A behavioural approach to time-varying linear systems

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Abstract

We develop a behavioural approach to linear, time-varying, differential algebraic systems. The analysis is "almost everywhere" in the sense that the statements hold on $\mathbb{R} \subset \mathbb{T}$, where $\mathbb{T}$ is a discrete set. Controllability, observability and autonomy is introduced and related to the behaviour of the system. Classical results on the behaviour of time-invariant systems are studied in the context of time-varying systems.

Keywords: Time-varying linear systems, behavioural approach, controllability, observability, autonomous system, adjoint system

Nomenclature

$\mathcal{A}$	the set of real analytic functions $f : \mathbb{R} \to \mathbb{R}$
$\mathcal{M}$	the vector space of real meromorphic functions $f : \mathbb{R} \to \mathbb{R}$
$\mathcal{A}[D]$, $\mathcal{M}[D]$	the skew polynomial ring of differential polynomials with coefficients in $\mathcal{A}, \mathcal{M}$ resp., indeterminate D, and multiplication rule $Df = fD + \dot{f}$
$\mathcal{C}^\infty(\mathbb{I}, \mathbb{R}^q)$	the real vector space of infinitely many times differentiable functions $f : \mathbb{I} \to \mathbb{R}^q$, $\mathbb{I} \subset \mathbb{R}$ an open interval
$\mathcal{C}^\omega(\mathbb{I}, \mathbb{R}^q)$	the real vector space of real analytic functions $f : \mathbb{I} \to \mathbb{R}^q$, $\mathbb{I} \subset \mathbb{R}$ an open interval
I_d	$:= \mathrm{diag}\{1, \ldots, 1\} \in \mathbb{R}^{d \times d}$
0_d	$:= (0, \ldots, 0)^T \in \mathbb{R}^d$
$\ker_{\mathcal{W}} r(\frac{\mathrm{d}}{\mathrm{d}t})$	$:= \left\{ w \in \mathcal{W} \,\middle|\, r(\frac{\mathrm{d}}{\mathrm{d}t})w = 0 \right\}$, for $r(D) \in \mathcal{M}[D]$ and $\mathcal{W}$ a suitable solution space to be specified.

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1 Introduction

Systems of differential algebraic equations play an important role in modelling multi-body systems, electric circuits, or coupled systems of partial differential equations, see [1, 9]. As an example, consider a simplified, linearized model of a two-dimensional, three-link constrained mobile manipulator [11] as depicted in Figure 1.

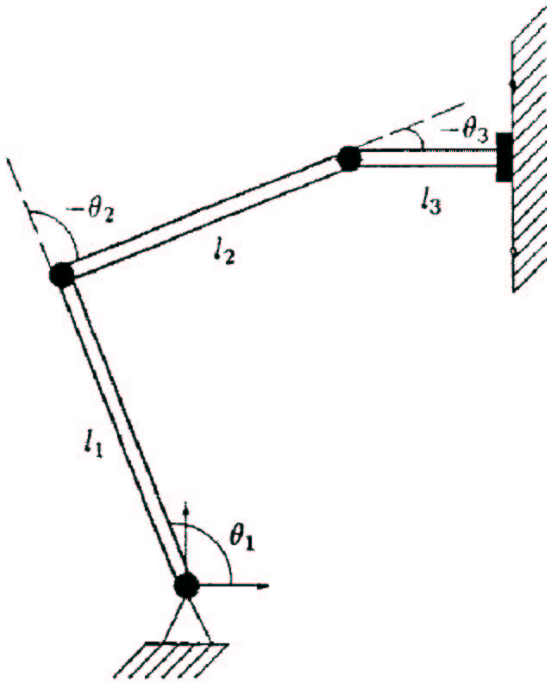

Figure 1: Three-link constrained mobile manipulator

The Lagrangian equations of motion take the form

$$\begin{aligned} M(\theta)\ddot{\theta} + D(\theta, \dot{\theta}) + K(\theta) &= u + F^T\mu, \\ \psi(\theta) &= 0, \end{aligned} \tag{1.1}$$

where $\theta = [\theta_1, \theta_2, \theta_3]^T$ is the vector of joint displacements, $u \in \mathbb{R}^3$ is the vector of control torques applied at the joints, the maps $M : \mathbb{R}^3 \to \mathbb{R}^{3\times3}$, $D : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^{3\times3}$, $K : \mathbb{R}^3 \to \mathbb{R}^3$ model the mass, centrifugal and Coriolis forces, gravity, respectively. $l_1, l_2, l_3, l > 0$ are the lengths of the robot arms, the constraint function is

$$\psi : \mathbb{R}^3 \times \mathbb{R}^2, \quad [\theta_1, \theta_2, \theta_3] \mapsto \begin{bmatrix} l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3)l_3 - l \\ \theta_1+\theta_2+\theta_3 \end{bmatrix}.$$

