blocks with vertices adjacent to z this leads to

$$d_G(z) > \sum_{u' \in V(B)} \mu(z, u') \geq 2t = d_G(u)$$

for all $u \in V(B) \setminus \{v_B\}$, a contradiction to $d_G(z) = \delta(G)$. This proves the claim. □

Note that Claim 2.6.3 together with the induction hypothesis particularly implies that each $B \in \mathfrak{B}$ is a tK_m with $t, m \geq 1$ or a tC_m for some $t \geq 1, m \geq 3$ odd.

Claim 2.6.4 *Each block from $\mathfrak{B}$ is a tK_m for some $t, m \geq 1$.*

Proof : Assume, by contrary, that there is a $B \in \mathfrak{B}$ such that $B = tC_m$ for some $t \geq 1, m \geq 5$ odd. Then, it follows from Claim 2.6.3 and from the induction hypothesis that either

(1) $f'(u) = (t, t, 0, \ldots, 0)$ for each $u \in V(B) \setminus \{v_B\}$,

(2) $f'(u) = (0, 2t, 0, \ldots, 0)$ for each $u \in V(B) \setminus \{v_B\}$, or

(3) $f'(u) = (0, t, t, 0, \ldots, 0)$ for each $u \in V(B) \setminus \{v_B\}$ (except for symmetry).

If (2) holds, Claim 2.6.3 implies that $f''(u) = (t, t, 0, \ldots, 0)$ for all $u \in V(B) \setminus \{v_B\}$ and, hence, $f(u) = (t, 2t, 0, \ldots, 0)$. However, this implies $\mu(z, u) = t$ for all $u \in V(B) \setminus \{v_B\}$. Since $|B| \geq 5$, it follows that $d_G(z) \geq 4t > 3t = d_G(u)$ for all $u \in V(B) \setminus \{v_B\}$. Thus, z is not of minimum degree, a contradiction. If (3) is satisfied, then $f''(u) = (t, 0, t, 0, \ldots, 0)$ for all $u \in V(B) \setminus \{v_B\}$ and, thus, $f(u) = (t, t, t, 0 \ldots, 0)$. Again, this implies $\mu(z, u) = t$ for all $u \in V(B) \setminus \{v_B\}$ and, consequently, z is not of minimum degree, a contradiction. Therefore,

(1) holds. By symmetry, it follows that $f''(u) = (t, t, 0, \ldots, 0)$ for all $u \in V(B) \setminus \{v_B\}$. But this is impossible, since the first two components of f are nowhere zero and since z is adjacent to at least one non-separating vertex of B. This shows the claim. □

Hence, each end-block $B \in \mathfrak{B}$ is a tK_m for some $t \geq 1, m \geq 3$. Let $B \in \mathfrak{B}$, say $B = tK_m$. By symmetry and by Claim 2.6.3, we may assume that f' restricted to $V(B) \setminus \{v_B\}$ has not less than two nowhere-zero coordinates. Hence, there are integers $m_1, m_2, \ldots, m_p \geq 0$ with at least two m_i different from zero such that $m_1 + m_2 + \ldots + m_p = m - 1$ and that $f'(u) = (tm_1, tm_2, \ldots, tm_p)$ for all $u \in V(B) \setminus \{v_B\}$. Then, regarding f'', there are also integers $m'_1, m'_2, \ldots, m'_p \geq 0$ with at least one $m'_i \geq 1$ such that $m'_1 + m'_2 + \ldots + m'_p = m - 1$ and that $f''(u) = (tm'_1, tm'_2, \ldots, tm'_p)$ for all $u \in V(B) \setminus \{v_B\}$. Since tm_2 is different from tm'_2 due to the reduction method, we conclude that there is a $k \geq 1$ such that $\mu(z, u) = k \cdot t$ for all $u \in V(B) \setminus \{v_B\}$, and, therefore,

$$d_{(B - v_B) + z}(z) = kt(m - 1). \tag{2.4}$$

Note that this implies as well

$$d_G(u) = t(m - 1) + kt \tag{2.5}$$

for all $u \in V(B) \setminus \{v_B\}$.

Since $|\mathfrak{B}| \geq 2$, we can choose two different end-blocks B and B' from $\mathfrak{B}$, such that $B = tK_m$ and that $B' = t'K_{m'}$ for some integers $t, t', m, m' \geq 1$. Then, by (2.4) and by (2.5), there are integers $k, k' \geq 1$ such that

$$d_G(z) \geq kt(m - 1) + k't'(m' - 1),$$

as well as

$$d_G(u) = t(m - 1) + kt \quad \text{and} \quad d_G(u') = t'(m' - 1) + k't'$$

for all $u \in V(B) \setminus \{v_B\}, u' \in V(B') \setminus \{v_{B'}\}$. By symmetry, we may assume

$$kt(m - 1) \geq k't'(m' - 1).$$

Since $k' \geq 1, m' \geq 3$, this implies $kt(m-1) \geq 2t'$, and hence,

$$\begin{aligned} d_G(u') &= t'(m'-1) + k't' \\ &\leq k't'(m'-1) + t' \\ &< k't'(m'-1) + 2t' \\ &\leq k't'(m'-1) + kt(m-1) \\ &\leq d_G(z), \end{aligned}$$

which is a contradiction to z having minimum degree. This completes the proof. ■

2.3 Applications

In this section we will present some applications of Theorem 2.2 following Borodin, Kostochka and Toft's depiction in [7].

Brooks's Theorem for List-Colorings

Recall from the introduction that the chromatic number, respectively the list-chromatic number of a graph G is always less or equal to the coloring number of G. In particular, it holds

$$\chi(G) \leq \chi^{\ell}(G) \leq \mathrm{col}(G) \leq \Delta(G) + 1.$$

Of course, this inequality raises the question in which case $\chi(G) = \Delta(G) + 1$ is satisfied. This problem was solved by Brooks [8] in 1941. His famous theorem states that complete graphs and odd cycles are the only connected graphs G with $\chi(G) = \Delta(G) + 1$. Erdős, Rubin, and Taylor [13] managed to extend Brooks's Theorem to list-colorings. They proved the following.

Theorem 2.7 (Erdős, Rubin, and Taylor). *Let G be a connected simple graph and let L be a list-assignment satisfying $|L(v)| \geq d_G(v)$ for all $v \in V(G)$. Then, G admits a proper L-coloring if and only if at least one block of G is neither an odd cycle nor a complete graph. As a consequence, $\chi^{\ell}(G) \leq \Delta(G) + 1$ and equality holds if and only if G is a complete graph or an odd cycle.*

What is the connection between the above theorem and Theorem 2.2? To see this, let G be a simple graph, and let L be a list-assignment of G with color set C. Assume that $|C| = p$. By renaming the colors we get $C = \{1, 2, \ldots, p\}$. Now let $f \in \mathcal{V}_p(G)$ such that for all $i \in C$ it holds $f_i(v) = 1$ if $i \in L(v)$ and $f_i(v) = 0$, otherwise. Then, for any proper L-coloring φ of G, we obtain that $(G_1, G_2, \ldots, G_p)$ with $G_i = G[\varphi^{-1}(i)]$ is an f-partition of G. On the other hand, given an f-partition $(G_1, G_2, \ldots, G_p)$ of G we can create a proper L-coloring φ of G just by setting $\varphi(v) = i$ if $v \in V(G_i)$, since G_i is edgeless by construction. Thus, G admits a proper L-coloring if and only if G is f-partitionable. As a consequence, Theorem 2.2 restricted to simple graphs immediately implies Theorem 2.7.

Additional Degree Constraints

Another extension of Brooks's Theorem was proven by Borodin [4], and, independently, by Bollobás and Manvel [3].

Theorem 2.8 (Borodin/Bollobás and Manvel). *Let G be a connected simple graph with maximum degree $\Delta(G) = \Delta \geq 3$ different from $K_{\Delta+1}$. Let also $k_1, k_2, \ldots, k_p$ be positive integers, $p \geq 2$, such that $k_1 + k_2 + \ldots + k_p \geq \Delta$. Then, there is a p-partition $(G_1, G_2, \ldots, G_p)$ of G such that* $\operatorname{col}(G_i) \leq k_i$ *whenever $1 \leq i \leq p$.*

Brooks's Theorem follows directly from the above theorem by setting

$$k_1 = k_2 = \ldots = k_p = 1.$$

Borodin [5] even generalized Theorem 2.8 by using a simple argument. The generalization was also proven indepently by Bollobás and Manvel [3].

Theorem 2.9 *Let G be a connected simple graph with maximum degree $\Delta(G) = \Delta \geq 3$ different from $K_{\Delta+1}$ Let also $k_1, k_2, \ldots, k_p$ be positive integers, $p \geq 2$, such that*

$$k_1 + k_2 + \ldots + k_p \geq \Delta.$$

Then, there is a p-partition $(G_1, G_2, \ldots, G_p)$ of G satisfying $col(G_i) \leq k_i$ and $\Delta(G_i) \leq k_i$ whenever $1 \leq i \leq p$.

While Borodin, Bollobás and Manvel only considered simple graphs, it is possible to prove some equivalent results for arbitrary graphs. Since Theorem 2.8 follows from Theorem 2.9, we will narrow down to mentioning the general case related to Theorem 2.9.

