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Output feedback stabilisation of minimum phase systems by delays

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Abstract

For linear, square, multi-input–multi-output, minimum-phase, relative-degree one or two systems, proportional output feedback controllers are studied. In particular, we show that derivative feedback can be replaced by delay feedback, essentially an Euler approximation of the derivative, if the delay is sufficiently small. Stability regions are determined and related.

Keywords: Delay-feedback; Minimum phase systems; Relative degree two; Output feedback stabilisation

1. Introduction

We preface this section by defining some notation. Let $\mathbb{C}_+$, $\mathbb{C}_-$ denote the open right-half, left-half, complex planes, respectively; for $x \in \mathbb{R}^N$, the Euclidean norm is $\|x\| := \sqrt{x^T x}$; for any interval $I \subset \mathbb{R}$, let $C(I;\mathbb{R}^N)$ denote the set of continuous functions from $I \to \mathbb{R}^N$; and let $\sigma(A)$ denote the spectrum of $A \in \mathbb{R}^{N\times N}$.

Consider finite-dimensional, real, linear, m-input ($u(t) \in \mathbb{R}^m$), m-output ($y(t) \in \mathbb{R}^m$) systems of the form

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x^0 \\ y(t) &= Cx(t) \end{aligned}\right\} \tag{1.1}$$

with $x^0 \in \mathbb{R}^n$, and $A \in \mathbb{R}^{n\times n}$, $B, C^T \in \mathbb{R}^{n\times m}$ satisfying the minimum phase condition

$$\det \begin{bmatrix} sI_n - A & B \\ C & 0 \end{bmatrix} \neq 0, \quad \forall s \in \mathbb{C} \setminus \mathbb{C}_- \tag{1.2}$$

and having either strict relative degree one and known "sign" of the high-frequency gain, i.e.

$$\sigma(CB) \subset \mathbb{C}_+ \quad \text{and} \quad n \geqslant m, \tag{1.3a}$$

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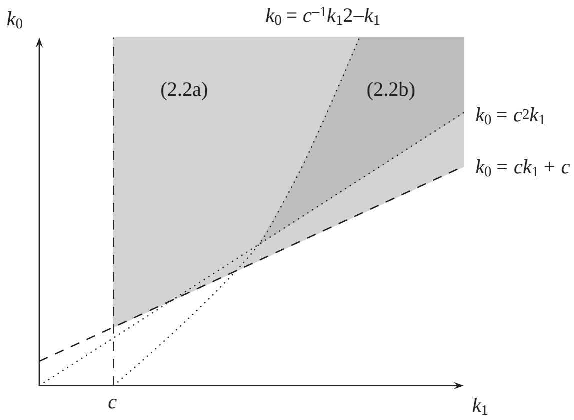

Fig. 1. Illustrating the regions (2.2a) and (2.2b).

or having strict relative degree two and known "sign" of the high-frequency gain, i.e.

$$CB = 0, \quad \sigma(CAB) \subset \mathbb{C}_+ \quad \text{and} \quad n \geqslant 2m. \tag{1.3b}$$

The main objective of the present note is to prove that the proportional output *derivative* feedback

$$u(t) = -k_0 y(t) - k_1 \dot{y}(t) \tag{1.4}$$

can be replaced by a proportional output *delay* feedback

$$u(t) = -k_0 y(t) - k_1 \frac{1}{h}[y(t) - y(t-h)], \tag{1.5}$$

provided the delay $h > 0$ is sufficiently small, and the gains k_0 and k_1 are sufficiently large. More precisely, we provide "stability regions" in terms of $h > 0$ and $k_0, k_1 \geqslant 0$ so that (1.4) as well as (1.5) applied to any multi-input, multi-output system (1.1) satisfying the minimum phase property (1.2) and having either relative degree one, i.e. (1.3a), or two, i.e. (1.3b), yields an exponentially stable closed-loop system.

The intuition behind the feedback strategy (1.5) is that for "small" $h > 0$, (1.5) is "close" to (1.4). In fact, the stability region for the delay feedback tends to the stability region for the derivative feedback as the delay $h > 0$ tends to zero. Moreover, the stability region for relative degree two systems is included in the stability region for relative degree one systems, as illustrated in Fig. 1. However, one difficulty when applying (1.5) to (1.1) is the factor h^{-1} in (1.5) which precludes setting $h = 0$ to allow consideration of an associated delay free system. To overcome this, we shall exploit Krasovskii–Lyapunov functionals and Razumikhin's approach to Lyapunov functions for delay differential equations, see for example [2, Chapter 5].

The derivative feedback (1.4) applied to any relative degree two, minimum phase system (1.1) yields exponential stability, provided the gain parameters are sufficiently large; this is well known and straightforward to show by invoking a root locus argument—at least in the single-input, single-output case. A different approach to stabilising relative degree two, minimum phase systems (1.1) is by invoking a filter; this approach also relies on an approximation of the output derivative. Although the feedback laws (1.4) or (1.5) are simple since they require only two or three parameters (in contrast to using B and C for derivative feedback as in [1, Theorem 4.5.1]), the delay controller (1.5) necessitates storing the history of the output signal y over the interval $[t-h, t]$. However, our primary interest lies, not in implementation or controller complexity, but in showing (i) that derivative feedback can be replaced by delay feedback, and (ii) how the stability regions for the two different feedbacks are related. One strategy to affect implementation would be a sampled data controller. Although it is clear how to derive a sampled data controller, showing that this works is not immediately obvious and is the subject of future research.