$F = \frac{\partial\psi}{\partial\theta}$, and $\mu \in \mathbb{R}^2$ represents the Lagrange multipliers and $F^T\mu$ is the generalized constraint force. Under suitable smoothness assumptions of the involved functions, it can be shown (see for example [22, p. 62]) that there exists a local (possibly global) solution $\theta(\cdot)$ of (1.1) on some open interval $\mathbb{I}$. Linearizing along this trajectory [3] and rewriting the system in Cartesian coordinates yields a model of the form

$$\begin{aligned} M_0(t)\,\ddot{z}(t) + D_0(t)\,\dot{z}(t) + K_0(t)\,z(t) &= S_0\,u(t) + F_0^T\,\mu \\ F_0\,z(t) &= 0\,, \end{aligned}$$

where $M_0, D_0, K_0 \in \mathcal{C}^\omega(\mathbb{I}, \mathbb{R}^{3\times 3})$ and $S_0, F_0^T \in \mathbb{R}^{3\times 2}$ with S_0 having full row rank. Introducing the 8 dimensional variable $x(t) = [z(t), \dot{z}(t), \mu(t)]$ results in the equivalent descriptor system description of the form

$$\begin{aligned} E(t)\,\tfrac{\mathrm{d}}{\mathrm{d}t}x(t) &= A(t)x(t) \quad + \quad B(t)u(t), \\ y(t) &= C(t)x(t)\,, \end{aligned} \tag{1.2}$$

where

$$E(t) := \begin{bmatrix} I_3 & 0 & 0 \\ 0 & M_0(t) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A(t) := \begin{bmatrix} 0 & I_3 & 0 \\ -K_0(t) & -D_0(t) & F_0^T \\ F_0 & 0 & 0 \end{bmatrix}, \quad B(t) := \begin{bmatrix} 0 \\ S_0 \\ 0 \end{bmatrix},$$

and $C(\cdot)$ denotes the output matrix with appropriate format, see [11] for explicit data.

The aim of the present paper is to develop a behavioural approach to linear time-varying systems described by differential-algebraic equations of the form

$$R(\tfrac{\mathrm{d}}{\mathrm{d}t})w = 0, \tag{1.3}$$

where $R(D)$ is a $g\times q$ polynomial matrix in the indeterminate D with real meromorphic coefficient matrices belonging to $\mathcal{M}^{g\times q}$; we use the notation $R(D) \in \mathcal{M}[D]^{g\times q}$.

Systems of the form (1.2) are covered by (1.3). Instead of considering the real analytic coefficients of $R(D)$ on the whole time axis $\mathbb{R}$, we also could develop the theory on some open interval $\mathbb{I} \subset \mathbb{R}$, this is omitted.

As D stands for the ordinary differential operator $\frac{\mathrm{d}}{\mathrm{d}t}$, the ring $\mathcal{M}[D]$ is endowed with the multiplication rule

$$Df = fD + \tfrac{\mathrm{d}}{\mathrm{d}t}f. \tag{1.4}$$

This is a consequence of assuming the associative rule $(Df)g = D(fg)$, which yields $(Df)(g) = \frac{\mathrm{d}}{\mathrm{d}t}f \cdot g + f \cdot \frac{\mathrm{d}}{\mathrm{d}t}g = \big(\frac{\mathrm{d}}{\mathrm{d}t}f \,+\, fD\big)\,(g)$. The non-commutativity of $\mathcal{M}[D]$, in contrast to the commutative ring $\mathbb{R}[D]$ in the time-invariant case, is crucial in the following.