Theorem 2.9' *Let G be a connected graph with maximum degree $\Delta(G) = \Delta \geq 1$ that is not a tK_n for some $t, n \geq 1$ and not a tC_n for $t \geq 1, n \geq 3$ odd. Let also $k_1, k_2, \ldots, k_p$ be positive integers, $p \geq 2$, such that $k_1 + k_2 + \ldots + k_p \geq \Delta$. Then, there is a p-partition $(G_1, G_2, \ldots, G_p)$ of G such that $col(G_i) \leq k_i$ and $\Delta(G_i) \leq k_i$ whenever $1 \leq i \leq p$.*

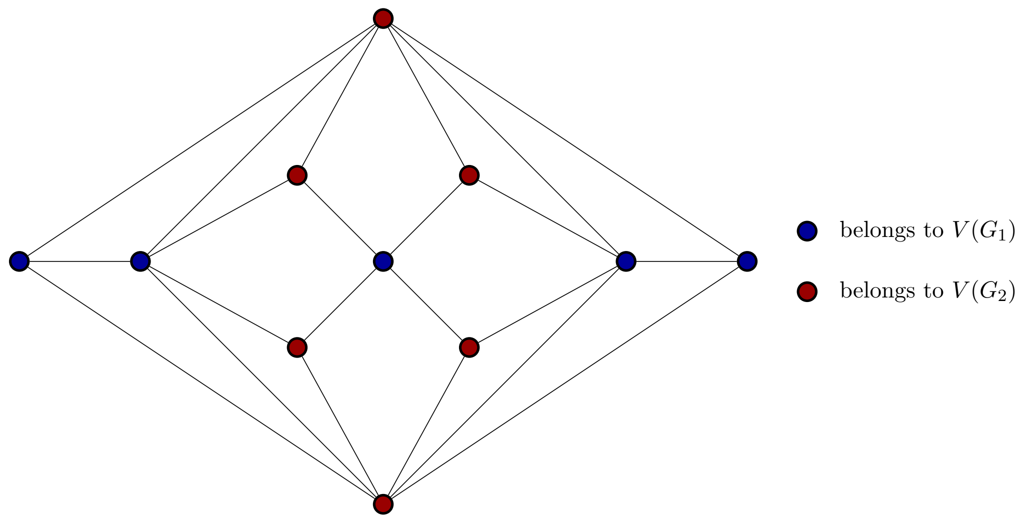

Figure 2.2: A partition (G_1, G_2) of G such that $\mathrm{col}(G_i) \leq 3$ and $\Delta(G_i) \leq 3$ for $i = 1, 2$.

Note that the condition $\Delta(G) \geq 3$ in the simple case ensures that G is not a block of type (H3). However, since the maximum degree of a tC_n may get arbitrarily high for huge t, we need to prohibit all those tC_n by hand. We will see that with the help of Theorem 2.2, the proof of Theorem 2.9' gets pretty simple. Nevertheless, first of all it is necessary to make the following useful observation.

Corollary 2.10 *If a graph G is f-partitionable for some $f \in \mathcal{V}_p(G)$ satisfying*

$$f_1(v) + f_2(v) + \ldots + f_p(v) \geq d_G(v)$$

for all $v \in V(G)$, then there is an f-partition $(G_1, G_2, \ldots, G_p)$ of G such that $d_{G_i}(v) \leq f_i(v)$ for all $v \in V(G)$ and $1 \leq i \leq p$.

Proof: For an arbitrary partition $(G_1, G_2, \ldots, G_p)$ of G let

$$W_{(G_1,G_2,\ldots,G_p)} = \sum_{i=1}^{p} \big(|E(G_i)| - \sum_{v \in V(G_i)} f_i(v)\big).$$

If there is a $v \in V(G)$ and $i \neq j$ such that $v \in V(G_i)$, $d_{G_i}(v) \geq f_i(v)$ and $d_{G_j}(v) < f_j(v)$, moving v from G_i to G_j decreases $W_{(G_1,G_2,\ldots,G_p)}$. To see this, assume $i < j$ (by symmetry), let $G'_i = G_i - v$, $G'_j = G_j + v$, and let $G'_k = G_k$ for $k \in \{1, 2, \ldots, p\} \setminus \{i, j\}$. Then, for $W = W_{(G_1, G_2, \ldots, G_p)}$ and for $W' = W_{(G'_1, G'_2, \ldots, G'_p)}$, we obtain

$$W - W' = -d_{G_i}(v) + f_i(v) + d_{G_j}(v) - f_j(v) < 0.$$

We claim that an f-partition $(G_1, G_2, \ldots, G_p)$ of G that minimizes $W_{(G_1, G_2, \ldots, G_p)}$ is the required partition. Assume that there is a $v \in V(G)$ such that $v \in V(G_i)$ for some $i \in \{1, 2, \ldots, p\}$ and $d_{G_i}(v) > f_i(v)$. Since $f_1(v) + f_2(v) + \ldots + f_p(v) \geq d_G(v)$, there is a $j \in \{1, \ldots, p\}$ such that $d_{G_j}(v) < f_j(v)$ and, thus, $G_j + v$ is still strictly f_j-degenerate. However, the above observation implies that moving v to G_j leads to a new f-partition $(G'_1, G'_2, \ldots, G'_p)$ with $W_{(G'_1, G'_2, \ldots, G'_p)} < W_{(G_1, G_2, \ldots, G_p)}$, a contradiction. ■

In particular, the above Corollary leads to a stronger version of Theorem 2.2.

Theorem 2.2' *Let G be a connected graph, and let $f \in \mathcal{V}_p(G)$ be a vector funtion with $p \geq 1$ such that $f_1(v) + f_2(v) + \ldots + f_p(v) \geq d_G(v)$ for all $v \in V(G)$. Then, there is an f-partition $(G_1, G_2, \ldots, G_p)$ of G such that $d_{G_i}(v) \leq f_i(v)$ for all $v \in V(G_i)$ and all $i \in \{1, 2, \ldots, p\}$ if and only if G is not f-hard.*

With that said, we are able to prove Theorem 2.9'.

Proof of Theorem 2.9' : Let $f_i(v) = k_i$ for all $v \in V(G)$ and $i \in \{1, \ldots, p\}$. Then, $f_1(v) + f_2(v) + \ldots + f_p(v) \geq \Delta \geq d_G(v)$ and $f_i(v) \geq 1$ for all $i \in \{1, 2, \ldots, p\}$ and all $v \in V(G)$. Since G is neither a tK_n nor a tC_n with $n \geq 3$ odd and since $p \geq 2$, we easily conclude that (G, f) is not a hard pair. As a consequence, Theorem 2.2' implies that G is f-partitionable, and, moreover, that there is an f-partition $(G_1, G_2, \ldots, G_p)$ of G such that $d_{G_i}(v) \leq f_i(v)$ for all $v \in V(G)$ and $i \in \{1, 2, \ldots, p\}$. Then, for all $i \in \{1, 2, \ldots, p\}$, the graph G_i is strictly f_i-degenerate and, thus, $\text{col}(G_i) \leq k_i$. ■

Point-Partition Number and List Point-Partition Number

Let G be a graph. The **point arboricity** of G is the least number k of forests forming a k-partition of G. Note that a k-partition $(F_1, F_2, \ldots, F_k)$ of G into forests can be seen as a coloring with k colors such that each color class induces an acyclic subgraph. We say that a coloring or list-coloring of G is **acyclic** if each color class induces a forest. Then, the point arboricity of G is the least number k such that G admits an acyclic k-coloring. Moreover, the **list point arboricity** of G is the minimum k such that there is an acyclic L-coloring of G for any list-assignment L satisfying $|L(v)| \geq k$ for all $v \in V(G)$. We can generalize those definitions even further. For a graph G, the **point-partition number** $\alpha_s(G)$ (with $s \geq 0$), introduced by Lick and White [20], is the minimum number k such that G admits a k-coloring in which each color class induces an s-degenerate subgraph of G. Hence, $\alpha_0(G)$ and $\alpha_1(G)$ correspond to the chromatic number and to the point arboricity of G, respectively. Furthermore, let $\alpha_s^\ell(G)$ be the **list point-partition number**, that is the least integer k such that for each list-assignment L fulfilling $|L(v)| \geq k$ for all $v \in V(G)$ there is an L-coloring of G in which each color class induces an s-degenerate subgraph of G. Contrary to the case of proper colorings, for the (list) point-partition number multiple edges play an important role.

Before we begin to characterize the point-partition number and the list point-partition number, we need to make the following observation.

Corollary 2.11 *Let G be a graph, let $f \in \mathcal{V}_p(G)$ and let $h : V(G) \to \mathbb{Z}_{\geq 1}$. If G is strictly h-degenerate and if $f_1(v) + f_2(v) + \ldots + f_p(v) \geq h(v)$ for all $v \in V(G)$, then G is f-partitionable.*

Proof: The proof is by induction on the number of vertices of G. For $|G| = 1$, we have $V(G) = \{v\}$. But then, $d_G(v) = 0 < h(v)$ and since $f_1(v) + f_2(v) + \ldots + f_p(v) \geq h(v)$, there is an $i \in \{1, 2, \ldots, p\}$ with $f_i(v) > 0$. Setting $G_i = \{v\}$ and $G_j = \varnothing$ for $j \neq i$ gives us the required partition. Now assume that $|G| \geq 2$. We choose a vertex v with $d_G(v) < h(v)$. Then, the induction hypothesis implies that there is an f-partition $(G_1, G_2, \ldots, G_p)$ of $G - v$. Since $d_G(v) < f_1(v) + f_2(v) + \ldots + f_p(v)$, there exists an $i \in \{1, 2, \ldots, p\}$ such that $d_{G_i}(v) < f_i(v)$. Obviously, by setting $G_i' = G[V(G_i) \cup \{v\}]$ we obtain an f-partition $(G_1, G_2, \ldots, G_{i-1}, G_i', G_{i+1}, \ldots, G_p)$. ∎

If we take a look back at Theorem 2.9', we obtain by setting $k_1 = k_2 = \ldots = k_p = s$, that the point-partition number $\alpha_s(G)$ is at most p if G is a connected graph with maximum degree $\Delta(G) = \Delta \geq 1$ different from tK_n with $t, n \geq 1$ and tC_n for $t \geq 1, n \geq 3$ odd such that $ps = k_1 + k_2 + \ldots + k_p \geq \Delta$. Those cases were originally solved for simple graphs by Kronk and Mitchem [19] and Mitchem [24]. If we take a look at the list point-partition number, we conclude the following.

