Ramsey Variant of the 2-Dimension of Posets

Master’s Thesis of

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Statement of Authorship

I hereby declare that this document has been composed by myself and describes my own work, unless otherwise acknowledged in the text.

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Abstract and Contents

The 2-dimension of a poset \( P \) is the smallest integer \( N \) such that \( P \) is a subposet of the Boolean lattice of dimension \( N \). While a wealth of research has been conducted on the topic, we consider a Ramsey-type variation that has so far been mostly ignored: What is the smallest integer \( N \) such that any colouring of the \( N \) dimensional Boolean lattice (usually with two colours) admits \( P \) as a monochromatic subposet?

We discuss the relation of this question to existing research and consider general posets as well as specific examples. A central question remains open: If \( P \) is the Boolean lattice of dimension \( n \), we show that \( N \) is at least linear and at most quadratic in \( n \), but we do not know the precise asymptotic behaviour.

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Part I.
Introduction

When one attempts to sketch what Ramsey Theory [Ram30], [GRS90] is all about, one might come to a generalised question like this:

*How large must a certain host-structure be, such that, no matter how it is partitioned into two parts, one of the parts will always contain a copy of a certain guest-structure?*

For instance, assume the “host” is a board similar to a chess-board of size $n \times n$. Each field is coloured either white or black, so, in a sense, the board is partitioned into a white part and a black part (we use colouring and partition synonymously). A “guest” is given by four fields that are the corners of an axis-aligned rectangle. For $n = 4$ there is a colouring of the $4 \times 4$ board (Figure 1 on the left) such that no choice of four black fields or four white fields are the corners of an axis-aligned rectangle. For $n = 5$, no such colouring exist: For every colouring there is a choice of four black fields or four white fields that are the corners of an axis-aligned rectangle (this is not obvious, but not hard either, see for instance [Fen+10]).

![Figure 1: Left: Each rectangle involves black and white fields. Right: A rectangle of only black fields exists.](image)

So the answer to this instance of the above question would be $n = 5$.

This thesis is about a *different* Ramsey question and it is time we reveal what “host”, “guest”, “colouring” and “contains” means in our case. The host is the Boolean lattice of dimension $n$, symbolised by $\mathbb{B}_n$. It is the set of all subsets of $[n] = \{1, 2, \ldots, n\}$, ordered by inclusion. We can therefore start to pose the question as follows:

*How high must the dimension of a Boolean lattice be...*

The Boolean lattice has $2^n$ elements and it is those elements which are partitioned into two parts – or equivalently: coloured with two colours.

*... such that no matter how we colour its elements with two colours...*
As guest we consider some partially ordered set $P$, i.e. a set $P$ accompanied by a reflexive, transitive, antisymmetric relation $\leq_P$ on $P$. For instance $P$ could be the set $\{a, b, c, d\}$ with $a < b$, $c < b$, $c < d$ and no further strict relations. We typically represent a poset by its Hasse diagram which is a collection of points and lines, the points corresponding to the elements of the poset, such that for two points $a, b$ we have $a \leq_P b$ if there is an upward-monotone path from $a$ to $b$. For the above poset we would write $P = \{b, a, d, c\}$ or simply $P = 1$. We say $P$ is contained in the Boolean lattice of dimension $n$ if there is a map $f : P \to 2^n$ such that $x \leq_P y \iff f(x) \subseteq f(y)$ (this just means that $P$ is an induced subposet of the lattice). If all elements in the range of $f$ are of the same colour, we say $P$ is contained monochromatically and this is what we ask for:

... it always contains $P$ monochromatically?

Figure 2 shows (again as Hasse diagrams), on the left, a colouring of the Boolean lattice of dimension 3 in red and blue that does not contain $N$ monochromatically and a colouring of the Boolean lattice of dimension 4 that does contain $N$ monochromatically (for instance $12, 124, 4, 134$ are suitable elements for the range of $f$). Here, commas and braces are omitted, e.g. $12 = \{1, 2\}$. In fact, for every colouring of the Boolean lattice

![Hasse diagrams](image)

Figure 2: A colouring of $S_3$ containing no monochromatic $N$, and a colouring of $S_4$ with a blue copy of $N$ as shown on the right.

of dimension 4, $N$ can be found monochromatically. So 4 is the number we asked for, and we would write $R(\emptyset) = 4$. Different posets $P$ will give different numbers $R(P)$.

Even though this particular flavour of Ramsey problem has not been considered in the literature so far, a rich collection of similar problems has. Part II of this thesis is spent reviewing these results, most importantly the uncoloured case which is already quite difficult: The smallest dimension of a Boolean lattice containing a certain poset is known as the 2-dimension of the poset. In Part III we present our results on $R(P)$.

We now give a section-by-section overview of the content.

**Part II: Survey: On 2-Dimension and other Related Results.**

**Section 1: Preliminaries.** We introduce posets, induced embeddings of posets, common examples of posets as well as Sperner’s and Dilworth’s Theorem.
Section 2: Basic Results on the 2-Dimension of Posets. We define and explain the notions of order dimension and 2-dimension and compute the 2-dimension of several posets that will be important later on.

Section 3: Hardness of Determining the 2-Dimension. It seems appropriate to review computational hardness of 2-dimension (since it naturally implies hardness of our Ramsey question). In a sense, we improve upon a non-approximability result by Habib et al. [Hab+04], if only by plugging in an improved version of non-approximability of the bipartite cover number.

Theorem 18. If \( P \neq \text{NP} \), then there is for no \( \varepsilon > 0 \) a polynomial-time algorithm approximating \( \text{dim}_2(P) \) for posets \( P \) within a factor of \( |P|^{1/3-\varepsilon} \).

Section 4: Forcing Posets in other Posets. We discuss three interesting variations of our Ramsey question:

Firstly, Nešetřil and Rödl [NR84] do not colour elements of posets, but fix some poset \( P \) and colour each occurrence of \( P \). This \( P \) is called Ramsey, if for every poset \( Q \) there is a poset \( R \) such that for any colouring of the copies of \( P \) in \( R \) there is a copy of \( Q \) in \( R \) such that each copy of \( P \) in that copy of \( Q \) has the same colour. The main result is:

Theorem 24 (Nešetřil and Rödl [NR84]). A poset is Ramsey iff it is a weak order.

Secondly, Kierstead and Trotter [KT87] consider arbitrary hosts instead of just Boolean lattices and try to bound their size, width or height instead of their 2-dimension.

Lastly, the extremal question deserves some attention, since it has spawned much research recently. Here we ask: How large can a subset of a Boolean lattice be without containing a fixed poset \( P \)?

Part III: Monochromatic Posets in Coloured Lattices.

We start by formally defining our Ramsey problem. Then:

Section 5: First Bounds on Ramsey Numbers. We derive general upper and lower bounds, most importantly the Blowup Lemma which states:

\[ R(P) \leq \text{dim}_2(P)(1 + h(P)), \]

where \( h(P) \) is the height of a poset \( P \) and \( \text{dim}_2(P) \) is its 2-dimension.

Section 6: Non-Approximability of Ramsey Numbers. We prove the inapproximability of Ramsey numbers by proving a stronger version of Theorem 18 from Section 4. While oblivious of some structural insights of Section 4, our proof is much simpler.

Section 7: Large Number of Colours. The thesis is mostly concerned with the case of two colours (or, at most, a constant number of colours). If, however, the number of colours \( c \) with which the host is coloured is asymptotically large, strong connections to the extremal setting emerge and we obtain linear behaviour:
Theorem 50. Let $P$ be a fixed poset that is not an antichain. Then

$$R_c(P) = \Theta(c).$$

Section 8: Ramsey Numbers of Cubes. We concentrate on the special case of Boolean lattices (as guest poset) which seems already quite difficult. We can make minor improvements on the general results from before, proving $R(\mathbb{B}_n) \leq n^2 + 1$ and consider a few small cases, showing $R(\mathbb{B}_3) \in \{7, 8\}$.

We also briefly consider random colourings, for instance the following simple result:

Theorem 63. Let $c \in \mathbb{N}$, $\varepsilon > 0$ and $n$ large. For $N = (1 + \varepsilon)n \log_2 n$ let the elements of $\mathbb{B}_N$ be $c$-coloured independently and uniformly at random. Then $\mathbb{B}_N$ contains a monochromatic layered copy (see Section 5.1) of $\mathbb{B}_n$ asymptotically almost surely.

Section 9: Asymmetric Ramsey Numbers and Sums of Posets. The asymmetric Ramsey number $R(P, Q)$ is the smallest $n$ such that any red/blue coloured Boolean lattice of dimension $n$ contains a red $P$ or a blue $Q$.

These numbers are closely connected to posets that are “horizontal sums” or “vertical sums” of smaller posets.

Each of the sections starts with a short summary of its content and we suggest the reader considers those for a still-brief-but-not-quite-as-brief overview of the thesis. A tabular representation of our quantitative results is given on page 45, but a glance at the preliminary section may be required to understand all notation used there.

As a last note in this introduction, let us give particular attention to a result from Section 8 since it indicates the extend of our understanding of Ramsey numbers, both in the positive and in the negative sense:

$$2n \leq R(\mathbb{B}_n) \leq n^2 + 1.$$ 

In words: $\mathbb{B}_{2n-1}$ can be coloured such that it does not contain $\mathbb{B}_n$ monochromatically but $\mathbb{B}_{n^2+1}$ cannot be coloured in this way. We could not close the gap between the quadratic upper bound and the linear lower bound any further and believe progress on this special case would significantly deepen our understanding.
Part II.
Survey: On 2-Dimension and other Related Results

1. Preliminaries

This section reviews the most basic notions and results surrounding partially ordered sets (posets). It can be skipped by readers who are familiar with the concepts. For their benefit, let us make a checklist of everything they need to know:

**Conventions.** $P$ and $Q$ are posets, $i, j, k, n, m, N$ are non-negative integers, $x, y, p$ are elements of posets. Unless otherwise stated, variables are universally quantified. By $[n]$ we denote the set $\{1, 2, \ldots, n\}$. Especially in figures, we tend to omit braces if all elements of a set are single digits, i.e. $134 = \{1, 3, 4\}$.

All structures are finite unless otherwise stated.

**Notions in posets.** height, width, subposets (induced), weak subposets (non-induced), joins and meets (need not exist), join-irreducible and meet-irreducible.

**Important posets.** The chain $I_n$, the antichain $\cdots_n$, the cube $\Diamond_n$, the standard example $S_n$.

**Theorems.** Sperner’s Theorem, Dilworth’s Theorem (Section 1.2).

**Definitions.** The Sperner number $\text{sp}(n) := \min\{N \mid \Diamond_N \text{ contains } \cdots_n\} \approx \log n$.

Trotter’s notion of critical pairs (Section 1.3).

1.1. Poset, Lattice, Cube, Chain, Antichain, Standard Example

The notions defined in the following are illustrated in Figure 3.

A partially ordered set or poset is a set $P$ with a reflexive, transitive, antisymmetric relation $\leq_P$ on $P$. Depending on the situation we may explicitly specify the pair $(P, \leq_P)$ or just talk about a poset $P$ assuming the presence of an accompanying relation implicitly. Instead of $(x, y) \in \leq_P$ we typically write $x \leq_P y$ or even $x \leq y$ if there is no ambiguity as to which ordering applies. To visualise a poset we draw the directed graph $G = (P, \{(x, y) \mid x \leq_P y\})$. However, to avoid visual clutter, reflexive and transitive edges are omitted and edges are implicitly directed upward. This is a Hasse diagram. By $x < y$ we mean $x \leq y$ and $x \neq y$ (we say: $x$ is less than $y$ if $x$ is at most $y$ but different from $y$). An element $x$ is minimal if there is no element that is less than $x$. It is a minimum, if it is less than every other element. An element $x$ is maximal if there is no element that is greater than $x$. It is a maximum if it is greater than every other element. The height of $P$ is the number of elements $h$ in
a longest strictly increasing sequence \( x_1 < x_2 < \ldots < x_h \). Two elements \( x \) and \( y \) are comparable if \( x \leq y \) or \( y \geq x \). The width of \( P \) is the number \( w \) of elements in a largest set \( \{a_1, \ldots, a_w\} \) were no two elements are comparable.

We say a poset \( P \) is a subposet of another poset \( Q \) (or, less formally, “\( P \) is (contained) in \( Q \)”) if there is a map \( f : P \to Q \) such that \( x \leq_P y \) if and only if \( f(x) \leq_Q f(y) \) for any \( x, y \in P \). The function \( f \) is an embedding. We also say \( P \) embeds into \( Q \) (via \( f \)). Note that embeddings are necessarily injective. The image of \( f \) is a copy of \( P \) in \( Q \). If \( C \) is a subset of the elements of \( P \), then \( C \) naturally becomes a poset by inheriting the ordering of \( P \). This is the poset induced by \( C \).

In the case that \( f : P \to Q \) is injective and \( x \leq_P y \) implies \( f(x) \leq_Q f(y) \) but not necessarily vice versa, we use the notions weak subposet, weak embedding and weak copy accordingly.\(^1\)

If \( f : P \to Q \) is a bijective embedding then \( f \) is an isomorphism (of posets). Note that in that case \( f^{-1} \) is an embedding as well. We say \( P \) and \( Q \) are isomorphic and write \( P \cong Q \).

An element \( u \in P \) is an upper bound of \( S \subseteq P \) if \( u \geq s \) for all \( s \in S \). It is the least upper bound of \( S \), also called join of \( S \), if for any other upper bound \( u' \) of \( S \) we have \( u \leq u' \). In the same way we define (greatest) lower bounds. A greatest lower bound is also called meet. We write \( a \lor b \) for the join of \( \{a, b\} \) and \( \lor S \) for the join of an arbitrary set \( S \). In the same way we use \( \land \) for meets. Note the special cases \( \lor \emptyset \) and \( \land \emptyset \) that are the minimum and maximum of \( P \), respectively, if they exist.

An element \( p \in P \) is join-irreducible if it cannot be written as the join of a set \( S \subseteq P \) with \( p \notin S \). Similarly, \( p \) is meet-irreducible if it cannot be written as the meet of \( S \subseteq P \) with \( p \notin S \). Note that if \( P \) has a minimum \( m \), then it is the join of

\(^1\)Some authors prefer to make weak subposets the default case, i.e. they say “subposet” when they mean “weak subposets” in our sense and “induced subposet” when they mean “subposet” in our sense.
the empty set and therefore not join-irreducible. Similarly, the maximum of \( P \) (if it exists) is the meet of the empty set. To show that some \( p \in P \) is join-irreducible, it suffices to show that it is neither a minimum nor the join of two elements different from \( p \). To show that some \( p \in P \) is meet-irreducible, it suffices to show that it is neither a maximum nor the meet of two elements different from \( p \).

A poset in which joins and meets exist for all \( S \subseteq P \) is a lattice.

The most important examples of posets are:

\( \mathbb{R} \). The real numbers are ordered by \( \leq_{\mathbb{R}} \), which is a total order since any two real numbers are comparable. Every subposet of \( \mathbb{R} \) is a total order as well. We are usually only interested in finite chains:

\( n \)-chain. We write \( I_n \) for the poset with elements \( [n] \) ordered by \( \leq_{\mathbb{R}} \). This poset is also referred to as a chain or more specifically the \( n \)-chain.

\( n \)-antichain We write \( \wedge_n \) for the poset with elements \( [n] \) and no non-trivial relations, i.e. the order relation is \( \{ (i, i) \mid i \in [n] \} \). This poset where no two distinct elements are comparable, is an antichain.

product order. Given two posets \( P \) and \( Q \), their product by the Cartesian product \( P \times Q \), ordered by the product order \( \leq_{P \times Q} \) which is defined as

\[
(p_1, q_1) \leq_{P \times Q} (p_2, q_2) \iff p_1 \leq_P p_2 \text{ and } q_1 \leq_Q q_2.
\]

We will see two important examples now.

\( \mathbb{R}^n \). The \( n \)-fold product of \( \mathbb{R} \) with itself yields the product order \( \leq_{\mathbb{R}^n} \) on \( \mathbb{R}^n \). For \( x, y \in \mathbb{R}^n \) with coordinate vectors \( (x_1, x_2, \ldots, x_n) \) and \( (y_1, y_2, \ldots, y_n) \) we have:

\[
x \leq_{\mathbb{R}^n} y \iff \forall i \in [n] : x_i \leq_{\mathbb{R}} y_i.
\]

Although \( \mathbb{R}^n \) is not an object we study itself, some posets we do study can be more easily defined as subposets of \( \mathbb{R}^n \). Unless otherwise stated, we assume that all subsets of \( \mathbb{R}^n \) are ordered by \( \leq_{\mathbb{R}^n} \).

\( n \)-cube. We write \( \mathbb{B}_n \) for the \( n \)-dimensional Boolean lattice \( 2^{[n]} \) ordered by \( \subseteq \). In other words, we consider the subsets of \( [n] = \{1, 2, \ldots, n\} \) ordered by inclusion. As the name suggests, this really is a lattice where the joins are the unions and the meets are the intersections. The set \( \{ S \subseteq [n] \mid \lvert S \rvert = i \} \) is the layer \( i \) of \( \mathbb{B}_n \). Note that the \( n \)-cube has \( n + 1 \) layers (layer 0 till layer \( n \)). Figure 4 shows two Hasse diagrams of \( \mathbb{B}_4 \).

Note that subset lattices of different \( n \)-element sets are isomorphic. Our choice of the ground set \( [n] \) is arbitrary.

The \( n \)-cube is isomorphic to the poset induced by \( \{0, 1\}^n \) in \( \mathbb{R}^n \): A binary sequence \( b = (b_1, b_2, \ldots, b_n) \in \{0, 1\}^n \) can be thought of as an incidence vector
Figure 4: Two drawings of the 4-cube. The left one nicely reflects the four-dimensional structure, showing how the 4-cube consists of two 3-cubes with a matching in between which are in turn 2-cubes with a matching in between. The right drawing captures the layered structure.

of a set $S_b \subseteq [n]$, where $i \in S_b$ if and only if $b_i = 1$. The isomorphism mapping $b \mapsto S_b$ for $b \in \{0,1\}^n$ allows us to freely switch perspective between sets and bit strings when convenient. Since $\{0,1\} \subseteq \mathbb{R}$ is a 2-chain, another way to define $\mathfrak{S}_n$ would be

$$\mathfrak{S}_n := 1 \times 1 \times \ldots \times 1,$$

where we represent the poset $\mathfrak{I}_2$ by its Hasse diagram 1.

**standard example** For $n \geq 3$, let $S_n$ be the poset induced by layer 1 and layer $n-1$ of $\mathfrak{S}_n$. That is, the elements of $S_n$ are $\{S \subseteq [n] \mid |S| = 1 \text{ or } |S| = n-1\}$. The Hasse diagram of $S_n$ is a complete bipartite graph minus a matching, see Figure 5.

Figure 5: Two drawings of the standard example $S_n$ for $n = 5$. The left drawing uses labels strictly according to definition. On the right we use $a_i := \{i\}$ and $b_i := [n] \setminus \{i\}$.

### 1.2. Sperner’s Theorem and Dilworth’s Theorem

We review two classical results due to Sperner (1928) and Dilworth (1950). Short and elegant proofs were given by Lubell [Lub66] and Galvin [Gal94] in one-page
papers respectively. We follow them closely.

**Theorem 1** (Sperner’s Theorem). *The largest antichains in \( \diamond_N \) have size \( \binom{N}{\lfloor N/2 \rfloor} \).*

**Proof.** Clearly, layer \( \lfloor N/2 \rfloor \) forms an antichain of the required size. We show that no other antichain \( A \) can be larger.

Any set \( T \subseteq [N] \) on layer \( |T| \) contains \( |T| \) sets from the previous layer and is contained in \( N - |S| \) sets in the next layer. By extension, the number of maximal chains (i.e. chains of size \( n + 1 \)) passing through some \( S \in A \) is \( |S|!(N - |S|)! \). Since any chain contains at most one element of \( A \), because any two elements of \( A \) are incomparable, and since there are \( N! \) different maximal chains, in total we get

\[
N! \geq \sum_{S \in A} |S|!(N - |S|)! \geq \sum_{S \in A} |N/2|!(N - |N/2|)! = |A|!\binom{N}{|N/2|}!(N - \lfloor N/2 \rfloor)!,
\]

which proves the claim. \( \square \)

With this result in mind, we define the Sperner number \( \text{sp}(n) \) to be the smallest \( N \) such that the \( N \)-cube contains \( \ldots \ldots n \), i.e.

\[
\text{sp}(n) := \min \left\{ N \mid \binom{N}{\lfloor N/2 \rfloor} \geq n \right\} = \min \left\{ N \mid \diamond_N \text{ contains } \ldots \ldots n \right\}. \quad (1)
\]

This number has been tightly approximated:

**Theorem 2** (Habib, Nourine, Raynaud, Thierry [Hab+04]). \( \text{sp}(n) \in \{ \lfloor \log_2(n) + \log_2 \log_2(n)/2 + 1 \}, \lfloor \log_2(n) + \log_2 \log_2(n)/2 + 2 \} \).

Note that it is easy to decompose a poset of height \( h \) into \( h \) antichains (i.e. layers).

The following result is the non-trivial analogue for width:

**Theorem 3** (Dilworth’s Theorem). *If \( P \) has width \( w \), then \( P \) can be covered by \( w \) chains.*

This result is tight: At least \( w \) chains are needed, since \( P \) contains an antichain of size \( w \) and no chain can cover two elements of an antichain.

**Proof.** We proceed by induction on the size of \( P \). Let \( a \in P \) be a maximal element and \( P' := P \setminus \{ a \} \). If \( P' \) has smaller width than \( P \) we can, by induction, cover \( P' \) with \( w - 1 \) chains and put \( a \) in a new 1-chain. Together, we covered \( P \) with \( w \) chains as desired.

Otherwise, if \( P' \) still has an antichain of size \( w \), cover \( P' \) by \( w \) chains \( C_1, \ldots, C_w \) (possible by induction) and consider the antichain \( A = \{ a_1, \ldots, a_w \} \) consisting of the elements \( a_i := \max \{ x \in C_i \mid x \text{ is part of a } w\text{-antichain in } P' \} \). Since \( A \) is a largest antichain in \( P \) and \( a \) is a maximal element of \( P \), we have \( a > a_{i_0} \) for some \( i_0 \in \{ 1, \ldots, w \} \).

Removing the chain \( \{ a \} \cup \{ x \in C_{i_0} \mid x \leq a_{i_0} \} \) from \( P \) yields a poset that contains no antichains of size \( w \) and can therefore be covered by \( w - 1 \) chains by induction. \( \square \)
When combined, Sperner’s and Dilworth’s Theorem assert that the $\mathcal{B}_N$ can be covered by $\binom{N}{\lfloor N/2 \rfloor}$ chains.

However, much stronger results are known for this case. A partition of $\mathcal{B}_N$ into symmetric chains can be constructed. A symmetric chain has its minimum in some layer $k$, its maximum in layer $n - k$ and length $n - 2k + 1$. Consider Figure 6 for an example. It has been known for a long time that such symmetric chain decompositions exist [GK76].

![Figure 6: Symmetric chain decompositions of $\mathcal{B}_1 = \emptyset$, $\mathcal{B}_2 = \emptyset$, $\mathcal{B}_3$ and $\mathcal{B}_4$.](image)

An even stronger result is fairly recent. Streib and Trotter [ST14] proved that a symmetric chain decomposition can be chosen such that the chains can be attached to one another to form a Hamiltonian cycle in the Hasse diagram of $\mathcal{B}_{n'}$.

### 1.3. Critical Pairs

Recall that a function $f : P \to Q$ between posets is an embedding, iff for any two elements $x, y \in P$ we have that $f$ respects the pair $(x, y)$ by which we mean

$$x \leq y \iff f(x) \leq f(y),$$

However, to prove that $f$ is an embedding, we need not check this condition for all pairs. If $f$ respects some important pairs, then the structure of $P$ and $Q$ assures that $f$ respects the remaining pairs as well. Call the pair $(x, y)$ of two elements from $P$

- **covering**, if $x < y$ and $\not\exists z : x < z < y$,
- **critical**, if $x \nleq y$ and $\not\exists x' : (x > x' \nleq y)$ and $\not\exists y' : (x \nleq y' > y)$.

Intuitively, a pair $(x, y)$ is covering if it is an edge of the Hasse diagram, i.e. it corresponds to an “upward step” that cannot be split into several upward steps. A pair $(x, y)$ is critical, if $x \nleq y$ but replacing $x$ with something smaller or $y$ with something bigger will yield “$\leq$”. Consider Figure 7 for examples.
In the poset on the right we have \( a < g \), but \((a, g)\) is not covering, since \( a < d < g \). The pairs \((a, d)\) and \((d, g)\) are covering. We have \( f \not< b \) but \((f, b)\) is not critical since \( f > c \not< b \). The pair \((c, b)\) is still not critical since \( c \not< d \not> b \).

The pair \((c, d)\) is critical.

Figure 7: In the poset on the right there are nine covering pairs (the edges that are drawn) and the following critical pairs: \((a, e), (b, c), (c, d), (d, f), (e, d), (e, f)\).

**Lemma 4.** Let \( f : P \to Q \) be a function. If \( f \) respects covering pairs and critical pairs, then \( f \) is an embedding.

**Proof.** We need to show both directions of (⋆) for arbitrary pairs \((x, y)\).

(⇒) Pick a pair \( x \leq y \). Since \( P \) is finite, we can find intermediate elements \( x = z_0 < z_1 < \ldots < z_n = y \) (potentially \( n = 0 \)) such that \((z_{i-1}, z_i)\) is covering \((i \in [n])\). Since \( f \) respects those pairs, we have:

\[
f(x) = f(z_0) \leq f(z_1) \leq \ldots \leq f(z_n) = f(y)
\]

so \( f(x) \leq f(y) \) by transitivity.