Note that we distinguish between the algebraic indeterminate D and the differential operator $\frac{\mathrm{d}}{\mathrm{d}t}$; for

$$R(D) = \sum_{i=0}^{n} R_i D^i \in \mathcal{M}[D]^{g\times q} \cong \mathcal{M}^{g\times q}[D],$$

equality in (1.3) means

$$\sum_{i=0}^{n} R_i(t) w^{(i)}(t) = 0 \qquad\qquad \text{for almost all } t \in \mathbb{R}.$$

We consider local solutions belonging to

$$\mathcal{C}_t^\infty(\mathbb{R}^q) \ := \ \big\{ w \in \mathcal{C}^\infty(\mathbb{I}, \mathbb{R}^q) \big| \, I \subset \mathbb{R} \text{ an open interval with } t \in I \big\} \qquad t \in \mathbb{R}\,. \tag{1.5}$$

Our approach generalizes results on the following sub-classes of systems.

(a) Time-varying state space systems of the form

$$\begin{array}{rcl} \frac{\mathrm{d}}{\mathrm{d}t}x(t) & = & A(t)x(t) \ + \ B(t)u(t), \\ y(t) & = & C(t)x(t) \ + \ F(t)u(t), \end{array} \tag{1.6}$$

with real analytic matrices $A \in \mathcal{A}^{n\times n}$, $B \in \mathcal{A}^{n\times m}$, $C \in \mathcal{A}^{p\times n}$ and $F \in \mathcal{A}^{p\times m}$, are well studied, see for example the standard monograph [26].

(b) Time-varying descriptor systems of the form

$$\begin{array}{rcl} E(t)\, \frac{\mathrm{d}}{\mathrm{d}t}x(t) & = & A(t)x(t) \ + \ B(t)u(t), \\ y(t) & = & C(t)x(t) \ + \ F(t)u(t), \end{array} \tag{1.7}$$

with $A \in \mathcal{A}^{\ell\times n}$, $B \in \mathcal{A}^{\ell\times m}$, $C \in \mathcal{A}^{p\times n}$, $F \in \mathcal{A}^{p\times m}$, where $E \in \mathcal{A}^{\ell\times n}$ is allowed to be singular in the sense that $\mathrm{rk}\ E(t) < \min\{l, n\}$ for some $t \in \mathbb{R}$, have been studied by different authors. In [4] controllability and observability has been studied in terms of derivative arrays. In [2] a first behaviour like approach to systems (1.7) with analytic coefficients has been discussed. A more general approach that allows for larger classes of coefficients and that can be implemented also numerically has been introduced in [17] and generalized partially to the nonlinear case in [16].

(c) In [13] time-varying polynomial systems of the form

$$\begin{array}{rcl} P(\frac{\mathrm{d}}{\mathrm{d}t})z(t) & = & Q(\frac{\mathrm{d}}{\mathrm{d}t})u(t), \\ y(t) & = & V(\frac{\mathrm{d}}{\mathrm{d}t})z(t) \ + \ W(\frac{\mathrm{d}}{\mathrm{d}t})u(t), \end{array} \tag{1.8}$$

where $P(D)$, $Q(D)$, $V(D)$ and $W(D)$ are matrices of size $r \times r$, $r \times m$, $p \times r$, $p \times m$, respectively, over $\mathcal{M}[D]$ are studied under the following assumptions:

- $P(D)$ represents a so called *full* operator, i.e. if z is a real analytic solution of $P(\frac{\mathrm{d}}{\mathrm{d}t})z = 0$ on some interval $\mathbb{I} \subset \mathbb{R}$, then this solution can be analytically extended to the whole of $\mathbb{R}$.
- For every $u \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ with bounded support to the left, there exist some $z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^r)$ and $y \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^p)$ so that (1.8) is satisfied.

Time-invariant polynomial (so called Rosenbrock) systems of the form (1.8), i.e. $P(D)$, $Q(D)$, $V(D)$ and $W(D)$ are matrices over $\mathbb{R}[D]$ and $\det P(\cdot) \neq 0$, were introduced in [24], and are well studied, see for example [10, 34].

(d) Time-invariant polynomial systems in the so called kernel representation

$$R(\tfrac{\mathrm{d}}{\mathrm{d}t})w(t) = 0, \qquad R(D) \in \mathbb{R}[D]^{g\times q} \tag{1.9}$$

have been introduced by Willems in [30]; see also[31, 32, 33] and the monograph [21].