Corollary 2.12 *Let G be a connected graph different from tK_n for all $t, n \geq 1$ and different from tC_n for all $t \geq 1, n \geq 3$ odd, and let $\Delta(G) = \Delta \geq 1$. Furthermore, let k, s be integers such that $k \cdot s \geq \Delta$, where $k \geq 2$, and let L be a list-assignment such that $|L(v)| \geq k$ for all $v \in V(G)$. Then, there is an L-coloring of G such that each color class induces a strictly s-degenerate subgraph.*

Proof: Let C be the set of colors used in the union of all $L(v), v \in V(G)$. By renaming the colors we may assume $C = \{1, 2, \ldots, p\}$. We define $f \in \mathcal{V}_p(G)$ as follows. Let

$$f_i(v) = \begin{cases} s & \text{if } i \in L(v), \\ 0 & \text{otherwise.} \end{cases}$$

First of all, we obtain

$$\sum_{i=1}^{p} f_i(v) \geq |L(v)|s \geq ks \geq \Delta$$

for all $v \in V(G)$. If G admits an f-partition $(G_1, G_2, \ldots, G_p)$, setting $\varphi(v) = i$ if $v \in V(G_i)$ clearly leads to the required list-coloring. Assume that G is not f-partitionable, and, therefore, that G is f-hard (by Theorem 2.2).

If G is not ks-regular, then G is strictly ks-degenerate. Thus, by Corollary 2.11, there is an f-partition $(G_1, G_2, \ldots, G_p)$ of G, a contradiction. Otherwise, G is ks-regular, and therefore, $ks = \Delta$. Since $k \geq 2$, there are no blocks of type (H1) with respect to f. By assumption, (G, f) is not of type (H2) nor (H3), as well. Thus, (G, f) is of type (H4) and each block is a tK_m for some $t, m \geq 1$ or a tC_m for some $t \geq 1, m \geq 3$ odd. However, in this case G cannot be Δ-regular, a contradiction. ■

Consider a graph G and an arbitrary list-assignment L. We define G to be **$L \times s$-choosable** if there is an L-coloring of G such that each color class induces a strictly s-degenerate subgraph. A natural extension of Corollary 2.12 will be expressed in the following theorem. It was proven for simple graphs by Borodin, Kostochka and Toft.

Theorem 2.13 *Let G be a connected graph. Then, G is $L \times s$-choosable for each list-assignment L satisfying $|L(v)| \geq d_G(v)/s$ for each $v \in V(G)$ if and only if at least one block of G is different from tK_n for all $t, n \geq 1$, from an s-regular graph, and from a tC_n with $t \geq 1, n \geq 3$ odd.*

Proof: Let again C be the set of colors used in the union of all $L(v), v \in V(G)$ and assume $C = \{1, 2, \ldots, p\}$. Furthermore let $f \in \mathcal{V}_p(G)$ such that

$$f_i(v) = \begin{cases} s & \text{if } i \in L(v), \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that G is $L \times s$-choosable if and only if G is f-partitionable (since each block of type (H1) is s-regular by construction of f). By applying Theorem 2.2 we obtain the proof's statement. ∎

Graphs on Surfaces

The last application of Theorem 2.2, which shall be mentioned here, deals with the point-partition number for graphs on surfaces. However, in this subsection we do not allow multiple edges. Consider a surface $\mathscr{S}$ with Euler characteristic ε. The maximum value of $\alpha_s(G)$ over all graphs embeddable on $\mathscr{S}$ is called the **point-partition number** $\alpha_s(\mathscr{S})$ of $\mathscr{S}$. In 1890, Heawood [17] proved that for each surface $\mathscr{S}$ of Euler genus ε there is a number $H(\varepsilon) = \lfloor (7 + \sqrt{49 - 27\varepsilon})/2 \rfloor$ such that $\chi(G) \leq H(\varepsilon)$ for all (simple or not simple) graphs G that are embeddable on $\mathscr{S}$. Thus $\alpha_0(G) \leq H(\varepsilon)$ for all graphs G that are embeddable on a surface with euler genus ε.

For simple graphs, Lick and White [20] found all values $\alpha_s(\mathscr{S})$ except from $s = 0$, which corresponds to the four color problem, and $s = 1$ and $s = 2$ for the Klein bottle $\mathscr{K}$. The cases $s = 1$ and $s = 2$ were later solved by Borodin [4]; in particular, it holds $\alpha_1(\mathscr{K}) = 3$ and $\alpha_2(\mathscr{K}) = 2$. Borodin, Kostochka and Toft [7] extended this even further.

Theorem 2.14 (Borodin, Kostochka and Toft). *Let G be a simple graph embedded on a surface $\mathscr{S}$ of euler genus ε other than the plane and let $f \in \mathcal{V}_p(G)$ for some $p \geq 1$ such that $f_1(v) + f_2(v) + \ldots + f_p(v) \geq H(\varepsilon)$ for all $v \in V(G)$. Then, G is f-partitionable. Moreover, if G is embeddable on the Klein bottle and $f_1(v) + \ldots + f_s(v) \geq 6$ for all $v \in V(G)$, then G is f-partitionable unless (G, f) has a subgraph which is a 6-regular block of type (H1).*

Proof: Note that each simple graph G embeddable on $\mathscr{S}$ other than the plane is strictly $H(\varepsilon)$-degenerate. By Corollary 2.11, it follows that G is f-partitionable. Furthermore, since a graph on the Klein bottle is either 6-regular (and not a K_7 by Franklin's Theorem [15]) or has a vertex of degree at most 5, applying Theorem 2.8 leads to the second part of the Theorem. ∎

It is important to notice that this theorem does not hold for graphs with multiple edges. In order to illustrate this, consider an embedding of the $2K_5$ on the torus, where $\varepsilon = 2$ and $H(2) = 7$. If we choose $p = 2$ and set $f_1(v) = f_2(v) = 4$ for all $v \in V(2K_5)$, then it holds $f_1(v) + f_2(v) = 8 \geq 7 = H(2)$ for all $v \in V(2K_5)$. On the other hand, if $f = (f_1, f_2)$, then (G, f) is a hard pair of type (H2) and so Theorem 2.2 implies that G is not f-partitionable.

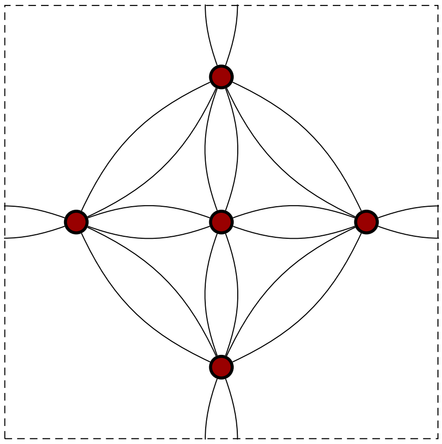

Figure 2.3: An embedding of the $2K_5$ on the torus.

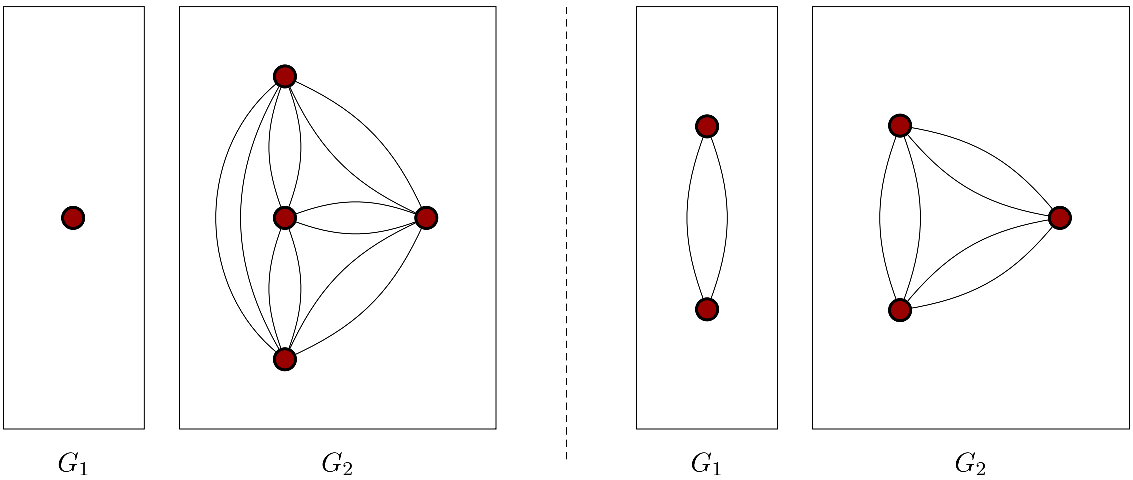

Figure 2.4: The two ways to partition a $2K_5$.

Chapter 3

Graph Partitions under Minimum Degree Constraints

The topic of the third chapter are partitions of graphs under minimum degree constraints. Started by a question of Gyöyi, in the last 35 years several mathematicians (consider [26], [25] for instance) found ways of how to partition graphs such that specific constraints for the minimum degree in each part of the partition are satisfied. However, most of the results were only proven for simple graphs until Amir Ban [1] managed to extend the basic case to weighted graphs. In the following section we will show, with the help of Ban's approach, how to transfer the most important results to graphs in general while obtaining some better bounds than Ban did.

3.1 Prior Research

In 1983, C. Thomassen [26] took up the question from Gyöyi if for each pair $s, t \geq 1$ of integers there exists an integer $g(s, t)$ such that each simple graph G with minimum degree $g(s, t)$ has a 2-partition (G_1, G_2) such that $\delta(G_1) \geq s$ and $\delta(G_2) \geq t$, respectively. He proved the following result.

Theorem 3.1 (Thomassen). *If $k \geq 1$ is an integer and G is a simple graph of minimum degree at least $12k$, then there is a 2-partition (G_1, G_2) of G such that G_1 and G_2 both have minimum degree at least k.*

However, Thomassen presumed that this boundary could still be improved by a sizeable margin and so he conjectured the following.