2. Output stabilisation

We preface this section by some notions and results pertaining to linear differential delay equations

$$\dot{x}(t) = Fx(t) + F_1 x(t-h), \tag{2.1}$$

where $F, F_1 \in \mathbb{R}^{n\times n}$, $h > 0$. It is well known, for example [2, Theorem 2.2.1], that for any initial condition $\phi \in C([-h, 0]; \mathbb{R}^n)$, there exists a unique solution $[-h, \infty) \to \mathbb{R}^n$, $t \mapsto x(t) = x(t; \phi)$ of (2.1); that is to say, $x(t)$ satisfies (2.1) for all $t \geqslant 0$, and satisfies $x(t;\phi) = \phi(t)$ for all $t \in [-h, 0]$.

The system (2.1) is said to be *exponentially stable* if, and only if, there exists $K, \varepsilon > 0$ such that

$$\|x(t;\phi)\| \leqslant K e^{-\varepsilon t} \sup_{s\in[-h,0]} \|\phi(s)\| \quad \forall t \geqslant 0,\ \forall \phi \in C([-h,0];\mathbb{R}^n).$$

For linear systems 'asymptotical stability', 'uniform asymptotic stability', and 'exponential stability' coincide, see for example [2, Lemma 5.1.1].

In the following results, we present qualitative regions for the gain parameters $k_0, k_1 \geqslant 0$ and the delay $h > 0$, to ensure exponential stability of the closed-loop system, first when derivative feedback, and secondly when delay feedback, is applied to relative degree one or two systems.

Proposition 2.1 (Derivative feedback). *For each system* (1.1) *satisfying* (1.2) *and* (1.3a) *or* (1.3b), *there exists a constant* $c = c(A, B, C) \geqslant 1$ *such that the closed-loop system* (1.1), (1.4) *is exponentially stable provided*

$$(k_1 = 0 \text{ and } k_0 > c) \quad \text{or} \quad (k_1 > c \text{ and } k_0 > ck_1 + c) \quad \text{in case (1.3a)}, \tag{2.2a}$$

$$k_1 > c + c\frac{k_0}{k_1} \quad \text{and} \quad \frac{k_0}{k_1} > c^2 \quad \text{in case (1.3b)}. \tag{2.2b}$$

Theorem 2.2 (Delay feedback). *For each system* (1.1) *satisfying* (1.2) *and* (1.3a) *or* (1.3b), *there exists a constant* $c = c(A, B, C) \geqslant 1$ *such that the closed-loop system* (1.1), (1.5) *is exponentially stable provided*

$$(k_1 = 0 \text{ and } k_0 > c) \quad \text{or} \quad k_0 > c\frac{k_1}{h} + c \quad \text{in case (1.3a)}, \tag{2.3a}$$

$$\left.\begin{aligned} & k_1 && > c + c\frac{k_0}{k_1} + hck_1[1+k_0+k_1]^2 \\ \text{and}\ & \frac{k_0}{k_1} && > c^2 + hck_1[1+k_0+k_1]^2 \\ \text{and}\ & 1 && > hck_1[1+k_0+k_1]^2 \end{aligned}\right\} \quad \text{in case (1.3b)}. \tag{2.3b}$$

Remark 2.3. We discuss the relationships between the "stability regions" defined by (2.2a), (2.2b), (2.3a), and (2.3b).

(i) The first observation is that the stability regions of the delay feedback (1.5) are included in the stability regions of the derivative feedback (1.4). That is to say, for $h > 0$ sufficiently small, (2.3a) implies (2.2a) and (2.3b) implies (2.2b).

(ii) The second observation is that the feedback controllers (1.4) and (1.5), which are designed for relative degree two systems, also stabilise relative degree one systems: specifically (2.2b) implies (2.2a), illustrated in Fig. 1, and (2.3b) implies (2.3a).

First note that inequalities (2.2b) admit feasible $k_0 \geqslant 0$ if, and only if,

$$k_1 > c(1 + c^2). \tag{2.4}$$

Secondly let $c^*(\approx 1.29)$ denote the unique real root of $c(1 + c^2) = (c - 1)^{-1}$ on $(1, \infty)$. Then for all $c > c^*$ we show that (2.2b) implies (2.2a). To see this, first note that

$$c(1 + c^2) > (c - 1)^{-1} \quad \forall c > c^*, \tag{2.5}$$

which gives

$$\begin{aligned}(2.2\text{b}) &\Rightarrow [k_1 \overset{(2.4)}{>} c(1 + c^2) \overset{(2.5)}{>} (c - 1)^{-1}] \\ &\Rightarrow [ck_1 > k_1 + 1] \\ &\Rightarrow k_0 \overset{(2.2\text{b})}{\geqslant} c^2 k_1 \overset{(2.6)}{>} ck_1 + c \quad \forall c > c^*. \end{aligned} \tag{2.6}$$

Furthermore,

$$k_1 \overset{(2.2\text{b})}{>} c + c\frac{k_0}{k_1} \Rightarrow k_1 > c \quad \forall c > c^*.$$

Lastly we note that for $h > 0$ sufficiently small, (2.3b) implies (2.3a). This proves the claim.

(iii) If (1.1) is minimum phase, i.e. (1.2), and of relative degree one, i.e. (1.3a), then a root locus analysis yields exponential stability of the closed-loop system (1.1), (1.4) for $k_1 = 0$ and k_0 sufficiently large. However, in Proposition 2.1 we prefer to include the derivative feedback in (1.4) and therefore assume that $k_1 > c \geqslant 1$.

(iv) In Proposition 2.1 and Theorem 2.2 we have assumed that $c \geqslant 1$, and hence $k_1 > 1$. A careful inspection of the proofs reveals that the assumption $k_1 > 1$ is not necessary. However, weakening this hypothesis yields a significantly more complicated analysis and consequent stability region, which in any case still does not allow $k_1 > 0$ to be arbitrarily small.

In the following Lemma 2.4, we give normal forms for systems (1.1) satisfying either (1.3a) or (1.3b). These forms are precursors for the proofs of Proposition 2.1 and Theorem 2.2, and also give insight into the minimum phase assumption (1.2).