(⇐) Pick two elements \( x, y \) with \( f(x) \leq f(y) \). Assume for contradiction that \( x \not< y \). If \((x, y)\) is not critical then we can make \( x \) smaller or \( y \) larger and still have “\( \not< \)”. We do this until we arrive at a critical pair. Formally, we find elements \( x_1, x_2, \ldots, x_k \) and \( y_1, y_2, \ldots, y_l \) \((k = 0 \text{ and } l = 0 \text{ allowed})\) such that:

\[
x = x_0 > x_1 > \ldots > x_k < y_l > \ldots > y_1 > y_0 = y
\]

where \((x_k, y_l)\) is a critical pair. Now we have, using that \( f \) fulfils (⇒),

\[
f(x_k) \leq f(x) \leq f(y) \leq f(y_l).
\]

This contradicts the fact that \( f \) respects the critical pair \((x_k, y_l)\).

In some posets critical pairs are rare, so the last claim significantly lightens our burden when proving that a function is an embedding. Take for instance the cube, where there are only \( n \) critical pairs, as the following result implies:

**Lemma 5.** If \((x, y)\) is a critical pair of a poset \( P \), then \( x \) is join-irreducible in \( P \) and \( y \) is meet-irreducible in \( P \).

**Proof.** If \( x \) were not join-irreducible, it would either be a minimum (which it clearly is not since \( x \not< y \)) or we have \( x = x_1 \lor x_2 \) for \( x \not\in \{x_1, x_2\} \). Since \((x, y)\) is critical and \( x_1, x_2 < x \), we have \( x_1 \leq y \) and \( x_2 \leq y \). Therefore, \( y \) is an upper bound of \( \{x_1, x_2\} \). Since \( x \) is the least upper bound of \( \{x_1, x_2\} \) we have \( x \leq y \), a contradiction.

Showing that \( y \) is meet-irreducible is analogous.
2. Basic Results on the 2-Dimension of Posets

In this section we examine the 2-dimension of \( P \), which is the smallest number \( n = \dim_2(P) \) such that \( P \) is a subposet of \( \mathbb{B}_n \). To our knowledge, this was first studied by Trotter [Tro75]. This parameter is hard to determine in general and has inspired much research, some of which will help us later on when \( \mathbb{B}_n \) will be coloured and the subposet \( P \) needs to be found monochromatically.

This section is simple but fundamental: Many results will be generalised in Part III. Most results that we fail to attribute to corresponding authors should be considered folklore.

Section 2.1: Different Notions of Dimension. We briefly motivate why 2-dimension is an interesting parameter and compare it to other notions of dimension, including the order dimension, the \( k \)-dimension and the encoding dimension.

Section 2.2: Order Dimension and Linear Orders. We briefly discuss how dimension is related to the intersection of linear orders.

Section 2.3: The 2-Dimension of Selected Posets. We determine the 2-dimension of \( \mathbb{I}_n \), \( \mathbb{S}_n \) and \( S_n \) precisely. We then bound the 2-dimension of \( \mathbb{B}_n(a,b) \), horizontal sums \( P \cup Q \) and vertical sums \( P \otimes Q \).

2.1. Different Notions of Dimension

Hasse diagrams work well enough when drawing small posets on a piece of paper. But to represent large posets, e.g. in computers, they are impractical.

The naïve alternative way of representing a poset \((P, \leq P)\) would be to make a list of all the pairs in \( \leq P \), as we did in Figure 3. This list, however, is a clumsy structure, potentially quadratic in the size of \( P \), and every time we want to know if some element \( x \) is smaller than another element \( y \), we would need to go through the list and determine if \((x, y)\) is contained in it.

A different strategy of representing \( P \) is to specify an embedding into \( \{0, 1\}^n \cong \mathbb{B}_n \), i.e. we specify a sequence of \( n \) bits for each element of \( P \). This is potentially more compact and, more importantly, the relationship between two elements can then be efficiently derived by comparing the sequences. The 2-dimension is the smallest \( n \) for which this works and therefore a measure for the efficiency of this representation of \( P \).

While the 2-dimension is certainly a natural notion and bit sequences are particularly convenient to handle in computers, one might argue that it is a somewhat arbitrary special case of the more general notion of the \( k \)-dimension of \( P \): The smallest \( n = \dim_k(P) \) such that \( P \) embeds into \( \{0, \ldots, k-1\}^n \).

There is also the (index-free) notion of dimension, often called order dimension: The smallest \( n = \dim(P) \) such that \( P \) embeds into \( \mathbb{R}^n \). When compared to \( k \)-dimension, the order dimension is arguably the more useful tool to understand the structure of posets. For example, all chains have order dimension 1 and all antichains have
order dimension 2. Order dimension has no lower bound in terms of the size of the poset: There are arbitrarily large posets of any constant order dimension, but not arbitrarily large posets of any constant \( k \)-dimension.

Note the relationship \( \dim_2(P) \leq \dim_3(P) \leq \ldots \leq \dim(P) \) and consider Figure 8 for an example.

![Figure 8](image)

If the goal is merely to find efficient representations of posets, embeddings into arbitrary products of chains can be considered, i.e. embeddings into \([k_1] \times [k_2] \times \ldots \times [k_l] \subseteq \mathbb{R}^l\) for \( k_i \in \mathbb{N} \). The efficiency of such an embedding, i.e. the number of bits per element, is then given by \( \sum_l \lceil \log_2(k_i) \rceil \). Minimising this yields the so-called encoding dimension \( \text{edim}(P) \) defined by Habib, Huchard, and Nourine [HHN95]. Similar to \( \dim \), \( \dim_2 \) and related properties, computing the encoding dimension of posets is NP-hard. For a proof, we refer to Raynaud and Thierry [RT10].

For some posets the encoding dimension is much smaller than the 2-dimension. A trivial example would be chains where \( \dim_2(\mathbb{I}_n) = n - 1 \gg \lceil \log_2 n \rceil = \text{edim}(\mathbb{I}_n) \).

Regardless, with the exception of Section 2.2 on order dimension, the remainder of this thesis focuses exclusively on \( \dim_2 \).

### 2.2. Order Dimension and Linear Orders

In contrast to what we said earlier, the order dimension \( \dim(P) \) of a poset \((P, \leq_P)\) is traditionally defined to be the smallest \( n \), such that the relation \( \leq_P \) is the intersection of \( n \) linear order relations on \( P \). Here, we view relations on \( P \) as subsets of \( P \times P \) and a linear order or total order is a relation on \( P \) where \( x \leq y \) or \( y \leq x \) holds for any \( x, y \in P \) (i.e. no two elements are incomparable). See Figure 9 for three simple examples.

The notion of order dimension has been extensively studied in the literature. An excellent source on the subject – and on posets in general – is William T. Trotter’s book “Combinatorics and Partially Ordered Sets: Dimension Theory” [Tro92].

Before we continue, let us quickly state and prove the equivalence of the two definitions of order dimension we have seen. We split the proof into two claims.

**Claim 6 (“\( \Rightarrow \)”**). If \( \leq_P \) is the intersection of \( n \) linear orders and \( P \) has size \( m \), then \( P \) can be embedded into the subposet of \( \mathbb{R}^n \) induced by \( [m]^n \).
Figure 9: Three examples of posets and their representations as intersections of linear orders. We have $x \leq y$ in the poset iff $x \leq y$ in each linear order. The order dimension of antichains $\cdots_n$ is 2, on the top right illustrated for $n = 4$. The order dimension of the standard example $S_n$ is $n$, on the bottom illustrated for $n = 3$.

Proof. We have $\leq_P = \leq_{l_1} \cap \ldots \cap \leq_{l_n}$ for $n$ linear orders $l_1, l_2, \ldots, l_n$ on $P$. We use the rank functions $r_i : P \to [m]$ defined as $r_i(x) := |\{y \in P \mid y \leq_{l_i} x\}|$, that map an element of $P$ to its position in the order $l_i$. With them we define the function:

$$f : P \to [m]^n \ni x \mapsto (r_1(x), \ldots, r_n(x)).$$

To see that $f$ is an embedding, observe

$$x \leq_P y \iff (x, y) \in \leq_P \iff \forall i : (x, y) \in l_i \iff \forall i : r_i(x) \leq r_i(y) \iff f(x) \leq f(y).$$

Claim 7 (“$\Leftarrow$”). The relation $\leq_R^n$ is the intersection of $n$ linear orders.

Note that this implies that $\leq_P$ is the intersection of (at most) $n$ linear orders if $P$ is a subposet of $\mathbb{R}^n$.

Proof. For any $i \in [n]$ we define the linear order $l_i$ as follows. If $x, y \in \mathbb{R}^n$ then:

$$x \leq_{l_i} y :\iff x = y \text{ or } x_i \leq_R y_i \text{ or } (x_i = y_i \text{ and } x_j \leq_R y_j),$$

where $j$ depends on $x$ and $y$ and is the smallest index with $x_j \neq y_j$ (choose $j$ arbitrary if $x = y$). It is easy to check that all $l_i$ are uniquely defined linear orders. We need to check that $\leq_R^n = \leq_{l_1} \cap \ldots \cap \leq_{l_n}$. If $(x, y) \in \leq_R$ then we have $x_i \leq y_i$ and $x_j \leq y_j$ for any $i$ and $j$ so clearly $(x, y) \in \leq_{l_i}$ for any $i$. Conversely, $(x, y) \in \leq_{l_i}$ implies $x_i \leq_R y_i$, so $(x, y) \in l_1 \cap \ldots \cap l_n$ implies $(x, y) \in \mathbb{R}^n$. \hfill \square

Together, Claim 6 and Claim 7 imply the following corollary:

Corollary 8. A relation $\leq_P$ of a poset $P$ is the intersection of $n$ linear orders if and only if $P$ can be embedded into $\mathbb{R}^n$. 

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2.3. The 2-Dimension of Selected Posets

We can immediately determine the 2-dimension of the following simple posets:

\[ \dim_2(I_n) = n - 1, \quad \dim_2(\cdots_1^n) = \text{sp}(n) \approx \log(n), \quad \dim_2(\square_n) = n. \]

The “−1” in the dimension of chains occurs because \( \square_{n-1} \) has \( n \) layers. The 2-dimension of antichains is a corollary to Sperner’s Theorem and Habib et al’s approximation. As for \( \square_n \), it clearly can be embedded into itself via the identity but not into any smaller cube.

Before we consider other specific posets, we make the following general observation:

\[ \log_2(|P|) \leq \dim_2(P) \leq |P|. \]

For the lower bound, observe that \( \square_n \) has only \( 2^n \) elements, so if a poset embeds into \( \square_n \) it can have at most \( 2^n \) elements. Therefore \( |P| \leq 2^{\dim_2(P)} \). The lower bound is tight for \( P = \square_{n} \) since \( \log_2(|\square_{n}|) = n = \dim_2(\square_{n}) \).

For the upper bound, consider the following embedding of \( P \) into \( 2^P \cong \binom{P}{|P|} \):

\[ f : P \to 2^P \]
\[ x \mapsto \{ z \in P \mid z \leq P \ x \} \]

To see that \( f \) is an embedding, note that in case of \( x \leq_P y \) every lower bound of \( x \) is by transitivity a lower bound of \( y \), so \( f(x) \subseteq f(y) \). If on the other hand \( x \not\leq_P y \) then \( x \in f(x) \setminus f(y) \), so \( f(x) \not\subseteq f(y) \).

The upper bound \( \dim_2(P) \leq |P| \) is tight for the \( n \)-element poset \( \cdot I_{n-1} \) consisting of an \((n − 1)\)-chain and an additional element incomparable to the others. To see this, assume that \( f \) is an embedding of \( \cdot I_{n-1} \) into \( \square_k \) for some \( k \in \mathbb{N} \). Then \( f \) cannot map any element to \( \emptyset \) or \([k]\) since \( \cdot I_{n-1} \) has no maximum or minimum. This leaves \( k - 1 \) layers of \( \square_k \) to be used. Since \( \cdot I_{n-1} \) contains a chain of size \( n - 1 \) we need \( k - 1 \geq n - 1 \), so \( k \geq n \).

Trotter [Tro75] showed that \( \cdot I_{n-1} \) is essentially the only example for a poset \( P \) with \( \dim_2(P) = |P| \). The exceptions are a few posets of 2-dimension at most \( 4 \), such as \( \cdots_4 \) and posets that are composite in the sense we introduce in Section 2.3.2, for instance \( \cdots_2 \otimes \cdots_2 \otimes \cdots_2 \).

2.3.1. Two Layers of \( \square_n \)

We define the poset induced by a **pair of layers** of a cube and the poset induced by a **range of layers** of a cube. For \( 0 < k < l < n \) let

\[ \square_n(k, l) := \{ X \subset [n] \mid |X| \in \{k, l\}\}, \]
\[ \square_n(k, \ldots, l) := \{ X \subset [n] \mid k \leq |X| \leq l\}. \]

Of course, \( \square_n(k, l) \) and \( \square_n(k, \ldots, l) \) inherit the order \( \subseteq \) from \( \square_n \).

We have already seen the special case \( S_n = \square_n(1, n-1) \). Since \( S_n \) contains only \( 2n \) elements out of \( 2^n \) elements of \( \square_n \), one might suspect that its 2-dimension is less than \( n \). This is not the case:
Lemma 9. Let \( 0 < k < l < n \) and \( N \in \mathbb{N} \). Any embedding \( f : \mathcal{O}_n(k,l) \to \mathcal{O}_N \) can be extended to an embedding of \( \mathcal{O}_n(k,\ldots,l) \) into \( \mathcal{O}_N \).

Proof. We need to extend the domain of \( f \) to the intermediate layers and define:

\[
 f(X) := \bigcup_{S \subseteq X, |S| = k} f(S) \quad \text{for } X \subset [n], k < |X| < l.
\]

It is easy to check that \( f \) is monotone, i.e. \( X \subseteq Y \) implies \( f(X) \subseteq f(Y) \).

By Lemma 4 we only need to check that \( X \not\subseteq Y \) implies \( f(X) \not\subseteq f(Y) \) for critical pairs \((X,Y)\). In that case we have by Lemma 5 that \( X \) is join-irreducible and \( Y \) is meet-irreducible. In \( \mathcal{O}_n(k,\ldots,l) \) only elements of layer \( k \) are join-irreducible: Any other element can be written as the union (i.e. join) of two smaller sets in \( \mathcal{O}_n(k,\ldots,l) \). Similarly, the meet-irreducible elements are exactly those of layer \( l \). So \( X,Y \in \mathcal{O}_n(k,l) \) and since \( f \) is known to be an embedding on this set we have \( f(X) \not\subseteq f(Y) \). \( \square \)

Actually, the preceding Lemma is a special case of the more involved insight that a poset has the same 2-dimension as the smallest lattice it fits into, the so-called Dedekind–MacNeille completion (see Proposition 13).

We now know that \( \dim_2(\mathcal{O}_n(k,l)) = \dim_2(\mathcal{O}_n(k,\ldots,l)) \). An upper bound on these numbers is clearly \( n \) since we are considering subposets of \( \mathcal{O}_n \). If \( k < n/2 < l \) then note that \( \mathcal{O}_n(k,\ldots,l) \) contains the middle layer of \( \mathcal{O}_n \), which is the largest antichain in \( \mathcal{O}_n \). So no embedding into cubes of dimension less than \( n \) can exist, since in those cubes we cannot find a large enough antichain. We have:

\[
0 < k < n/2 < l < n \implies \dim_2(\mathcal{O}_n(k,l)) = \dim_2(\mathcal{O}_n(k,\ldots,l)) = n,
\]

and in particular

\[
\dim_2(S_n) = n.
\]

But what about other ranges of layers, i.e. what is the 2-dimension of \( \mathcal{O}_N(k,l) \) for general \( k \) and \( l \)? We do not know a general result but a bound in the case of \( k = 1 \) and \( l \ll \sqrt{N} \) is given by the following probabilistic argument.

Proposition 10 (Trotter [Tro75]). Let \( 2 \leq k \ll \sqrt{N} \). Then

\[
\dim_2(\mathcal{O}_N(1,k)) \leq L := (k + 1)^2 \cdot e \cdot \ln N = \mathcal{O}(k^2 \log N).
\]

Proof. We use a well-known analytic result:

\[
\sup_{x \geq 1} \left( 1 - \frac{1}{x} \right)^x = \frac{1}{e} = \inf_{x \geq 1} \left( 1 - \frac{1}{x + 1} \right)^x \quad (\ast)
\]

Let \( (\mathbf{1}_{n,l})_{n \in [N], l \in [L]} \) be a family of independent indicator random variables with probability \( p = \frac{1}{k+1} \) each, i.e. each \( \mathbf{1}_{n,l} \) equals 1 with probability \( \frac{1}{k+1} \) and 0 with probability \( \frac{k}{k+1} \).
From this we define the random sets $A_1, \ldots, A_L \subseteq [N]$ as

$$A_l := \{ n \in [N] \mid 1_{n,l} = 1 \}, \quad \text{for } l \in [L].$$

In other words, $A_1, \ldots, A_L$ are random sets where the events “$n \in A_l$” ($n \in [N], l \in [L]$) are independent and have probability $\frac{1}{k+1}$.

Now consider the following (random) map:

$$f : \mathcal{N}(1,k) \to 2^{[L]} \quad S \mapsto \{ l \mid S \cap A_l \neq \emptyset \}$$

This construction of $f$ ensures that $S \subseteq T$ implies $f(S) \subseteq f(T)$. Using the results from Section 1.3 we now examine the probability that $f$ respects all critical pairs.

Let $(S = \{s\}, T = \{t_1, \ldots, t_k\})$ with $s \notin T$ be such a pair. The probability of the “bad event” that $S \nsubseteq T$ is not respected by $f$ is

$$\Pr[f(S) \subsetneq f(T)] = \Pr[\exists l \in [L]: S \cap A_l \neq \emptyset \land T \cap A_l = \emptyset]$$

$$= (1 - p \cdot (1 - p)^k)^L \left( 1 - \frac{1}{k+1} \cdot \left( 1 - \frac{1}{k+1} \right)^k \right)^L$$

$$\leq \left( 1 - \frac{1}{(k+1)e} \right)^L \left( 1 - \frac{1}{(k+1)e} \right)^{(k+1)^2 e \ln(N)}$$

$$< \left( \frac{1}{e} \right)^{(k+1) \ln(N)} = e^{-(k+1) \ln(N)} = N^{-(k+1)}.$$

Let $C$ be the set of all critical pairs. The size of $C$ is $|C| = \binom{N}{k} \cdot (N - k) < N^{k+1}$. We apply the union bound to see that the probability that all critical pairs are respected is positive:

$$\Pr[\forall (S,T) \in C: f(S) \subsetneq f(T)] \geq 1 - \sum_{(S,T) \in C} \Pr[f(S) \subsetneq f(T)]$$

$$> 1 - N^{k+1} \cdot N^{-(k+1)} = 0.$$

So there is some choice for $A_l$ such that $f$ respects all critical pairs. In that case $f$ is an embedding of $\mathcal{N}(1,k)$ into $\mathcal{L}$ which proves the claim.

### 2.3.2. Vertical and Horizontal Sums of Posets

We consider two natural ways of building a bigger poset out of two (disjoint) smaller posets $P$ and $Q$ (ordered by $\leq_P$ and $\leq_Q$). In both cases the new poset will have element set $P \cup Q$. In the case of the *vertical sum*, denoted by $P \oplus Q$, each element of $P$ is considered smaller than any element of $Q$. In the case of the *horizontal sum*,

\[^{2}\text{Trotter calls this the *join* of posets. To prevent confusion with "\lor" in posets we avoid this terminology.}\]
denoted by $P \uplus Q$, elements of $P$ are unrelated to elements of $Q$. Formally the orders $\leq_{P \uplus Q}$ and $\leq_{P \cup Q}$ are defined as:

\[
\begin{align*}
\leq_{P \uplus Q} & := \leq_P \cup \leq_Q \cup (P \times Q), \\
\leq_{P \cup Q} & := \leq_P \cup \leq_Q.
\end{align*}
\]  

(2)

Note that in order to roughly double the height of $\bigotimes_n$ we need to roughly double the dimension. If we want to roughly double the width however, it suffices to increase the dimension by 1. It should therefore come as no surprise that putting things next to one another requires less “spare dimensions” than putting them above one another.

For vertical sums, Trotter proves:

**Proposition 11** (Trotter [Tro75]). Let $P$ and $Q$ be posets. Then

\[
\dim_2(P \uplus Q) = \dim_2(P) + \dim_2(Q) + b
\]

where $b$ is 1 if $P$ has a maximum and $Q$ has a minimum and 0 otherwise.

**Proof.** Let $n := \dim_2(P)$ and $m := \dim_2(Q)$.

“≤” Let $f : P \to 2^{\{1,\ldots,n\}}$ and $g : Q \to 2^{\{n+1,\ldots,n+m\}}$ be embeddings. Note that $2^{\{n+1,\ldots,n+m\}} \cong \bigotimes_m$. Then consider the function

\[
\begin{align*}
h : P \uplus Q & \to [n+m] \\
p & \mapsto f(p), \quad \text{for } p \in P \\
q & \mapsto [n] \cup g(q), \quad \text{for } q \in Q.
\end{align*}
\]

It is an embedding of $P \uplus Q$, unless $f(x) = [n]$ and $g(y) = \emptyset$ for some $x$ and $y$: In that case $h$ is not injective. But then $x$ is necessarily a maximum of $P$ and $y$ a minimum of $Q$. An embedding is then given by $h' : P \uplus Q \to [n+m+1]$ where $h'(p) = h(p)$ for $p \in P$ and $h'(q) = h(q) \cup \{n+m+1\}$ for $q \in Q$.

“≥” Let $f : P \uplus Q \to \bigotimes_N$ be an embedding and $S := \bigcup f(P)$. Then $f$ must send elements of $P$ to subsets of $S$ and elements of $Q$ to supersets of $S$. The set of subsets of $S$ is isomorphic to $\bigotimes_{|S|}$ and the set of supersets of $S$ is isomorphic to $\bigotimes_{N-|S|}$. From this we see that $|S| \geq n$ and $N - |S| \geq m$, so $N \geq n+m$. It is not hard to see that if $P$ has a maximum $x$ and $Q$ has a minimum $y$, this can be improved by 1: The image of $x$ is necessarily $S$ so the image $S' := f(y)$ will be a proper superset of $S$. So the elements of $Q$ must be mapped to supersets of $S'$. Therefore $N - |S| \geq N - |S'| + 1 \geq m + 1$ and $N \geq n + m + 1$. □

For horizontal sums note that if we have embeddings $f : P \to \bigotimes_n$ and $g : Q \to \bigotimes_n$, then an embedding of $h : P \cup Q \to \bigotimes_{n+2}$ is given by

\[
\begin{align*}
h : P \cup Q & \to [n+2] \\
p & \mapsto f(x) \cup \{n+1\}, \quad \text{for } p \in P \\
q & \mapsto g(y) \cup \{n+2\}, \quad \text{for } q \in Q.
\end{align*}
\]
Since an image of \( p \in P \) contains \( n + 1 \) but not \( n + 2 \) and an image of \( q \in Q \) contains \( n + 2 \) but not \( n + 1 \), the images of \( p \) and \( q \) are incomparable. This shows that

\[
\max(\dim_2(P), \dim_2(Q)) \leq \dim_2(P \cup Q) \leq \max(\dim_2(P), \dim_2(Q)) + 2
\]

which is actually an application of the following more general insight:

**Proposition 12.** \( \dim_2(\bigotimes_{w \text{ times}} n_{
}) \leq n + \text{sp}(w) \).

![Figure 10: An example for the construction of Proposition 12 with \( w = 6 \) and thus \( \text{sp}(w) = 4 \). Let \( N := n + 4 \) and \( |N| = abcd \cup [n] \) (we omit braces and commas as usual). For each element of the six-element antichain highlighted on the left, we identify one copy of \( \bigotimes_n \) within \( \bigotimes_N \) as shown on the right. For instance, one copy of \( \bigotimes_n \) is formed by the sets in between \( ab \) and \( ab \cup [n] \).]

**Proof.** Let \( N := n + \text{sp}(w) \) and \( |N| = [n] \cup H \) the corresponding partition. Since \( |H| = \text{sp}(w) \), there is an antichain \( \{S_1, \ldots, S_w\} \) in \( 2^H \). We find the \( w \) copies of \( \bigotimes_n \) as:

\[
\bigotimes_n^{(i)} := \{X \mid S_i \subseteq X \subseteq S_i \cup [n]\}, \quad 1 \leq i \leq w.
\]

Elements from different copies are unrelated as required: If \( X \in \bigotimes_n^{(i)} \) and \( Y \in \bigotimes_n^{(j)} \) for \( i \neq j \), then \( S_i \subseteq X \) but \( S_j \not\subseteq Y \). So \( X \not\subseteq Y \).

As a corollary, note that for large \( w \) and a poset \( P \) of 2-dimension \( n \) we have

\[
\dim_2(\bigcup_{w \text{ times}} P \cup \ldots \cup P) \leq \dim_2(\bigotimes_{w \text{ times}} n_{
}) \leq n + \text{sp}(w) = \dim_2(P) + O(\log w).
\]

so getting many unrelated copies of a fixed poset \( P \) is “cheap” in terms of 2-dimension.

Much more precise results than this are known. For instance, Katona and Nagy [KN15] determine the asymptotic behaviour of the number \( k \) of incomparable copies of a fixed poset \( P \) that can be found in a cube of dimension \( n \).

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3. Hardness of Determining the 2-Dimension

This thesis is not primarily concerned with computational complexity. But understanding that some quantity is difficult to approximate on a fundamental level (and why) can provide valuable insights and help with managing expectations. Familiarity with approximation algorithms is not required to follow the reduction argument but may potentially be helpful to appreciate the result. We refer the reader to a standard volume on algorithms [Cor+09] that addresses approximation algorithms in Section 35 and to Vazirani [Vaz01] for a more detailed account.

In this section we give a self-contained proof of an argument by Habib, Nourine, Raynaud and Thierry [Hab+04] that shows an equivalence between the problem of determining the 2-dimension of posets and the problem of determining the biclique cover number of bipartite graphs. Plugging together this result and an inapproximability result for bipartite cover by Holzer and Gruber [GH07] yields:

Theorem 18. If \( P \neq NP \), then there is for no \( \varepsilon > 0 \) a polynomial-time algorithm approximating \( \dim_2(P) \) for posets \( P \) within a factor of \( |P|^{1/3-\varepsilon} \).

So not only is 2-dimension NP-hard to calculate, it is also NP-hard to approximate. This implies, for instance, that any statement of the form \( \forall P : \dim_2(P) \leq \varphi(P) \) where \( \varphi \) is a function only involving easily computed parameters (like, height, width, size, \ldots) will necessarily (assuming \( P \neq NP \)) be imprecise for some \( P \).