Conjecture 3.2 (Thomassen). *It holds $g(s,t) = s + t + 1$ for all s and t.*

In 1996, Stiebitz [25] solved the conjecture by using a generalized approach. For the sake of aesthetics, in the further progress we will consider partitions of the vertex set of a graph and not partitions of graphs anymore. Given a graph G and two functions $a, b : V(G) \to \mathbb{R}_{\geq}0$, a partition (A, B) of $V(G)$ is called an (a, b)**-feasible partition** (of the vertex set) of G if

(1) $d_{G[A]}(v) \geq a(v)$ for all $v \in A$, and

(2) $d_{G[B]}(v) \geq b(v)$ for all $v \in B$.

Thus, by putting $a(v) = s$ and $b(v) = t$ for all $v \in V(G)$, an (a, b)-feasible partition (A, B) is a partition of $V(G)$ that induces two subgraphs $G[A]$ and $G[B]$ such that each vertex in $G[A]$ has at least degree s and each vertex in $G[B]$ has at least degree t.

Theorem 3.3 (Stiebitz). *Let G be a simple graph, and let $a, b : V(G) \to \mathbb{Z}_{\geq 0}$ be two functions. Assume that $d_G(v) \geq a(v) + b(v) + 1$ for every vertex $v \in V(G)$. Then, there is an (a, b)-feasible partition of G.*

Stiebitz also mentioned some examples to show that the boundary in Theorem 3.3 is sharp. The first graph he mentioned is the K_{s+t+1}; if we set $a(v) = s$ and $b(v) = t$ for all $v \in V(K_{s+t+1})$, it is obvious, that an (a, b)-feasible partition does not exist. Another example is the graph of the icosahedron (See figure 3.1) with $a(v) = 1$ and $b(v) = 4$ for all vertices v. Then, no (a, b)-feasible partition exists due to the fact that the icosahedron is 5-regular, that $a(v) = 1$ for all $v \in V(G)$ and that $A, B \neq \varnothing$ requires A to contain at least 2 adjacent vertices u, v that have a common neighbor w in B. But then, $d_B(w) \leq 3$, a contradiction.

Throughout the last 20 years, many mathematicians addressed Thomassen's Conjecture and tried to improve Stiebitz's bound for graphs with spezified characteristics. Kaneko [18] as well as Bazgan, Tuza and Vanderpooten [2] examined the case, that G is a triangle-free simple graph. While Kaneko only considered constant functions a and b, Bazgan, Tuza and Vanderpooten generalized Kaneko's result to the case of variable functions.

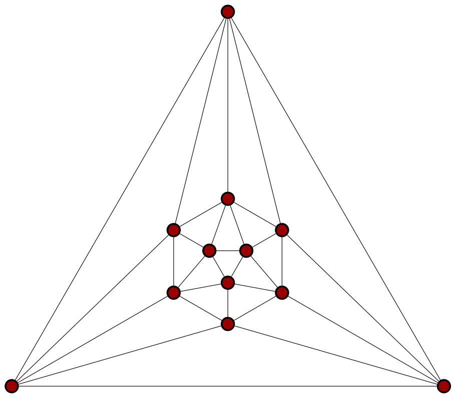

Figure 3.1: The graph of the icosahedron.

Theorem 3.4 (Bazgan, Tuza and Vanderpooten). *Let G be a triangle-free simple graph, and let $a, b : V(G) \to \mathbb{Z}_{\geq 1}$ be two functions. Assume that $d_G(v) \geq a(v) + b(v)$ for all $v \in V(G)$. Then, G has an (a, b)-feasible partition.*

In 2017, Liu and Xu [22] proved that one may obtain the same boundary as in the triangle-free case for K_4^--free graphs, whereby K_4^- denotes the graph that results from the K_4 by removing one edge.

Theorem 3.5 (Liu and Xu). *Let G be a K_4^--free simple graph, and let $a, b : V(G) \to \mathbb{Z}_{\geq 1}$ be two functions. Assume that $d_G(v) \geq a(v) + b(v)$ for all $v \in V(G)$. Then, G has an (a, b)-feasible partition.*

By forbidding cycles of length 3 and 4, Diwan [12] as well as Gerber and Kobler [16] managed to soften the degree-condition even more for constant, respectively variable functions a, b. Recall that $g(G)$ denotes the **girth** of a graph G, that is, the minimum integer $\ell \geq 3$ such that G contains a cycle of length ℓ.

Theorem 3.6 (Gerber and Kobler). *Let G be a simple graph with $g(G) \geq 5$, and let $a, b : V(G) \to \mathbb{Z}_{\geq 2}$ be two functions. Assume that $d_G(v) \geq a(v) + b(v) - 1$ for all $v \in V(G)$. Then, G has an (a, b)-feasible partition.*

Li and Xu [22] also managed to soften the condition for the above theorem. They proved the following.

Theorem 3.7 (Liu and Xu). *Let G be a triangle-free simple graph of which each vertex is contained in at most one cycle of length 4. Moreover, let $a, b : V(G) \to \mathbb{Z}_{\geq 2}$ be two* *functions, and assume that* $d_G(v) \geq a(v) + b(v) - 1$ *for all* $v \in V(G)$. *Then,* G *has an* (a, b)*-feasible partition.*

It shows that most of these theorems can be extended to arbitrary graphs. In the next few sections we will show how this may be done, whereas first of all we will motivate why it seems natural to consider graphs with multiple edges, as well.

3.2 Weighted Graph Partitions

With a similar approach as Stiebitz's, Ban [1] proved a related result for weighted graphs. Let G be a simple graph, and let $w : E(G) \to \mathbb{R}_{>0}$ be a **weight function** for G. Then,

$$d_G(v) = \sum_{e \in E_G(v)} w(e) \text{ and } W_G(v) = \max_{e \in E_G(v)} w(e)$$

is the **weighted degree** of v in G, respectively the **maximum weight** of an edge incident to v in G. Ban obtained the following result.

Theorem 3.8 (Ban). *Let* G *be a simple graph, and let* $w : E(G) \to \mathbb{R}_{>0}$ *be a weight function for* G. *Furthermore, let* $a, b : V(G) \to \mathbb{R}_{\geq 0}$ *be two functions. Assume that* $d_G(v) \geq a(v) + b(v) + 2W_G(v)$ *for every vertex* $v \in V(G)$. *Then, there is an* (a, b)*-feasible partition of* G.

Ban furthermore showed that his boundary is sharp. This can be seen by considering K_9 with unit edge weights, i. e. $w(e) = 1$ for all $e \in E(K_9)$. By setting $a(v) = b(v) = 3 + \varepsilon$, no (a, b)-feasible partition exists for any $\varepsilon > 0$. He additionally mentioned a nice application that shall be presented here, as well. For this to happen, it is necessary to generalize his theorem to weighted graph with loops. Let $w(v)$ be the weight of the loop incident to the vertex v (multiple loops get merged to one loop).

Corollary 3.9 (Ban). *Let* G *be a weighted graph with loops, and let* $a, b : V(G) \to \mathbb{R}_{\geq 0}$ *be two functions. Assume that* $d_G(v) \geq a(v) + b(v) + 2W_G(v) - 2w(v)$ *for every vertex* $v \in V(G)$. *Then, there is an* (a, b)*-feasible partition of* G.

This follows directly from applying Theorem 3.8 to a graph resulting from G by deleting all loops. Now let V be a set of grid squares of size 1 in $\mathbb{R}^2$, and let $r > 0$ be a fixed radius. Ban states the question if there exists a partition (A, B) of V such that

(1) For each $v \in A$, a circle of radius r drawn around the center of v covers at least as much area in A as in B, and

(2) For each $v \in B$, a circle of radius r drawn around the center of v covers at least as much area in B as in A.

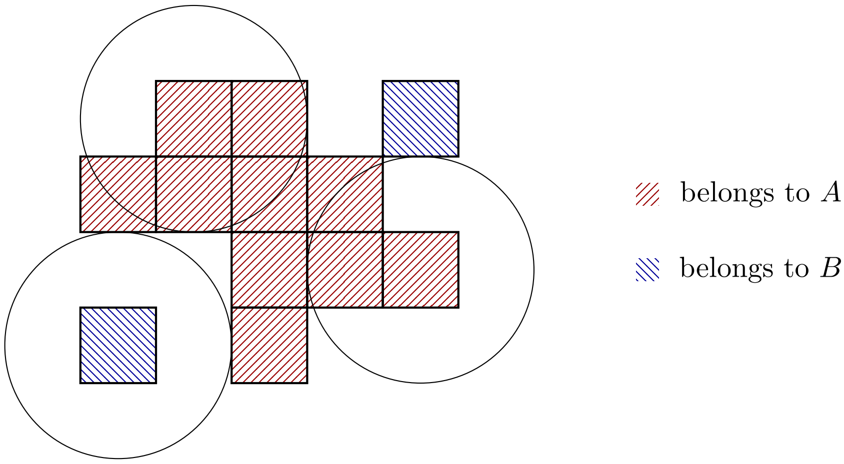

Figure 3.2: A trivial partition (A, B) that satisfies (1) and (2) for $r = 1.5cm$.

Ban's answer is as follows. For $r \leq \sqrt{\frac{2}{\pi}}$ and for an arbitrary $v \in V$, the square v covers the majority of the circle of radius r drawn around its centre. Thus, any partition is satisfactory. Nevertheless, for $r > \sqrt{\frac{2}{\pi}}$, we create a weighted graph G. Let $V(G) = V$, $E(G) = \{uv \mid u, v \in V(G)\}$, and for $e = uv \in E(G)$ let $w(e)$ be the area of the part of v whose distance from u's centre is at most r. Obviously, $w(e) \leq 1$ for every $e \in E(G)$. Due to the fact that $r > \sqrt{\frac{2}{\pi}} > \sqrt{\frac{1}{2}}$, it holds $w(v) = 1$ for each $v \in V(G)$. As a consequence, $W_G(v) \leq 1$ for all $v \in V(G)$. Thus, the existence of the required partition follows from Corollary 3.9 by setting $a(v) = b(v) = d_G(v)/2$ for each $v \in V(G)$.