Time-varying descriptor systems (1.7) or, if $E = I_n$ and $n = l$, state space systems (1.6) are a special case of time-varying Rosenbrock systems (1.8): set

$$R(D) = \begin{bmatrix} ED - A, & -B, & 0 \\ -C, & -F, & I_p \end{bmatrix}, \quad \text{and} \quad w = \left(x^T, u^T, y^T\right)^T. \tag{1.10}$$

Furthermore, time-varying Rosenbrock systems of the form (1.8) are a special case of systems in kernel representation (1.3): set $w = \left(x^T, u^T, y^T\right)^T$ and

$$R(D) = [R_1(D), R_2(D)], \qquad R_1(D) = \begin{bmatrix} P(D) \\ V(D) \end{bmatrix}, \quad R_2(D) = \begin{bmatrix} -Q(D), & 0 \\ W(D), & -I_p \end{bmatrix}. \tag{1.11}$$

In the following, we present some prototypical scalar differential equations which illustrate how time-varying coefficients may effect the solutions in very different ways.

Example 1.1

(i) Consider $r(D) = tD + 1$. The function $t \mapsto w(t) = t^{-1}$ is a meromorphic solution of $r(\frac{\mathrm{d}}{\mathrm{d}t})w = t\frac{\mathrm{d}}{\mathrm{d}t}w + w = 0$. The point 0 is the only zero of the leading coefficient $t \mapsto t$ of $r(D)$, and 0 is also a pole of $t \mapsto w(t)$. Therefore,

$$\ker_{\mathcal{A}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) = \ker_{\mathcal{C}^\infty(\mathbb{R},\mathbb{R})} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) = \{0\},$$

but, for every interval $\mathbb{I} \subset \mathbb{R}$ with $0 \notin \mathbb{I}$,

$$\dim\ \ker_{\mathcal{M}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) = \dim\ \ker_{\mathcal{A}_{|_{\mathbb{I}}}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) = 1 = \deg r(D).$$

In this example, in the meromorphic case the dimension of the solution space equals the degree of $r(D)$. This is not true in general as illustrated by the following example.

(ii) Consider $r(D) = t^2D + 1$. The function $t \mapsto w(t) = e^{1/t}$ solves $r(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. The point 0 is again the only zero of the leading coefficient $t \mapsto t^2$ of $r(D)$, and 0 is also a pole of $t \mapsto w(t)$. But w is not meromorphic and the singularity at $t = 0$ differs from (i) as follows: No matter whether the solution w in (i) approaches 0 from the left or right, the limit at $t = 0$ does not exist; whereas, for the solution w in the present example, we have $\lim_{t\to 0-} w(t) = 0$ and $\lim_{t\to 0+} w(t) = \infty$. Hence,

$$\ker_{\mathcal{M}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) = \{0\}.$$

For every interval $\mathbb{I} \subset \mathbb{R}$ with $0 \notin \mathbb{I}$ we have

$$\dim\ \ker_{\mathcal{M}_{|_{\mathbb{I}}}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) = 1 = \deg r(D).$$

(iii) Consider $r(D) = tD - 1$. The function $t \mapsto w(t) = t$ solves $r(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ and

$$\dim\ \ker_{\mathcal{A}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) = 1 = \deg r(D).$$

Note that again the point $t = 0$ is the only zero of the leading coefficient $t \mapsto t$ of $r(D)$, but this time the zero does not produce a pole of the solution, the solution w is even a real analytic function on $\mathbb{R}$. However, the solution is not as arbitrary as for time-invariant systems, since $w(0) = 0$ is the only value at $t = 0$.