Section 3.1: Completions and Reductions of Posets. A poset can be stripped to a core (its join- and meet-irreducible elements) or blown up to a lattice (the Dedekind–MacNeille completion) without affecting the set of lattices it embeds into. In particular, both operations do not affect the 2-dimension, despite the fact that there may be an exponential difference in size. This is needed in the reduction that follows, but also interesting on its own.

Section 3.2: Relationship to Bipartite Cover, Computational Complexity. We examine the problem of covering bipartite graphs with bicliques and see how it is equivalent to computing 2-dimension of posets. This allows the transfer of hardness results from bipartite cover.

3.1. Completions and Reductions of Posets

When trying to embed a poset \( P \) into a lattice \( L \), some elements of \( P \) may be tricky to embed while others just “fall into place” as soon as the tricky part is done. We will see a precise formulation and proof of this insight in the following.

We start with an analogy to geometry and talk about fitting shapes into squares instead of posets into lattices.

Let \( P \) be a polygon and \( Q = [0, 1] \times [0, 1] \) the unit square. We say that a geometric object fits into \( Q \) if it can be rotated and shifted to be a subset of \( Q \). Say a point \( p \in P \) is convex-irreducible if it cannot be written as a convex combination of other
points in $P$, i.e. there is no $p_1, p_2 \in P \setminus \{p\}, 0 < d < 1$ such that $p = d \cdot p_1 + (1-d) \cdot p_2$. Then the following statements are equivalent (see Figure 11):

(i) The set of convex-irreducible points of $P$ fits into $Q$.

(ii) $P$ fits into $Q$.

(iii) The convex hull of $P$ fits into $Q$.

Note that it is important that $Q$ is convex.

When embedding a poset $P$ into a lattice $L$ there is a similar effect: There is some core $\text{JM}(P) \subseteq P$, the join- and meet-irreducible elements of $P$, and some hull $\text{DM}(P)$, the Dedekind–MacNeille completion of $P$, and both can be embedded into $L$ if and only if $P$ can be embedded into $L$. See Figure 11 for a sketch.

![Figure 11: Whether or not a shape fits into a square is not affected by adding or removing points to the shape that are convex combinations of other points. Similarly, whether or not a poset can be embedded into a lattice is not affected by adding or removing elements that are joins and meets of other elements.](image)

Of course, we need to make this more formal and for our convenience, we introduce new notation accompanying a poset $P$:

- $J(P)$, the set of join-irreducible elements of $P$.
- $M(P)$, the set of meet-irreducible elements of $P$.
- $\text{JM}(P) := J(P) \cup M(P)$. This is a poset: It inherits the order from $P$. 

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DM(P), the Dedekind–MacNeille completion of P, the smallest lattice containing P. It is named after MacNeille [Mac37] who first defined and constructed it. We formally define and construct it later.

We will prove several claims that together yield the following proposition:

**Proposition 13** (See e.g. [DP02], [Hab+04]). Let P be a poset and L a lattice. Then the following statements are equivalent:

(i) JM(P) can be embedded into L.

(ii) P can be embedded into L.

(iii) DM(P) can be embedded into L.

Note that because JM(P) is a subposet of P, the implication (ii) ⇒ (i) is clear. For the same reason (iii) ⇒ (ii) will become clear, as soon as we manage to construct DM(P) and its properties later.

**Claim** ((i) ⇒ (ii)). Let $f : JM(P) \rightarrow L$ be an embedding. Then an embedding of P into L is

$$g : P \rightarrow L$$

$$x \mapsto \bigvee \{ f(j) \mid j \in J(P), j \leq x \}.$$  

**Proof.** Since L is a lattice, the joins of subsets of L and therefore g is well-defined. For $a, b \in P$ we see that $a \leq b$ implies $g(a) \leq g(b)$ since, when computing $g(b)$, the join is taken over a superset compared to the join taken when computing $g(a)$.

In light of the results of Section 1.3, we need to check that $a \not\leq b$ implies $g(a) \not\leq g(b)$ only for join-irreducible $a$ and meet-irreducible $b$. In that case, using that $f$ is an embedding of JM(P):

$$g(a) = \bigvee \{ f(j) \mid j \in J(P), j \leq a \}^{a \in J(P)} f(a).$$

$$g(b) = \bigvee \{ f(j) \mid j \in J(P), j \leq b \}^{f(j) \leq f(b)} f(b).$$

Now the relation $g(a) \leq g(b)$ would imply $f(a) = g(a) \leq g(b) \leq f(b)$ which contradicts the fact that $f$ was an embedding of JM(P). So indeed $g(a) \not\leq g(b)$ and we are done.

The idea behind the construction of DM(P) is that we add new elements that take the role of missing joins and meets in P. This process will terminate and we

---

3Note the asymmetry in the definition. We could have just as well defined the embedding of P in terms of meets of meet-irreducible elements.
will arrive at a unique lattice. First, we introduce notation for the sets of upper and lower bounds of $S \subseteq P$:

$$
S^\uparrow := \{ x \in P \mid \forall s \in S : x \geq s \}
$$

$$
S^\downarrow := \{ x \in P \mid \forall s \in S : x \leq s \}.
$$

We will make use of the following simple facts in the next claim:

$$(i) X \supseteq X' \Rightarrow X^\uparrow \subseteq X'^\uparrow,$$

$$(ii) X \subseteq X^\uparrow \downarrow, X \subseteq X^\downarrow \uparrow.$$

In words: $(i)$ Upper bounds of a set $X$ are also upper bounds of each subset $X' \subseteq X$ and $(ii)$ every $x \in X$ is a lower bound of the upper bounds of $X$ and an upper bound of the lower bounds of $X$.

**Claim.** Assume $P$ is not a lattice. Then there are sets $S$ and $T$ with $S^\uparrow = T$ and $T^\downarrow = S$ where $S$ has no maximum and $T$ has no minimum.

![Diagram](image)

**Figure 12:** The set $X$ does not have a join. We set $T := X^\uparrow$ and $S := T^\downarrow$. We then put a new element $x$ in between $S$ and $T$.

**Proof.** Since $P$ is not a lattice, some set $X \subseteq P$ lacks a meet or a join. Say the join of $X \subseteq P$ does not exist (if a meet does not exist, the argument is similar). Define $T := X^\uparrow$ and $S := T^\downarrow$. Using only $(i)$ and $(ii)$ from above we find that

$$
X^\uparrow \subseteq (X^\uparrow)^\downarrow = S^\uparrow = (X^\downarrow)^\uparrow \subseteq X^\uparrow.
$$

This shows $S^\uparrow = X^\uparrow = T$ as desired.

We know that $T$ has no minimum since that would be the join of $X$. Since $X \subseteq S$, a maximum of $S$ would be an upper bound of $X$ and less than any element of $T$ (the set of upper bounds of $X$): That would also be a join of $X$. So $S$ has no maximum. □ (claim)

We can now construct a lattice as follows: As long as $P$ is not a lattice we find $S, T \subseteq P$ as stated above. We add a new element $x$ to $P$ that is considered to be bigger than any $s \in S$ and smaller than any $t \in T$ (and unrelated to other elements). Formally, we consider the set $P' := P \cup \{ x \}$ with the order relation

$$
\leq_{P'} := \leq_P \cup (S \times \{ x \}) \cup (\{ x \} \times T) \cup \{(x, x)\}.
$$
It is easy to check that $\leq P'$ is indeed reflexive, anti-symmetric and transitive. Therefore, $P'$ is a poset with subposet $P$. By construction, the new element $x$ is the join of $S$ and the meet of $T$. In particular, $P$ and $P'$ have the same join- and meet-irreducible elements. This implies that the construction terminates since every new element $x$ is the join of some set $S$ and therefore also the join of $J(P) \cap S$. Since there is only a finite number of subsets of $J(P)$, eventually each of them will have a join. The construction may take very long, though: When starting with the standard example $S_n$ of size $2n$ we will end up with $\bigotimes_n$, which has a size $2^n$.

As soon as all joins and meets are present, we arrived at a lattice which we call $\text{DM}(P)$. Note that we have not specified an order in which missing joins and meets are addressed. Assume, for now, that we address them in an arbitrary, but well defined, way. We will soon see that the order does not matter.

We now complete the proof of Proposition 13. We may already use "(i) ⇔ (ii)".

**Claim (ii) ⇒ (iii).** If a poset $P$ can be embedded into a lattice $L$, then $\text{DM}(P)$ can be embedded into $L$.

**Proof.** Let $P' := \text{DM}(P)$. Note that $\text{JM}(P') = \text{JM}(\text{DM}(P)) = \text{JM}(P)$. Since $P$ can be embedded into $L$, so can its subposet $\text{JM}(P)$. Now, using "(i) ⇒ (ii)", since $\text{JM}(P')$ can be embedded into $L$ so can $P'$ and we are done.

Now it is also clear that $\text{DM}(P)$ is well defined and any choices we might have thought we had during the construction where no real choices at all: If $\text{DM}'(P)$ is another lattice containing $P$ and fulfilling the last claim, then $P$ can be embedded into both $\text{DM}(P)$ and $\text{DM}'(P)$. Using the claim, we can find an embedding of $\text{DM}(P)$ into $\text{DM}'(P)$ and vice versa. This means $\text{DM}(P)$ and $\text{DM}'(P)$ are in fact isomorphic. So $\text{DM}(P)$ is the unique lattice fulfilling the claim, we call it the Dedekind–MacNeille completion.

An alternative, but equivalent, construction of $\text{DM}(P)$ in one step is:

$$\text{DM}(P) := (\{S \subseteq P \mid S \uparrow \downarrow = S\}, \subseteq).$$

The canonical embedding of $P$ to $\text{DM}(P)$ would then map $x \in P$ to $\{x\} \uparrow \downarrow$. While this undoubtedly looks more elegant, the author thinks the iterative construction presented here is a bit more transparent.

### 3.2. Relationship to Bipartite Cover, Computational Complexity

Recall the order dimension of posets as discussed in Section 2.2. Determining it has long been known to be computationally hard even in very restricted settings. Yannakakis showed in 1982 [Yan82] that it is NP-complete to decide if a poset has order dimension at most 3. Felsner, Mustata and Pergel [FMP15] were able to show that this holds even for posets of height 2.

In the case of the 2-dimension however, a bound on the 2-dimension implies a bound on the size. For instance, posets of 2-dimension 3 have size at most 8. Consequently, NP-hardness results must necessarily be weaker in that regard.
In the following, we review a construction by Habib et al. [Hab+04] which in turn uses results from [MN96] and [DP02]. We show how we obtain for each poset $P$ a bipartite graph $B$ and vice versa, such that the 2-dimension of $P$ equals the biclique cover number of $B$. The goal is to transfer NP-hardness results and non-approximability results between the corresponding computational problems.

We now define the relevant new notions.

**Definition 14.** (i) A bipartite graph $B = (S, T, E)$ is given by two sets $S$ and $T$ and a set of edges $E \subseteq S \times T$. In our figures, $S$ will always be at the bottom and $T$ at the top.

(ii) For a bipartite graph $B$, let the biclique cover number $\text{BicCov}(B)$ be the smallest number $n$, such that $B$ is the union of $n$ complete bipartite graphs, i.e.

$$E = (S_1 \times T_1) \cup (S_2 \times T_2) \cup \ldots \cup (S_n \times T_n),$$

where $S_i \subseteq S, T_i \subseteq T$ $(i \in [n])$.

We call each $S_i \times T_i$ for $i \in [n]$ a biclique and say $E$ is covered by them.

(iii) For a poset $P$, define $\text{Bip}(P) := (J(P), M(P), \not\preceq_P)$. In other words, $\text{Bip}(P)$ is the bipartite graph with join- and meet-irreducible elements of $P$ as bottom and top sets, respectively. It contains an edge $(j, m)$ if and only if $j \not\preceq_P m$ (for $j \in J(P), m \in M(P)$).

Note that strictly speaking we would have to write $\not\preceq_P \cap (J(P) \times M(P))$ for the edge set to restrict the relation $\not\preceq_P$ to the appropriate ground set.

See Figure 13 for an example. It also hints at what comes next: We show that the problem of finding the 2-dimension of a poset $P$ is equivalent to the problem of finding the bipartite cover number of $\text{Bip}(P)$.

![Figure 13](image)

**Figure 13:** A poset $P$ with $J(P) = \{a, b, d\}$, $M(P) = \{a, c, d\}$. The corresponding bipartite graph $\text{Bip}(P)$ can be covered with the three bicliques $\{d\} \times \{a, c\}$, $\{a\} \times \{d\}$ and $\{b\} \times \{a\}$. From this we can obtain an embedding of $P$ into $\mathbb{X}_3$ and vice versa, as shown in Proposition 15.

**Proposition 15** (Habib et al. [Hab+04]). Let $P$ be a poset. From an embedding of $P$ into $\mathbb{X}_n$ we can efficiently obtain a covering of $\text{Bip}(P)$ with $n$ bicliques and vice

---

4We mean there is a polynomial-time algorithm, but we will not make this precise.
versa. In particular we have

\[ \text{BicCov}(\text{Bip}(P)) = \text{dim}_2(P). \]

**Proof.** **Embedding to Covering.** Let \( f : P \to \mathbb{H}_n \) be an embedding of \( P \). We cover \( \text{Bip}(P) \) with bicliques \( S_1 \times T_1, \ldots, S_n \times T_n \) where:

\[ S_i := \{ x \in J(P) \mid i \in f(x) \}, \quad T_i := \{ x \in M(P) \mid i \notin f(x) \}, \quad i \in [n]. \]

To see that \( \bigcup_{i \in [n]} S_i \times T_i = \not\subseteq_P (J(P) \times M(P)) \), we show both inclusions:

- For \( i \in [n] \) and any \( s \in S_i, t \in T_i \) we have \( i \in f(s) \setminus f(t) \), so \( f(s) \not\subseteq f(t) \) and \( s \not\subseteq_P t \) (since \( f \) is an embedding). The set \( S_i \times T_i \) is therefore a subset of \( \not\subseteq_P \).

- Conversely, if \( (j, m) \in \not\subseteq_P (J(P) \times M(P)) \), then \( j \not\subseteq_P m \) so \( f(j) \not\subseteq f(m) \) since \( f \) is an embedding. This allows us to find \( i \in [n] \) with \( i \in f(j) \setminus f(m) \) and therefore \( (j, m) \in S_i \times T_i \).

**Covering to Embedding.** Let \( \not\subseteq_P (J(P) \times M(P)) = (S_1 \times T_1) \cup \ldots \cup (S_n \times T_n) \) be a covering of \( \text{Bip}(P) \). Then an embedding of \( \text{JM}(P) \) is given by

\[ f : \text{JM}(P) \to \mathbb{H}_n, \quad x \mapsto \{ i \in [n] \mid \exists j \in S_i : j \leq x \}. \]

It is clear that \( x \leq_P y \) implies \( f(x) \subseteq f(y) \). In light of the results of Section 1.3 we need to only check that \( j \not\subseteq m \) implies \( f(j) \not\subseteq f(m) \) for join-irreducible \( j \) and meet-irreducible \( m \). In that case \((j, m)\) is an edge of \( \text{Bip}(P) \) so \((j, m) \in S_i \times T_i \) for some \( i \). Since \( S_i \times T_i \subseteq \not\subseteq_P \) we have \( j' \not\subseteq m \) for each \( j' \in S_i \). The definition of \( f \) now gives \( i \in f(x) \setminus f(y) \) and thus \( f(x) \not\subseteq f(y) \).

By Proposition 13, this embedding of \( \text{JM}(P) \) can be extended to an embedding of \( P \) and we are done.

Proposition 15 allows us to convert a poset \( P \) into a bipartite graph \( B = \text{Bip}(P) \) such that \( \text{dim}_2(P) = \text{BicCov}(B) \). However, this is not the direction that allows us to prove hardness of computing 2-dimension. For that, we have to start with an arbitrary bipartite graph and obtain a corresponding poset. We do so in two steps: First go from \( B = (S, T, E) \) to a reduced bipartite graph \( \text{Red}(B) \) that has the same cover number. From this reduced graph, build a poset \( \text{Pos}(\text{Red}(B)) \) with \( \text{Bip}(\text{Pos}(\text{Red}(B))) \cong \text{Red}(B) \).

Since it is easy to get lost in the following argument we try to give some life to the abstract notions without sacrificing rigour. The following notions should be thought of as helpful synonyms for objects in an arbitrary bipartite graph \( B(D, F, E) \), not merely fuzzy analogies.

You have a set of friends \( F \) and know how to cook a set of dishes \( D \). There is an edge \( (d, f) \) iff your friend \( f \in F \) likes \( d \in D \). You throw dinner parties where you cook a subset \( D' \subseteq D \) of the dishes and invite a subset \( F' \subseteq F \) of your friends,
but you want that on those parties every $f \in F'$ likes every $d \in D'$. In this setting, $\text{BicCov}(B)$ is the smallest number of parties you have to throw such that each of your friends gets to try every dish he likes.

Say a friend is less picky than another friend, if he likes a superset of dishes. Say a dish is more popular than another dish, if it is liked by a superset of friends\(^5\).

Note that when you throw a party and are committed to inviting a friend $f$, then you make no mistake by inviting all friends that are less picky since they will like everything if $f$ likes everything. Similarly, if you cook a dish $d$ then you can also cook all dishes that are more popular, since, if everybody likes $d$, then everybody will also like the more popular dishes. In short, you can abide by the following rules, without increasing the number $\text{BicCov}(B)$ of parties you have to throw:

\begin{itemize}
  \item[(⋆1)] If a dish is cooked, then each more popular dish is cooked as well.
  \item[(⋆2)] If a friend is invited, then each less picky friend is invited as well.
\end{itemize}

From now on assume you follow (⋆1) and (⋆2).

Assume there is a dish $d$ such that for every friend $f$ who likes $d$, there is a less popular dish $d'$ that $f$ likes as well (where $d'$ may depend on $f$). Then rule (⋆1) ensures that $f$ gets to taste $d$ when he gets to taste $d'$. In other words: We can ignore the dish $d$ entirely and any set of parties that we come up with for the simplified problem will work for the original problem just by rule (⋆1). Similarly, if there is a friend $f$ such that every dish that he likes is also liked by someone more picky than $f$, then we can ignore $f$ and (⋆2) ensures that $f$ gets to taste every dish he likes. See Figure 14 for an example.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{diagram1.png}
\caption{Example for a reduction of a bipartite graph. In the situation shown above, consider dish $d_3$: By rule (⋆) it is served whenever the less popular dishes $d_1, d_2$ or $d_4$ are served. If we make sure – as we are supposed to – that $f_1$ and $f_2$ get to taste $d_1$ and $f_3$ gets to taste $d_4$, rule (⋆) makes sure that they all get to taste $d_3$ as well. Therefore, $d_3$ does not contribute to the problem in a meaningful way and can be removed. By a similar argument, $f_2$ can be removed as well. Note that $d_1$ and $d_2$ are liked by the same set of friends and we can remove any one of them but not both.}
\end{figure}

\(^5\)The supersets need not be proper, and in slight abuse of language, we permit that two friends are mutually “less picky” if they like the same set of dishes and two dishes to be mutually “more popular” if the same set of friends likes them.
Ignoring dishes and friends means, that we iteratively\(^6\) remove vertices from the bipartite graph and get a reduced bipartite graph \(\text{Red}(B)\). It has the same cover number and from a covering of \(\text{Red}(B)\) we can easily construct a covering of \(B\) with the same number of bicliques.

We now assume that \(B\) is already reduced, by construction this means:

- For every dish there is a friend who likes that dish but no less popular dish.
- Every friend likes a dish that none of the more picky friends like.

In a second step, we now construct a poset \(\text{Pos}(B)\) with \(\text{Bip}(\text{Pos}(B)) = B\).

You decide that it is a bit silly that you do not invite a friend just because there is a dish he does not like. From now on, you invite friends if you are cooking at least one dish they like. For some reason\(^7\) you only consider to cook the following sets of dishes:

- For a dish \(d \in D\): You just cook \(d\). This allows you to invite a set \(J_d \subseteq F\) of friends – those who like \(d\).
- For a friend \(f \in F\): Cook every dish except the ones that \(f\) likes. This allows you to invite a set \(M_f \subseteq F\) of friends – everyone except for \(f\) and those that are more picky than \(f\).

Given these sets, we define

\[
\mathcal{J} := \{J_d \mid d \in D\} \quad \text{and} \quad \mathcal{M} := \{M_f \mid f \in F\} \quad \text{as well as} \quad \text{Pos}(B) := \mathcal{J} \cup \mathcal{M}, \text{ ordered by inclusion.}
\]

See Figure 15 for a few examples. Note that for two dishes \(d_1 \neq d_2\) we have \(J_{d_1} \neq J_{d_2}\) since \(B\) is reduced so no two dishes can be liked by the same set of friends. Similarly, for two friends \(f_1 \neq f_2\) we have \(M_{f_1} \neq M_{f_2}\), otherwise we would have \(f_1 \notin M_{f_1} = M_{f_2}\) so \(f_1\) dislikes all dishes that \(f_2\) dislikes and, similarly, \(f_2\) dislikes all dishes that \(f_1\) dislikes, so they like the same set of dishes which is forbidden since \(B\) is reduced. Note that \(\mathcal{J}\) and \(\mathcal{M}\) need not be disjoint, though. The names “\(\mathcal{J}\)” and “\(\mathcal{M}\)” are chosen to suggest that they are the sets of join- and meet-irreducible elements in \(\text{Pos}(B)\). Before we prove this, note one simple fact:

If \(f \in F\) likes \(d \in D\), then \(f \in J_d \setminus M_f\), so \(J_d \nsubseteq M_f\). If \(f \in F\) does not like \(d \in D\), then “cooking all dishes except the ones that \(f\) likes” implies cooking \(d\), so \(J_d \subseteq M_f\). Together:

\[
f \text{ likes } d \iff J_d \nsubseteq M_f. \quad (\Delta)
\]

Claim. The set of join-irreducible elements in \(\text{Pos}(B)\) is \(J(\text{Pos}(B)) = \mathcal{J}\) and the set of meet-irreducible elements in \(\text{Pos}(B)\) is \(M(\text{Pos}(B)) = \mathcal{M}\).

\(^6\)If there are no vertices with identical neighbourhoods we need not do the reduction iteratively but can do all in one step.

\(^7\)Mostly because you are too lazy to consider an exponential number of sets of dishes.
Proof. If \( X \in \text{Pos}(B) \) is a set of friends that are invited when cooking a set \( Y \subseteq D \) of dishes (all elements of \( \text{Pos}(B) \) are of this form), then \( X = \bigcup_{d \in Y} J_d \), so \( X \) is the join of some elements in \( \mathcal{J} \). This shows that \( J(\text{Pos}(B)) \subseteq \mathcal{J} \). We now take any \( J_d \in \mathcal{J} \) and show that it is join-irreducible. Assume that \( J_d \) is an upper bound of some elements \( \{J_{d_1}, \ldots, J_{d_k}\} \subseteq \mathcal{J} \) all of which are proper subsets of \( J_d \). Then the dishes \( d_1, \ldots, d_k \) are all less popular than \( d \). By reducedness of \( B \) there is a friend \( f \) that likes \( d \) but none of \( d_1, \ldots, d_k \). Then \( (\Delta) \) implies that \( J_{d_i} \subseteq M_f \) for all \( i \in [k] \). So \( M_f \) is an upper bound of \( \{J_{d_1}, \ldots, J_{d_k}\} \). However, \( J_d \not\subseteq M_f \) so \( J_d \) is not a least upper bound (i.e. join) of \( \{J_{d_1}, \ldots, J_{d_k}\} \). This concludes the proof of \( J(\text{Pos}(B)) = \mathcal{J} \).

The proof for the meet case is similar: If \( d \) is a dish then

\[
J_d \subseteq \bigcap_{f \in F \setminus J_d} M_f \subseteq \bigcap_{f \in F \setminus J_d} F \setminus \{f\} \subseteq F \setminus (F \setminus J_d) = J_d.
\]

Therefore, any \( J_d \) can be written as a meet of elements from \( \mathcal{M} \).

To see conversely, that every \( M_f \in \mathcal{M} \) is meet-irreducible, assume that \( M_f \) is the lower bound of \( \{M_{f_1}, \ldots, M_{f_k}\} \) where each \( M_{f_i} \) (for \( i \in [k] \)) is a proper superset of \( M_f \). Since \( f_i \not\in M_f \), we need \( f_i \not\in M_f \) as well, but this means that \( f_i \) is more picky than \( f \) (i.e. \( i \in [k] \)). By reducedness of \( B \), there is some dish \( d \) that \( f \) likes but that is not liked by any of \( f_1, \ldots, f_k \). By \( (\Delta) \) this means \( J_d \subseteq M_{f_i} \) for any \( i \in [k] \). So \( J_d \) is a lower bound of \( \{M_{f_1}, \ldots, M_{f_k}\} \). However, \( J_d \not\subseteq M_f \) so \( M_f \) is not a greatest lower bound of \( \{M_{f_1}, \ldots, M_{f_k}\} \). \( \square \)
With the definition of $\text{Bip}(\cdot)$ this implies
\[
\text{Bip}(\text{Pos}(B)) = (J(\text{Pos}(B)), M(\text{Pos}(B)), \not\subseteq_{\text{Pos}(B)}) = (J, M, \not\subseteq).
\]

We claim that $B$ is isomorphic to $(J, M, \not\subseteq)$ as witnessed by mapping $d \mapsto J_d$ and $f \mapsto M_f$ ($d \in D, f \in F$). Indeed, we already checked in $(\Delta)$ that $(d, f) \in E \iff J_d \not\subseteq M_f$. Now we get
\[
\dim_2(\text{Pos}(B)) \overset{\text{Prop15}}{=} \text{BicCov}(\text{Bip}(\text{Pos}(B))) = \text{BicCov}(B).
\]

We summarise our findings in the following Theorem.

**Theorem 16** (Habib et al [Hab+04]). For any bipartite graph $B = (S, T, E)$ there is a poset $P = \text{Pos}(\text{Red}(B))$ such that

(i) $\text{BicCov}(B) = \dim_2(\text{Pos}(B))$

(ii) $P$ can be computed in polynomial time from $B$ and $|P| \leq |S| + |T|$.