3.3 Main Results

If we set $w(e) = 1$ for all $e \in E(G)$ and assume that $a, b : V(G) \to \mathbb{Z}_{\geq 0}$, then Ban's Theorem states the same as Stiebitz's Theorem, except that Ban requires $d_G(v) \geq a(v) + b(v) + 2$ for all $v \in V(G)$, whereas Stiebitz only needs $d_G(v) \geq a(v) + b(v) + 1$ for all $v \in V(G)$. Hence, one may wonder if at least in the case of a graph with integer weights, i. e. in the case of a graph with multiple edges, $d_G(v) \geq a(v) + b(v) + 2w_G(v) - 1$ for all $v \in V(G)$ is sufficient for the existence of an (a,b)-partition of G, whereby

$$w_G(v) = \max_{u \in V(G) \setminus \{v\}} \mu_G(v,u)$$

denotes the **weight** of the vertex v. That this is indeed the case, states the following theorem.

Theorem 3.10 *Let G be a graph with $\delta(G) \geq 1$, and let $a, b : V(G) \to \mathbb{Z}_{\geq 0}$ be two functions such that $d_G(v) \geq a(v) + b(v) + 2w_G(v) - 1$ for every $v \in V(G)$. Then, there is an (a,b)-feasible partition of G.*

Our proof is strongly based on Ban and therefore on Stiebitz's proof. For the triangle-free, respectively the K_4^--free case, we obtain the following.

Theorem 3.11 *Let G be a triangle-free graph with $\delta(G) \geq 1$, and let $a, b : V(G) \to \mathbb{Z}_{\geq 1}$ be two functions such that $d_G(v) \geq a(v) + b(v) + 2w_G(v) - 2$ for every $v \in V(G)$. Then, there is an (a,b)-feasible partition of G.*

Theorem 3.12 *Let G be a K_4^--free graph with $\delta(G) \geq 1$, and let $a, b : V(G) \to \mathbb{Z}_{\geq 1}$ be two functions such that $d_G(v) \geq a(v) + b(v) + 2w_G(v) - 2$ for every $v \in V(G)$. Then, there is an (a,b)-feasible partition of G.*

Note that this theorem obviously implies Theorem 3.11, so we abstain from giving an extra proof of the triangle-free case.

It shows that it is not easily possible to adjust the proof neither of Theorem 3.6 nor Theorem 3.7 in order to obtain the related statements for graphs in general. However, the evidence available indicates that the related statement is true, so we feel confident to conjecture the following. Recall that $g(G)$ denotes the girth of a graph G.

Conjecture 3.13 *Let G be a graph with $g(G) \geq 5$, and let $a, b : V(G) \to \mathbb{Z}_{\geq 2}$ be two functions such that $d_G(v) \geq a(v) + b(v) + 2w_G(v) - 3$ for every $v \in V(G)$. Then, there is an (a,b)-feasible partition of G.*

Conjecture 3.14 *Let G be a triangle-free graph of which each vertex is contained in at most one cycle of length 4. Moreover, let $a, b : V(G) \to \mathbb{Z}_{\geq 2}$ be two functions such that $d_G(v) \geq a(v) + b(v) + 2w_G(v) - 3$ for every $v \in V(G)$. Then, there is an (a,b)-feasible partition of G.*

3.4 Preliminary Considerations

In this section we shall prove two auxiliary results. Firstly, we need some notation. Let G be a graph, and let $f : V(G) \to \mathbb{Z}_{\geq 0}$ be a function. If $X \subseteq V(G)$ and $v \in V(G)$, then we write $d_X(v)$ for the degree of v in $G[X \cup \{v\}]$ (note that $v \in X$ may or may not hold). A subset $X \subseteq V(G)$ is said to be **f-meager** in G, if for every non-empty subset Y of X there is a vertex $v \in Y$ such that $d_Y(v) \leq f(v) + w_G(v) - 1$. A set $X \subseteq V(G)$ is called **f-nice** in G, if $d_X(v) \geq f(v)$ for all $v \in X$.

Now let $a, b : V(G) \to \mathbb{R}_{\geq}0$ be two functions. Note that a partition (A, B) of $V(G)$ is an (a, b)-feasible partition of G if and only if A is a-nice in G and B is b-nice in G. We say that a pair (A, B) is an **(a, b)-feasible pair** of G, if A and B are disjoint subsets of $V(G)$ such that A is a-nice and B is b-nice. Furthermore, a partition (A, B) of $V(G)$ is called an **(a, b)-meager** partition of G if A is a-meager in G and B is b-meager in G.

In the proofs of Theorem 3.10 and Theorem 3.12, we will need the following two observations related to [25].

Proposition 3.15 *Let G be a graph with $\delta(G) \geq 1$, and let $a, b : V(G) \to \mathbb{Z}_{\geq 0}$ be two functions such that $d_G(v) \geq a(v) + b(v) + 2w_G(v) - 3$ for all $v \in V(G)$. If there exists an (a, b)-feasible pair, then there exists an (a, b)-feasible partition of G, too.*

Proof: Note that $\delta(G) \geq 1$ implies $w_G(v) \geq 1$ for all $v \in V(G)$. Consider an (a, b)-feasible pair (A, B) such that $A \cup B$ is maximal. All we need to show is that $A \cup B = V(G)$. Assume, in contrary, that $C = V(G) \setminus (A \cup B)$ is non-empty. Since $A \cup B$ is maximal, this implies that $(A, B \cup C)$ is not (a, b)-feasible. Thus, there exists a vertex $v \in C$ such that $d_{B \cup C}(v) \leq b(v) - 1$. Due to the fact that $d_G(v) \geq a(v) + b(v) + 2w_G(v) - 3$, we conclude $d_A(v) \geq a(v) + 2w_G(v) - 2 \geq a(v)$. But then $(A \cup \{v\}, B)$ is an (a, b)-feasible pair, in contradiction to the maximality of $A \cup B$. ■

Proposition 3.16 *Let G be a graph with $\delta(G) \geq 1$, and let $a, b : V(G) \to \mathbb{Z}_{\geq 1}$ be two functions. Moreover, assume that*

$$d_G(v) \geq a(v) + b(v) + 2w_G(v) - 2$$

for all $v \in V(G)$. Then, G has an (a, b)-feasible partition, provided that there is no (a, b)-meager partition of G.

Proof: Since $\delta(G) \geq 1$, we have $|G| \geq 2$ and $w_G(v) \geq 1$ for all $v \in V(G)$. We choose a non-empty subset A of $V(G)$ such that

(a) A is a-nice, and

(b) $|A|$ is minimum subject to (a).

Let $B = V(G) \setminus A$. Obviously, $V(G) \setminus \{v\}$ satisfies (a) for each vertex v, therefore, A exists and B is non-empty. Because of (b), for every proper subset A' of A that is non-empty, we find a vertex $u \in A'$ fulfilling $d_{A'}(u) \leq a(u) - 1$. Since A' can be chosen such that $|A'| = |A| - 1$, this implies the existence of a vertex $u \in A$ such that $d_A(u) \leq a(u) + w_G(u) - 1$. Thus, A is a-meager. Since there is no (a, b)-meager partition of G, B is not b-meager and, therefore, there is a non-empty subset B' of B such that $d_{B'}(v) \geq b(v) + w_G(v) \geq b(v)$ for all $v \in B'$. Hence, (A, B') is an (a, b)-feasible pair and Proposition 3.15 implies the existence of an (a, b)-feasible partition of G. ■

Let G be a graph, let $a, b : V(G) \to \mathbb{Z}_{\geq 0}$ be two functions, and let (A, B) be a partition of $V(G)$. We define the (a, b)**-weight** $w(A, B)$ as

$$w(A, B) = |E_G[A]| + |E_G[B]| + \sum_{v \in A} b(v) + \sum_{v \in B} a(v).$$

If $|A| \geq 2$ we can choose an arbitrary vertex $x \in A$ and $(A \setminus \{x\}, B \cup \{x\})$ remains a partition of $V(G)$. In particular, it holds

$$w(A \setminus \{x\}, B \cup \{x\}) - w(A, B) = d_B(x) - d_A(x) + a(x) - b(x). \tag{3.1}$$

Similarly, if $|B| \geq 2$ we may choose a vertex $y \in B$ and $(A \cup \{y\}, B \setminus \{y\})$ is also a partition of $V(G)$ satisfying

$$w(A \cup \{y\}, B \setminus \{y\}) - w(A, B) = d_A(y) - d_B(y) + b(y) - a(y). \tag{3.2}$$

If $v \in A$, then $d_{A \setminus \{v\}}(u) = d_A(u) - \mu(u, v)$ for all $u \in A \setminus \{v\}$ and, hence,

$$d_{A \setminus \{v\}}(u) \geq d_A(u) - w_G(u) \quad \text{and} \quad d_{A \setminus \{v\}} \geq d_A(u) - w_G(v). \tag{3.3}$$

3.5 Proof of Theorem 3.10

Let G be a graph with $\delta(G) \geq 1$, and let $a, b : V(G) \to \mathbb{Z}_{\geq 0}$ be two functions such that

$$d_G(v) \geq a(v) + b(v) + 2w_G(v) - 1 \tag{3.4}$$

for all $v \in V(G)$. Our aim is, to prove that there is an (a,b)-feasible partition of G. Since $\delta(G) \geq 1$, we obtain that $w_G(v) \geq 1$ for all $v \in V(G)$. As a conclusion, if there exists a vertex v such that $a(v) = 0$ or $b(v) = 0$, then equation (3.4) implies that $(\{v\}, V(G) \setminus \{v\})$ or $(V(G) \setminus \{v\}, \{v\})$, respectively, is an (a,b)-feasible partition of G. Thus, in the following we may assume that

$$a(v) \geq 1 \text{ and } b(v) \geq 1 \tag{3.5}$$

for every vertex $v \in V(G)$. If there is no (a,b)-meager partition of $V(G)$, then we conclude from Proposition 3.16 that there is an (a,b)-feasible partition of G, and we are done. It remains to consider the case that there is an (a,b)-meager partition of G. Then, we choose an (a,b)-meager partition (A,B) of G such that $w(A,B)$ is maximum. Since A is a-meager, there is a vertex $x \in A$ fulfilling $d_A(x) \leq a(x) + w_G(x) - 1$. Together with equations (3.4) and (3.5), this implies $d_B(x) \geq b(x) + w_G(x) \geq w_G(x) + 1$. Hence, $|B| \geq 2$. By symmetry, we conclude that $|A| \geq 2$, too.