Lemma 2.4. (i) *For each system* (1.1), *there exists an invertible* $S \in \mathbb{R}^{n\times n}$ *such that the coordinate transformation* $S^{-1}x(t)$ *converts* (1.1) *into*

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} + \begin{pmatrix} CB \\ 0 \end{pmatrix} u(t) \quad \textit{in case } (1.3\text{a}), \tag{2.7a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y(t) \\ \dot{y}(t) \\ z(t) \end{pmatrix} = \begin{bmatrix} 0 & I_m & 0 \\ A_5 & A_6 & A_7 \\ A_8 & 0 & A_9 \end{bmatrix}\begin{pmatrix} y(t) \\ \dot{y}(t) \\ z(t) \end{pmatrix} + \begin{bmatrix} 0 \\ CAB \\ 0 \end{bmatrix} u(t) \quad \textit{in case } (1.3\text{b}) \tag{2.7b}$$

for suitable $A_1 \in \mathbb{R}^{m\times m}$, A_2, $A_3^{\mathrm{T}} \in \mathbb{R}^{m\times(n-m)}$, $A_4 \in \mathbb{R}^{(n-m)\times(n-m)}$, $A_5, A_6 \in \mathbb{R}^{m\times m}$, $A_7, A_8^{\mathrm{T}} \in \mathbb{R}^{m\times(n-2m)}$, $A_9 \in \mathbb{R}^{(n-2m)\times(n-2m)}$.

(ii) *The closed-loop system* (1.1), (1.4) *may be represented as*

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = \begin{bmatrix} [I_m + k_1 CB]^{-1}[A_1 - k_0 CB] & [I_m + k_1 CB]^{-1}A_2 \\ A_3 & A_4 \end{bmatrix}\begin{pmatrix} y(t) \\ z(t) \end{pmatrix}, \quad \textit{in case } (1.3\mathrm{a}), \tag{2.8a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y(t) \\ \dot{y}(t) \\ z(t) \end{pmatrix} = \begin{bmatrix} 0 & I_m & 0 \\ A_5 - k_0 CAB & A_6 - k_1 CAB & A_7 \\ A_8 & 0 & A_9 \end{bmatrix}\begin{pmatrix} y(t) \\ \dot{y}(t) \\ z(t) \end{pmatrix}, \quad \textit{in case } (1.3\mathrm{b}). \tag{2.8b}$$

where we have assumed, in the case of (1.3a),

$$k_1 > \|(CB)^{-1}\| \tag{2.9}$$

so that the inverse matrix in (2.8a) *exist.*

(iii) *If* (1.1) *satisfies* (1.2), *then* $\sigma(A_4) \subset \mathbb{C}_-$, $\sigma(A_9) \subset \mathbb{C}_-$, *respectively.*

3. Proofs

We preface this section by some simple but useful inequalities, which will be used frequently. For any $R, M \in \mathbb{R}^{N\times N}$, with R positive-definite and symmetric, and $\varepsilon > 0$, we have

$$-2\xi^{\mathrm{T}} M\zeta \leqslant \xi^{\mathrm{T}} MR^{-1}M^{\mathrm{T}}\xi + \zeta^{\mathrm{T}} R\zeta \quad \forall \xi, \zeta \in \mathbb{R}^N, \tag{3.1}$$

$$2ab \leqslant \frac{1}{\varepsilon}a^2 + \varepsilon b^2 \quad \forall a, b \in \mathbb{R}, \tag{3.2}$$

$$\|R^{-1}\|^{-1}\|w\|^2 \leqslant w^{\mathrm{T}} Rw \leqslant \|R\|\|w\|^2 \quad \forall w \in \mathbb{R}^N. \tag{3.3}$$

Eq. (3.1) follows from expanding $\|R^{-1/2}M^{\mathrm{T}}\xi + R^{1/2}\zeta\|^2 \geqslant 0$; (3.2) is a special case of (3.1); (3.3) follows from the singular-value decomposition.

Proof of Lemma 2.4. (i) is proved in [1, Lemma 4.5.3]. To see (ii), note that by

$$[I_m + k_1 CB]^{-1}CB = k_1^{-1}[I_m + (k_1 CB)^{-1}]^{-1} \tag{3.4}$$

the matrix $[I_m + k_1 CB]$ is invertible. Now (ii) is a straightforward calculation. Finally, (iii) is also proved in [1, Lemma 4.5.3]. □

Proof of Proposition 2.1. (a) Suppose a system (1.1) satisfies (1.2). Then it is easy to show that for $k_1 = 0$ and k_0 sufficiently large, in terms of the entries of (1.1), the closed-loop system (1.1), (1.4) becomes exponentially stable. So it remains to consider the case "$k_1 > c$ and $k_0 > ck_1 + c$" in (2.2a). Suppose (1.1) satisfies (1.2), (1.3a), is in the form (2.7a), and let

$$c_1 \geqslant 3\|(CB)^{-1}\|. \tag{3.5}$$

If $k_1 > c_1$, then by Lemma 2.4, the closed-loop system (1.1), (1.4) is of the form (2.8a). Let $Q \in \mathbb{R}^{m\times m}$ be the positive-definite, symmetric solution of

$$A_4^{\mathrm{T}} Q + QA_4 = -2I_m.$$

Differentiation of the Lyapunov-function candidate

$$t \mapsto V(t) := \tfrac{1}{2} y(t)^{\mathrm{T}} y(t) + \tfrac{1}{2} z(t)^{\mathrm{T}} Q z(t)$$

along any solution of (2.8a) yields, for all $t \geqslant 0$ (where we omit the argument t for brevity),