(iii) From a covering of $B$ with $k$ bicliques we can compute an embedding of $P$ into $\otimes_k$ and vice versa, both in polynomial time.

We did not formally prove “polynomial time” but all our constructions were explicit and converting them to polynomial algorithms is straightforward.

Theorem 16 allows to transfer NP-hardness and any non-approximability results from biclique-cover to 2-dimension. By citing non-approximability of graph colouring and its relation to BicCov, Habib et al establish a non-approximability result for $\dim_2$, noting that any progress on the graph colouring end will carry over to their result – and progress has indeed happened since.

Indeed, using non-approximability of graph colouring by Zuckerman [Zuc06], Gruber and Holzer [GH07] established:

**Theorem 17** (Gruber and Holzer [GH07]). If $P \neq \text{NP}$, then there is for no $\varepsilon > 0$ a polynomial-time algorithm approximating the biclique cover number of every bipartite $B = (S, T, E)$ within a factor of $(|S| + |T|)^{1/3-\varepsilon}$.

By plugging together Theorem 16 and Theorem 17 we obtain:

**Theorem 18.** If $P \neq \text{NP}$, then there is for no $\varepsilon > 0$ a polynomial-time algorithm approximating $\dim_2(P)$ for posets $P$ within a factor of $|P|^{1/3-\varepsilon}$.

Note what this implies: For any polynomial-time algorithm that takes posets $P$ as input and gives some upper bound on $\dim_2(P)$ as output (optionally accompanied by a corresponding embedding), there will be some poset $P^*$ for which the upper bound exceed the actual 2-dimension by a factor of $|P|^{1/3-\varepsilon}$. In particular, there is no “good” bound on the dimension that only uses simple parameters like, height, width and size, since those parameters can be determined in polynomial time. Any upper or lower bound will either involve some difficult to calculate parameters (of course, we could always write $\dim_2(P) \leq \dim_2(P)$) or will be far off for some $P$.
4. Forcing Posets in other Posets

In this section, we review results that are related to our main object of study but answer **slightly different** questions. The beginning of this section also serves as a brief introduction to Ramsey Theory.

**Section 4.1: Ramsey Theory: Notation and Examples.** We introduce the notation \((X^Y)\) for the subobjects of \(X\) that are isomorphic to \(Y\) – the copies of \(Y\) in \(X\). Based on this, we define what it means for \(Z\) to *force* \(X\) when colouring copies of \(Y\), the notation being \(Z \rightarrow X\). If for each \(X\) there exists such a \(Z\), then \(Y\) is called *Ramsey*. We give a few examples, most importantly we cite the following result which will be used in *Section 4.2:*

**Theorem 22 ([GRS90]).** In the class of \(d\)-dimensional grids, any grid is Ramsey.

**Section 4.2: Posets are Ramsey iff they are Weak Orders.** Consider a poset \(Z\). In the rest of the thesis we would colour the *elements* of \(Z\), so subposets of size 1. In this section however, we colour all copies of a fixed poset \(Y\) in \(Z\) and then look for a subposet \(X\) of \(Z\) such that each copy of \(Y\) in \(X\) has the same colour. Nešetřil and Rödl [NR84] completely answered which posets \(Y\) have the Ramsey property (as defined in Section 4.1):

**Theorem 24 (Nešetřil and Rödl [NR84]).** A poset is Ramsey iff it is a weak order.

This validates the quantitative questions we address later as the single-element poset is a weak order, therefore Ramsey. This means every poset \(X\) can be forced by some poset \(Z\) when single elements of \(Z\) are coloured. So it makes sense to ask for a *particularly small* poset \(Z\) with this property.

**Section 4.3: Size Ramsey Numbers.** In this section we colour elements of posets. But instead of trying to minimise the 2-dimension of a host poset \(Z\) forcing a poset \(X\) as we do in Part III, Kierstead and Trotter [KT87] tried to minimise its width, height or size.

**Section 4.4: Extremal Questions.** Instead of studying when *colourings* of a Boolean lattice \(Q\) necessarily contain a monochromatic \(P\), we ask how large a subposet \(Q' \subset Q\) can be without containing \(P\). This size is typically studied for asymptotically large Boolean lattices \(Q\). The result, that will turn out to be most useful for us, is a recent one by Méroueh [Mé15]. It states that the so-called Lubell-mass of \(Q'\) is bounded by a constant \(\lambda^*(P)\) only depending on \(P\), not on the dimension of \(Q\).

**Theorem 34 (Méroueh [Mé15]).** \(\lambda^*(P)\) exists for any \(P\).
4.1. Ramsey Theory: Notation and Examples

Ramsey-type questions can emerge whenever we have a class \( \mathcal{U} \) of objects in which we have some notion of subobject and isomorphism. This gives for two objects \( N, K \in \mathcal{U} \) a set \( \binom{N}{K} \) of those subobjects of \( N \) that are isomorphic to \( K \), the set of copies of \( K \) in \( N \). Table 1 illustrates the notions of copy of \( K \) in \( N \) is the classes we consider in the following.

<table>
<thead>
<tr>
<th>class</th>
<th>( N )</th>
<th>( K )</th>
<th>( \binom{N}{K} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set</td>
<td>{a, b, c, d}</td>
<td>{x, y}</td>
<td>{{a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}}</td>
</tr>
<tr>
<td>Grid</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ind-Graph</td>
<td>[] []</td>
<td>[] []</td>
<td></td>
</tr>
<tr>
<td>Poset</td>
<td>[] [] []</td>
<td>[] [] []</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Examples illustrating the notion of copies of \( K \) in \( N \) in various settings.

We introduce the notation \( Z \uparrow X \) and say \( Z \) forces \( X \) when \( c \)-colouring copies of \( Y \), if for every colouring of the copies of \( Y \) in \( Z \) with \( c \in \mathbb{N} \) colours, there is a copy of \( X \) in \( Z \) such that all copies of \( Y \) in that copy of \( X \) have the same colour. Formally:

\[
Z \uparrow X : \Leftrightarrow \forall (\text{col}: \binom{Z}{Y} \rightarrow \mathbb{N}): \exists X^* \in \binom{Z}{X}: \text{col is constant on } \binom{X^*}{Y}.
\]

Since the number of colours is of minor interest to us in this section, we declare \( c = 2 \) to be the default and drop the index in that case. Note that the ‘\(^*\)’ as in \( X^* \) is no operator. On a syntactic level \( X \) and \( X^* \) are just different symbols. But we use it to suggest that \( X^* \) and \( X \) are isomorphic and \( X^* \) is subobject of a bigger structure.

We say that an object \( Y \in \mathcal{U} \) is Ramsey in \( \mathcal{U} \) if, in the context of colouring copies of \( Y \), every \( X \) can be forced by some \( Z \), i.e.

\[
Y \text{ is Ramsey } \Leftrightarrow \forall X \in \mathcal{U} : \exists Z \in \mathcal{U} : Z \uparrow X.
\]

We now give examples for classes and objects that are Ramsey within these classes.

**Example 19** (Pigeonhole Principle for two Holes). Consider the class of finite sets. Sets are isomorphic iff they have the same cardinality and the subobjects of a set are its subsets. Then a copy \( K^* \in \binom{N}{K} \) is just a subset of \( N \) with a size of \( |K| \).

If we pick \( Y = \{p\} \) to be the singleton set (containing a single “pigeon”), then 2-colouring copies of \( Y \) in a set just means 2-colouring elements of the set, i.e. partitioning “pigeons” into two “holes”.

We claim that \( Y \) is Ramsey. Indeed, for a set \( X \) of size \( x \) we can pick \( Z \) to be a set of size \( 2x - 1 \) and have \( Z \uparrow X \). Admittedly, this is a rather convoluted way to state a very simple thing.

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Example 20 (Ramsey’s Theorem). Still in the class of finite sets, the set $Y$ of size $y$ is Ramsey by Ramsey’s Theorem [Ram30]. Here, colouring copies of $Y$ means colouring subsets of size $y$. We have stated Ramsey’s Theorem in more classical terms as Theorem 37.

A well known example is $y = 2$ which means edges of a complete graph are coloured and large monochromatic cliques can be found. The fact $6 \rightarrow (3,3)$ is popularised as:

In a group of six people ($|Z| = 6$) where any two people ($|Y| = 2$) are either acquaintances or strangers ($c = 2$), there will be a group of three people ($|X| = 3$) that are either pairwise acquaintances or pairwise strangers.

Example 21 (Product Ramsey Theorem). Consider the class of finite $d$-dimensional grids, a grid $S$ being the product $S = S_1 \times S_2 \times \ldots \times S_d$ of finite sets $S_1, S_2, \ldots, S_d$. A subobject of that grid (a subgrid) is given as $T := T_1 \times T_2 \times \ldots \times T_d$ where $T_i \subseteq S_i$. Two grids $S$ and $T$ are isomorphic if their sizes coincide, i.e. if $|S_i| = |T_i|$ for all $i$.

The following result on grids is known (e.g. Trotter [Tro99]) as the Product Ramsey Theorem and may seem surprising at first.

Theorem 22 ([GRS90]). In the class of $d$-dimensional grids, any grid is Ramsey.

For a proof refer to [GRS90, pp 113, Theorem 5].

Example 23 (Induced Graph Setting). Consider the class of finite graphs where subobjects are given as induced subgraphs. In particular, for two graphs $G = (V,E)$ and $H$, the copies of $H$ in $G$ are given as

$$\binom{G}{H} = \{(V',E') \mid V' \subseteq V, E' = E \cap (V' \times V'), (V',E') \cong H\}.$$

Here, the graph consisting of a single edge is Ramsey, meaning that for every graph $H$ we can find a larger graph $G$ such that for any colouring of the edges of $G$ we find an induced copy of $H$ with monochromatic edges, see [Die10, Chapter 9.3]. The result was originally proved around 1973 by Deuber, Erdős, Hajnal, Posa, Rödl.

4.2. Posets are Ramsey iff they are Weak Orders

We now consider the class of finite posets where the copies $\binom{P}{Q}$ of a poset $Q$ in $P$ are given by the induced subposets of $P$ that are isomorphic to $Q$. There is a subtlety, so we give a formal definition:

$$\binom{P}{Q} = \{f(Q) \mid f \text{ is an embedding of } Q \text{ into } P\}.$$

In yet other words, a copy of $Q$ in $P$ is a subset of the elements of $P$ that induces a poset isomorphic to $Q$. Of course, each copy inherits the ordering from $P$ and is not just a set but a poset. See also Table 1.

---

8The set of copies is not the set of all embeddings of $Q$ into $P$. In fact, if $Q$ has a non-trivial automorphism, then there are different embeddings of $Q$ into the same copy of $Q$, so more embeddings than copies.
A result that is of interest to this thesis was first proved by Nešetřil and Rödl [NR84] and written down more clearly by Paoli and Trotter [PTW85].

**Theorem 24** (Nešetřil and Rödl [NR84]). *A poset is Ramsey iff it is a weak order.*

Here, we say a poset $P$ is a weak order if there is a function $f : P \to \mathbb{N}$ such that for two elements $x, y$ we have $x \lessdot_P y \iff f(x) < f(y)$.

An example is shown on the right. Note how there is a complete bipartite relation between adjacent “layers”.

To get a better intuition, observe two equivalent characterisations of weak orders:

**Lemma 25.** For a poset $P$ the following are equivalent.

1. $P$ is a weak order.
2. $P$ is the vertical sum (defined in Section 2.3.2) of several antichains, i.e. there is $k$ and $n_1, \ldots, n_k$ such that $P = \cdots \oplus n_1 \oplus \cdots n_2 \oplus \cdots \oplus \cdots n_k$.

(iii) $P$ does not contain $1_2 \cup 1_1 = \mathbf{1}$ as an induced subposet.

**Proof.** *(ii) ⇒ (i).* If $P$ is the vertical sum of $k$ antichains, then the map $f : P \to [k]$ sending elements of the $i$-th antichain to $i$ shows that $P$ is a weak order.

*(iii) ⇒ (i).* If $\mathbf{1}$ is an induced subposet of $P$, we can find $a, b, c \in P$ such that $a < b$ and $c$ is incomparable to both $a$ and $b$. If $P$ were a weak order we would need $f : P \to \mathbb{N}$ with $f(a) < f(b)$ as well as $f(a) = f(c)$ (since $a$ and $c$ are incomparable) and $f(b) = f(c)$ (since $b$ and $c$ are incomparable. Clearly, this is not possible.

*(iii) ⇒ (ii).* Assume that $P$ does not contain $\mathbf{1}$ as an induced subposet. Let $M$ be the set of minimal elements of $P$. Note that $M$ is an antichain. Assume that for some $m \in M$ and some $x \in P \setminus M$ we have $m \not\lessdot x$. Then, because $x$ is not minimal, we can find $m' \in M$ with $m' < x$ and $\{m', x, m\}$ forms a copy of $\mathbf{1}$, contradiction our assumption to the contrary. Therefore, any $m \in M$ is in fact smaller than any $x \in P \setminus M$. This means $P$ is of the form $P = M \oplus P'$. Now since $P'$ still does not contain $\mathbf{1}$, the claim follows by induction on size.

We will not prove Theorem 24 here, but provide two results that make it seem plausible while dodging the majority of the involved technicalities.

**Proposition 26.** The poset $\mathbf{1}$ is not Ramsey.
Proof. We claim that \( \mathbb{1} \) cannot be forced when copies of \( \mathbb{1} \) are coloured, i.e. there is no poset \( Z \) such that \( Z \not\rightarrow \mathbb{1} \). To show this, we construct for arbitrary \( Z \) a red/blue-colouring of \( (\mathbb{Z}) \), such that any copy \( \mathbb{1}^* \in (\mathbb{N}) \) contains a red and a blue copy of \( \mathbb{1} \).

Let the elements of \( Z \) be labelled as \( z_1, \ldots, z_n \), take for instance a linear extension of \( Z \). Colour a copy \( \mathbb{1}^* \in (\mathbb{Z}) \) red, if the element of \( \mathbb{1}^* \) that is incomparable to the other two – call it the loner – has the smallest label among the three elements. If one of the other two elements has the smallest label, colour \( \mathbb{1}^* \) blue.

Now consider any copy \( \mathbb{1}^* \in (\mathbb{Z}) \) consisting of elements \( \mathbb{1}^* = \{z_{i_1}, z_{i_2}, z_{i_3}, z_{i_4}\} \) for some \( i_1 < i_2 < i_3 < i_4 \). There are exactly four copies of \( \mathbb{1} \) in \( \mathbb{1}^* \) as each element \( z_{i_1}, z_{i_2}, z_{i_3}, z_{i_4} \) can assume the role of the loner. We therefore have \( \mathbb{1}_r^* \in (\mathbb{1}^*) \) where \( z_{i_1} \) is used for the loner (making \( \mathbb{1}_r^* \) red) and \( \mathbb{1}_b^* \in (\mathbb{1}^*) \) where \( z_{i_1} \) is one of the two non-loner elements (making \( \mathbb{1}_b^* \) blue).

If \( Y \) is an extension of \( \mathbb{1} \), it is more difficult to show that \( Y \) is not Ramsey, but the idea is similar. First note that \( Y \) has substantially different linear extensions. For example has three linear extensions (the loner can occur as first, second or third element). Weak orders, by contrast, have essentially only one linear extension, up to automorphism. Given some large poset \( Z \) and a linear extension \( L = \{z_1 \leq \ldots \leq z_n\} \) of \( Z \), then \( L \) induces a linear extension \( L_Y^* \) on each \( Y^* \in (\mathbb{Z}) \). Depending on which of the linear extensions of \( Y \) that corresponds to, we colour \( Y^* \) red or blue. Then for a carefully chosen poset \( X \) (depending only on \( Y \)), any copy of \( X \) in \( Z \) will contain differently coloured copies of \( Y \). For details refer to [PTW85].

We will not show here that general weak orders are Ramsey and only focus on chains. In this special case the argument given in [PTW85] can be simplified substantially:

**Proposition 27.** For \( y \in \mathbb{N} \), the chain \( \mathbb{1}_y \) is Ramsey.

Proof. Let \( X \) be a poset, \( d \) its order dimension and \( x \) its size. For \( s \in \mathbb{N} \) we denote by \( G_s \) the grid

\[
G_s := [s] \times \ldots \times [s] \subset \mathbb{N}^d.
\]

Ordered by \( \leq_{gd} \) every grid is also a poset, so we can talk about induced subposets as well as subgrids (defined in Example 21). We use the notation \( (\cdot) \) when talking about induced subposets and \( [\cdot] \) when talking about subgrids.

We use the grids \( G_y, G_x \) and \( G_z \) where \( z \in \mathbb{N} \) is a large integer we fix later. We fix any colouring \( \text{col} : (G_z) \rightarrow [2] \) of the copies of chains in \( G_z \) and will argue that we can find \( X^* \in (G_y) \) such that \( \text{col} \) is monochromatic on \( (X^*) \).

We define a colouring of grids \( \text{col}_G : (G^*_y) \rightarrow [2] \) where \( G_y^* \in [G_y] \) is assigned the colour \( \text{col}_G(G_y^*) := \text{col}(\text{diag}(G_y^*)) \). Here \( \text{diag}(G_y^*) \) is the unique chain of length \( y \) with

\(^9\)Note that "\( \mathbb{1}^* \)" is just some a variable name, not some operation \( * \) applied to \( \mathbb{1} \).
strictly increasing coordinates in each dimension that lies diagonally in $G_y^*$ as shown in Figure 16.

![Figure 16](image_url)

Figure 16: We illustrate the correspondence between chains and grids in $d = 2$ dimensions. For each grid $G_x^* \in [G_x^*_{G_y}]$ there is a unique 3-chain $\text{diag}(G_x^*) \in (G_x^*_{G_y})$ with $\text{diag}(G_x^*) \subset G_x^*$ and strictly increasing coordinates in each dimension. The inverse map $\text{diag}^{-1}$ is well-defined as well.

The Product Ramsey Theorem ensures that, provided that $z$ is large enough, we find a copy $G_x^* \in [G_x^*_{G_y}]$ such that $[G_x^*_{G_y}]$ is monochromatic with respect to $\text{col}_G$.

Since $X$ is of order dimension $d$ it can be embedded into $\mathbb{R}^d$. Since $|X| = x$, we can even find an embedding into $[x]^d$ that uses each coordinate of each dimension exactly once (see e.g. Claim 6). This gives a copy $X^* \in \left(\frac{G_x^*}{X}\right)$ where every chain in $X^*$ has strictly increasing coordinates in each dimension. So for any $i_y^* \in (X^*)_{G_y}$ we have that $\text{diag}^{-1}(i_y^*) \in \left(\frac{G_x^*}{G_y}\right)$ is well defined and $\text{col}(i_y^*) = \text{col}_G(\text{diag}^{-1}(i_y^*))$. Since $\text{col}_G$ was monochromatic on $\left(\frac{G_x^*}{G_y}\right)$ we know that $\text{col}_G$ is monochromatic on $(X^*)_{G_y}$. □

Theorem 24 completely answers which posets are Ramsey. But only the existence of a $Z$ forcing $X$ is asserted. We now turn to the problem of minimising the size, width and height of $Z$.

### 4.3. Size Ramsey Numbers

In Section 4.2 we established that chains are Ramsey. This implies that the single-element poset $Y = \cdot$ is Ramsey, i.e. for any poset $P$ there exists a poset $Z$ such that $Z \rightarrow P$. In the following, we write $Z \rightarrow P$ when we mean $Z \rightarrow P$. What remains to be answered is how large $Z$ has to be, in terms of $P$, for some notion of “largeness”.

Define for posets $P$ and $Q$ the blowup $P \times Q$ of $P$ with $Q$ as $P \times Q$ ordered by:

$$(p_1, q_1) \leq_{P \times Q} (p_2, q_2) :\iff (p_1 \leq_P p_2 \text{ or } (p_1 = p_2 \text{ and } q_1 \leq_Q q_2)).$$

In this context we call for $x \in P$ the set $B_x := \{(x, y) \in P \times Q \mid y \in Q\}$ the blob of $x$. Consider Figure 17 for an example.

It is not hard to see that $P \times P$ forces $P$:

**Claim 28** (Kierstead and Trotter [KT87]). $(P \times P) \rightarrow P$.

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Figure 17: Blowup of a three-element poset with a five element poset.
In addition to the relations within the three blobs (black) there are complete relations (grey) between blobs $B_x$ and $B_y$ if $x \leq y$ in $\Lambda$. We did not bother to remove transitive edges.

Proof. Consider any colouring of $P \times P$ with red and blue. If one blob $B_{x_0}$ for $x_0 \in P$ is monochromatic, then this is a monochromatic copy of $P$ and we are done. Otherwise, choose one red element $r_x \in B_x$ for each $x \in P$. Now observe that the red set $\{r_x \mid x \in P\}$ is a copy of $P$.

Note that if $P$ has width $w$, height $h$ and size $s$, then $P \times P$ has width $w^2$, height $h^2$ and size $s^2$. In other words: No matter if by “largeness” we mean width, height or size, it is possible to choose a poset $Z$ that is only “quadratically large” compared to $P$ such that $Z$ forces $P$. Sometimes smaller $Z$ may suffice, but there is an easy lower bound as well:

Claim 29. Let $P$ be a poset of size $s$, height $h$ and width $w$ and $Z$ a poset of size $s'$, height $h'$ and width $w'$. If $Z \rightarrow P$, then:

- $s' \geq 2s - 1$,
- $h' \geq 2h - 1$,
- $w' \geq 2w - 1$.

Proof. Assume that $Z$ has smaller size, height or width than stated. We show that it does not force $P$ by constructing a colouring: We partition the “resources” in $Z$ into a red and a blue class of almost even size. These resources are

- In the case of size: The elements, at most $2s - 2$.
- In the case of height: The layers, at most $2h - 2$.
- In the case of width: Chains covering $P$, at most $2w - 2$.

In all cases, neither the red colour class nor the blue colour class contains a copy of $P$ since the class has size at most $s - 1$, height at most $h - 1$ or width at most $w - 1$, respectively.

While these trivial bounds can be obtained simultaneously for width, height and size, the parameters do behave differently when looking more closely.

Theorem 30 (Kierstead and Trotter [KT87], Nešetřil and Rödl [NR84]).

(i) For any poset $P$ of height $h$, there exists a poset $Z$ of height $2h - 1$ forcing $P$. 

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There is a poset \( P \) of size \( s \) such that any poset \( Z \) forcing \( P \) has size at least \( s^2/4 \).

There is a poset \( P \) of width \( w \) such that any poset \( Z \) forcing \( P \) has width at least \( 2w \).

This settles the question for height: To force a height \( h \) poset, a host of height \( 2h - 1 \) is both necessary and sufficient.

Considering size, there are posets of size \( s \) where hosts of size \( O(s^2) \) are both necessary and sufficient (the poset \( P := \cdots \cup s/2 \cup \cdots \) is an example).

For width, the problem seems to be wide open. It already took Trotter and Kierstead four pages to prove Theorem 30(iii) which improves the trivial lower bound merely by 1. Trotter [Tro99] claims that the lower bound was further improved to \( 5w/2 \) by Kierstead, but it seems to still be unknown if posets of width \( w \) exist that are only forced by posets \( Z \) of super-linear width.

The contribution of this thesis is parallel to the efforts just presented, trying to bound the 2-dimension of \( Z \) instead of height, width or size.

4.4. Extremal Questions

Many researchers have considered questions of the form “How large can a family of subsets of \([n]\) be without containing \( X \)?”, for instance for \( X \) being a Boolean Algebra of dimension 2 [EK71], a Boolean Algebra of higher dimension [GRS99] (see also [Bar+11]), certain intersection patterns [Lon15] and so-called \( d \)-dimensional projections [PZ98].

Most relevant for us is, of course, the case where \( X \) is a poset \( P \), i.e. we search for large set families not containing \( P \) as a subposet. Such extremal considerations will help us in Section 7 to bound Ramsey numbers for an asymptotically large number of colours.

Let \( \text{La}(N,P) \) and \( \text{La}^*(N,P) \) be the largest \( k \) such that a family of \( k \) elements of \( \binom{N}{P} \) exists that does not contain \( P \) as a weak subposet and induced subposet, respectively. Note the obvious relationship \( \text{La}(N,P) \leq \text{La}^*(N,P) \).

For instance, note that a family of sets does not contain \( 1 \) if and only if it is an antichain, so Sperner’s Theorem gives us \( \text{La}(N, 1) = \text{La}^*(N, 1) = \binom{N}{N/2} \). Erdős [Erd45] generalised Sperner’s result for paths of any length in 1945, and lots of research has since been conducted on other posets.

There seems to be an important connection to \( e(P) \), the largest number \( k \) such that \( P \) cannot be weakly embedded into \( k \) consecutive layers of any cube. Using our notation from Section 2.3.1 we formally define:

\[
e(P) := \max \{ k \in \mathbb{N} \mid \forall N, l: P \text{ is not weak subposet of } \binom{N}{l, \ldots, l+k-1} \}.
\]

As examples consider for instance \( e(\emptyset) = n, e(\forall) = 2, e(\exists) = 2 \).