Next, we claim the existence of a non-empty subset $\tilde{A} \subseteq A$ such that $d_{\tilde{A}}(v) \geq a(v)$ for all $v \in \tilde{A}$. Otherwise, A is $(a-1)$-degenerate and, by (3.3), we conclude that $A \cup \{y\}$ is a-meager for all $y \in B$. Since B is b-meager, there exists a vertex $y \in B$ such that $d_B(y) \leq b(y) + w_G(y) - 1$ and, by (3.4), we have $d_A(y) \geq a(y) + w_G(y)$. Because of $|B| \geq 2$, the pair $(A \cup \{y\}, B \setminus \{y\})$ is an (a,b)-meager partition of G such that

$$w(A \cup \{y\}, B \setminus \{y\}) - w(A,B) = d_A(y) - d_B(y) + b(y) - a(y) \geq 1$$

(by (3.2)), in contradiction to the maximality of $w(A,B)$. This proves the claim. By symmetry, there is also a non-empty subset $\tilde{B} \subseteq B$ such that $d_{\tilde{B}} \geq b(v)$ for all $v \in \tilde{B}$ and, hence, $(\tilde{A}, \tilde{B})$ is an (a,b)-feasible pair. By Proposition 3.15, this implies the existence of an (a,b)-feasible partition of G, and the proof is complete.

3.6 Proof of Theorem 3.12

Let G be a K_4^--free graph with $\delta(G) \geq 1$, and let $a, b : V(G) \to \mathbb{Z}_{\geq 1}$ be two functions such that

$$d_G(v) \geq a(v) + b(v) + 2w_G(v) - 2 \tag{3.6}$$

for all $v \in V(G)$. If G has an (a,b)-feasible pair, then we are done by Proposition 3.15. Otherwise, G contains no (a,b)-feasible pair. Firstly, we claim that $|G| \geq 3$. Otherwise,

$|G| = 2$ and, thus, $V(G) = \{u, v\}$. Since $\delta(G) \geq 1$, equation (3.6) implies

$$d_G(v) \geq a(v) + b(v) + 2w_G(v) - 2 \geq 2w_G(v) = 2d_G(v),$$

which is not possible. As a consequence, $|G| \geq 3$. Secondly, we show that for each edge $uv \in E(G)$ it holds

$$a(u) + a(v) \geq 3 \quad \text{and} \quad b(u) + b(v) \geq 3. \tag{3.7}$$

Assume, to the contrary, that there is an edge $uv \in E(G)$ that does not fulfill the above equation. By symmetry, we can assume that $b(u) = b(v) = 1$. Together with equation (3.6), this implies that $d_G(u) \geq a(u) + w_G(u)$ and that $d_G(v) \geq a(v) + w_G(v)$. If $N_G(u) \cap N_G(v)$ is empty, we claim that for $A = V(G) \setminus \{u, v\}$ and $B = \{u, v\}$, the pair (A, B) is an (a, b)-feasible partition. This follows from the fact that

$$d_A(w) \geq d_G(w) - w_G(w) \geq a(w)$$

for all $w \in A$ (by (3.6)), $d_B(w) \geq 1 = b(w)$ for all $w \in B$, and that A is non-empty because of $|G| \geq 3$. Hence, $N_G(u) \cap N_G(v)$ is non-empty. Due to the fact that G is K_4^--free, $N_G(u) \cap N_G(v)$ consists of exactly one vertex w and each vertex of $V(G) \setminus \{u, v, w\}$ has at most one neighbor within $\{u, v, w\}$. If $b(w) = 1$, then $(V(G) \setminus \{u, v, w\}, \{u, v, w\})$ is an (a, b)-feasible partition of $V(G)$ (by a similar argument as before). Otherwise, $b(w) \geq 2$ and, by (3.6),

$$d_G(w) \geq a(w) + 2w_G(w).$$

However, this leads to $(V(G) \setminus \{u, v\}, \{u, v\})$ being an (a, b)-feasible partition, a contradiction. This proves the claim that (3.7) holds.

If G has no (a, b)-meager partition, then we are done by Proposition 3.16. Hence, we only need to consider the case that there is an (a, b)-meager partition of G, say (A, B). Since G has no (a, b)-feasible pair and by symmetry, we can assume that B is $(b-1)$-degenerate and, therefore, $(b-1)$-meager. We choose an $(a, b-1)$-meager partition (A, B) such that

(1) $w(A, B)$ is maximum and

(2) $|A|$ is minimum subject to (1).

Firstly, we show that B contains at least two vertices. Otherwise, $B = \{y\}$. Since A is a-meager, there is an $x \in A$ such that $d_A(x) \leq a(x) + w_G(x) - 1$, and, by (3.6),

$$d_B(x) \geq b(x) + w_G(x) - 1 \geq w_G(x).$$

Since B consists of only one vertex, this implies that $d_B(x) = w_G(x)$ and, hence, $b(x) = 1$. By (3.7), $b(y) \geq 2$ and, thus,

$$d_{B\cup\{x\}}(y) = w_G(x) = 2 + w_G(x) - 2 \leq b(y) + w_G(y) - 2.$$

Since $|G| \geq 3$, this implies that $(A \setminus \{x\}, B \cup \{x\})$ is also an $(a, b-1)$-meager partition and, by (3.1),

$$w(A \setminus \{x\}, B \cup \{x\}) - w(A, B) = d_B(x) - d_A(x) + a(x) - b(x) \geq 0,$$

contradicing the choice of (A, B). Hence, $|B| \geq 2$.

Since B is $(b-1)$-meager we can choose a $y \in B$ such that $d_B(y) \leq b(y) + w_G(y) - 2$. Now we claim that there are $x, z \in A$ fulfilling $d_A(x) \leq a(x) + w_G(x) - 1$ and $d_A(z) \leq a(z) + w_G(z) - 1$ such that $G[\{x, y, z\}]$ contains a triangle. Due to the fact that $d_B(y) \leq b(y) + w_G(y) - 2$, we get

$$d_A(y) \geq a(y) + w_G(y) \geq w_G(y) + 1,$$

implying that $|A| \geq 2$. Let $A' = A \cup \{y\}$ and let $B' = B \setminus \{y\}$. By (3.2),

$$\begin{aligned} w(A', B') - w(A, B) &= d_A(y) - d_B(y) + b(y) - a(y) \\ &\geq a(y) + w_G(y) - b(y) - w_G(y) + 2 + b(y) - a(y) \\ &\geq 2. \end{aligned}$$

Since B' is still $(b-1)$-meager, this implies that A' is not a-meager since otherwise this would contradict the choice of (A, B). Thus, there exists a subset $\tilde{A}$ of A such that $\tilde{A} \cup \{y\}$ is $(a + w_G)$-nice and, therefore, $\tilde{A}$ is a-nice. If A is not a-nice, then there is an $x' \in A \setminus \tilde{A}$ such that $d_A(x') \leq a(x') - 1$ and, by (3.6), $d_B(x') \geq b(x') + 2w_G(x') - 1$. Let $A'' = A \setminus \{x'\}$ and $B'' = B \cup \{x'\}$. Then, A'' is still a-meager and, since $\tilde{A} \subseteq A''$ and due to the fact that G contains no (a, b)-feasible pair, B'' is $(b-1)$-degenerate. Hence, (A'', B'') is an $(a, b-1)$-meager partition and, by (3.1),

$$\begin{aligned} w(A'', B'') - w(A, B) &= d_B(x') - d_A(x') + a(x') - b(x') \\ &\geq b(x') + 2w_G(x') - 1 - a(x') + 1 + a(x') - b(x') \\ &\geq 2, \end{aligned}$$

contradicting the choice of (A,B). Consequently, A is a-nice. Let

$$C = \{u \in A \mid d_A(u) \leq a(u) + w_G(u) - 1\}.$$

Since A is a-meager, C is non-empty. We claim that $C \subseteq \tilde{A}$. Otherwise, there is an $x' \in C \setminus \tilde{A}$. Then, by (3.6), $d_B(x') \geq b(x') + w_G(x') - 1 \geq b(x')$. Let again $A'' = A \setminus \{x'\}$, and $B'' = B \cup \{x'\}$. Since $\tilde{A} \subseteq A''$ and $\tilde{A}$ is a-nice, the set B'' must be $(b-1)$-meager and, hence, (A'', B'') is an $(a, b-1)$-meager partition. Furthermore, by (3.1), it follows that

$$w(A'', B'') - w(A,B) = d_B(x') - d_A(x') + a(x') - b(x') \geq 0.$$

Since $|A''| < |A|$, this contradicts the choice of (A,B). Hence, the claim that $C \subseteq \tilde{A}$ is proven. Since $\tilde{A} \cup \{y\}$ is $(a + w_G)$-nice, it follows that $C \subseteq N_G(y)$, which leads to $C \subseteq \tilde{A} \cap N_G(y)$.

Let $x \in C$. If $N_G(x) \cap C = \varnothing$, then let $A'' = A \setminus \{x\}$ and $B'' = B \cup \{x\}$. Since A is a-nice, and since $d_G(v) \geq a(v) + w_G(v)$ for all $v \in A \setminus C$, A'' is still a-nice and, thus, B'' is $(b-1)$-meager. However, (3.1) implies

$$\begin{aligned} w(A'', B'') - w(A,B) &= d_B(x) - d_A(x) + a(x) - b(x) \\ &\geq b(x) + w_G(x) - 1 - a(x) - w_G(x) + 1 + a(x) - b(x) \geq 0. \end{aligned}$$

Since $|A''| < |A|$, this contradicts the choice of (A,B). Hence, there is a vertex $z \in N_G(x) \cap C$ and $G[\{x,y,z\}]$ contains a triangle, as claimed.