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t} V(t)\right|_{(2.8\mathrm{a})} &= y^{\mathrm{T}}[[I_m + k_1 CB]^{-1}[A_1 - k_0 CB]y + [I_m + k_1 CB]^{-1} A_2 z] + z^{\mathrm{T}} Q[A_3 y + A_4 z] \\ &\leqslant -k_0 y^{\mathrm{T}} [I_m + k_1 CB]^{-1} CB y - \|z\|^2 + \|[I_m + k_1 CB]^{-1} A_1\| \|y\|^2 \\ &\quad + [\|[I_m + k_1 CB]^{-1} A_2\| + \|QA_3\|] \|y\| \|z\|. \end{aligned} \tag{3.6}$$

Substituting

$$\begin{aligned} -y^{\mathrm{T}}[I + k_1 CB]^{-1} CB y &\overset{(3.4)}{=} -k_1^{-1} y^{\mathrm{T}} \sum_{i \geqslant 0} (-k_1 CB)^{-i} y \\ &\leqslant -k_1^{-1} \|y\|^2 + k_1^{-1} \sum_{i \geqslant 1} \|(k_1 CB)^{-1}\| \|y\|^2 \\ &= -k_1^{-1} \left[1 - \left(\frac{1}{1 - \|(k_1 CB)^{-1}\|} - 1 \right) \right] \|y\|^2 \\ &\overset{(3.5)}{\leqslant} -\frac{1}{2k_1} \|y\|^2 \quad \forall y \in \mathbb{R}^m, \end{aligned}$$

and

$$\begin{aligned} \|[I + k_1 CB]^{-1}\| &\overset{(3.4)}{=} k_1^{-1} \|[I + (k_1 CB)^{-1}]^{-1} (CB)^{-1}\| \\ &\leqslant \frac{1}{k_1} \frac{\|(CB)^{-1}\|}{1 - \|(k_1 CB)^{-1}\|} \leqslant \frac{1}{k_1} \frac{3}{2} \|(CB)^{-1}\| \end{aligned}$$

into (3.6) yields, for all $t \geqslant 0$,

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t} V(t)\right|_{(2.8\mathrm{a})} &\overset{(3.2)}{\leqslant} - \left[\frac{k_0}{2k_1} - \frac{3}{2k_1} \|(CB)^{-1}\| \|A_1\| - 2(\|[I + k_1 CB]^{-1} A_2\| QA_3\|)^2 \right] \|y\|^2 - \tfrac{1}{2} \|z\|^2 \\ &\leqslant -\frac{1}{k_1} \left[\frac{k_0}{2} - \tfrac{3}{2} \|(CB)^{-1}\| \|A_1\| - \frac{9}{k_1} \|(CB)^{-1}\|^2 \|A\|^2 - 4k_1 \|QA_2\|^2 \right] \|y\|^2 - \tfrac{1}{2} \|z\|^2 \\ &\leqslant -\frac{1}{k_1} \left[\frac{k_0}{2} - \tfrac{3}{2} \|(CB)^{-1}\| (\|A_1\| + 2\|A_2\|^2) - 4k_1 \|QA_3\|^2 \right] \|y\|^2 - \tfrac{1}{2} \|z\|^2. \end{aligned}$$

Defining

$$c := \max\{c_1, 3\|(CB)^{-1}\|(\|A_1\| + 2\|A_2\|^2), 8\|QA_3\|^2\},$$

choose k_0, k_1 so that the second condition in (2.2a) holds. Then there exists $\varepsilon > 0$ such that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} V(t)\right|_{(2.8\mathrm{a})} \leqslant -\varepsilon V(t) \quad \forall t \geqslant 0,$$

whence exponential stability of (2.8a) (respectively (1.1), (1.4)).

(b) Suppose a system (1.1) satisfies (1.2),(1.3b), and is in the form (2.7b). By Lemma 2.4, the closed-loop system (1.1), (1.4) is of the form (2.8b). By writing $G := CAB$,

$$\begin{pmatrix} w(t) \\ v(t) \\ z(t) \end{pmatrix} := T \begin{pmatrix} y(t) \\ \dot{y}(t) \\ z(t) \end{pmatrix},$$

$$T := \begin{bmatrix} k_1 G & I_m & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_{n-2m} \end{bmatrix} \quad \text{for } k_1 > 0, \tag{3.7}$$

and

$$\hat{\Delta} := T \begin{bmatrix} 0 & I & 0 \\ A_5 - k_0 G & A_6 - k_1 G & A_7 \\ A_8 & 0 & A_9 \end{bmatrix} T^{-1} = \begin{bmatrix} k_1^{-1} A_5 G^{-1} - \dfrac{k_0}{k_1} I & \dfrac{k_0}{k_1} I + A_6 - k_1^{-1} A_5 G^{-1} & A_7 \\ k_1^{-1} A_5 G^{-1} - \dfrac{k_0}{k_1} I & \dfrac{k_0}{k_1} I + A_6 - k_1^{-1} A_5 G^{-1} - k_1 G & A_7 \\ k_1^{-1} A_8 G^{-1} & -k_1^{-1} A_8 G^{-1} & A_9 \end{bmatrix}, \tag{3.8}$$

and so (2.8b) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} w(t) \\ v(t) \\ z(t) \end{pmatrix} = \hat{\Delta} \begin{pmatrix} w(t) \\ v(t) \\ z(t) \end{pmatrix}. \tag{3.9}$$

