KOLLOQUIUM ÜBER KOMBINATORIK – 08. UND 09. NOVEMBER 2013
DISKRETE MATHEMATIK UND ALGEBRA – TU ILMENAU

Canonical tree-decompositions, k-blocks and tangles

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Ever since the first graph was decomposed into its blocks by the now well-known block/cutvertex tree, people have wondered how this might be generalized to higher k: can we identify, somehow, 'the k-connected pieces' in a $(k-1)$-connected graph, and decompose it in a tree-like way into those pieces?

Tutte did this for $k = 3$, and some attempts were made for $k = 4$. But there the problem rested for a long time.

In the 1990s, Robertson and Seymour suggested the concept of a *tangle* (of order k) as a notion of a 'k-connected piece' of a graph, and proved that every graph has a tree-decomposition that 'distinguishes' all its maximal tangles. Unlike Tutte's decomposition for $k = 3$, these were not *canonical* in the sense that they depend only on the structure of the graph, but depended on some fixed vertex enumeration.

In the 1970s, Mader had suggested the notion of a *k-block* of a graph: a maximal set of (at least k) vertices no two of which can be separated by $< k$ vertices. Tutte's theorem can be elegantly rephrased by saying that every 2-connected graph has a canonical tree-decomposition that distinguishes all its 3-blocks.

We can now prove this for arbitrary k, and graphs of any connectivity: every finite graph has a canonical tree-decomposition that distinguishes all its k-blocks. In fact, the decomposition can be chosen so as to distinguish all the graph's tangles of order k as well, strengthening the Robertson-Seymour result.

For matroids we have a similar theorem: every finite matroid has a canonical tree-decomposition, one that is invariant under its automorphisms, which distinguishes all its maximal tangles. This strengthens a recent result of Geelen, Gerards and Whittle (who proved this with non-canonical tree-decompositions).

The notion of a k-block suggests many new questions. For example: if $\beta(G)$ denotes the largest k such that G has a k-block, how does β interact with other graph invariants? What natural assumptions will force G to have a k-block? How long will it take to find all the k-blocks of G, or to decide whether one exists?