To complete the proof, let $A'' = A \setminus \{x,z\}$ and let $B'' = B \cup \{x,z\}$. We claim that (A'', B'') is an $(a, b-1)$-meager partition. Firstly, we show that $A'' \neq \varnothing$. Otherwise, since $\{x,z\} \subseteq C \subseteq \tilde{A} \subseteq A$, we have $C = \tilde{A} = A$. Thus, $A' = A \cup \{y\} = \{x,y,z\}$ is $(a + w_G)$-nice. Since G contains no K_4^-, each vertex of $B' = B \setminus \{y\}$ has at most one neighbor within $\{x,y,z\}$. Thus, $d_{B'}(v) \geq d_G(v) - w_G(v)$ for all $v \in B'$ and, by (3.6), we get

$$d_{B'}(v) \geq a(v) + b(v) + w_G(v) - 2 \geq b(v)$$

for all $v \in B'$, which implies that (A', B') is an (a,b)-feasible partition of $V(G)$, a contradiction. As a consequence, $A'' \neq \varnothing$. Since G contains no K_4^- and since $C \subseteq N_G(y)$, $N_G(x) \cap C = \{z\}$ and $N_G(z) \cap C = \{x\}$. Thus, if $v \in A'' \cap C$, then v has no neighbour in $\{x,z\}$, and therefore $d_{A''}(v) \geq a(v)$ (since A is a-nice). If $v \in A'' \setminus C$, then v has at most one neighbor in $\{x, z\}$. Thus, we get

$$d_{A''}(v) \geq d_A(v) - w_G(v) \geq a(v) + w_G(v) - w_G(v) \geq a(v).$$

As a consequence, A'' is a-nice (and non-empty). Since G has no (a, b)-feasible partition, this implies that B'' is $(b-1)$-meager and (A'', B'') is an $(a, b-1)$-meager partition. On the other hand,

$$\begin{aligned} w(A'', B'') - w(A, B) &\geq d_B(x) + d_B(z) - d_A(x) - d_A(z) + 2 \\ &\quad -b(x) - b(z) + a(x) + a(z) \\ &\geq b(x) + w_G(x) - 1 + b(z) + w_G(z) - 1 \\ &\quad -a(x) - w_G(x) + 1 - a(z) - w_G(z) + 1 + 2 \\ &\quad -b(x) - b(z) + a(x) + a(x) \geq 2, \end{aligned}$$

which contradicts the choice of (A, B). This completes the proof.

3.7 Concluding Remarks

It follows from a simple induction that Theorems 3.10, 3.11 and 3.12 can be extended to partitions of more than two sets.

Corollary 3.17 *Let G be a graph, and let $f_1, f_2, \ldots, f_p : V(G) \to \mathbb{Z}_{\geq h-1}$ be p functions with $p \geq 2$ and $h \in \mathbb{Z}_{\geq 1}$. Assume that $\delta(G) \geq 1$ and*

$$d_G(v) \geq f_1(v) + f_2(v) + \cdots + f_p(v) + (p-1)(2w_G(v) - h)$$

for all $v \in V(G)$. Then, there is a partition $(A_1, A_2, \ldots, A_p)$ of $V(G)$ such that $d_{A_i}(v) \geq f_i(v)$ for every $i \in \{1, 2, \ldots, p\}$ and every $v \in A_i$, provided that either $h = 1$, or $h = 2$ and G is K_4^--free.

If we renounce the condition $\delta(G) \geq 1$ in Theorem 3.10, it may happen that G only consists of isolated vertices. But then, if $a(v) = 1$ for all $v \in V(G)$ and if $b(v) = 0$ for all $v \in V(G)$, the only possible choice would be $A = \varnothing$ and $B = V(G)$, which is not a partition. However, demanding G to have at least one edge is sufficient. In this case we can delete all isolated vertices and Theorem 3.10 implies the existence of an (a, b)-feasible partition for the remaining graph. Adding each isolated vertex v to the set A if $a(v) = 0$ and to B if

$b(v) = 0$ will do the trick then. For Theorem 3.11 and 3.12 it is obvious that we cannot give up the condition $\delta(G) \geq 1$.

If we take a look back at Thomassen's Conjecture, we obtain an analogue for arbitrary graphs by considering only constant functions. Recall that $\mu(G)$ denotes the maximum multiplicity of a graph G.

Corollary 3.18 *Let G be a graph and let $s, t \geq 0$ be integers. Assume that $\delta(G) \geq s + t + 2\mu(G) - 1 \geq 1$. Then, there is a 2-partition (G_1, G_2) of G such that $\delta(G_1) \geq s$ and $\delta(G_2) \geq t$.*

The next question standing to reason is as follows. Is it really necessary to require $d_G(v) \geq a(v) + b(v) + 2w_G(v) - 1$ in Theorem 3.10, or can we maybe save another value one. To answer this, consider the case $G = tK_3$, that is, G is the graph on 3 vertices in which each two vertices are joined by t edges. By requesting only $d_G(v) \geq a(v) + b(v) + 2t - 2$, setting $a(v) = 1$ and $b(v) = 1$ for all $v \in V(G)$ leads to a counter-example, since obviously there is no partition (A, B) such that $d_A(v) \geq 1$ for all $v \in A$ and $d_B(v) \geq 1$ for all $v \in B$, although $d_G(v) = 2t$ is fulfilled.

For another counter-example consider the graph H shown in Figure 3.3. If we set $G = tH$ for some $t \geq 1$, it holds $4t = d_G(v)$ for all $v \in V(G)$. By requiering only $d_G(v) \geq a(v) + b(v) + 2t - 2$, setting $a(v) = 1$ for all $v \in V(G)$ and $b(v) = 2t + 1$ gives us a counter-example. This is due to the fact that if (A, B) is an (a, b)-feasible partition of G, then A consists of at least two adjacent vertices that have a common neighbour v in B. But then, $d_B(v) \leq 2t$, a contradiction. If H is the icosahedron and if $G = tH$ for some $t \geq 1$, then, by the same argumentation as above, setting $a(v) = 1$ and $b(v) = 3t + 1$ leads to a third counter-example. Note that these examples prove that the boundary in Corollary 3.18 is sharp, as well.

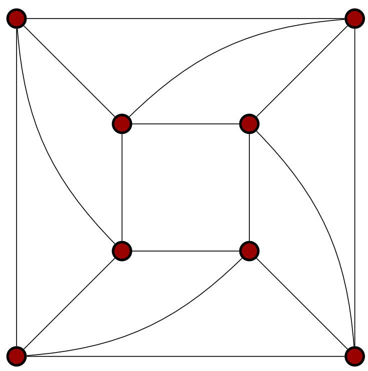

Figure 3.3: The graph H serves as a starting point to design a counter-example.

Chapter 4

Graph Partitions under Further Degree Constraints

In the last chapter, we will take a look at other constraints under which it may be reasonable to decompose graphs, as well. We will begin with partitions under average degree constraints; those were examined by Csóka, Lo, Norin, Wu and Yepremyan [10]. Their researches lead to some open problems which will be mentioned in the first section. After that, we will give a short overview about graph partitions under color degree constraints with respect to edge-colorings, following Li, Fujita and Wang [21]. The chapter ends with some results on graph partitions under σ_2-constraints, proven by Chiba and Lichiardopol [9].

4.1 Graph Partitions under Average Degree Constraints

As mentioned in the introduction already, the **average degree** of a graph G is defined by

$$av(G) = \frac{1}{|G|} \sum_{v \in V(G)} d_G(v) = \frac{2|E(G)|}{|G|}.$$

The question we want to address in this section is if there is a function f such that for any real numbers $s, t > 0$ each graph G with $av(G) \geq f(s, t)$ admits a partition (G_1, G_2)

satisfying $av(G_1) \geq s$ and $av(G_2) \geq t$. In 2017, Csóka et al. [10] proved a related result for simple graphs.

Theorem 4.1 (Csóka, Lo, Norin, Wu and Yepremyan). *Let $s, t \geq 1$ be real, and let G be a non-empty simple graph such that $|E(G)| > (s+t+1)(|V(G)|-1)$. Then, there exist two vertex disjoint non-empty subgraphs G_1 and G_2 of G such that $|E(G_1)| > s(|V(G_1)|-1)$ and $|E(G_2)| > t(|V(G_2)|-1)$.*

However, this theorem does not yet answer our question. First of all, the graphs G_1 and G_2 do not necessarily have to form a partition of G and, secondly, the boundaries mentioned are just closely related but not identical to the average degree. While Csóka et al. did not manage to fully solve this, they conjectured the following.

Conjecture 4.2 (Csóka et al.). *Let $s, t > 0$ be real and let G be a non-empty simple graph such that $av(G) \geq (s+t+2)$. Then, there exists a partition (G_1, G_2) of G such that $av(G_1) \geq s$ and $av(G_2) \geq t$.*

By adjusting the parameters used in the proof of Theorem 4.1, Wu [28] managed to prove at least some parts of the conjecture.

Theorem 4.3 (Wu). *Conjecture 4.2 holds if $s = t$, or if $av(G) \geq (s+t+3)$.*

Nevertheless, large parts of the conjecture still remain to be shown. Also, it is an open problem how multiple edges may affect the boundary.