It remains to show exponential stability of (3.9). Let $P \in \mathbb{R}^{m \times m}$ and $Q \in \mathbb{R}^{(n-2m)\times(n-2m)}$ be the positive-definite solutions to

$$G^{\mathrm{T}} P + PG = 2I_m, \quad A_9^{\mathrm{T}} Q + QA_9 = -2I_{n-2m},$$

and set

$$R := \frac{1}{2} \begin{bmatrix} P & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & Q \end{bmatrix}, \tag{3.10}$$

$$c_1 := \|PA_5 G^{-1}\| + 4(\|PA_6\| + 2\|P\|\|A_5 G^{-1}\|)^2 + 4(\|PA_7\| + \|QA_8 G^{-1}\|)^2, \tag{3.11}$$

$$c_2 := \tfrac{1}{2} + \|PA_6\| + \|PA_5 G^{-1}\| + 4(\|PA_7\| + \|QA_8 G^{-1}\|)^2, \tag{3.12}$$

$$c = \max\{c_1, c_2, \|P\|, \|P^{-1}\|\}.$$

Now invoking (3.2) and (3.3) yields, for $k_1 \geqslant 1$,

$$(w^{\mathrm{T}}, v^{\mathrm{T}}, z^{\mathrm{T}})[R\hat{\Delta} + \hat{\Delta}^{\mathrm{T}} R] \begin{pmatrix} w \\ v \\ z \end{pmatrix}$$

$$
\begin{aligned}
&\leqslant k_1^{-1}\|PA_5G^{-1}\|\|w\|^2 - \frac{k_0}{k_1}w^{\mathrm{T}}Pw + [\|PA_6\| + 2k_1^{-1}\|P\|\|A_5G^{-1}\|]\|w\|\|v\| - \tfrac{1}{2}\|z\|^2 - k_1\|v\|^2 \\
&\quad + [\|PA_7\| + k_1^{-1}\|QA_8G^{-1}\|]\|z\|(\|w\| + \|v\|) + \left[\frac{k_0}{k_1}\|P\| + \|PA_6\| + k_1^{-1}\|PA_5G^{-1}\|\right]\|v\|^2 \\
&\leqslant -\left[\frac{k_0}{k_1}\|P^{-1}\|^{-1} - \frac{\|PA_5G^{-1}\|}{k_1} - 4(\|PA_6\| + 2\|P\|\|A_5G^{-1}\|)^2 - 4(\|PA_7\| + \|QA_8G^{-1}\|)^2\right]\|w\|^2 \\
&\quad - \left[k_1 - \frac{k_0}{k_1}\|P\| - \tfrac{1}{2} - \|PA_6\| - \|PA_5G^{-1}\| - 4(\|PA_7\| + \|QA_8G^{-1}\|)^2\right]\|v\|^2 - \tfrac{1}{2}\|z\|^2 \\
&\leqslant -\left[\frac{k_0}{k_1}c^{-1} - c\right]\|w\|^2 - \left[k_1 - \frac{k_0}{k_1}c - c\right]\|v\|^2 - \tfrac{1}{2}\|z\|^2 \\
&\qquad \forall (w,v,z) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{n-2m}.
\end{aligned}
\tag{3.13}
$$

Finally, differentiation of the Lyapunov-function candidate

$$
t \mapsto V(t) := (w(t)^{\mathrm{T}}, v(t)^{\mathrm{T}}, z(t)^{\mathrm{T}})R\begin{pmatrix} w(t) \\ v(t) \\ z(t) \end{pmatrix}
$$

along any solution of (3.9) yields,

$$
\frac{\mathrm{d}}{\mathrm{d}t}V(t)\bigg|_{(3.9)} \leqslant -\left(\frac{k_0}{k_1}c^{-1} - c\right)\|w(t)\|^2 - \left(k_1 - \frac{k_0}{k_1}c - c\right)\|v(t)\|^2 - \tfrac{1}{2}\|z(t)\|^2 \quad \forall t \geqslant 0. \tag{3.14}
$$

Taking k_0, k_1 satisfying (2.2b), yields negativity of the right-hand side of (3.14), whence exponential stability of (3.9) (respectively (1.1), (1.4)). This completes the proof of the proposition. □

Proof of Theorem 2.2. (a) Suppose (1.1) satisfies (1.2) and (1.3a). If the condition "$k_1 = 0$ and $k_0 > c$" of (2.3a) is satisfied, then exponential stability follows from Proposition 2.1. Therefore, it remains to consider the closed-loop system (1.1), (1.5) for $k_1 > 1$. By invoking (2.7a), (1.1) may be written in the form

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = \begin{bmatrix} A_1 - \left(k_0 + \dfrac{k_1}{h}\right)CB & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} + \begin{bmatrix} \dfrac{k_1}{h}CB & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} y(t-h) \\ z(t-h) \end{pmatrix} \tag{3.15}
$$

for arbitrary initial conditions $\phi \in C([-h,0];\mathbb{R}^n)$.

Since $\sigma(CB) \subset \mathbb{C}_+$ and $\sigma(A_4) \subset \mathbb{C}_-$, we may choose the unique positive-definite, symmetric solutions $P_1 \in \mathbb{R}^{m\times m}$, $P_2 \in \mathbb{R}^{(n-m)\times(n-m)}$ to

$$
(CB)^{\mathrm{T}}P_1 + P_1CB = I_m \quad \text{and} \quad A_4^{\mathrm{T}}P_2 + P_2A_4 = -I_{n-m}.
$$

Define

$$
c := \max\{2\|P_1A_1\| + 4\|P_1A_2\|^2 + 4\|P_2A_3\|^2, \|P_1CB\|^2\}.
$$

By the second assumption of (2.3a), we may choose $h, k_0, k_1 > 0$ and $\alpha > 0$ such that

$$
\alpha \in \left(\frac{k_1}{h}, k_0 + \frac{k_1}{h}(1-c) - c\right). \tag{3.16}
$$

Define the (Lyapunov–Krasovskii) functional $W : C([-h,0];\mathbb{R}^m) \times C([-h,0];\mathbb{R}^{(n-m)}) \to \mathbb{R}$ as

$$(\phi,\psi) \mapsto W(\phi,\psi) := \phi^{\mathrm{T}}(0)P_1\phi(0) + \psi^{\mathrm{T}}(0)P_2\psi(0) + \alpha\int_{-h}^{0}\|\phi(s)\|^2\,\mathrm{d}s.$$