By taking the middle \( e(P) \) layers of \( \binom{N}{P} \) we know that \( \text{La}(N,P) \) is at least the sum of the \( e(P) \) largest binomial coefficients. Results so far suggests that, asymptotically,
taking middle layers is the best one can do. To make this precise, define:

\[
\pi(P) := \limsup_{N \to \infty} \frac{\text{La}(N, P)}{\binom{N}{N/2}}.
\]

While \(\pi(P) \geq e(P)\) is clear, equality is a conjecture attributed to Griggs and Lu and explicitly stated for example by Patkós [Pat15]:

**Conjecture 31** (Griggs and Lu [GL09]). \(\pi(P) = e(P)\).

Considerable effort has been made to establish this for various posets:

**Theorem 32** (De Bonis, Bukh, Erdős, Griggs, Katona, Kramer, Martin, Young).

i) \(\pi(I_n) = e(I_n) = n - 1\). [Erd45] (Sperner for \(n = 2\))

ii) \(\pi([n]) = e([n]) = 2\). [DBKS05]

iii) \(\pi(\wedge \cdots \wedge \vee) = e(\wedge \cdots \wedge \vee) = 1\) [DBK07]

iv) \(\pi(\cdots \wedge \cdots \vee \cdots \wedge \cdots) = e(\cdots \wedge \cdots \vee \cdots \wedge \cdots) = 2\). [DBK07]

v) \(\pi([n]) = e([n]) = 1\). [GK08]

vi) \(\pi(T) = e(T) = h(T) - 1\) if the Hasse diagram of \(T\) is a tree of height \(h(T)\). [Buk09]

vii) \(2 = e(\{\}) \leq \pi(\{\}) \leq 2.25\). [KMY13]

Note that any poset is a weak subposet of a path, so \(\pi(P) \leq |P| - 1\) is clear from Theorem 32(i). In particular: \(\pi(P) < \infty\), i.e. \(\pi(P)\) exists for each \(P\). The existence of the induced variant \(\pi^*(P)\) using \(\text{La}^*(N, P)\) in its definition is, in contrast, not as simple. Only recently Methuku and Pálvölgyi established:

**Theorem 33** (Methuku and Pálvölgyi [MP14]).

The limit \(\pi^*(P) := \limsup_{N \to \infty} \frac{\text{La}^*(N, P)}{\binom{N}{N/2}}\) exists for any \(P\).

A conjecture analogous to Conjecture 31 has been made by [Pat15], suspecting \(\pi^*(P) = e^*(P)\) where \(e^*(P)\) is the induced analogue of \(e(P)\). On a related note, bounds on the Lubell-mass \(l(F)\) of families \(F \subseteq 2^{[N]}\) avoiding \(P\) as an induced subposet are considered.

\[
l(F) := \sum_{S \in F} 1/|S|,
\]

\[
\lambda^*(P) := \limsup_{N \to \infty} \left\{ l(F) \middle| F \subseteq 2^{[N]} \text{ contains no induced } P \right\}.
\]

It is clear that \(\pi^*(P) \leq \lambda^*(P)\). The following theorem is therefore stronger than Theorem 33. It was conjectured by Lu and Milans [LM14] and proved in a recent not yet published paper by Méroueh.

**Theorem 34** (Méroueh [Mé15]). \(\lambda^*(P)\) exists for any \(P\).

We will make use of this in Section 7.
Part III.
Monochromatic Posets in Coloured Lattices

As of now we are concerned with finding the smallest Boolean lattice such that any colouring of its elements admits a monochromatic copy of a given poset $P$. Let us make this precise.

**Definition 35.** For a poset $P$ and an integer $c$, the Ramsey number $R_c(P)$ is the smallest $N$ such that for any map $\text{col} : \mathbb{N}_N \to [c]$ there exists an embedding $f : P \to \mathbb{N}_N$ such that $\text{col} \circ f$ is constant. We sometimes omit the index if we are concerned with $c = 2$, i.e. $R(P) := R_2(P)$.

Note that our notion of Ramsey numbers generalises the notion of 2-dimension since $\dim_2(P) = R_1(P)$. Section 8.1 might help to develop an intuition for the notion by discussing small examples. Section 5.2 on the other hand contains the general result that, directly or indirectly, is the basis for most of our upper bounds. An overview of our results is given in Table 2.

<table>
<thead>
<tr>
<th>Poset</th>
<th>Result</th>
<th>Proved in</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$ not antichain</td>
<td>$c \cdot e(P) \leq R_c(P) \leq \dim_2(P) \cdot \frac{h(P)^c - 1}{h(P) - 1}$</td>
<td>Lemma 36, 41</td>
</tr>
<tr>
<td>$I_n$</td>
<td>$R_c(I_n) = c(n-1)$</td>
<td>Theorem 42</td>
</tr>
<tr>
<td>$\cdots n$</td>
<td>$R_c(\cdots n) = \text{sp}(c(n-1) + 1)$</td>
<td>Theorem 43</td>
</tr>
<tr>
<td>$S_n$</td>
<td>$c(n-2) + 2 \leq R_c(S_n) \leq (2^c - 1)n$</td>
<td>Theorem 45</td>
</tr>
<tr>
<td>$\mathfrak{N}_n$</td>
<td>$cn \leq R_c(\mathfrak{N}_n) \leq (n + 1)^c - 1$</td>
<td>Theorem 46</td>
</tr>
<tr>
<td>$P$ not antichain</td>
<td>$R_c(P) = \Theta(c)$ (for large $c$)</td>
<td>Theorem 50</td>
</tr>
<tr>
<td>$\mathfrak{N}_n$</td>
<td>$2n \leq R(\mathfrak{N}_n) \leq n^2 + 1$</td>
<td>Theorem 64</td>
</tr>
<tr>
<td>$\wedge$</td>
<td>$R(\wedge) = 3$</td>
<td>Theorem 52</td>
</tr>
<tr>
<td>$N$</td>
<td>$R(N) = 4$</td>
<td>Theorem 53</td>
</tr>
<tr>
<td>$\hat{\diamond}$</td>
<td>$R(\hat{\diamond}) = 4$</td>
<td>Theorem 54</td>
</tr>
<tr>
<td>$S_3$</td>
<td>$R(S_3) = 6$</td>
<td>Theorem 56</td>
</tr>
<tr>
<td>$\hat{\hat{\diamond}}$</td>
<td>$R(\hat{\hat{\diamond}}) \in {7, 8}$</td>
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</tr>
<tr>
<td>$P \cup Q$</td>
<td>$R_c(P, Q) \leq R_c(P \cup Q) \leq R_c(P, Q) + \text{sp}(c + 1)$</td>
<td>Theorem 72</td>
</tr>
</tbody>
</table>

Table 2: Bounds on Ramsey numbers derived in this thesis.

With the exception of Sections 6 and 7 (which are very short), each of the sections starts with an overview of its content.
5. First Bounds on Ramsey Numbers

In the following, we establish the existence of Ramsey numbers $R_c(P)$ and some upper and lower bounds.

Section 5.1: Bounds Using Layered Colourings and Ramsey’s Theorem.

We derive bounds on Ramsey numbers from first principles and Ramsey’s Theorem. The upper bounds are essentially superceded by Section 5.2, except for the fact that we find monochromatic embeddings that are also layered.

Section 5.2: Upper Bounds using a “Blowup Lemma”. We derive an upper bound on Ramsey numbers in terms of 2-dimension and height:

$$R_c(P) \leq \dim_2(P) \cdot \frac{h(P)^c - 1}{h(P) - 1},$$

so in particular for $c = 2$

$$R(P) \leq \dim_2(P) \cdot (h(P) + 1).$$

At the core of these bounds lies the simple observation that the blowup $P \otimes_m$ has 2-dimension at most $\dim(P) + h(P) \cdot m$ and the observation that any red/blue colouring of that blowup contains a red $P$ or a blue $\otimes_m$.

Section 5.3: Application of Bounds. We apply our bounds to our favourite posets, namely chains, antichains, the standard example and cubes. Note that cubes are also discussed in more detail in Section 8.

5.1. Bounds Using Layered Colourings and Ramsey’s Theorem

Even with little insight into the structure of Boolean lattices, some bounds on $R_c(\otimes_n)$ can be derived using basic Ramsey Theory alone.

We consider embeddings, copies and colourings which we call layered, by which we mean the following.

- An embedding $f : \otimes_n \to \otimes_N$ is layered if
  $$\forall S_1, S_2 \subseteq [n] : |S_1| = |S_2| \Rightarrow |f(S_1)| = |f(S_2)|.$$
  In words: Sets from the same layer of $\otimes_n$ are mapped to the same layer of $\otimes_N$.

- The image of a layered embedding of $\otimes_n$ is a layered copy of $\otimes_n$.

- A colouring $\text{col} : \otimes_N \to [c]$ is layered if
  $$\forall S_1, S_2 \in \otimes_N : |S_1| = |S_2| \Rightarrow \text{col}(S_1) = \text{col}(S_2).$$
  In words: Sets of the same layer get the same colour. More weakly, we say a colouring is layered when restricted to $X \subseteq \otimes_N$ if the above need only hold for $S_1, S_2 \in X$. 
Lower Bounds using the Pigeonhole Principle. Recall that if $P$ is a poset, then $e(P)$ is the largest $k$ such that $P$ cannot be embedded into $k$ consecutive layers of any cube (see Section 4.4). Therefore, by ensuring that colour classes do not span more than $e(P)$ layers, we avoid a monochromatic $P$. Formally:

**Lemma 36.**

(i) $R_c(P) \geq c \cdot e(P)$.

(ii) $R_c(P) \geq c \cdot e(P) + 2$, if $P$ has no maximum or minimum.

**Proof.** Consider the layered colouring of $c \cdot e(P) - 1$, in which we colour the $c \cdot e(P)$ layers such that the layers $(i-1) \cdot e(P)$ till $i \cdot e(P) - 1$ receive colour $i$ ($1 \leq i \leq c$).

By definition of $e(P)$, there cannot be a monochromatic copy of $P$ in any of the $c$ colours. This proves (i).

To see (ii), observe that if an embedding of $P$ into $N$ maps some $x \in P$ to the empty set, then $x$ must be a minimum of $P$. Similarly, only the maximum of $P$ can be mapped to the universal set $[N]$. So if $P$ has no minimum or maximum, then the first and last layer of $N$ cannot be used in any embedding. The argument for (i) can be refined accordingly, colouring the $c \cdot e(P)$ “interesting” layers of $c \cdot e(P) + 1$ in a layered way that avoids $P$.

Upper Bounds Using Ramsey’s Theorem. Recall the following classical result:

**Theorem 37 (Ramsey’s Theorem [Ram30]).** For $n, c, k \in \mathbb{N}$ there is a number $N := R(n, k, c)$ such that for any $c$-colouring of $\{T \subseteq [N] \mid |T| = k\}$ there is $S \subseteq [N]$ of size $n$ such that all sets in $\{T \subseteq S \mid |T| = k\}$ have the same colour.

Ramsey’s Theorem will allow us to find in any sufficiently large coloured cube a subcube that has a layered colouring.

Since Ramsey numbers are very large (see e.g. [CFS10]) the results we get from this are very inefficient: We need super-polynomial $N$ to find a monochromatic copy of $\square_n$ in $\square_N$, even though $R_c(\square_n) \in O(n^c)$ as we will see later. On the upside, the copy of $\square_n$ will be layered.

**Lemma 38.** For any numbers $n$ and $c$, there is a number $N$ such that for any $c$-colouring of all subsets of $[N]$ there is a set $S \subseteq [N]$ of size $n$ such that the colouring restricted to subsets of $S$ is layered.

**Proof.** Define $N_n := n$ and $N_i := R(N_{i+1}, i + 1, c)$ for $0 \leq i \leq n - 1$. Let $N := N_0$ and fix a colouring col of $2^{|N|} = \square_N$. We prove the following claim by induction.

**Claim.** For $0 \leq i \leq n$, there is a subset $S_i \subseteq [N]$ of size $N_i$ such that col is layered when restricted to $\{X \subseteq S_i \mid |X| \leq i\}$.

$i = 0$. Clearly, $S_0 := [N]$ works since the condition on the colouring is trivial: Only the empty set has size 0 so the colouring is layered when restricted to that single-element-layer.
\(i \to i + 1\). By induction hypothesis, we got a set \(S_i \subseteq [N]\) of size \(N_i\) such that its subsets of size at most \(i\) are coloured in a layered way. By choice of \(N_i\) as \(R(N_{i+1}, i + 1, c)\), there is a set \(S_{i+1} \subseteq S_i\) of size \(N_{i+1}\) such that all subsets of \(S_{i+1}\) of size \(i + 1\) have the same colour. Therefore, the subsets of \(S_{i+1}\) of size at most \(i + 1\) are coloured in a layered way – inheriting the property from \(S_i\) for layers 0 till layer \(i\).

After the last step of the induction we have the claim for \(i = n\), so we found a set \(S := S_n\) of size \(n = N_n\) such that its subsets ("of size at most \(n\" which is no restriction) are coloured in a layered way. This is just what we wanted.

**Corollary 39.** For any \(n, c \in \mathbb{N}\), there is an \(N \in \mathbb{N}\) such that any \(c\)-colouring of \(\mathbb{N}\) admits a monochromatic layered copy of \(\mathbb{N}\).

**Proof.** We apply the previous lemma for \(n \leftarrow n \cdot c\) and \(c \leftarrow c\). This gives an \(N\) such that for any \(c\)-colouring of \(\mathbb{N}\) (i.e. a \(c\)-colouring of subsets of \([N]\)) there is a set \(S \subseteq [N]\) of size \(c \cdot n\) such that the colouring restricted to the subsets of \(S\) is layered. This subset lattice is isomorphic to \(\mathbb{N}\).

Now we merely need to show that a layered \(c\)-colouring of \(\mathbb{N}\) admits a monochromatic layered copy of \(\mathbb{N}\).

Since \(\mathbb{N}\) has \(cn + 1\) layers, there is, by the pigeonhole principle, one colour \(c_0\) that is used for \(n + 1\) layers with indices \(l_0 < l_1 < \ldots < l_n\). We pick sets \(S_0 \subseteq \ldots \subseteq S_n \subseteq [cn] \setminus [n]\) where \(|S_i| = l_i - i, 1 \leq i \leq n\). Then consider the following map

\[
f : \mathbb{N} \to \mathbb{N}
\]

\[
X \mapsto X \cup S_{|X|}.
\]

It is easily seen to be an embedding. It is layered since a set \(X\) in layer \(0 \leq |X| \leq n\) is mapped to a set of size \(|X \cup S_{|X|}| = |X| + l_{|X|} - |X| = l_{|X|}\). And lastly, it is monochromatic by choice of \(l_0, \ldots, l_n\).

As an iterated Ramsey number, the \(N\) required by our approach is astronomical. We will not try to bound it in terms of \(n\) and \(c\). Perhaps surprisingly, a randomised analysis (see Section 8.3) suggests that, at least on average, layered monochromatic copies of \(\mathbb{N}\) can be found already for \(N \sim n \log n\).

Even though the implications for the problems of this thesis are unclear, we refer the interested reader to a curious result on layered subcubes: Pudlák [Pud90] proves that any 2-colouring \(col : \{0, 1\}^n \to \{0, 1\}\) of the Boolean lattice that avoids certain monochromatic layered subcubes must have a high formula size complexity, i.e. \(col\) cannot be computed by a short Boolean formula.

### 5.2. Upper Bounds using a “Blowup Lemma”

Before we get started we introduce and recall a few concepts.
Cube Isomorphisms. We already discussed in Section 1 that \( \mathcal{O}_n \cong \{0,1\}^n \) in the following sense: The cube \( \mathcal{O}_n \) can be seen as the set of all binary strings of length \( n \) ordered by “\( x \leq y \) iff \( y \) has a 1 everywhere where \( x \) has a 1” for \( x, y \in \{0,1\}^n \). For \( 0 < k < n \) we obviously have \( \{0,1\}^n = \{0,1\}^k \times \{0,1\}^{n-k} \) and therefore

\[
\mathcal{O}_n \cong \mathcal{O}_k \times \mathcal{O}_{n-k},
\]

where, of course, the order on the product is the intersection of the orders on the components, i.e. we have

\[
(x_1, y_1) \leq \mathcal{O}_k \times \mathcal{O}_{n-k} (x_2, y_2) \iff (x_1 \leq \mathcal{O}_k x_2 \text{ and } y_1 \leq \mathcal{O}_{n-k} y_2).
\]

Switching perspective this way will be convenient sometimes. For instance we sometimes want to distinguish some elements \( S \subset [n] \) in notation, if those elements serve a certain purpose. In the string setting, when given any permutation \( \pi \) of \([n]\), the map that sends each binary string \((b_1, b_2, \ldots, b_n)\) to \((b_\pi(1), b_\pi(2), \ldots, b_\pi(n))\) is clearly an automorphisms of the cube. This insight, when applied in the set setting, guarantees us for any set \( S \subset [n] \) of size \( k \) an isomorphism between \( \mathcal{O}_n \) and \( \mathcal{O}_k \times \mathcal{O}_{n-k} \) such that \( S \) is mapped to \(([k], \emptyset)\). We hope the reader will not be confused when we use relabelling-arguments of this kind in the following without explicitly constructing the corresponding morphisms.

Height. Recall that for a poset \( P \), its height \( h(P) \) is the size of the largest chain in \( P \). We define additionally for an element \( x \in P \) its height \( h(x) \) as the size of the largest chain that has \( x \) as its maximum (see Figure 18).

\[
\begin{array}{c}
\text{← elements with height 3} \\
\text{← elements with height 2} \\
\text{← elements with height 1}
\end{array}
\]

Figure 18: A poset of height 3 has elements with height 1, 2 and 3.

Blowup. In Section 4.3 we defined the blowup \( P \bowtie Q \) of \( P \) with \( Q \) as the poset with elements \( P \times Q \) and ordering

\[
(p_1, q_1) \leq_{P \bowtie Q} (p_2, q_2) \iff (p_1 \leq_P p_2 \text{ or } (p_1 = p_2 \text{ and } q_1 \leq_Q q_2)).
\]

The blowup \( P \bowtie Q \) contains for every element \( x \in P \) a blob \( B_x := \{(x, y) \mid y \in Q\} \) and each \( B_x \) is isomorphic to \( Q \).

Blowup Lemma. We now bound the 2-dimension of \( P \bowtie \mathcal{O}_m \) and show that a red/blue-colouring of this blowup admits a red \( P \) or a blue \( \mathcal{O}_m \).

Lemma 40 (Blowup Lemma). Let \( P \) be a poset.
(i) For $m \in \mathbb{N}$ we have $\dim_2(P \times \bigotimes_m) \leq \dim_2(P) + h(P) \cdot m$.

(ii) For any poset $Q$, any red/blue-colouring of $P \times Q$ admits a red copy of $P$ or a blue copy of $Q$.

![Diagram](image)

Figure 19: A poset $P$ (left), an embedding $f : P \rightarrow \{0, 1\}^3$ (middle) and a copy of $P \times Q$ (right) in $\bigotimes_{\dim_2(P) + h(P) \cdot 2} \cong \{0, 1\}^9$ as identified by the Blowup Lemma. For instance, the Blob $B_a$ consists of the four strings fitting the pattern 110 ?? 00 00, where ? can be 0 or 1. The first three binary digits are just $f(a) = 110$. The height $h(a) = 1$ determines the position of the question marks among the three groups of two digits afterwards.

Proof. (i) Set $n := \dim_2(P)$ and $N := \dim_2(P) + h(P) \cdot m$. We pick an embedding $f : P \rightarrow \bigotimes_n$ and view $N$ as the product poset

$$\bigotimes_n \cong \bigotimes_n \times \bigotimes_m \times \cdots \times \bigotimes_m.$$

An embedding of $P \times \bigotimes_m$ into this poset is given as

$$g : P \times \bigotimes_m \rightarrow \bigotimes_n \times \bigotimes_m^{h(P)},$$

$$(p, S) \mapsto (f(p), [m], \ldots, [m], S, \emptyset, \ldots, \emptyset).$$

This is easily checked to be an embedding: Consider $(p_1, S_1), (p_2, S_2) \in P \times \bigotimes_m$. If $p_1 <_P p_2$ then, because of $f(p_1) \subset f(p_2)$ and $h(p_1) < h(p_2)$ we have $g(p_1, S_1) \leq g(p_2, S_2)$ as well. If $p_1 \not<_P p_2$, then $f(p_1) \not\subset f(p_2)$ ensures $g(p_1, S_1) \not\leq g(p_2, S_2)$. If $p_1 = p_2$, then the outcome depends on whether $S_1 \subseteq S_2$, as it should. An example is shown in Figure 19.

(ii) We distinguish two cases.

**Case 1.** Some blob $B_x$ contains no red element. Then this blob is a blue copy of $Q$ and we are done.
Case 2. Every blob $B_x$ contains a red element $r(x)$. Then $r : P \to P \ltimes Q$ is a monochromatic (red) embedding of $P$. \qed

We now condense the insights from the Blowup Lemma into a formula and generalise it to an arbitrary number of colours.

**Corollary 41 (Iterated Blowup Lemma).** Let $P$ be a poset and $c \in \mathbb{N}$. Then

(i) $R(P) \leq \dim_2(P)(1 + h(P))$.

(ii) $R_{c+1}(P) \leq \dim_2(P) + h(P) \cdot R_c(P)$.

(iii) $R_c(P) \leq \dim_2(P) \cdot \sum_{0 \leq i < c} h(P)^i$.

(iv) $R_c(P) \leq \dim_2(P) \cdot \frac{h(P)^c - 1}{h(P) - 1}$, if $P$ is not an antichain.

**Proof.** (i) This is a special case of (ii) but we prove it separately for clarity. Consider $N := \dim_2(P) + h(P) \cdot \dim_2(P)$ and any red/blue-colouring of $N$. By Lemma 40(i) we find $P \ltimes \dim_2(P)$ in $N$ which is, likewise, red/blue-coloured.

Now Lemma 40(ii) gives either a red copy of $P$ or a blue copy of $\dim_2(P)$ which clearly contains a blue copy of $P$. In both cases we found a monochromatic copy $P$.

(ii) Consider $N := \dim_2(P) + h(P) \cdot R_c(P)$ and any $(c + 1)$-colouring of $N$. By Lemma 40(i) we find $P \ltimes \dim_2(P)$ in $N$ which is, likewise, $(c + 1)$-coloured.

We say that an element is *red* if it has colour 1 and *blue* if it has one of the other $c$ colours.

Now apply Lemma 40(ii). This gives either a red $P$ which corresponds to a copy of $P$ in colour 1, in which case we are done. Or it gives a “blue” copy of $\dim_2(P)$, which is actually a $c$-coloured copy of $\dim_2(P)$. In it we find, by definition of $R_c(P)$ a monochromatic copy of $P$. This proves the claim.

(iii) This is merely an iterative application of (ii), i.e. we use induction. For $c = 1$, the inequality clearly holds since $R_1(P) = \dim_2(P)$. For $c > 1$ we have

\[
R_c(P) \leq \dim_2(P) + h(P) \cdot R_{c-1}(P) \leq \dim_2(P) + h(P) \cdot \sum_{0 \leq i < c-1} h(P)^i = \dim_2(P) + \dim_2(P) \cdot \sum_{1 \leq i < c} h(P)^i = \dim_2(P) \cdot \sum_{0 \leq i < c} h(P)^i.
\]

(iv) If $P$ is not an antichain, then $h(P) > 1$ and we can restate (iii) using the well-known identity for geometric sequences. \qed

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5.3. Application of Bounds

The tools we have seen are fairly simple: We used layered colourings and $e(P)$ to obtain lower bounds (see Section 5.1) and our “Blowup Lemma” to obtain upper bounds. We now apply our knowledge to our favourite posets, namely chain, antichain, standard example and cube. Our tools will work in some cases but reveal their bluntness against a very natural example: The cube. We fail to determine precise asymptotics of $R(\square_n)$, learning only that it grows at least linear and at most quadratic in $n$. This is the starting point for Section 8 but, frustratingly, remains open in the end.

**Theorem 42 (n-Chain).** $R_c(\mathbb{I}_n) = c(n-1)$.

**Proof.** $\leq$ We need to find a monochromatic $\mathbb{I}_n$ in the cube $\square_{c(n-1)}$. Since $\square_{c(n-1)}$ has $c(n-1) + 1$ layers, we can find a chain with $c(n-1) + 1$ elements. In any colouring with $c$ colours there is, by the pigeonhole principle, one colour that is used for $n$ elements of that chain. Those elements induce a monochromatic $\mathbb{I}_n$ as desired.

$\geq$ Note that $e(\mathbb{I}_n) = n - 1$ and apply Lemma 36. $\square$

**Theorem 43 (n-Antichain).**

$$R_c(\cdots_n) = \text{sp}(c(n-1) + 1) = \log_2(n) + \log_2(c) + O(\log \log(\max\{n,c\})).$$

**Proof.** Recall the definition of Sperner numbers, i.e., that $N := \text{sp}(c(n-1) + 1)$ is the smallest number $N$ such that the middle layer of the $\square_N$ has size at least $c(n-1) + 1$. The right identity is given in Theorem 2, we only prove the left one.

$\leq$ We need to find a monochromatic copy of $\cdots_n$ in a $c$-coloured $\square_N$. The middle layer of $\square_N$ contains at least $c(n-1) + 1$ elements. Those elements form an antichain. By the pigeonhole principle, one of the colour classes contains at least $n$ elements from that antichain, therefore forming the monochromatic $\cdots_n$ we want.

$\geq$ If $N' < N$ then the middle layer of the $\square_{N'}$ has size $M \leq c(n-1)$. By Sperner’s Theorem the middle layer forms the largest antichain in $\square_{N'}$, and therefore, using Dilworth’s Theorem, we can cover $\square_{N'}$ with $M$ disjoint chains $C_1, \ldots, C_M$. We now colour the elements of the first $n-1$ chains $C_1, \ldots, C_{n-1}$ with the first colour, the elements of the next $n-1$ chains $C_n, \ldots C_{2n-2}$ with the second colour and so on, using each colour for at most $n-1$ chains.

Since an antichain can only contain one element from any chain, and there are only $n-1$ chains of each colour, we avoided monochromatic antichains of size $n$. $\square$

To handle the lower bound of the standard example we first proof a Lemma.
Lemma 44. $e(S_n) = n - 2$.

Proof. The canonical embedding of $S_n$ into $n - 1$ layers of $\mathbb{S}_n$ shows that $e(S_n) < n - 1$. Now consider an embedding $f : S_n \to \mathbb{S}_N$ for some $N$. By Lemma 9 it can be extended to an embedding $f : \mathbb{S}_n(1, \ldots, n - 1) \to \mathbb{S}_N$. The chain $I_{n-1} \cong \{[1], [2], \ldots, [n-1]\}$ in $\mathbb{S}_n(1, \ldots, n - 1)$ must be mapped to a chain in $\mathbb{S}_N$, so $f([1])$ and $f([n-1])$ span at least $n - 1$ layers of $\mathbb{S}_N$. This proves $e(S_n) \geq n - 2$. \qed

Theorem 45 (Standard Example). $c(n-2) + 2 \leq R_c(S_n) \leq cn$ for $n \geq 3$.

Proof. $\geq$ By Lemma 44, $e(S_n) = n - 2$. Applying Lemma 36(ii) immediately gives the desired lower bound.