4.2 Graph Partitions under Color Degree Constraints

Let G be a graph. An **edge-coloring** of G is a function $\psi : E(G) \to \mathbb{Z}_{\geq 1}$. Consider an arbitrary edge-coloring ψ of G. Then, $d_G^{\psi}(v)$ denotes the **color degree** of a vertex $v \in V(G)$, that is, the number of colors used for edges which are incident to v, i. e.

$$d_G^{\psi}(v) = |\{\psi(e) \mid e \in E_G(v)\}|.$$

Moreover, the **minimum color degree** of G is defined by

$$\delta^{\psi}(G) = \min_{v \in V(G)} d_G^{\psi}(v).$$

Li, Fujita and Wang [21] considered the problem of how to partition a graph under minimum color degree constraints. Therefore, they defined the following. Given a graph

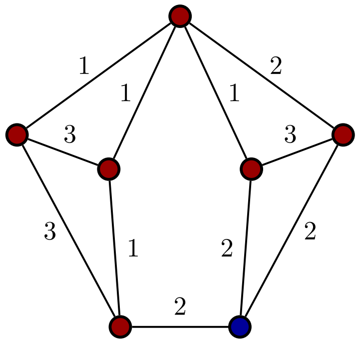

Figure 4.1: A graph G with an edge-coloring ψ such that $\delta^{\psi}(G) = 1$.

G and integers $a \geq b \geq 1$, a pair (A, B) is called (a, b)**-feasible** if A and B are disjoint, non-empty subsets of $V(G)$ such that $\delta^{\psi}(G[A]) \geq a$ and $\delta^{\psi}(G[B]) \geq b$. If A and B also form a partition of $V(G)$, we say that (A, B) is an (a, b)-feasible partition of G. Li, Fujita and Wang conjectured the following.

Conjecture 4.4 (Li, Fujita and Wang). *Let a, b integers with $a \geq b \geq 2$, let G be a simple graph, and let ψ be an edge-coloring of G such that $\delta^{\psi}(G) \geq a + b + 1$. Then, G admits an (a, b)-feasible partition.*

Although the conjecture still remains open, Li et al. managed to prove the following.

Theorem 4.5 (Li, Fujita and Wang). *Conjecture 4.4 is true for $a = b = 2$.*

4.3 Graph Partitions under σ-Constraints

Let G be a graph. By $\sigma(G)$ we denote the minimum degree sum of two non-adjacent vertices, more formally,

$$\sigma(G) = \min\{d_G(u) + d_G(v) \mid u, v \in V(G), u \neq v, E_G(u, v) = \varnothing\}.$$

If there are no non-adjacent vertices, i.e. if G contains a complete graph on $|G|$ vertices, we set $\sigma(G) = +\infty$. Chiba and Lichiardopol [9] raised the question if it is possible to prove σ-versions of Stiebitz's and Kaneko's Theorems from chapter 2. They showed the following.

Theorem 4.6 *Let $s_1, s_2 \geq 2$ be integers, and let G be a non-complete simple graph. If $\sigma(G) \geq 2(s_1 + s_2 + 1) - 1$, then there exist two vertex-disjoint induced subgraphs G_1 and G_2 of G such that $\sigma(G_i) \geq 2s_i - 1$ and $|G_i| \geq s_i + 1$ for $i \in \{1, 2, \}$.*

Theorem 4.7 *Let s_1, s_2 be integers, and let G be a simple graph of order at least 3. If $\sigma(G) \geq 2(s_1 + s_2) - 1$ and $g(G) \geq 4$, then there exist two disjoint induced subgraphs G_1 and G_2 of G such that $\sigma(G_i) \geq 2s_i - 1$ and $|G_i| \geq 2s_i$ for $i \in \{1, 2\}$.*

Note that if we renounce the condition $|G_i| \geq s_i + 1$ and $|G_i| \geq 2s_i$ in Theorem 4.6 and 4.7, the problem gets pretty simple since for each edge $uv \in E(G)$ setting $G_1 = G[\{u, v\}]$ and $G_2 = (G - u) - v$ obviously does the job in both cases. If $s_i = 1$ for some $i \in \{1, 2\}$, then the solution can easily be obtained, as well. Chiba and Lichiardopol also elaborated that the σ-conditions in Theorems 4.6 and 4.7 are best possible. To this end, let G be a balanced complete multipartite graph with $r + 1$ (≥ 4) partite sets of size $s \geq 2$, that is, a graph G such that $V(G) = V_1 \cup V_2 \cup \ldots \cup V_{r+1}$, $|V_i| = s$ for all $i \in \{1, 2, \ldots, r + 1\}$, $V_i \cap V_j = \varnothing$ for all $i \neq j \in \{1, 2, \ldots, r + 1\}$ and $E(G) = \{v_i v_j \mid v_i \in V_i, v_j \in V_j, i \neq j \in \{1, 2, \ldots, r + 1\}\}$. Then, $\sigma(G) = 2rs = 2((rs + r + 1) + (r - 1) + 1) - 2$ and it is an easy exercise to see that G contains no partition as in Theorem 4.6 for $(s_1, s_2) = (rs - r + 1, r - 1)$. Hence, the condition $\sigma(G) \geq 2(s_1 + s_2 + 1) - 1$ is best possible. To obtain the same for Theorem 4.7, consider the complete bipartite graph $K_{s_1+s_2-1, s_1+s_2}$. Then, $\sigma(G_2) = 2(s_1 + s_2) - 2$ and it is obvious that G does not contain a partition as required.

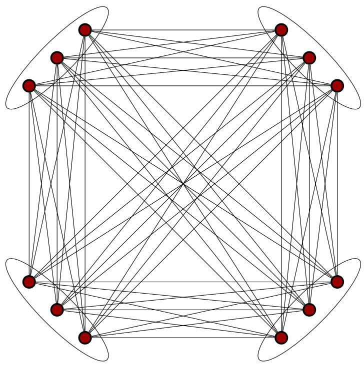

Figure 4.2: A balanced complete multipartite graph with 4 partite sets of size 3.

However, it is an open problem if for any graph G that fulfills the requirements of Theorem 4.6 or Theorem 4.7 there is a partition (G_1, G_2) of G that does the job. Also, Chiba and Lichiardopol considered only simple graphs; the influence of multiple edges remains unclear.

References

[1] Ban, A., Decomposing Weighted Graphs. arXiv:1702.00205

[2] Bazgan, C., Tuza, Zs. and Vanderpooten, D., Efficient algorithms for decomposing graphs under degree constraints. *Discrete Appl. Math.* **155** (2017) 979–988.

[3] Bollobás, B. and Manvel, B., Optimal vertex partition. *Bull. London Math. Soc.* **11** (1979) 113–116.

[4] Borodin, O.V., On decomposition of graphs into degenerate subgraphs. *Metody Diskret. Analiz* **28** (1976) 3–11 (in Russian).

[5] Borodin, O.V., Problems of colouring and of covering the vertex set of a graph by induced subgraphs. Ph.D.Thesis, Novosibirsk, 1979 (in Russian).

[6] Borodin, O.V. and Kostochka, A.V., On an upper bound of a graph's chromatic number, depending on the graph's degree and density. *J. Combin. Theory Ser. B* **23** (1977) 247–250.

[7] Borodin, O.V., Kostochka, A.V. and Toft, B., Variable degeneracy: extensions of Brooks' and Gallai's theorems. *Discrete Math.* **214** (2000) 101–112.

[8] Brooks, R. L., On colouring the nodes of a network. *Proc. Cambridge Philos. Soc., Math. Phys. Sci.* **37** (1941) 194–197.

[9] Chiba, S. and Lichiardopol, N., Vertex-disjoint subgraphs with high degree sums. *Electr. Notes in Discr. Math.* **49** (2015) 359–366.

[10] Csóka, E., Lo, I., Norin, S., Wu, H. and Yepremyan, L., The extremal function for disconnected minors. *J. Combin. Theory Ser. B* Available online 15 May 2017

[11] Diestel, R., *Graph Theory* 5th Edition. Springer GTM 173 (2016)(English Edition).

[12] Diwan, A. , Decomposing graphs with girths at least five under degree constraints. *J. Graph Theory* **33** (2000) 237–239.

[13] Erdős, P., Rubin, A. L. and Taylor, H. , Choosability in graphs. *Congr. Numer.* **XXVI** (1979) 125–157.

[14] Erdős, P. and Hajnal, A. , On the chromatic number of graphs and set-systems. *Acta Math. Acad. Sci. Hungar.* **17** (1966) 61–99.

[15] Franklin, F. , A six-color theorem. *J. Math. Phys.* **13** (1934) 363–369.

[16] Gerber, M. U. and Kobler, D. , Classes of graphs that can be partitioned to satisfy all their vertices. *Aust. J. Combin.* **29** (2004) 201–214.

[17] Heawood, P.J. , Map-colour theorem. *Quart J. Math. Oxford Ser.* **24** (1890) 332–338.

[18] Kaneko, A. , On decomposition of triangle-free graphs under degree constraints. *J. Graph Theory* **27** (1998) 7–9.

[19] Kronk, H.V. and Mitchem, J. , Critical point-arboritic graphs. *J. London Math. Soc.* **9** (1975) 459–466.

[20] Lick, D.R. and White, A.T. , The point partition numbers of closed 2-manifolds. *J. London. Math. Soc.* **4** (1972) 577–583.

[21] Li, R., Fujita, S. and Wang, G. , Decomposing edge-colored graphs under color degree constraints. arXiv:1701.03007

[22] Liu, M. and Xu, B. , On partitions of graphs under degree constraints. *Discrete Appl. Math.* Available online 28 April 2017.

[23] Lovász, L. , On decomposition of graphs. *Studia Sci. Math. Hungar.* **1** (1966), 237–238.

[24] Mitchem, J. , An extension of Brooks' theorem to n-degenerate graphs. *Discrete Math.* **17** (1977) 291–298.

[25] Stiebitz, M. , Decomposing graphs under degree constrains. *Journal of Graph Theory* **23** (1996) 321–324.

[26] Thomassen, C. , Graph decompositions with constraints on the connectivity and minimum degree. *J. Graph Theory* **7** (1983) 165–167.

[27] Vizing, V. G., Vertex coloring with given colors (auf Russisch). *Diskret. Analiz.* **29** (1976) 3–10.

[28] Wu, H., Private communication.