(iv) Consider $r(D) = 2tD - 1$. The functions $t \mapsto w_+(t) = \sqrt{t}$ and $t \mapsto w_-(t) = \sqrt{-t}$ solve $r(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ on $(0,\infty)$, $(-\infty, 0)$, respectively. For every interval $\mathbb{I} \subset \mathbb{R}$ with $0 \notin \mathbb{I}$, we have

$$\dim \ \ker_{\mathcal{A}_{|_{\mathbb{I}}}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) = 1 = \ \deg r(D).$$

However,

$$\ker_{\mathcal{M}} r(\tfrac{\mathrm{d}}{\mathrm{d}t}) = \{0\}.$$

The real analytic solution w_+ on $(0,\infty)$ cannot be continued to $(-\varepsilon, \infty)$ for any $\varepsilon > 0$. This also proves that the attempt to connect real analytic solutions between critical points by cutting the neighbourhood and going into the complex sphere, as suggested by Ilchmann et al. [12], does not work. * □

Our results are related to the following results in the literature. In [7], a behavioural approach for time-varying linear systems of the form

$$\begin{array}{rclcl} \frac{\mathrm{d}}{\mathrm{d}t}x(t) & = & A(t)x(t) & + & B(t)u(t), \\ y(t) & = & C(t)x(t) & + & F(t)u(t), \end{array} \tag{1.12}$$

has been introduced, where $\mathbb{I} \subset \mathbb{R}$ is an open interval and the entries of the time-varying coefficient matrices $R_i(\cdot)$ are rational analytic functions

$$\frac{f(\cdot)}{g(\cdot)}, \quad f, g \in \mathbb{C}[t] \qquad \text{and} \quad g(t) \neq 0 \text{ for all } t \in \mathbb{I}.$$

The solution space $\mathcal{B}$ for w solving (1.12) is allowed to be the space of Sato's hyperfunctions on $\mathbb{I}$; $\mathcal{B}$ is also called the behaviour of $R(D)$. One motivation for the rather large solution space of hyperfunctions is that a categorial one-to-one correspondence between behaviours and finitely generated modules over a suitable ring of differential operators can be proved, see [19]. Hyperfunctions are generalized distributions and capture the singularities. The dimension of the solution space of an ordinary differential equation is the degree plus order of the singularities. However, in [7] these zeros are excluded and the analysis is restricted to intervals $\mathbb{I}$ where $g(t) \neq 0$.

Time-varying Rosenbrock systems of the form (1.10) have been introduced and studied in [13]. The solution space is the set of $\mathcal{C}^\infty$-functions on the whole time axis, but this is ensured by the assumption that $\operatorname{im} Q(\frac{\mathrm{d}}{\mathrm{d}t}) \subset \operatorname{im} P(\frac{\mathrm{d}}{\mathrm{d}t})$ and, most importantly, that $P(D)$ is a "full" operator, i.e. every local analytic solution of $P(\frac{\mathrm{d}}{\mathrm{d}t})z = 0$ is extendable to a global analytic solution on the whole of $\mathbb{R}$.

In [6] matrices over the ring of linear differential operators $k[D]$ are considered, where k is a differential field. Linear dynamics are finitely generated left $k[D]$-modules. This contribution is rather on the algebraic side, the solution space is not specified.

In [27] contributions to duality of systems in the set-up of [6] for systems in generalized state space representation are given, however the solution space is not specified.

Important early contributions to time-varying systems in polynomial descriptions are given in [36, 37, 14]; however, the assumptions on the system classes are rather restrictive.

*We are indebted to the anonymous referee of an earlier version of the present paper for pointing out this example to us.

In [12] a first approach in the spirit of the present paper is presented for scalar systems.

A completely different approach results from the study of differential-algebraic equations, see [1, 8]. A general solvability theory for nonsquare linear time-varying systems was first given in [15] and analysed for control problems in a behavioural context in [2, 17, 23], see also [16] for the general nonlinear case.

This paper is organized as follows. In Section 2, the concept of local behaviour is defined and the Teichmüller-Nakayama normal form is cited and certain consequences for the present system class are shown. In Section 3, we introduce and characterise algebraically the concept of controllability for the kernel and image representation. The relationships between controllability, behaviour and autonomous behaviour are studied in Section 4. Finally, in Section 5, observability is defined, it is related via the adjoint of the kernel representation to the controllable behaviour, and it is characterized algebraically. We conclude with a summary and some direction of future research.

2 Behaviour

In this section we introduce the concept of local controllability as a generalization of the behaviour concept introduced by Willems [30], see also [21]. The Teichmüller-Nakayama normal form is then the main tool to characterize controllability algebraically and to show that it is a generalization of well known controllability concepts for less general system classes.