$\leq$ Note that the Blowup Lemma, or more precisely Corollary 41(iv), would only give $R_c(S_n) \leq (2^c - 1)n$. However, it is easy to show the improved claim. The case $c = 1$ is trivial so let $c \geq 2$ and $N := cn$. The first layer of $\mathbb{S}_N$ contains $cn$ elements, so in particular $n$ elements of the same colour. Without loss of generality (by relabelling) assume those elements are $\{1\}, \{2\}, \ldots, \{n\} \in \mathbb{S}_N$ and their colour is 1. Now define blobs

$B_i := \{X \mid [n] \setminus \{i\} \subseteq X \subseteq [N] \setminus \{i\}, \quad i \in [n].$

Either every Blob $B_i$ contains an element $X_i$ of colour 1 in which case $\{\{i\} \mid i \in [n]\} \cup \{X_i \mid i \in [n]\}$ is a copy of $S_n$. Or there is a blob without an element of colour 1, then this blob is a $(c-1)$-coloured copy of $\mathbb{S}_{N-n} = \mathbb{S}_{(c-1)n}$. By induction it contains a monochromatic copy of $S_n$. \qed

Theorem 46 (Cube). $n \cdot c \leq R_c(\mathbb{S}_n) \leq (n + 1)^c - 1.$

Proof. $\geq$ Apply Lemma 36 using $e(\mathbb{S}_n) = n$.

$\leq$ Apply Corollary 41(iv) using $h(\mathbb{S}_n) = n + 1$ and $\text{dim}_2(\mathbb{S}_n) = n$. \qed

For $c = 2$, this gives $2n \leq R(\mathbb{S}_n) \leq n^2 + 2n$. Note that in Section 8 we manage to improve the upper bound to $R(\mathbb{S}_n) \leq n^2 + 1$.

6. Non-Approximability of Ramsey Numbers

Since the 2-dimension of posets is hard to approximate, it is no big surprise that the Ramsey number of posets is hard to approximate as well. Even though we believe an intuitive understanding of “approximation” is sufficient for what we do in the following, we give a more formal account as well to avoid misunderstandings:

Given some family $\mathcal{X}$ of objects, we say an algorithm approximates some quantity $q : \mathcal{X} \to \mathbb{N}$ within a factor of $f : \mathcal{X} \to [1, \infty)$, if for each input triple $(x, a, b) \in \mathcal{X} \times \mathbb{N} \times \mathbb{N}$ with $\frac{b}{a} \geq f(x)$ the algorithm distinguishes between the two cases:

(i) $q(x) \leq a$ and (ii) $b \leq q(x).$
The output is therefore a bit, specifying whether (i) or (ii) is true. In the case of \( a < q(x) < b \), the behaviour of the algorithm is unspecified, i.e. any output is permitted.

For instance, if \( \mathcal{X} \) is the set of all finite posets and the quantity to approximate is \( q = \dim_2 \), then being able to approximate \( \dim_2 \) within a factor of 2 means that one is able to answer questions such as

\[
\text{Is } \dim_2(\mathcal{X}) \leq 3 \text{ or } \dim_2(\mathcal{X}) \geq 6? 
\]

Such a problem is called a *gap problem* or “promise problem” since a correct answer is only expected if one of the two given alternatives is true. The factor \( f \) is the constant 2 here, but later, we allow for larger errors in case of larger instances, i.e. the factor \( f \) depends on the instance.

Below we show that the gap problem for 2-dimension and Ramsey numbers cannot be decided in polynomial time. This implies in particular, that the more complicated tasks of finding corresponding embeddings is at least as hard.

We shall derive a strengthening of Theorem 18 from Section 3.2, the change being an added “of height 2”:

**Theorem 47.** If \( P \neq \text{NP} \), then there is for no \( \varepsilon > 0 \) a polynomial-time algorithm approximating \( \dim_2(P) \) for posets \( P \) of height 2 within a factor of \( |P|^{1/3 - \varepsilon} \).

Once this is proven, the hardness of approximating Ramsey numbers follows quickly: For a poset \( P \) of height 2 the Blowup Lemma implies:

\[
\dim_2(P) \leq R(P) \leq \dim_2(P)(1 + h(P)) = 3 \dim_2(P).
\]

So any approximation of \( R(P) \) is also an approximation of \( \dim_2(P) \) and vice versa, we only lose a constant factor of 3 when converting the estimates. This constant vanishes in the \( \varepsilon \)-considerations since \( 3|P|^{1/3 - \varepsilon} \leq |P|^{1/3 - \varepsilon'} \) for \( \varepsilon' < \varepsilon \) and large \( P \). Therefore, Theorem 47 implies

**Corollary 48.** If \( P \neq \text{NP} \), then there is for no \( \varepsilon > 0 \) a polynomial-time algorithm approximating \( R(P) \) for posets \( P \) of height 2 within a factor of \( |P|^{1/3 - \varepsilon} \).

To prove Theorem 47, we use ideas and arguments similar to those in Section 3.2. From there, recall the definition of a bipartite graph \( B = (S,T,E) \) with bottom set \( S \), top set \( T \) and edges \( E \subseteq S \times T \) as well as the definition of \( \text{BicCov}(B) \) as the least number \( n \) of bicliques \( B_1, B_2, \ldots, B_n \subseteq S \times T \) with \( \bigcup_{i=1}^n B_i = E \). Also recall Theorem 17 (Gruber and Holzer [GH07]) stating that \( \text{BicCov}(B) \) cannot be efficiently approximated within a factor of \( (|S| + |T|)^{1/3 - \varepsilon} \) unless \( P = \text{NP} \).

Deviating from Section 3.2, we construct a poset \( P_B \) of height 2 from \( B \) in a very simple way. Let \( P_B = S \cup T \) where \( x \leq_P y \) if \( x = y \) or \( (x,y) \in S \times T \setminus E \).

We do not necessarily have \( \dim_2(P_B) = \text{BicCov}(B) \) as in Section 3.2, but there still is a close connection:
Lemma 49. For any bipartite graph $B = (S, T, E)$ we have

$$\text{BicCov}(B) \leq \dim_2(P_B) \leq \text{BicCov}(B) + \mathcal{O}(\log(\max\{|S|, |T|\})).$$

Proof. Lower Bound. Assume $f : P_B \to \otimes_n$ is an embedding. Then $B$ can be covered by the following $n$ bicliques

$$B_i := \{s \in S \mid i \in f(s)\} \times \{t \in T \mid i \notin f(t)\}, \quad i \in [n].$$

These bicliques cover $E$ since for $(s, t) \in S \times T$ we have

$$(s, t) \in \bigcup_{i \in [n]} B_i \iff \exists i : (s, t) \in B_i \iff \exists i : i \in f(s) \setminus f(t)$$

$$(s, t) \in E \iff f(s) \not\subseteq f(t) \iff s \not\in P_B t \iff (s, t) \in E.$$

Upper Bound. Assume that $E = (S_1 \times T_1) \cup \ldots \cup (S_n \times T_n)$ is a covering of $E$ with $n$ bicliques. We show $\dim_2(P) \leq n + \mathcal{O}(\log(\max\{|S| + |T|\}))$. The first building block of the embedding is the map

$$f_1 : S \cup T \to \otimes_n$$

$$s \mapsto \{i \in [n] \mid s \in S_i\} \text{ for } s \in S,$$

$$t \mapsto \{i \in [n] \mid t \notin T_i\} \text{ for } t \in T.$$

Observe that for $(s, t) \in S \times T$ we have

$$s \not\in P_B t \iff (s, t) \in E \iff \exists i : (s, t) \in S_i \times T_i$$

$$\iff \exists i : i \in f_1(s) \setminus f_1(t) \iff f_1(s) \not\subseteq f_1(t).$$

The reason that $f_1$ is not yet an embedding is that it might allow for $f_1(x) \subseteq f_1(y)$ for $(x, y) \in (S \times S) \cup (T \times T) \cup (T \times S)$ which must not be the case unless $x = y$. To fix this, we spend a few auxiliary dimensions. By Proposition 11 and Sperner’s Theorem we have

$$k := \dim_2(\ldots \in S \otimes \ldots |T|) \leq \dim_2(\ldots |S| \otimes \ldots |T|) + \dim_2(\ldots |T|)$$

$$= \mathcal{O}(\log(\max\{|S|, |T|\})).$$

We use the canonical function $f_2 : P_B \to \otimes_k$ mapping $S$ to the “lower” antichain in and $T$ to the “upper” antichain in $\ldots |S| \otimes \ldots |T|$, which gives $f_2(x) \subseteq f_2(y)$ iff $x = y$ or $(x, y) \in S \times T$.

Now consider the function $f : P_B \to \otimes_n \times \otimes_k$ mapping $x \mapsto (f_1(x), f_2(x))$. The order on $\otimes_n \times \otimes_k$ is the product order as usual. From our analysis of $f_1$ and $f_2$ it is now clear that $f$ is an embedding. Since $\otimes_n \times \otimes_k \cong \otimes_{n+k}$, we are done. 

\qed
The preceding lemma gives a close approximation (closer than any polynomial factor) of $\text{BicCov}(B)$ in terms of $\dim_2(P_B)$. Since $\text{BicCov}(B)$ cannot be approximated by Theorem 17 within a factor of $|S + T|^{1/3 - \varepsilon}$, we now also know, that $\dim_2(P_B)$ cannot be approximated by Theorem 17 within a factor of $|S + T|^{1/3 - \varepsilon} - \varepsilon$. Of course, the factor of 2 we added in front of $\varepsilon$ is irrelevant once we remember that the inapproximability result for BicCov was true regardless of $\varepsilon$. This proves Theorem 47.

7. Large Number of Colours

In most parts of this thesis we examine $R_c(P)$ where $P$ is a family of posets, sometimes parametrised, for instance $P = \bigotimes_n$ and the number of colours $c$ is typically a small constant, for instance $c = 2$.

Things become very different if we consider $R_c(P)$ for a poset $P$ of constant size and a large number of colours $c$. In particular, $R_c(P)$ will be large compared to $\dim_2(P)$. In that situation we immediately get results from purely extremal considerations.

Recall the definitions and results from Section 4.4 and in particular Theorem 34. It implies, for every poset $P$, the existence of a constant $\eta(P)$ such that the Lubell mass $l(F)$ of any $P$-free family $F$ is bounded by $\eta(P)$.

**Theorem 50.** Let $P$ be a fixed poset that is not an antichain. Then

$$R_c(P) = \Theta(c).$$

**Proof.** When colouring the $c$ layers of $\bigotimes_{c-1}$ with one colour each, then each colour class forms an antichain. Then there is no monochromatic copy of $P$, so $R_c(P) \geq c$. In particular $R_c(P)$ grows at least linearly in $c$.

To see that it grows at most linearly, we use Theorem 34. Consider any number $c$ of colours and a $P$-free colouring of $\bigotimes_{R_c(P)-1}$. Each of the colour classes $F_1, \ldots, F_c$ has Lubell-mass at most $\eta(P)$, but together they have Lubell-mass $R_c(P)$, since $l(\bigotimes_{R_c(P)-1}) = R_c(P)$. This means $R_c(P) = l(F_1 \cup \ldots \cup F_c) \leq c \cdot \eta(P)$ so in particular $R_c(P) \leq \mathcal{O}(c)$. □

For large $c$ this is clearly superior to our previous bounds, such as $R_c(\bigotimes_n) \leq (n+1)^c$ from Theorem 46. For small $c$, however, the constant $\eta(P)$ may be prohibitively large.

8. Ramsey Numbers of Cubes

Even though we only slightly improve our bounds on $R(\bigotimes_n)$, we still think that the observations in this section on related or specialised questions are interesting and, possibly, helpful to guide future research.

**Section 8.1: $\bigotimes_3$, $\bigotimes_4$, and Other Small Examples.** To better understand how colourings avoiding cubes may look like and how strategies for finding cubes may
proceed, we consider a series of small examples, the largest being $R(\hat{S}_3)$, which is already surprisingly complicated.

Section 8.2: Number and Structure of Embeddings from $\hat{S}_n$ to $\hat{S}_N$. To get a better intuition of how small cubes can lie in bigger cubes, we analyse the structure of embeddings of $\hat{S}_n$ into $\hat{S}_N$, which also leads to a way of counting them fairly precisely.

Section 8.3: Random Colourings. In random colourings monochromatic cubes quickly occur with high probability. A randomly coloured $\hat{S}_{n \log n}$ contains a layered monochromatic $\hat{S}_n$ asymptotically almost surely, and this $\hat{S}_n$ can be found by a simple blob-type approach. In other words, monochromatic cubes are hard to avoid “on average”.

Section 8.4: Improving the Blowup Lemma. The Blowup Lemma is a clumsy tool. For cubes we show a slight improvement and outline in what way this may be generalised. We then briefly discuss why the task at hand seems so tricky and what properties a stronger result would have to exploit.

Note that our efforts concerning Ramsey numbers of cubes also continue in Section 9.1, where we examine asymmetric Ramsey numbers such as $R_{1,1}(\hat{S}_2, \hat{S}_n)$.

8.1. ◇, $\hat{S}_3$ and Other Small Examples

Determining Ramsey numbers for small posets is already surprisingly hard. In the following we take a look at $\hat{S}_2 = \Diamond$, $\hat{S}_3$, $S_3$ and related posets.

We will frequently omit braces and commas in notation, e.g. write 13 when talking about $\{1, 3\}$.

We will also frequently assume that colourings have certain properties “without loss of generality”. To justify these steps, let us formally state the simple insights that allow us to make them:

Lemma 51. Let $\text{col} : \hat{S}_n \to [c]$ be a colouring and $P$ a poset. Furthermore, let $g : \hat{S}_n \to \hat{S}_n$ be any automorphism of the cube, $\pi : [c] \to [c]$ any permutation of the colours and $\text{inv}$ the function mapping any $S \subseteq [n]$ to its complement $[n] \setminus S$. Then

(i) $\pi \circ \text{col}$ admits a monochromatic $P$ iff $\text{col}$ does.

(ii) $\text{col} \circ g$ admits a monochromatic $P$ iff $\text{col}$ does.

(iii) Let $\hat{P}$ be the poset with the same elements as $P$ and reversed order relation $\leq_P := \geq_P$. Then $\text{col} \circ \text{inv}$ admits a monochromatic $\hat{P}$ iff $\text{col}$ admits a monochromatic $P$.

(iv) If $\hat{P} \cong P$, then $\text{col} \circ \text{inv}$ admits a monochromatic $P$ iff $\text{col}$ does.

Proof. (i) If $\text{col}$ admits a copy of $P$ of colour $r \in [c]$, then the same copy is monochromatic and of colour $\pi(r)$ in the colouring $\pi \circ \text{col}$ and vice versa.
(ii) If the colouring \( \text{col} \circ g \) admits a monochromatic copy of \( P \) via the embedding \( f : P \to \mathbb{H}_n \) (meaning \( \text{col} \circ g \circ f \) is constant), then \( f' = g \circ f \) is also an embedding of \( P \) into \( \mathbb{H}_n \) and monochromatic with respect to \( \text{col} \). The reverse holds as well.

(iii) If the colouring \( \text{col} \circ \text{inv} \) admits a monochromatic copy of \( \hat{P} \), i.e. \( \text{col} \circ \text{inv} \circ f \) is constant for some embedding \( f \) of \( \hat{P} \), then \( \text{inv} \circ f \) is an embedding of \( P \) and monochromatic with respect to \( \text{col} \). The reverse holds as well.

(iv) This is a simple consequence of (iii).

We showed that, if a colouring avoids some structure, then lots of “similar” colourings avoid the structure as well. We can pick one with special properties to simplify arguments, relabelling colours and elements of \([n]\) and, in some cases, turn the colouring upside down.

**Theorem 52.** \( R(\wedge) = 3 \).

**Proof.** \( \geq \) Obviously, \( \mathbb{H}_2 = \bigcirc \) can be coloured such that two elements are red and two are blue to avoid a monochromatic \( \wedge \).

\( \leq \) Consider an arbitrary colouring of \( \mathbb{H}_3 \). We claim it contains a monochromatic \( \wedge \). Assume without loss of generality that the colour of 123 is red. If there are two incomparable red elements, then together with 123 we have a red \( \wedge \). If on the other hand the red elements form a chain, then this chain is subset of a maximal chain, which is without loss of generality \( \{\emptyset, 1, 12, 123\} \). Then the other colour class induces a copy of \( \wedge \) via \( \{2, 3, 23\} \).

**Theorem 53.** \( R(\mathbb{N}) = 4 \).

**Proof.** \( \geq \) In \( \mathbb{H}_4 \) colour the upper two layers red and the lower two layers blue. This avoids \( \mathbb{N} \).

\( \leq \) Consider an arbitrary colouring of \( \mathbb{H}_4 \). We claim it contains a monochromatic \( \mathbb{N} \). We can assume that two elements of layer 1 are red (otherwise switch the roles of blue and red) and that those elements are 1 and 2 (otherwise apply an appropriate automorphism).

**Case 1: 12 or 123 is red.** Let \( r \in \{12, 123\} \) be red. Note that \( \{1, 2, r\} \) forms a red \( \wedge \) and if an element in \( \{14, 24, 134, 234\} \) is also red, we have a red \( \mathbb{N} \). So assume all of \( \{14, 24, 134, 234\} \) are blue. A blue 3 or 34 will complete a blue \( \mathbb{N} \), so assume 3 and 34 are both red. If \( r = 123 \), then \( \{1, 123, 3, 34\} \) is a red \( \mathbb{N} \), so assume 123 is red, which means \( r = 12 \) is red. If 23 is red, then \( \{1, 12, 2, 23\} \) forms a red \( \mathbb{N} \), otherwise if 23 is blue. Now \( \{123, 23, 234, 24\} \) is a blue \( \mathbb{N} \).

**Case 2: 124 is red.** Switch the roles of 3 and 4 and apply Case 1.
Case 3: 12, 123 and 124 are blue. This forms a blue $\bigvee$. If any element from $\{3, 13, 23\}$ is also blue, we have a blue $\bigwedge$. If all of $\{3, 13, 23\}$ is red, then together with 2 we have a red $\bigwedge$. 

**Theorem 54.** $R(\bigvee) = 4$.

**Proof.** ≥ In $\mathcal{H}_3$ we can colour the upper two layers red and the lower two layers blue to avoid a monochromatic $\bigvee$.

≤ Consider any colouring of $\mathcal{H}_4$. Assume without loss of generality that $\emptyset$ is red.

**Case 1: 1234 is red.** If there are two incomparable red elements, then, together with $\emptyset$ and 1234, they form a red $\bigvee$. If, on the other hand, the red elements are subset of a chain, without loss of generality $\{\emptyset, 1, 12, 123, 1234\}$, then everything else is blue and $\{2, 23, 24, 234\}$ forms a blue $\bigwedge$.

**Case 2: 1234 is blue.** By Theorem 53, $\mathcal{H}_4$ contains a monochromatic $\bigwedge$. Note that this $\bigwedge$ will not make use of $\emptyset$ or 1234, since $\bigwedge$ has neither maximum nor minimum. If the copy of $\bigwedge$ is blue, we restrict it to a blue $\bigwedge$ and add 1234 to obtain a blue $\bigwedge$. If it is red, we restrict it to a red $\bigwedge$ and add $\emptyset$ to obtain a red $\bigwedge$. 

**Theorem 55.** $R(\mathcal{H}_3) \in \{6, 7, 8\}$.

**Proof.** The general bounds shown elsewhere give $6 \leq R(\mathcal{H}_3) \leq 15$ (Theorem 46) and $R(\mathcal{H}_3) \leq 10$ (Theorem 64). What remains to be shown here is $R(\mathcal{H}_3) \leq 8$. Assume for contradiction that there is a red/blue-colouring $\text{col}$ of $\mathcal{H}_8$ that avoids a monochromatic $\bigwedge$. Let $\text{COL}$ be set of all colourings that are identical to $\text{col}$ up to an isomorphism of the cube, swapping the roles of red and blue, turning the colouring upside-down or a combination of all three. By Lemma 51 all colourings in $\text{COL}$ avoid a monochromatic $\mathcal{H}_3$.

**Claim.** There is some colouring $\text{col}_0 \in \text{COL}$ that admits a monochromatic embedding $f : P \to \mathcal{H}_8$ where $P$ is the poset shown on the right and $f$ fulfils:

\[
f(x) = \{i_x\} \cup H \quad \text{for } x \in \{a, b, c\} \\
|f(m) \setminus H| \geq 6
\]

for some $i_a, i_b, i_c \in [8]$ and a set $H$ of size 0 or 1.

**Proof of the claim.**

**Case 1:** Any three consecutive layers of $\mathcal{H}_8$ are monochromatic in col (say red).

This contradicts the assumption that col does not admit a monochromatic $\mathcal{H}_3$:

If there is any additional red element $r_0$ above (or below) the consecutive red layers, then choose $f([3]) := r_0$ (or $f(\emptyset) := r_0$) and use the three consecutive red layers in $\mathcal{H}_8$ to extend $f$ to a monochromatic embedding of $\mathcal{H}_3$. 

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If no such $r_0$ exists, then six out of nine layers of $\mathcal{G}_8$ are blue, and a blue embedding of $\mathcal{G}_3$ is possible.

**Case 2:** The two bottom or two top layers of $\mathcal{G}_8$ are monochromatic in col (but no three consecutive layers). Pick $\text{col}_0 \in \text{COL}$ such that the bottom two layers are red.

Since the top three layers are not monochromatic (as in Case 1), pick any red set $M \in \mathcal{G}_8$ of size at least 6. Fix $f(m) := M$, $f(o) = H := \emptyset$, pick three distinct elements $i_a, i_b, i_c \in M$, and set $f(a) = \{i_a\}, f(b) = \{i_b\}, f(c) = \{i_c\}$. This is a red copy of $P$ satisfying the claim.

**Case 3:** Neither the two bottom layers nor the two top layers of $\mathcal{G}_8$ are monochromatic. Pick $\text{col}_0 \in \text{COL}$ such that $\emptyset$ is blue (swap roles of colours), 1 is red (first two layers not monochromatic) and $12, 13, 14, 15$ are of the same colour $c \in \{\text{red}, \text{blue}\}$ (use pigeonhole on $12, 13, \ldots, 18$ and permute $2, \ldots, 8$ suitably).

Now consider two cases:

**Case 3.1:** $[8]$ is of colour $c$ or $[8] \setminus i$ is of colour $c$ for some $i \in \{2, 3, 4, 5\}$.

Then define the set $M$ to be either $[8]$ or $[8] \setminus i$, whatever is of colour $c$, choose $f(m) := M, f(o) \in \{\emptyset, 1\}$ (whatever is of colour $c), H = \{1\}$ and use three elements out of $\{12, 13, 14, 15\}$ for $f(a), f(b), f(c)$, but avoid $1i$ if $[8] \setminus i$ was chosen for $M$.

These choices satisfy the claim.

**Case 3.2:** All elements of $\{[8], [8] \setminus 2, [8] \setminus 3, [8] \setminus 4, [8] \setminus 5\}$ are not of colour $c$.

Then consider the colouring $\text{col}_0' := \text{col}_0 \circ \text{inv}$ that colours each set $S$ with $\text{col}_0([8] \setminus S)$. This colouring is also in $\text{COL}$. We can then pick $f(o) := \emptyset, f(a) := 2, f(b) := 3, f(c) = 4$ and $f(m) \in \{[8], [8] \setminus 1\}$, which are all of the colour other than $c$ with respect to $\text{col}_0'$. ($\Box$ of claim)

Assume we have an embedding $f : P \to \mathcal{G}_8$ as stated in the claim. Consider the following blobs $B_{ab}, B_{bc}, B_{ca}$ defined as

$$B_{xy} := \{S \mid \{i_x, i_y\} \cup H \subseteq S \subseteq f(m) \setminus \{i_z\}\}, \quad \text{for} \quad \{x, y, z\} = \{a, b, c\}.$$  

Each blob is isomorphic to a cube of dimension

$$|(f(m)| - 1) - (2 + |H|) = -3 + |f(m) \setminus H| \geq 6 - 3 = 3.$$  

A monochromatic blob would therefore immediately give a monochromatic embedding of $\mathcal{G}_3$. Otherwise, we can extend the embedding of $P$ (say it was red) to an embedding of $\mathcal{G}_3$ by picking one red element from each of $B_{ab}, B_{bc}, B_{ca}$. $\Box$

**Theorem 56.** $R(S_3) = 6$. 

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Proof. From Theorem 45 we know $4 \leq R(S_3) \leq 6$, so it suffices to improve the lower bound by giving a colouring of $\mathbb{S}_5$ avoiding a monochromatic copy of $S_3$. The two colour classes are shown in Figure 20, omitting the sets $\emptyset$ and $\{12345\}$ since no embedding of $S_3$ can make use of them anyway. Note that the two colour classes induce posets that are isomorphic after reversing the order relation. Since $S_3$ is self-isomorphic in that sense, it suffices to argue that one of the colour classes does not contain $S_3$.

We show that the colour class $C$ on the right of Figure 20 does not contain a copy of $S_3$, i.e. there is no embedding $f : S_3 \to C$. We proceed in several steps, depicted in Figure 21, each time removing elements that cannot contribute to a copy of $S_3$.

(1) $\to$ (2) No copy of $S_3$ can contain $123$, $124$, $134$ or $234$ since of those elements has only one element in $C$ that is greater and one element in $C$ that is less. Every element of $S_3$ on the other hand has either two elements that are greater or two elements that are less.

(2) $\to$ (3) The elements of the upper layer of $S_3$ clearly cannot be represented by the minimal elements $12$, $5$ or $34$. So they are all represented by elements containing
5. Since in $S_3$ no element is less than each element of the upper layer, 5 cannot be used.

(3)$\rightarrow$ (4) No embedding can use 12 and 125 at the same time since that would either result in a chain of size 3 or in an isolated chain of size 2. If only one of them is used, then 12 and 125 are equivalent, as they have the same relationships to the other elements. So any embedding of $S_3$ can without loss of generality refrain from using 12 and, by the same argument 34.