Then there exist $w_1, w_2 > 0$ such that, for all $(\phi,\psi) \in C([-h,0];\mathbb{R}^m) \times C([-h,0];\mathbb{R}^{(n-m)})$,

$$w_1(\|\phi(0)\|^2 + \|\psi(0)\|^2) \leqslant W(\phi,\psi) \leqslant w_2(\|\phi(0)\|^2 + \|\psi(0)\|^2) + h\alpha \sup_{s\in[-h,0]}\|\phi(s)\|^2.$$

Let $(y(\cdot)^{\mathrm{T}}, z(\cdot)^{\mathrm{T}})^{\mathrm{T}} : [-h,\infty) \to \mathbb{R}^n$ denote the unique solution of (3.15), with initial conditions $\phi \in C([-h,0];\mathbb{R}^n)$. Adopting the notation

$$y_t(s) := y(t+s) \quad \text{and} \quad z_t(s) := z(t+s) \quad \forall t \geqslant 0,\ \forall s \in [-h,0],$$

differentiation of $t \mapsto W(t) := W(y_t, z_t)$ along solutions of (3.15) yields, for all $t \geqslant 0$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}W(t)\bigg|_{(3.15)} &= 2y(t)^{\mathrm{T}}P_1\dot{y}(t) + 2z(t)^{\mathrm{T}}P_2\dot{z}(t) + \alpha y(t)^{\mathrm{T}}y(t) - \alpha y(t-h)^{\mathrm{T}}y(t-h)\\
&= -\left(k_0 + \frac{k_1}{h} - \alpha\right)\|y(t)\|^2 - \|z(t)\|^2 - \alpha\|y(t-h)\|^2 + 2y(t)^{\mathrm{T}}P_1A_1y(t)\\
&\quad +2y(t)^{\mathrm{T}}(P_1A_2 + A_3^{\mathrm{T}}P_2)z(t) + \frac{2k_1}{h}y(t)^{\mathrm{T}}P_1CBy(t-h)\\
&\overset{(3.2)}{\leqslant} -\left(k_0 + \frac{k_1}{h} - \alpha\right)\|y(t)\|^2 - \tfrac{1}{2}\|z(t)\|^2 - \left(\alpha - \frac{k_1}{h}\right)\|y(t-h)\|^2\\
&\quad + \left(4\|P_1A_2\|^2 + 4\|P_2A_3\|^2 + 2\|P_1A_1\| + \frac{k_1}{h}\|P_1CB\|^2\right)\|y(t)\|^2\\
&\leqslant -\left(k_0 + \frac{k_1}{h}(1-c) - \alpha - c\right)\|y(t)\|^2 - \tfrac{1}{2}\|z(t)\|^2 - \left(\alpha - \frac{k_1}{h}\right)\|y(t-h)\|^2.
\end{aligned}$$

Define

$$\varepsilon := \min\left\{\tfrac{1}{2}, k_0 + \frac{k_1}{h}(1-c) - \alpha - c, \alpha - \frac{k_1}{h}\right\}.$$

By (3.16) it follows that $\varepsilon > 0$, and since

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t)\bigg|_{(3.15)} \leqslant -\varepsilon(\|y(t)\|^2 + \|z(t)\|^2 + \|y(t-h)\|^2) \quad \forall t \geqslant 0.$$

We may apply [2, Theorem 5.2.1] to conclude uniform asymptotic, and hence exponential, stability of the zero solution of (3.15).

(b) Suppose (1.1) satisfies (1.2) and (1.3b). Define $G = (g_{ij}) := CAB$, then by (2.7b), the closed-loop system (1.1), (1.4) may be represented as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y(t)\\ \dot{y}(t)\\ z(t)\end{pmatrix} = \begin{bmatrix} 0 & I_m & 0\\ A_5 - k_0G & A_6 & A_7\\ A_8 & 0 & A_9\end{bmatrix}\begin{pmatrix} y(t)\\ \dot{y}(t)\\ z(t)\end{pmatrix} - \frac{k_1}{h}\begin{bmatrix}0\\ G\\ 0\end{bmatrix}(y(t) - y(t-h)). \tag{3.17}$$

Define

$$\Delta_s := \begin{bmatrix} 0 & G & 0 \\ 0 & G & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Delta := T \begin{bmatrix} 0 & I_m & 0 \\ A_5 - k_0 G & A_6 & A_7 \\ A_8 & 0 & A_9 \end{bmatrix} T^{-1}. \tag{3.18}$$

Applying the coordinate transformation (3.7) to (3.17) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi(t) = \Delta\xi(t) - \frac{k_1}{h}\Delta_s \begin{pmatrix} 0_{m\times 1} \\ y(t) - y(t-h) \\ 0_{(n-2m)\times 1} \end{pmatrix}, \quad \xi(t) = \begin{pmatrix} w(t) \\ v(t) \\ z(t) \end{pmatrix}. \tag{3.19}$$

To prove exponential stability of (3.17), equivalently (3.19), we will apply Razumikhin's Theorem as given in [2, Theorem 5.4.2] to the Lyapunov-function candidate

$$\xi \mapsto W(\xi) := \xi^{\mathrm{T}} R \xi \quad \forall \xi \in \mathbb{R}^n, \text{ for } R \text{ as in (3.10).}$$

Condition (4.2) in [2, p. 152] is obvious.

Henceforth denote, for arbitrary but fixed initial conditions $\phi \in C([-h,0];\mathbb{R}^n)$, the solution of (3.17), (3.19) by $t \mapsto (y(t), \dot{y}(t), z(t))$, $t \mapsto \xi(t)$, respectively.