Definition 2.1 For $R(D) \in \mathcal{M}[D]^{g\times q}$, the *local behaviour* of the system $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ at $t \in \mathbb{R}$ is defined as

$$\mathfrak{B}_R^{\ker}(t) \;:=\; \left\{ w \in \mathcal{C}_t^{\infty}(\mathbb{R}^q) \,\middle|\, R(\tfrac{\mathrm{d}}{\mathrm{d}t})w(t) = 0 \;\forall t \in \operatorname{dom} w \right\}. \tag{2.1}$$

The set $\mathfrak{B}_R^{\ker}(t)$ becomes a *real vector space* if endowed, for $w_1, w_2 \in \mathfrak{B}_R^{\ker}(t)$, with addition

$$(w_1 + w_2)(\tau) \;:=\; w_1(\tau) + w_2(\tau) \qquad \forall \tau \in \operatorname{dom} w_1 \cap \operatorname{dom} w_2\,,$$

and obvious scalar multiplication. The dimension of this vector space is defined as

$$\dim \mathfrak{B}_R^{\ker}(t) \;:=\; \sup \left\{ k \in \mathbb{N} \,\middle|\, \exists\, w_1, \ldots, w_k \in \mathfrak{B}_R^{\ker}(t) \text{ linearly independent on } \bigcap_{i=1}^{k} \operatorname{dom} w_i \right\}.$$

□

Crucial for the analysis of the local behaviour is the following Teichmüller-Nakayama normal form. To this end we recall some results on matrices over the skew polynomial ring $\mathcal{M}[D]$; a standard reference for this is [5]. $\mathcal{M}[D]$ is *simple*, i.e., the only ideals which are right and left ideals at the same time are the trivial ones; the rank of a matrix over $\mathcal{M}[D]$ is unambiguous, since column rank and row rank coincide; the Teichmüller-Nakayama normal form is the analogue of the Smith normal form for matrices over the commutative ring $\mathbb{R}[D]$, it is simpler for matrices over $\mathcal{M}[D]$, since the class of transformations is larger. $W(D) \in \mathcal{M}[D]^{n\times n}$ is called *unimodular* if, and only if, there exists some $W(D)^{-1} \in \mathcal{M}[D]^{n\times n}$ such that $W(D)\,W(D)^{-1} = I_n$; two elements $q_1, q_2 \in \mathcal{M}[D]$ are *similar* if, and only if, $q_1 a = b q_2$ for some $a, b \in \mathcal{M}[D]$ for which q_1 and b (q_2 and a) are called left (right) coprime, respectively.

Lemma 2.2 (*Teichmüller-Nakayama normal form*)
Any $R(D) \in \mathcal{M}[D]^{g\times q}$ with $\mathrm{rk}_{\mathcal{M}[D]} R(D) = l$ can be factorized into

$$R(D) \;=\; U(D)^{-1}\begin{bmatrix} I_{l-1} & 0 & 0 \\ 0 & r(D) & 0 \\ 0 & 0 & 0_{(q-l)\times(q-l)} \end{bmatrix} V(D)^{-1}, \tag{2.2}$$

where $U(D)$ and $V(D)$ are $\mathcal{M}[D]$-unimodular matrices of sizes g and q, respectively, and $r(D) \in \mathcal{M}[D]$ is non-zero, unique up to similarity, and of unique degree.

Proof: A proof and an interesting historical description of the development of the normal form (2.2) can be found in [5, Ch. 8]. In [20] it is shown that the degree of similar polynomials is equal. $\square$

Remark 2.3 Let $R(D) \in \mathcal{M}[D]^{g\times q}$ and consider the factorization (2.2). Let $\mathbb{T} = \mathbb{T}(R, U, V, r)$ denote the union of all zeros and poles of the meromorphic coefficients in all entries of $U(D)$, $U(D)^{-1}$, $V(D)$, $V(D)^{-1}$, and $r(D)$.