(4)$\rightarrow$ $\Box$ Note that the Hasse diagram is a cycle of length 8 alternating between the two layers. Obviously, we cannot remove elements and still have a cycle. Since $S_3$ is a cycle of length 6, this concludes the proof. $\Box$

**Theorem 57.** $R(\mathcal{G}_S) > 6$.

**Proof.** We give a colouring of $\mathcal{G}_6$ that avoids a monochromatic $\mathcal{G}_3$. We give an illustration (but not the Hasse diagram) of the red colour class in Figure 22.

We now describe the structure underlying this colouring. The set $[6]$ is divided into three pairs, namely 12, 34 and 56. We order these pairs cyclically like this: $12 \leftrightarrow 34 \leftrightarrow 56 \leftrightarrow 12$, saying for instance that the pair 34 is left of the pair 56 and 12 is right of 56. We now list the red elements, giving them types along the way.

Type 0 : The empty set.

Type 1 : All singleton sets.

Type 2 : The three pairs.

Type 3 : All sets $\{g_1, g_2, r\}$ where $G = \{g_1, g_2\}$ is a pair and $r$ is an element from the pair to the right of $G$.

Type 3* : All sets $\{a, b, c\}$ with one element from each pair and odd sum $a+b+c$. 

Figure 22: The red colour class of a colouring of $\mathcal{G}_6$ avoiding a monochromatic $\mathcal{G}_3$. To keep the picture simple, we omit edges spanning multiple layers and edges involving the special elements 135, 146, 236, 245 of Type 3*. 

We now describe the structure underlying this colouring. The set $[6]$ is divided into three pairs, namely 12, 34 and 56. We order these pairs cyclically like this: $12 \leftrightarrow 34 \leftrightarrow 56 \leftrightarrow 12$, saying for instance that the pair 34 is left of the pair 56 and 12 is right of 56. We now list the red elements, giving them types along the way.

Type 0 : The empty set.

Type 1 : All singleton sets.

Type 2 : The three pairs.

Type 3 : All sets $\{g_1, g_2, r\}$ where $G = \{g_1, g_2\}$ is a pair and $r$ is an element from the pair to the right of $G$.

Type 3* : All sets $\{a, b, c\}$ with one element from each pair and odd sum $a+b+c$. 

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Type 4 : All \{g_1, g_2, l, r\} where \(G = \{g_1, g_2\}\) is a pair and \(l\) and \(r\) come from the pair left and right of \(G\) respectively.

All other elements should be coloured blue. Now it is easy to check that

(i) If \(S_i\) is the set of elements of type \(i\) then we have
\[
|S_0| = 1, \quad |S_1| = 6, \quad |S_2| = 3, \quad |S_3| = 6, \quad |S_3\ast| = 4, \quad |S_4| = 12,
\]
so in total 32 elements – exactly half of \(\mathcal{P}_b\).

(ii) If an element \(S \subseteq [6]\) is red then \([6] \setminus S\) is blue. Either check this manually or observe what taking complements does on the individual types. Check, for instance, that type 2 elements are not turned into type 4 elements and vice versa. To see that type 3 elements are not turned into type 3 elements note that taking complements changes “right” to “left” in the definition and concerning type 3* elements, “odd” changes to “even”.

(iii) The blue colour class therefore induces the same poset, except that it is upside down. So the blue class contains \(\mathcal{P}_3\) if and only if the red class contains \(\mathcal{P}_3\). We may thus focus on the red class.

Now assume that there is an embedding \(f\) from \(\mathcal{P}_3\) into the red class.

**Claim.** \(f(123)\) is not of type 4.

**Proof of claim.** Assume the contrary. Then \(f(123) = \{g_1, g_2, l, r\}\) where \(\{g_1, g_2\}\) is a pair and \(l\) and \(r\) are elements from the pair to the left and to the right, respectively. We now look for candidates for the elements \(f(12), f(13), f(23)\). They cannot be of type 0 or 1 since they must allow for a chain of size 2 to be beneath them. They must also be less than \(f(123)\). The elements less than \(f(123)\) of the remaining types are

Type 2 : Only \(\{g_1, g_2\}\).

Type 3 : Only \(\{g_1, g_2, r\}\).

Type 3* : Only \(\{g, l, r\}\) where \(g\) is either \(g_1\) or \(g_2\), whatever makes the sum odd.

So we got three candidates for three elements. However \(\{f(12), f(23), f(13)\}\) must form an antichain which the elements we just found do not. This contradicts the assumption that \(f(123)\) is of type 4. \(\Box\) of claim

Therefore, \(f\) must embed \(\mathcal{P}_3\) into the bottom four layers of \(\mathcal{P}_6\), using only elements of type 0, 1, 2, 3 and 3*. This requires that \(f\) preserves size, in particular \(f(12), f(23)\) and \(f(13)\) are all of size 2. Only type 2 elements are of size 2 so all three of them must be used by \(f\). This is clearly impossible since \(f(123)\) would have to be above all three, so \(f(123) \supseteq 12 \vee 34 \vee 56 = 123456\), which is not possible, since 123456 is blue. So \(f\) does not exist and our colouring avoids monochromatic \(\mathcal{P}_3\). \(\Box\)
In conclusion, $R(\mathfrak{G}_3) \in \{7, 8\}$. The correct answer seems to be $R(\mathfrak{G}_3) = 7$, which we found with an exhaustive computer search. Since this did not allow us to grasp the “reason” as to why a colouring of $\mathfrak{G}_7$ cannot avoid $\mathfrak{G}_3$, we refrain from stating this as a theorem.

For now we content ourselves with the observation that the lower bound of $R(\mathfrak{G}_n) \geq 2n$ is not tight. It would be interesting to see if the above example can be generalised for $n > 3$.

8.2. Number and Structure of Embeddings from $\mathfrak{G}_n$ to $\mathfrak{G}_N$

In the following we examine the family of all embeddings from $\mathfrak{G}_n$ to $\mathfrak{G}_N$ for $N \geq n$. For convenience, we make a few definitions:

**Definition 58.** Let $n, N \in \mathbb{N}$ with $n \leq N$. Then

(i) If $P$ is a poset, then $U \subseteq P$ is upward-closed if

$$\forall u \in U, x \in P : u \leq x \Rightarrow x \in U.$$

(ii) In $\mathfrak{G}_n$ the upper planes $1\uparrow, 2\uparrow, \ldots, n\uparrow$ are:\n
$$i\uparrow := \{S \subseteq [n] \mid i \in S\}, \quad i \in [n].$$

(iii) In the context of an embedding $f : \mathfrak{G}_n \rightarrow \mathfrak{G}_N$, define its fibres

$$\overleftarrow{f}_j := f^{-1}(j\uparrow) = \{S \subseteq [n] \mid j \in f(S)\}, \quad j \in [N],$$

meaning $\overleftarrow{f}_j \subseteq \mathfrak{G}_n$ is the set of sets that are mapped into $j\uparrow$.

As a first example, consider the embedding of $\mathfrak{G}_2$ into $\mathfrak{G}_3$ shown in Figure 23 and its three fibres. (Note that we omit braces and commas in notation, i.e. we write $123$ instead of $\{1, 2, 3\}$.)

We give a second example that is harder to visualise but more expressive. Consider the embedding $f : \mathfrak{G}_2 \rightarrow \mathfrak{G}_5$ with mappings

$$\emptyset \mapsto 1, \quad 1 \mapsto 123, \quad 2 \mapsto 134, \quad 12 \mapsto 1234.$$

Its fibres are:

$$\overleftarrow{f}_1 = \mathfrak{G}_2, \quad \overleftarrow{f}_2 = \{1, 12\}, \quad \overleftarrow{f}_3 = \emptyset, \quad \overleftarrow{f}_4 = \{2, 12\} = 2\uparrow.$$

Intuitively, $\overleftarrow{f}_2$ and $\overleftarrow{f}_4$ “do the important work”, ensuring that $f(1)$ and $f(2)$ are incomparable, since

$$2 \in f(1) \setminus f(2), \quad 4 \in f(2) \setminus f(1).$$

\[10\]With the notation from Section 3.1 we would write $\{i\}\uparrow$ instead of $i\uparrow$. 

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The other dimensions of $\mathbb{N}_n$ are “wasted” since they do not take care of critical pairs. This is particularly obvious for 1, which is used for every element, and 5, which is not used at all.

This is no coincidence: The next lemma shows that in an embedding $g : \mathbb{N}_n \rightarrow \mathbb{N}_N$ there are always $n$ dimensions of $\mathbb{N}_N$ “doing all the work”.

**Lemma 59.** Let $f : \mathbb{N}_n \rightarrow \mathbb{N}_N$ be an embedding.

(i) $f$ is uniquely determined by its fibres.

(ii) Each fibres of $f$ is an upward-closed set.

(iii) For all $i \in [n]$, the upper plane $i^\uparrow$ occurs as a fibre of $f$.

(iv) There are no further conditions on fibres, i.e. any sequence of $N$ upward-closed sets of $\mathbb{N}_n$ in which all upper planes of $\mathbb{N}_n$ occur, is the sequence of fibres for some embedding of $\mathbb{N}_n$ into $\mathbb{N}_N$.

**Proof.** (i) This is clear since, given the fibres, we can write $f$ as

$$f(S) = \{ j \in [N] \mid S \in \overleftarrow{f_j} \}.$$

(ii) Since $f$ is an embedding we have for two sets $S \subseteq T$

$$S \in \overleftarrow{f_j} \iff j \in f(S) \Rightarrow j \in f(T) \iff T \in \overleftarrow{f_j}.$$

(iii) Pick $i \in [n]$. Since $\{i\} \notin [n] \setminus \{i\}$ there is some $j \in f(\{i\}) \setminus f([n] \setminus \{i\})$. Thus for any $S \subseteq [n]$ we have

$$i \in S \Rightarrow \{i\} \subseteq S \Rightarrow j \in f(\{i\}) \subseteq f(S),$$

$$i \notin S \Rightarrow S \subseteq [n] \setminus \{i\} \Rightarrow j \notin f([n] \setminus \{i\}) \supseteq f(S).$$

Therefore $i \in S \iff j \in f(S) \iff S \in \overleftarrow{f_j}$, so $\overleftarrow{f_j} = i^\uparrow$. 

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(iv) Let \((U_1, \ldots, U_N)\) be any sequence of upward-closed sets of \(\mathcal{H}_n\) such that \(i^\uparrow\) occurs in the sequence for each \(i \in [n]\). Define \(g\):

\[
g : \mathcal{H}_n \to \mathcal{H}_N
\]

\[
S \mapsto \{ j \mid S \in U_j \}
\]

Since \(U_j\) are upward-closed, \(S \subseteq T\) implies \(f(S) \subseteq f(T)\). The critical pairs (see Section 1.3) are of the form \((\{i\}, [n] \setminus \{i\})\). For such a pair, pick \(j\) such that \(i^\uparrow = U_j\). This ensures \(j \in g(\{i\}) \setminus g([n] \setminus \{i\})\). So \(g\) respects critical pairs and is an embedding by Lemma 4. It clearly has \((U_1, \ldots, U_N)\) as its fibres.

We emphasise the key observation of the last lemma yet again: The fact that every upper plane is a fibre means that for every \(i \in [n]\) there is some \(j_i \in [N]\) such that \(i \in S \iff j_i \in f(S)\). This means that the projection of \(f\) to the dimensions \(j_1, j_2, \ldots, j_n\) looks essentially like the identity embedding. The freedom that remains for \(f\) is merely to be an arbitrary monotone function with respect to the remaining \(N - n\) dimensions.

We capture this rigidity of \(f\) on \(n\) dimensions and the freedom on the others in the following corollary.

**Corollary 60.**

(i) If \(f : \mathcal{H}_n \to \mathcal{H}_N\) is an embedding, then there is a permutation \(\pi : [N] \to [N]\) (corresponding to an automorphism of \(\mathcal{H}_N\)), such that

\[\forall S \subseteq [n] : [n] \cap (\pi \circ f)(S) = S.\]

(ii) If \(a(n)\) is the number of distinct upward-closed sets in \(\mathcal{H}_n\), then for the number \(e(n, N)\) of distinct embeddings of \(\mathcal{H}_n\) into \(\mathcal{H}_N\) we have

\[
\frac{N!}{(N-n)!}(a(n) - n)^{N-n} \leq e(n, N) \leq \frac{N!}{(N-n)!}a(n)^{N-n}.
\]

**Proof.**

(i) Let \((U_1, \ldots, U_N)\) be the fibres of \(f\). Since every upper plane \(i^\uparrow\) occurs as a fibre of \(f\), there is a permutation \(\pi : [N] \to [N]\) of the fibres such that \(U_{\pi^{-1}(i)} = i^\uparrow\) (for \(i \in [n]\)). Now check for \(S \subseteq [n]\) and \(i \in [n]\):

\[i \in S \iff S \in i^\uparrow \iff S \in U_{\pi^{-1}(i)} \iff \pi^{-1}(i) \in f(S) \iff i \in (\pi \circ f)(S).\]

So \((\pi \circ f)(S)\) contains the same elements as \(S\), except maybe for additional elements outside of \([n]\).

(ii) By Lemma 59(iv) there are just as many embeddings from \(\mathcal{H}_n\) into \(\mathcal{H}_N\) as there are sequences \((U_1, \ldots, U_N)\) of upward-closed sets of \(\mathcal{H}_n\) where each upper plane of \(\mathcal{H}_n\) occurs at least once.

There are \(N!/(N-n)!\) ways to choose the positions for the upper planes in the sequence and then \(a(n)^{N-n}\) possibilities to fill the remaining positions. All valid sequences can be constructed in this way, which proves the upper bound.
However, if we count this way, sequences that contain some upper plane more than once will be counted several times. Our lower bound is therefore given by the number of sequences that contain each upper plane exactly once. Then we only have \( a(n) - n \) choices for each position in the sequence not corresponding to an upper plane, which proves the lower bound.

Lemma 61 (Counting Embeddings).

(i) There is a natural bijection between upward-closed sets and antichains.

(ii) For the number \( a(n) \) of distinct antichains in \( \mathcal{P}_n \) we have

\[
2^{\binom{n}{\lfloor n/2 \rfloor}} \leq a(n) \leq (n + 2)^{\binom{n}{\lfloor n/2 \rfloor}}.
\]

Proof. (i) For an upward-closed set \( U \), the minimal elements of \( U \) form an antichain. For an antichain \( A \), the upward-closure, i.e. the set \( \{ x \mid \exists a \in A : a \leq x \} \) is upward-closed. These mappings are inverse to one another so we found a bijection.

(ii) For the lower bound, just consider all subsets of the middle layer, each of them is an antichain.

For the upper bound, cover the cube with \( \binom{n}{\lfloor n/2 \rfloor} \) chains (possible due to Sperner’s Theorem and Dilworth’s Theorem). An antichain \( A \) contains at most one element from each of these chains and is fully determined if we capture for each chain \( C \) whether it is disjoint from \( A \) (one possibility) or, if not, which element of \( C \) is in \( A \) (\(|C|\) possibilities). Since each chain has length at most \( n + 1 \), this gives at most \( n + 2 \) possibilities per chain.

The number \( a(n) \) is called the Dedekind Number and precise estimates are known – far more precise than the ad-hoc bounds we gave. Consider the following theorem due to Kleitman and Markowsky [KM75].

Theorem 62 (Kleitman and Markowsky [KM75]). \( a(n) = 2^{\binom{n}{\lfloor n/2 \rfloor}(1+O(\log(n)/n))} \).

Together with Corollary 60 this implies that for the number \( e(n, N) \) of embeddings from \( \mathcal{P}_n \) to \( \mathcal{P}_N \) we have

\[
e(n, N) = 2^{(N-n)\binom{n}{\lfloor n/2 \rfloor}(1+O(\log n/n))},
\]

where the factor \( N!/(N - n)! \) is hidden in the \( O(\cdot) \)-term.

8.3. Random Colourings

One of the reasons we suspect that our upper bounds on \( R_c(\mathcal{P}_n) \) are wasteful is that random colourings easily permit monochromatic embeddings.
Theorem 63. Let \( c \in \mathbb{N}, \varepsilon > 0 \) and \( n \) large. For \( N = (1 + \varepsilon)n \log_2 n \) let the elements of \( \bigotimes_N \) be \( c \)-coloured independently and uniformly at random. Then \( \bigotimes_N \) contains a monochromatic layered copy (see Section 5.1) of \( \bigotimes_n \) asymptotically almost surely.

Proof. We view \( \bigotimes_N \) as the following product poset (ignoring rounding issues)

\[
\bigotimes_N \cong \bigotimes_n \times \bigotimes_m \times \ldots \bigotimes_m
\]

where \( m = (1 + \varepsilon') \log_2 n \) for some appropriate \( \varepsilon' \in (0, \varepsilon) \). Assume without loss of generality that \( m \) is even. Let \( A \) be the middle layer of \( \bigotimes_m \). By Theorem 2 \( A \) has size \( n^{1+\varepsilon''} \) for some \( \varepsilon'' > 0 \).

Call one of the colours “red”. For each \( S \subseteq \left[ n \right] \) try to find a red element \( f(S) \) from the set

\[
\text{Strip}(S) := \left\{ \left( S, [m], \ldots, [m], X, \emptyset, \ldots, \emptyset \right) \mid X \in A \right\}
\]

If this works, then \( f \) is a red embedding of \( \bigotimes_n \). It is layered since all sets of size \( s \) are mapped to sets of size \( s + m \cdot s + m/2 \). We fail to find \( f \) if there is some \( S \subseteq \left[ n \right] \) such that \( \text{Strip}(S) \) contains no red elements, let \( \{ E_S \mid S \subseteq \left[ n \right] \} \) be the corresponding “bad events”.

By the union bound, the probability that none of these bad events occur is at least:

\[
1 - \Pr\left[ \bigcup_{S \subseteq \left[ n \right]} E_S \right] \geq 1 - \sum_{S \subseteq \left[ n \right]} \left( \frac{c-1}{c} \right)^{|A|} \geq 1 - 2^n \left( \frac{c-1}{c} \right)^{n^{1+\varepsilon''}} \xrightarrow{n \to \infty} 1.
\]

8.4. Improving the Blowup Lemma

When tracking how the bound \( R_c(\bigotimes_n) \leq (n+1)^c - 1 \) is established from the Blowup Lemma in Theorem 46, it can be seen fairly quickly that some improvement is possible by handling the first two layers of \( \bigotimes_n \) differently. We show this in the following and sketch some other potential angles of improvements.

8.4.1. Find a Partial Embedding by Hand

Without further ado, we give a slight improvement of Theorem 46, that will, however, still be of the form \( R_c(\bigotimes_n) \leq n^c + o(n^c) \).

Theorem 64. \( R_c(\bigotimes_n) \leq \frac{(n-1)^{c+2} - (n-1)^2 - c(n-2)}{(n-2)^2} \), for \( n \geq \max\{3, c\} \).

Proof. For \( c = 1 \) the claim holds since the right side evaluates to \( n \). Now let \( c \geq 2 \). We show that \( R_c(\bigotimes_n) \leq c - 1 + n + (n-1)R_{c-1}(\bigotimes_n) \) from which the claim follows by induction. For convenience, define \( N' := R_{c-1}(\bigotimes_n) \) and \( N := c - 1 + n + (n-1)N' \). Assume \( \bigotimes_N \) is \( c \)-coloured. We need to find a monochromatic \( \bigotimes_n \).
Claim. There are disjoint sets $H_1, H_2, X \subset [N]$ with $|H_1| + |H_2| \leq c - 1$, $|X| = n$ such that $H_1$ and all sets $H_1 \cup H_2 \cup \{x\}$ for $x \in X$ have the same colour, say red.

Proof of Claim: The construction may require several steps. Before the $k$-th step we have sets $\emptyset = R_0 \subset R_1 \subset \ldots \subset R_{k-1}$ with $|R_i| = i$ such that the colours of all $R_i$ are distinct. We call this a rainbow path. Clearly, before the first step (i.e. $k = 1$), $R_0 = \emptyset$ fulfils this requirement. Note also that $k$ can never exceed $c$. Now when doing the $k$-th step there are two cases as shown in Figure 24.

Figure 24: Consider the candidates to extend a rainbow path. Either a new colour occurs (left case), then the rainbow path can be extended, here using the yellow element. We forget about the rest (greyed out). If no new colours occur (right case), then pigeon-holing guarantees that one colour already in the path (here blue) occurs often. This yields a monochromatic $\mathcal{V}$, here a blue $\mathcal{V}$.

Case 1: There is a set $R_k \supset R_{k-1}$ of size $k$ and of a colour that was not yet used in the rainbow path. Then we can extend the rainbow path with $R_k$ and continue with the next step.

Case 2: The $N - (k - 1)$ sets from $M := \{ R_{k-1} \cup \{x\} \mid x \in [N] \setminus R_{k-1} \}$ all have colours that were already used in the rainbow path. Then there is a colour, call it red, that was already used on some $R_i$ and that is used for at least

$$\frac{N - (k - 1)}{c} > \frac{(n - 1)N'}{c} \geq \frac{(n - 1)n(c - 1)}{c} \geq n \cdot 2 \cdot \frac{c - 1}{c} \geq n$$

elements from $M$, allowing us to pick a red set $M' \subseteq M$ of size $n$. Then $H_1 := R_i$, $H_2 := R_{k-1} \setminus R_i$ and $X := \{ x \in [N] \mid R_{k-1} \cup \{x\} \in M' \}$ fulfils the requirement.

The claim essentially gave us the first two layers of a cube in red. We now proceed in the same way we did in the Blowup Lemma. For convenience (as in Section 5.2) we rewrite the elements of $\mathcal{H}_N$ using the isomorphism

$$\mathcal{H}_N \cong \mathcal{H}_{c-1} \times \mathcal{H}_N \times \overbrace{\mathcal{H}_{N'} \times \ldots \times \mathcal{H}_{N'}}^{n-1 \text{ times}}.$$ 

where $H_1, H_2 \in \mathcal{H}_N$ corresponds to $(H'_1, \emptyset, \ldots, \emptyset)$ and $(H'_2, \emptyset, \ldots, \emptyset)$, respectively, and $X$ corresponds to $(\emptyset, [n], \emptyset, \ldots, \emptyset)$. Note that after this switch of perspective the
red elements are now \((H_1, \emptyset, \ldots, \emptyset)\) as well as \((H_1 \cup H_2, \{i\}, \emptyset, \ldots, \emptyset)\) for \(i \in [n]\). We identify for \(S \subseteq [n]\) with \(|S| \geq 2\) the blob
\[
B_S := \{(H_1 \cup H_2, S, [N'], \ldots, [N'], Y, \emptyset, \ldots, \emptyset) \mid Y \subseteq [N']\}_{|S| \cdot 2 \text{ times}}^{n-|S|}.
\]
Now either there is a blob that has no red element. In that case that blob is a \((c-1)\)-coloured copy of \(\mathcal{H}_{N'}\). Recall that \(N' = R_{c-1}(n)\), meaning we can find a monochromatic \(\mathcal{H}_n\) in that blob and are done. If on the other hand every blob \(B_S\) contains a red element \(g(S)\), then a red embedding \(f\) of \(\mathcal{H}_n\) is given as
\[
f : \mathcal{H}_n \to \mathcal{H}_n \times \mathcal{H}_n \times \mathcal{H}_{R_c(n-1)} \times \ldots \times \mathcal{H}_{R_c(n-1)} \cong \mathcal{H}_N
\]
\[
\emptyset \mapsto (H_1, \emptyset, \ldots, \emptyset)
\]
\[
\{i\} \mapsto (H_1 \cup H_2, \{i\}, \emptyset, \ldots, \emptyset) \quad \text{for } i \in [n]
\]
\[
S \mapsto g(S) \quad \text{for } S \subseteq [n] \text{ of size at least 2}.
\]
Note that for \(c = 2\) the result can be simplified to \(R(\mathcal{H}_n) \leq n^2 + 1\). We exploited that blob-approaches allow us to choose the colour in which to find a structure. So we found an embedding of the first two layers of a cube by hand in some colour (we didn’t care which) and then used a blob-argument to find the rest of the cube in the same colour.

How about finding the first \(k\)-layers by hand and using the blob-approach only for \(n - k\) layers? Recall from Proposition 10 and Lemma 9 that
\[
dim_2(\mathcal{H}_n(1, \ldots, k)) \leq O(k^2 \log N),
\]
which means, by Corollary 41, that \(R(\mathcal{H}_n(1, \ldots, k)) \leq O(k^3 \log n)\). The Blowup Lemma, in contrast, would allocate \(n \cdot k\) dimensions for the first \(k\) layers. Can we therefore improve our bounds on \(R(\mathcal{H}_n)\) this way, by choosing, say, \(k = n^{1/4}\)? As it turns out, it is not that simple.

Consider for instance the following function:
\[
f : \emptyset \mapsto \{\emptyset\}, \{1\} \mapsto \{12\}, \{2\} \mapsto \{23\}, \{3\} \mapsto \{13\}.
\]
This is an embedding of \(\mathcal{V}\), i.e. the bottom two layers of \(\mathcal{H}_3\), but it is impossible to extend \(f\) to an embedding of \(\mathcal{H}_3\). This is because \(f(\{1, 2\})\) would have to be a superset of both \(f(1) = \{1, 2\}\) and \(f(2) = \{2, 3\}\), but not a superset of \(f(3) = \{1, 3\}\).

Clearly such a set does not exist.

Therefore, a general improvement of the Blowup Lemma along those lines may require a better understanding of partial embeddings that can be extended to complete embeddings, and corresponding Ramsey numbers.
8.4.2. More Substantial Improvements?

**Dedekind–MacNeille Completion.** In Proposition 13 we have seen that \( \dim_2(P) = \dim_2(\text{DM}(P)) \), i.e. a poset \( P \) and the smallest lattice containing \( P \) have the same 2-dimension. Does this carry over to the coloured case? Well, no. In Section 8.1 we have seen for instance that \( 3 = R(\cdot \cdot) < R(\text{DM}(\cdot \cdot)) = R(\Diamond) = 4 \) and also that \( R(S_3) = 6 < R(\text{DM}(S_3)) = R(\Diamond_{S_3}) \). Of course, there may still be a useful connection between \( R(P) \) and \( R(\text{DM}(P)) \) that we just fail to see.

**Global Approaches.** Recall that the Blowup Lemma tries to find a monochromatic \( \Diamond_n \) in red/blue-coloured cubes by looking for a blowup \( \Diamond_n \times \Diamond_n \) in what is basically \( \Diamond_{n^2} \). It then assumes that none of the \( 2^n \) blobs is blue and finds a red \( \Diamond_n \).