To apply [2, Theorem 5.4.2], it suffices to consider differentiation of $t \mapsto W(\xi(t))$ along the solution merely at points t belonging to

$$\mathscr{T} := \{t \geqslant 0 \,|\, \xi(t+s)^{\mathrm{T}} R \xi(t+s) \leqslant 2\xi(t)^{\mathrm{T}} R \xi(t) \ \forall s \in [-h, 0]\}. \tag{3.20}$$

We will show that there exists $c > 0$ such that

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t} W(t)\right|_{(3.19)} \leqslant & -\left(\frac{k_0}{k_1}c^{-1} - c\right)\|w(t)\|^2 - \left(k_1 - \frac{k_0}{k_1}c - c\right)\|v(t)\|^2 \\ & -\tfrac{1}{2}\|z(t)\|^2 + hck_1(1 + k_0 + k_1)^2\|\xi(t)\|^2 \quad \forall t \in \mathscr{T}. \end{aligned} \tag{3.21}$$

If (3.21) holds, then for all k_0, k_1, h satisfying (2.3b) exponential stability of (3.17) (respectively (1.1), (1.5)) will follow from [2, Theorem 5.4.2]. The remainder of the proof establishes (3.21).

Define, for all $i = 1, \ldots, m$,

$$\Delta_i := \left[\begin{array}{c|c|c} 0_{m\times(m+i-1)} & \begin{matrix} g_{1i} \\ \vdots \\ g_{mi} \end{matrix} & 0_{m\times(n-m-i)} \\ \hline 0_{m\times(m+i-1)} & \begin{matrix} g_{1i} \\ \vdots \\ g_{mi} \end{matrix} & 0_{n\times(n-m-i)} \\ \hline 0_{(n-2m)\times(m+i-1)} & 0_{(n-2m)\times 1} & 0_{(n-2m)\times(n-m-i)} \end{array}\right], \tag{3.22}$$

which gives $\Delta_s = \sum_{i=1}^m \Delta_i$. Note that

$$\Delta \overset{(3.18)}{=} \begin{bmatrix} k_1^{-1}A_5G^{-1} - \dfrac{k_0}{k_1}I & \dfrac{k_0}{k_1}I + A_6 - k_1^{-1}A_5G^{-1} + k_1G & A_7 \\ k_1^{-1}A_5G^{-1} - \dfrac{k_0}{k_1}I & \dfrac{k_0}{k_1}I + A_6 - k_1^{-1}A_5G^{-1} & A_7 \\ k_1^{-1}A_8G^{-1} & -\dfrac{1}{k_1}A_8G^{-1} & A_9 \end{bmatrix}$$

$$= \begin{bmatrix} k_1^{-1}A_5G^{-1} - \dfrac{k_0}{k_1}I & \dfrac{k_0}{k_1}I + A_6 - k_1^{-1}A_5G^{-1} & A_7 \\ k_1^{-1}A_5G^{-1} - \dfrac{k_0}{k_1}I & \dfrac{k_0}{k_1}I + A_6 - k_1^{-1}A_5G^{-1} - k_1G & A_7 \\ k_1^{-1}A_8G^{-1} & -k_1^{-1}A_8G^{-1} & A_9 \end{bmatrix} + k_1\Delta_s$$

$$\overset{(3.8)}{=} \hat{\Delta} + k_1\Delta_s = \hat{\Delta} + k_1\sum_{i=1}^m \Delta_i. \tag{3.23}$$

Note further the existence of some $c_3 > 0$ such that

$$\|\Delta\| \leqslant c_3[1 + k_0 + k_1] \quad \forall k_0 \geqslant 0,\ k_1 \geqslant 1. \tag{3.24}$$

Since each component of $y(\cdot)$ in the solution to (3.17) is differentiable, we may apply the Mean Value Theorem and obtain mappings

$$[0,\infty) \to (0,h),\ t \mapsto h_t^i \quad \text{such that} \quad \frac{y_i(t) - y_i(t-h)}{h} = \dot{y}_i(t - h_t^i) \quad \forall i = 1,\ldots,m. \tag{3.25}$$

Notice that each Δ_i is defined precisely to pick out the ith component so that we may write

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi(t) \overset{(3.19)}{=} \Delta\xi(t) - k_1\sum_{i=1}^m \Delta_i \begin{pmatrix} 0_{m\times 1} \\ \dot{y}_1(t - h_t^1) \\ \vdots \\ \dot{y}_m(t - h_t^m) \\ 0_{(n-2m)\times 1} \end{pmatrix}$$

$$= \Delta\xi(t) - k_1\sum_{i=1}^m \Delta_i\xi(t - h_t^i) \tag{3.26}$$

$$\overset{(3.23)}{=} (\Delta - k_1\Delta_s)\xi(t) + k_1\sum_{i=1}^m \Delta_i(\xi(t) - \xi(t - h_t^i))$$

$$\overset{(3.23)}{=} \hat{\Delta}\xi(t) + k_1\sum_{i=1}^m \Delta_i \int_{-h_t^i}^0 \dot{\xi}(t+s)\,\mathrm{d}s$$

$$\overset{(3.26)}{=} \hat{\Delta}\xi(t) + k_1\sum_{i=1}^m \Delta_i \int_{-h_t^i}^0 \left[\Delta\xi(t+s) - k_1\sum_{j=1}^m \Delta_j\xi(t - h_t^j + s)\right]\mathrm{d}s. \tag{3.27}$$

Note, that the delay differential equations (3.26), etc. are formulated in m time-varying delays $t \mapsto h_t^i$, where nothing has been said about the regularity of these maps. However, existence of the solution to (3.19) has already been established, and in (3.26), etc. we have only rewritten the right-hand side of (3.19) in a form convenient for the ensuing estimates of the Lyapunov function $W(\cdot)$.