There are several ways to state the shortcoming of this approach:

1. The Blowup Lemma does not harness the full power of the assumption “there is no blue cube”, considering only \( 2^n \) disjoint configurations for them. Red cubes, in contrast, may be found in \( (2^n)^2 \) different configurations.

2. The Blowup Lemma is non-adaptive. A more dynamic strategy, that employs some branching and backtracking may have better guarantees. It may also be helpful to switch the colour to focus on if a promising partial cube arises. If there is no blue blob (and blue blobs are intuitively rare) the Blowup Lemma basically only looks for red cubes.

3. We have already seen that randomness in the colouring makes finding monochromatic cubes much easier, so it may be that suitable randomness in the embedding strategy can have have a similar effect. As a first measure, we could pick a random automorphism of the cube before trying to find an embedding. After such an automorphism, every blob (that can be chosen to be much smaller than before), will contain a red element almost surely – unless the colouring was greatly imbalanced in the first place, with almost monochromatic layers. This latter circumstance, however, may be exploitable through different means. The goal would of course be to show, that the embedding strategy works with positive probability which shows that an embedding exists.

Of course we have no clear picture of how these vague ideas can be turned into working upper bounds, but it may very well be that we missed a reasonably simple angle of attack – it sometimes feels as though we must have.

9. Asymmetric Ramsey Numbers and Sums of Posets

The concept of asymmetric Ramsey numbers is a very natural generalisation in which there is a different structure associated with each colour class. For instance, using an asymmetric variation of the pigeonhole principle, if it takes 5 mealy potatoes to make mashed potatoes and 7 waxy potatoes to make French fries, then if you have
5 + 7 − 1 = 11 potatoes, you will be able to make one of the two. A bit less obvious is the asymmetric Graph-Ramsey statement that given 10 people on a party there will be 3 people that are pairwise acquainted or 4 people that are pairwise strangers. Compare for instance, [GRS90, Page 2].

In the poset setting, asymmetric Ramsey numbers are closely related to certain compositions of posets, but we believe they are also interesting on their own.

**Section 9.1: Asymmetric Ramsey Numbers.** We formally introduce asymmetric Ramsey numbers \( R_{c_1,c_2}(P,Q) \) for posets. For instance, \( R_{1,1}(P,Q) \) is the smallest \( N \) such that any red/blue colouring of \( 
abla_N \) contains a red copy of \( P \) or a blue copy of \( Q \). We examine how \( R_{1,1}(P,Q) \) relates \( R(P) \) and \( R(Q) \) and consider a special case where \( P \) and \( Q \) are cubes. We prove

**Theorem 70.** \( R_{1,1}(\nabla, \nabla_n) \leq 2n + 1. \)

**Section 9.2: Ramsey Numbers of** \( P \overset{\cup}{\lor} Q \) **and** \( P \otimes Q \). We examine Ramsey numbers of horizontal and vertical compositions, i.e. \( R_c(P \overset{\cup}{\lor} Q) \) and \( R_c(P \otimes Q) \). We manage to tightly approximate the former in terms of asymmetric Ramsey numbers.

### 9.1. Asymmetric Ramsey Numbers

**Definition 65.** For two posets \( P,Q \) and \( c_1,c_2 \in \mathbb{N}_0 \), let \( R_{c_1,c_2}(P,Q) \) be the smallest integer \( N \) such that for any colouring \( \text{col} : N \to [c_1 + c_2] \) there is a monochromatic copy of \( P \) with a colour from \( \{1, \ldots, c_1\} \) or a monochromatic copy of \( Q \) with a colour from \( \{c_1 + 1, \ldots, c_1 + c_2\} \) (or both).

For instance, \( R_{1,1}(P,Q) \) is the smallest \( N \) such that any red/blue colouring of \( \nabla_N \) admits a red \( P \) or a blue \( Q \) (or both). We start with a few simple observations.

**Lemma 66.** Let \( P,Q \) be posets and \( c,c_1,c_2 \in \mathbb{N}_0 \) and \( c_1 + c_2 = c \). Then

(i) \( R_{c_1,c_2}(P,P) = R_c(P). \)

(ii) \( R_{c_1,c_2}(P,Q) = R_{c_2,c_1}(Q,P). \)

(iii) If \( P \) is a subposet of \( Q \), then

\[
R_c(P) = R_{c,0}(P,Q) \leq R_{c-1,1}(P,Q) \leq \ldots \leq R_{0,c}(P,Q) = R_c(Q).
\]

**Proof.**

(i) Clearly, there is a copy of \( P \) in a colour from \([c_1]\) or a copy of \( P \) in a colour from \([c] \setminus [c_1]\) if and only if there is a copy of \( P \) in an arbitrary colour.

(ii) The claim follows simply from relabelling the colours. It is easy to overlook that this claim is actually useful: In a red/blue-coloured cube of dimension \( R_{1,1}(P,Q) \) we can find either a red \( P \) or a blue \( Q \), but additionally we can also find either a blue \( P \) or a red \( Q \).
(iii) To see $R_{c_1,c_2}(P,Q) \leq R_{c_1-1,c_2+1}(P,Q)$ assume some colouring avoids $P$ in colours $\{1, \ldots, c_1\}$ and avoids $Q$ in colours $\{c_1 + 1, \ldots, c_1 + c_2\}$. Then clearly it avoids $P$ in colours $\{1, \ldots, c_1 - 1\}$ and $Q$ in colours $\{c_1, \ldots, c_1 + c_2\}$, in the case of $c_1$ because $P$ is a subposet of $Q$.

Note that we already have a result for asymmetric numbers, although we did not state it as such at the time. Namely, the Blowup Lemma asserts

$$R_{1,1}(P, \hat{\mathcal{Y}}_m) \leq \dim_2(P \ltimes \hat{\mathcal{Y}}_m) \leq \dim_2(P) + h(P) \cdot m,$$

which implies $R_{1,1}(P,Q) \leq \dim_2(P) + h(P) \cdot \dim_2(Q)$.

Seeing Lemma 66(iii), one might wonder if asymmetric Ramsey numbers $R_{c_1,c_2}(P,Q)$ are always between $R_{c_1+c_2}(P)$ and $R_{c_1+c_2}(Q)$. We show that this is not always the case by giving an example with $c_1 = c_2 = 1$.

**Proposition 67.** There are posets $P$ and $Q$ with $R_{1,1}(P,Q) > \max\{R(P), R(Q)\}$.

**Proof.** For suitable $x$ and $y$, $R_{1,1}(\hat{1}_x, \hat{\ldots}_y)$ is bigger than both $R(\hat{1}_x)$ and $R(\hat{\ldots}_y)$. In the following computation we ignore all rounding issues and $\approx$ means equality up to factors that are asymptotically negligible in $n$.

We pick $x = \beta n + 1$ and $y = \left(\frac{n}{\alpha n}\right) + 1$ for $\alpha, \beta \in (0,1)$ to be determined later and fulfilling $2\alpha + \beta = 1$.

Consider a cube of dimension $n$ (for sufficiently large $n$). Colour the middle $\beta n$ layers red, the rest of the cube blue, like this:

$$\hat{\mathcal{Y}}_n = \begin{cases} \text{\alpha n blue layers} \\ \beta n \text{ red layers} \\ \text{\alpha n blue layers} \end{cases}$$

It is easy to see that there is neither a red chain of length $x$, nor a blue antichain of size $y$. Therefore, we showed $R_{1,1}(\hat{1}_x, \hat{\ldots}_y) \gtrsim n$.

However, we will find that, for appropriate choices of $\alpha$ and $\beta$, the Ramsey numbers of the involved chain and antichain are both smaller:

$$R(\hat{1}_x) \approx 2\beta n$$

$$R(\hat{\ldots}_y) = \sp(y) \approx \log_2\left(\frac{n}{\alpha n}\right) = \frac{1}{\ln 2}\left[(\ln n) - \ln(\alpha n)! - \ln((1 - \alpha)n)\right] \approx \frac{1}{\ln 2}\left[(n \ln(n) - n - \alpha n \ln(\alpha n) + \alpha n - (1 - \alpha)n \ln((1 - \alpha)n) + (1 - \alpha)n)\right]$$

$$= (n \log_2(n) - \alpha n \log_2(\alpha n) - (1 - \alpha)n \log_2((1 - \alpha)n))$$

$$= (n \log_2(n) - \alpha n \log_2(\alpha) - \alpha n \log_2(n) - (1 - \alpha)n \log_2(1 - \alpha) - (1 - \alpha)n \log_2(n))$$

The ansatz $R(\hat{1}_x) = R(\hat{\ldots}_y)$ yields approximately $\beta = 0.431$, in particular there are choices for $x$ and $y$ such that $R(\hat{1}_x) \approx R(\hat{\ldots}_y) \approx 0.862n$. Hence, we found a family of examples with $\frac{R_{1,1}(P,Q)}{\max\{R(P), R(Q)\}} \approx \frac{1}{0.862} > 1.$

□
It is easy to find examples where \( R_{1,1}(P, Q) \) is strictly between \( R(P) \) and \( R(Q) \), for instance if \( P = \mathbb{I}_n, Q = \mathbb{I}_{2n} \), then \( R(P) \approx 2n < R_{1,1}(P, Q) \approx 3n < R(Q) \approx 4n \).

It might be interesting to find examples where \( R_{1,1}(P, Q) < \min\{R(P), R(Q)\} \) or examples with \( \max\{R(P), R(Q)\} = \Omega(1) \) – or proof that those do not exist.

As a variation on our search for \( R(n) \), we now consider \( R_{1,1}(m, n) \) for \( m \in \{1, 2\} \), i.e. we analyse \( R_{1,1}(1, n) \) and \( R_{1,1}(\emptyset, n) \). The first case is handled by a simple application of the following Lemma.

**Lemma 68.** If \( A \subseteq \mathbb{I}_{n+1} \) is an antichain, then we can find a copy of \( \mathbb{I}_n \) in \( \mathbb{I}_{n+1} \) that is disjoint from \( A \).

**Proof.** We specify an embedding

\[
\begin{align*}
f : \mathbb{I}_n &\to \mathbb{I}_{n+1} \\
S &\mapsto \begin{cases} 
S \cup \{n + 1\} & \text{if } \exists A \in A: A \subseteq S, \\
S & \text{otherwise.}
\end{cases}
\end{align*}
\]

With the results from Section 8.2 it is obvious that \( f \) is an embedding, just check that \( \{S \subseteq \mathbb{I}_n \mid n + 1 \in f(S)\} \) is upward-closed. It is also easy to see that no set is mapped into \( A \). In the first case of the map, the image \( S \cup \{n + 1\} \) has a proper subset \( A \in A \) and can therefore not be part of \( A \) itself. The second case is only applied if \( \forall A \in A: A \not\subseteq S \) which implies \( S \not\in A \).

**Theorem 69.** \( R_{1,1}(1, \mathbb{I}_n) = n + 1 \).

**Proof.** Let \( \mathbb{I}_{n+1} \) be coloured with red and blue. We need to find a red 1 or a blue \( \mathbb{I}_n \). Assume there is no red 1, this means the red elements form an antichain. The preceding Lemma asserts that there is an embedding of \( \mathbb{I}_n \) into \( \mathbb{I}_{n+1} \) not using any of the red elements, so we found a blue copy of \( \mathbb{I}_n \).

We note without proof that this can be generalised to \( R_{1,1}(1, P) \leq R(P) + n - 1 \).

**Theorem 70.** \( R_{1,1}(\emptyset, \mathbb{I}_n) \leq 2n + 1 \).

**Proof.** Note that the Blowup Lemma only gives \( R_{1,1}(\emptyset, \mathbb{I}_n) \leq 3 \cdot n + 2 \), so we need to do something smarter.

Let \( \mathbb{I}_{2n+1} \) be coloured with red and blue and assume that there is no red copy of \( \emptyset \).

We focus on a subposet \( \mathbb{I}_n \times L \subseteq \mathbb{I}_n \times \mathbb{I}_{n+1} \cong \mathbb{I}_{2n+1} \) where \( L \) is a generalised diamond with an antichain of size \( \left(\frac{n}{2}\right) + 1 \), like this:

\[
L := \begin{array}{c}
l_1 \quad l_2 \quad l_3 \\
\vdots \\
\emptyset \quad \cdots \\
\end{array}
\]

\[
\begin{array}{c}
\cdot \\
[n + 1] \\
\end{array}
\]

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Clearly, $L$ is a subposet of $\mathcal{H}_{n+1}$ as required. We call the elements of $L$ labels, $\emptyset$ and $[n+1]$ are special labels, the others are ordinary labels. In the following we attempt to construct an embedding of the form

$$f : \mathcal{H}_n \to \mathcal{H}_n \times L,$$

$$S \mapsto (S, l(S)).$$

such that $f$ only uses blue elements and $l(S)$ are labels to be determined. Let

$$B := \{S \in \mathcal{H}_n \mid \forall S' \subseteq S : (S', \emptyset) \text{ is blue}\},$$

$$T := \{S \in \mathcal{H}_n \mid \forall S' \supseteq S : (S', [n+1]) \text{ is blue}\},$$

$$\mathcal{M} := \mathcal{H}_n \setminus (B \cup T).$$

For $S \in B$ we define $l(S) := \emptyset$, and for $S \in T \setminus B$ we define $l(S) := [n+1]$.

Claim. Let $M_1, M_2 \in \mathcal{M}$ with $M_1 \subseteq M_2$ and let $l_1 \neq l_2 \in L \setminus \{\emptyset, [n+1]\}$ be two distinct ordinary labels. Then $(M_1, l_1)$ and $(M_2, l_2)$ are not both red.

Proof of Claim. Assume they are. Then because $M_1 \notin B$ there is $S \subseteq M_1$ such that $(S, \emptyset)$ is red. And because $M_2 \notin T$ there is $S' \supseteq M_2$ such that $(S', [n+1])$ is red. Then $\{(S, \emptyset), (M_1, l_1), (M_2, l_2), (S', [n+1])\}$ is a red copy of $\mathcal{Q}$, contradicting the assumption to the contrary. Here we use that $l_1$ and $l_2$ are incomparable. (□ of claim)

Let $\mathcal{M}_0 := \{M \in \mathcal{M} \mid (M, l) \text{ is red for some ordinary label } l\}$ and let $\mathcal{A}$ be a maximal antichain in $\mathcal{M}_0$. For an illustration consider Figure 25. Now define the set of bad ordinary labels $L_{\text{bad}} := \{l \in L \setminus \{\emptyset, [n+1]\} \mid \exists A \in \mathcal{A} : (A, l) \text{ is red}\}$. Since each $A \in \mathcal{A}$ can only be responsible for one bad label (apply the claim for $A = M_1 = M_2$)

![Figure 25: Illustration for the proof of Theorem 70. The set $B$ is downward-closed, $T$ is upward-closed (they may intersect) and $\mathcal{M}$ is the remaining part (red). The set $\mathcal{M}_0$ (dots) contains those $M \in \mathcal{M}$ for which an ordinary label $l$ exists such that $(M, l)$ is red. We choose a maximal antichain $\mathcal{A} \subseteq \mathcal{M}_0$ (big dots) in $\mathcal{M}_0$.]

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we have $|L_{\text{bad}}| \leq |A| \leq \binom{n}{n/2}$ so there is at least one ordinary label $l^*$ left that is not bad.

We define $l(M) := l^*$ for each $M \in \mathcal{M}$. With this definition $f$ is an embedding since the labels clearly preserve inclusion. Assume however, that $(M, l^*)$ is red for some $M \in \mathcal{M}$. This means $M \in \mathcal{M}_0$ and since $\mathcal{A}$ was a maximal antichain in $\mathcal{M}_0$, $M$ is comparable to some element $M' \in \mathcal{A} \subseteq \mathcal{M}_0$. Let $l'$ be the ordinary label such that $(M', l')$ is red (exists by definition of $\mathcal{M}_0$) and by definition of $L_{\text{bad}}$ we know $l'$ is bad, so different from $l^*$. Now $(M, l^*)$ and $(M', l')$ contradict the claim which completes the proof. \hfill $\square$

Our results do not reach far, but it may be that more general bounds on $R(\bigotimes_m, \bigotimes_n)$ could be a first step in reaching better bounds on $R(\bigotimes_n)$.

### 9.2. Ramsey Numbers of $P \cup Q$ and $P \otimes Q$

Recall from Section 2.3.2 that the horizontal sum $P \cup Q$ of two posets is a copy of $P$ and a copy of $Q$ with no relation in between.

Before considering $P \cup Q$ in general, we examine the special case $P = Q$. Recall from Equation (1) that $\text{sp}(w)$ is the smallest number $N$ such that the middle layer of $\bigotimes_N$ has size at least $w$.

**Theorem 71.** (i) $R(P) \leq R(P \cup P) \leq R(P) + 3$.

(ii) $R_c(P) \leq R_c(P \cup \ldots \cup P) \leq R_c(P) + \text{sp}(c(k-1) + 1) \approx R_c(P) + \log_2(ck)$.

**Proof.** The lower bounds follow from monotonicity and (i) is a special case of (ii), so it suffices if we prove the upper bound of the latter.

In the cube of dimension $R_c(P) + \text{sp}(c(k-1) + 1)$ we find, by Proposition 12, the horizontal sum of $c(k-1) + 1$ copies of $\bigotimes_{R_c(P)}$, i.e. $\bigotimes_{R_c(P)} \cup \ldots \cup \bigotimes_{R_c(P)}$.

In each copy we find a monochromatic $P$. They are not necessarily of the same colour but, by the pigeonhole principle, one colour occurs at least $k$ times, giving us a monochromatic $P \cup \ldots \cup P$ consisting of $k$ copies of $P$. \hfill $\square$

So finding many independent copies of $P$ in one colour is “not much harder” than finding a single monochromatic copy. We only need logarithmically many additional dimensions. This is not particularly surprising, given that the width of a Boolean lattice grows exponentially in its dimension, so horizontal space is readily available.

For similar reasons, $R_c(P \cup Q)$ is almost equal to $R_c(P, Q)$, which we define as

$$R_c(P, Q) := \max_{0 \leq c_1, c_2 \leq c} R_{c_1, c_2}(P, Q).$$

Put differently, $R_c(P, Q)$, it is the smallest $N$ such that for any colouring of $\bigotimes_N$ and any partition of the colours $[c]$ into two sets $C_P$ and $C_Q$, there is either a copy of $P$ with a colour from $C_P$ or a copy of $Q$ with a colour from $C_Q$. In the case of $c = 2$
this simplifies to \( R(P, Q) := R_2(P, Q) = \max\{R(P), R(Q), R_{1,1}(P, Q)\} \). As seen in Proposition 67, any of the three numbers can be the maximum.

**Theorem 72.**

(i) \( R(P, Q) \leq R(P \cup Q) \leq R(P, Q) + 3 \).

(ii) \( R_c(P, Q) \leq R_c(P \cup Q) \leq R_c(P, Q) + \text{sp}(c + 1) \approx R_c(P, Q) + \log_2(c) \).

**Proof.** Since (i) is a special case of (ii), we only proof the latter.

**Lower Bound.** Showing \( R_c(P \cup Q) \geq R_c(P, Q) \) is easy: Consider a \( c \)-coloured cube of dimension \( R_c(P, Q) \) and any partition \( [c] = C_P \cup C_Q \). We need to find a monochromatic copy of \( P \) in a colour from \( C_P \) or a monochromatic copy of \( Q \) in a colour from \( C_Q \).

By the choice of the dimension, we find a monochromatic \( P \cup Q \) in the cube. It has some colour \( c_0 \in C \).

If \( c_0 \in C_P \), then restrict the copy of \( P \cup Q \) to \( P \).

If \( c_0 \in C_Q \), then restrict the copy of \( P \cup Q \) to \( Q \).

In both cases we got what we wanted.

**Upper Bound.** To show \( R_c(P \cup Q) \leq R_c(P, Q) + \text{sp}(c + 1) \), set \( n := R_c(P, Q) \) and \( N := n + \text{sp}(c + 1) \). We need to find a monochromatic copy of \( P \cup Q \) in \( \otimes_N \).

By Proposition 12 we first find the horizontal sum of \( c + 1 \) copies of \( \otimes_n \), i.e. \( \otimes_n \cup \ldots \cup \otimes_n \). Then we apply the following \((c + 1)\)-step algorithm:

**Initially:** \( \text{Col} := \emptyset \)

**Invariant:** If after the \( i \)-th step we have not found a monochromatic copy of \( P \cup Q \) then the set \( \text{Col} \) of colours has size \( i \) and for each \( \text{col} \in \text{Col} \) there is a copy of \( P \) in colour \( \text{col} \) within the first \( i \) cubes.

**Step:** Assume we have not found a copy of \( P \cup Q \) yet. The \((i + 1)\)-st cube has dimension \( R_c(P, Q) \). We take the partition \( C_Q := \text{Col}, C_P := [c] \setminus \text{Col} \) and find either:

- **a copy of** \( Q \) **of a colour** \( c_q \in \text{Col} \): In this case we have a copy of both \( P \) and \( Q \) in colour \( c_q \), in different cubes, and we can combine them to a copy of \( P \cup Q \).

- **a copy of** \( P \) **in a colour** \( c_p \notin \text{Col} \): We add \( c_p \) to \( \text{Col} \) and proceed with the next step.

The algorithm necessarily terminates with a copy of \( P \cup Q \) since when \( \text{Col} = [c] \) the second case of the step cannot occur.

Similar ideas can be applied in principle to find upper bounds on the Ramsey numbers of vertical compositions, we state them in the case of \( c = 2 \) only. Recall from Section 2.3.2 that the vertical composition \( P \otimes Q \) of two posets consist of a copy of \( P \) and a copy of \( Q \) where every element from \( P \) is less than every element from \( Q \).
Theorem 73. \( R(P \otimes Q) \leq R(P) + R_{1,1}(P, Q) + R(Q) + 2 \).

Proof. Set \( N := R(P) + R_{1,1}(P, Q) + R(Q) + 2 \). We first identify a copy of 
\( \mathbb{R}_{R(P)} \otimes \mathbb{R}_{R_{1,1}(P, Q)} \otimes \mathbb{R}_R \) in \( \mathbb{R}_N \) as given by \( S_1 \cup S_2 \cup S_3 \) with

\[
S_1 := \{ S \mid S \subseteq [R(P)] \}, \\
S_2 := \{ S \mid [R(P) + 1] \subseteq S \subseteq [R(P) + R_{1,1}(P, Q) + 1] \}, \\
S_3 := \{ S \mid [R(P) + R_{1,1}(P, Q) + 2] \subseteq S \subseteq [R(P) + R_{1,1}(P, Q) + R(Q) + 2] \}.
\]

Then find a monochromatic \( P \) in \( S_1 \cong \mathbb{R}_{R(P)} \) and a monochromatic \( Q \) in \( S_3 \cong \mathbb{R}_{R(Q)} \). If they are both of the same colour, then together they form a monochromatic \( P \otimes Q \) as desired. Otherwise, say \( P \) is blue and \( Q \) is red, we find either a red \( P \) or a blue \( Q \) in \( S_2 \cong \mathbb{R}_{R_{1,1}(P, Q)} \). Therefore, either \( S_1 \cup S_2 \) contains a blue \( P \otimes Q \) or \( S_2 \cup S_3 \) contains a red \( P \otimes Q \). \( \square \)

Note that while in Theorem 72(i) the lower and upper bound on \( R(P \cup Q) \) were equal up to a constant of 3, Theorem 73 may leave a significant gap even in simple cases, for example \( P = Q = 1_n \). In this case \( R(1_n \otimes 1_n) = R(1_{2n}) = 4n - 2 \) while Theorem 73 only gives a bound of \( 6n - 4 \). We mostly stated it for completeness sake as an afterthought.
Part IV.
Conclusion and Open Problems

Relevance. In Part II we collected research that either underlies or runs parallel to this thesis, hoping to convince the reader, that, given the interesting topics in its vicinity, our question is also well worth studying:

The underlying notion of 2-dimension presented in Section 2 goes back to Trotter [Tro75]. The existence of Ramsey numbers of posets was verified in Nešetřil and Rödl [NR84] and similar flavours of quantitative Ramsey questions were studied by Kierstead and Trotter [KT87], both of which we discussed in Section 4. We have also seen connections to extremal considerations on posets that were presented in Section 4.4, which helped to solve our question to satisfaction in the special case of many colours, see Section 7.

Open Question. In Part III we studied the Ramsey numbers of several posets, but considering the computational intractability of the general problem shown in Section 6, it seemed appropriate to focus on special cases. One stood out in particular: The Ramsey number of the $n$-cube, i.e. $R(\mathcal{O}_n)$. Not only is it natural to use the same candidate for the “host”-poset that is coloured and the “guest”-poset that we want to find monochromatically. Upper bounds on the Ramsey numbers of cubes may carry over to upper bounds for other interesting posets as well.

It turns out that determining the precise asymptotic behaviour of $R(\mathcal{O}_n)$, let alone precise numbers, were (at least so far) beyond the reach of the author. We only know $2n \leq R(\mathcal{O}_n) \leq n^2 + 1$. We think the most interesting open questions are therefore:

- **What is the asymptotic behaviour of $R(\mathcal{O}_n)$? In particular:**
- **Does $R(\mathcal{O}_n)$ grow super-linear in $n$?**
- **Does $R(\mathcal{O}_n)$ grow sub-quadratic in $n$?**

What to try next? Considering that $\mathcal{O}_n$ can be found monochromatically in a randomly coloured cube of dimension $n \log n$ almost surely, we believe that a randomised argument might be key to improving the upper bound of $n^2 + 1$. We discussed this in Section 8. As a simpler problem that might lead to valuable insights, we propose examining asymmetric Ramsey numbers $R_{1,1}(\mathcal{O}_m, \mathcal{O}_n)$, as we did in Section 9, albeit only for $m \in \{1, 2\}$.

The author’s obsession with $R(\mathcal{O}_n)$ should not distract from the fact that the Ramsey numbers of other families of posets may be just as interesting. For instance, Habib et al. [Hab+04] approximate the 2-dimension of tree-posets, so why not try to extend their result to an approximation of Ramsey numbers of tree-posets?
References