Differentiation of $t \mapsto W(\xi(t))$ along solutions of (3.19), yields

$$\frac{\mathrm{d}}{\mathrm{d}t} W(t)\bigg|_{(3.19)} \overset{(3.27)}{=} \xi(t)^{\mathrm{T}}[\hat{\Delta}^{\mathrm{T}}R + R\hat{\Delta}]\xi(t) + 2k_1 \xi(t)^{\mathrm{T}} R \sum_{i=1}^{m} \Delta_i \int_{-h_t^i}^{0} \left[\Delta \xi(t+s) - k_1 \sum_{j=1}^{m} \Delta_j \xi(t - h_t^j + s) \right] \mathrm{d}s$$

$$= \xi(t)^{\mathrm{T}}[\hat{\Delta}^{\mathrm{T}}R + R\hat{\Delta}]\xi(t) + 2k_1 \sum_{i=1}^{m} \int_{-h_t^i}^{0} \xi(t)^{\mathrm{T}} R \Delta_i \Delta \xi(t+s)\,\mathrm{d}s$$

$$-2k_1^2 \sum_{i=1}^{m} \Delta_i \int_{-h_t^i}^{0} \sum_{j=1}^{m} \xi(t)^{\mathrm{T}} R \Delta_j \xi(t - h_t^j + s)\,\mathrm{d}s \quad \forall t \geqslant 0. \tag{3.28}$$

First, define $c_4 := \|R\|^2 \|R^{-1}\| \|G\|^2 + \|R\|$, and estimate, for all $t \in \mathscr{T}$,

$$2k_1 \sum_{i=1}^{m} \int_{-h_t^i}^{0} \xi(t)^{\mathrm{T}} R \Delta_i \Delta \xi(t+s)\,\mathrm{d}s \overset{(3.1)}{\leqslant} k_1 \sum_{i=1}^{m} \int_{-h_t^i}^{0} [\xi(t+s)^{\mathrm{T}} R \xi(t+s) + \xi(t)^{\mathrm{T}} R \Delta_i \Delta R^{-1} \Delta^{\mathrm{T}} \Delta_i^{\mathrm{T}} R \xi(t)]\,\mathrm{d}s$$

$$\leqslant k_1 m \int_{-h}^{0} \xi(t+s)^{\mathrm{T}} R \xi(t+s)\,\mathrm{d}s + hk_1 \sum_{i=1}^{m} \xi(t)^{\mathrm{T}} R \Delta_i \Delta R^{-1} \Delta^{\mathrm{T}} \Delta_i^{\mathrm{T}} R \xi(t)$$

$$\overset{(3.20)}{\leqslant} 2hk_1 m \|\xi(t)\|^2 \|R\| + hk_1 m \|\xi(t)\|^2 c_4 \|\Delta\|^2$$

$$\overset{(3.24)}{\leqslant} hk_1 m c_4 (2 + c_3^2 [1 + k_0 + k_1]^2) \|\xi(t)\|^2.$$

Secondly define $c_5 := \|R\|^2 \|R^{-1}\| \|G\|^4$ and estimate, for all $t \in \mathscr{T}$,

$$-2k_1^2 \sum_{i=1}^{m} \int_{-h_t^i}^{0} \sum_{j=1}^{m} \xi(t)^{\mathrm{T}} R \Delta_i \Delta_j \xi(t - h_t^j + s)\,\mathrm{d}s$$

$$\overset{(3.1)}{\leqslant} k_1^2 \sum_{i=1}^{m} \int_{-h_t^i}^{0} \sum_{j=1}^{m} [\xi(t - h_t^j + s)^{\mathrm{T}} R \xi(t - h_t^j + s) + \xi(t)^{\mathrm{T}} R \Delta_i \Delta_j R^{-1} \Delta_j^{\mathrm{T}} \Delta_i^{\mathrm{T}} R^{\mathrm{T}} \xi(t)]\,\mathrm{d}s$$

$$\leqslant k_1^2 \sum_{i=1}^{m} \sum_{j=1}^{m} \int_{-h}^{0} [\xi(t - h_t^j + s)^{\mathrm{T}} R \xi(t - h_t^j + s)]\,\mathrm{d}s + hk_1^2 \sum_{i=1}^{m} \sum_{j=1}^{m} \xi(t)^{\mathrm{T}} R (\Delta_i \Delta_j R^{-1} \Delta_j^{\mathrm{T}} \Delta_i^{\mathrm{T}}) R^{\mathrm{T}} \xi(t)$$

$$\overset{(3.20)}{\leqslant} hk_1^2 m^2 (4 + c_5) \|\xi(t)\|^2.$$

Thirdly we take c_1 as in (3.11), c_2 as in (3.12) and define

$$c := \max\{c_1, c_2, mc_3^2 c_4, 2mc_4, m^2(4 + c_5), \|P\|, \|P^{-1}\|\}.$$

Estimate, for all $t \geqslant 0$,

$$\xi(t)^{\mathrm{T}}[\hat{\Delta}^{\mathrm{T}}R + R\hat{\Delta}]\xi(t) \overset{(3.13)}{\leqslant} -\left(\frac{k_0}{k_1} c^{-1} - c\right) \|w(t)\|^2 - \left(k_1 - \frac{k_0}{k_1} c - c\right) \|v(t)\|^2 - \tfrac{1}{2} \|z(t)\|^2.$$

Hence (3.28) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t)\bigg|_{(3.19)} \leqslant -\left(\frac{k_0}{k_1}c^{-1}-c\right)\|w(t)\|^2-\left(k_1-\frac{k_0}{k_1}c-c\right)\|v(t)\|^2-\tfrac{1}{2}\|z(t)\|^2$$
$$+hk_1m[2c_4+c_3^2c_4(1+k_0+k_1)^2+k_1m(4+c_5)]\|\xi(t)\|^2 \quad \forall t\in\mathcal{T},$$

and (3.21) follows. The proof of the theorem is therefore complete. □

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