(i) Certainly, $\mathbb{T}$ is a discrete set, and hence it follows that the local behaviour is non-trivial almost everywhere, i.e.

$$\left\{ t \in \mathbb{R} \,\middle|\, \mathfrak{B}_R^{\ker}(t) \neq \{0\} \right\} \qquad \text{is discrete.}$$

(ii) Furthermore,

$$\dim \mathfrak{B}_R^{\ker}(t) \;=\; \begin{cases} \deg r(D) & \text{for a.a. } t \in \mathbb{R}, \quad \text{if } \mathrm{rk}\, R(D) = q \\ \infty & \text{for all } t \in \mathbb{R}, \quad \text{if } \mathrm{rk}\, R(D) < q. \end{cases}$$

The latter is a simple consequence of (2.2) and the fact that the set of t where $r(\frac{\mathrm{d}}{\mathrm{d}t})\varphi(t) = 0$ does not have a solution, is a subset of $\{t \in \mathbb{R} \mid r_N(t) = 0\}$, where $r(D) = \sum_{i=0}^{N} r_i(t) D^i$, $r_N \not\equiv 0$. To see this use canonical transformation to a vector-valued differential equation of first order, see for example [29, Ch. IV].

(iii) Note that in the case of time-varying state space systems or time-invariant Rosenbrock systems the set $\mathbb{T}$ is empty, the system is defined on the whole time axis.

$\square$

Remark 2.4 Suppose that $R(D)$ has constant coefficients, i.e. $R(D) \in \mathbb{R}[D]^{g\times q}$. If the class of unimodular transformations for the computation of the normal form (2.2) is restricted to $\mathbb{R}[D]$-unimodular matrices, then we arrive at the Smith normal form

$$R(D) \;=\; U(D)^{-1}\begin{bmatrix} \mathrm{diag}\,\{r_1(D), \ldots, r_l(D)\} & 0_{l\times(q-l)} \\ 0_{(q-l)\times l} & 0_{(q-l)\times(q-l)} \end{bmatrix} V(D)^{-1}, \tag{2.3}$$

where $U(D)$ and $V(D)$ are $\mathbb{R}[D]$-unimodular matrices of sizes g and q, respectively, and $r_i(D) \in \mathbb{R}[D]$ are non-zero monic polynomials with $r_i | r_{i+1}$, $i = 1, ..., l-1$, where $l = \mathrm{rk}_{\mathbb{R}[D]} R(D)$ and $r_i(D) = \psi_i(D)/\psi_{i-1}(D)$, $\psi_0(\cdot) \equiv 1$ and $\psi_i(D)$ is the greatest common divisor of minors of order i of $R(D)$; see for example [25, pp. 91-93].

Note that due to the smaller class of transformations, the Smith nomral form is less simple than the Teichmüller-Nakayama normal form.

Suppose additionally that $\mathrm{rk}_{\mathbb{R}[D]} R(D) = q$. Then every local solution $w \in \mathcal{C}_t^N(\mathbb{R}^q)$ of $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, where N is sufficiently large depending on $\deg R(D)$ and the degrees of the transformation matrices, can be continued to a global solution on $\mathbb{R}$ and it is even real analytic. This follows immediately from the Smith normal form (2.3) and the theory of linear time-invariant differential equations. Therefore,

$$\mathfrak{B}_R^{\ker}(t) \;=\; \mathfrak{B}_R^{\ker}(s) \qquad \text{for all } t, s \in \mathbb{R}$$

and

$$\dim \mathfrak{B}_R^{\ker}(t) \;=\; \sum_{i=0}^{q} \deg r_i(D) \qquad \text{for all } t \in \mathbb{R}.$$

If $R(D)$ is factorized as in (2.2), then $\deg r(D) = \sum_{i=0}^{q} \deg r_i(D)$. □

Remark 2.5 Suppose that $R(D) \in \mathcal{M}[D]^{g\times q}$ is of full rank $g \le q$. Let $R(D)$ be factorized as in (2.2) and differently into

$$R(D) \;=\; \bar{U}(D)^{-1} \left[\begin{array}{cc|c} I_{g-1} & & \\ & \bar{r}(D) & 0_{g\times(q-g)} \end{array}\right] \bar{V}(D)^{-1}. \tag{2.4}$$

Then a simple algebraic manipulation shows that

$$\bar{V}(D)^{-1} V(D) = \begin{bmatrix} W_1(D) & 0 \\ W_3(D) & W_4(D) \end{bmatrix}, \tag{2.5}$$

where $W_1(D) \in \mathcal{M}[D]^{g\times g}$, $W_4(D) \in \mathcal{M}[D]^{(q-g)\times(q-g)}$ are unimodular and $W_3(D) \in \mathcal{M}[D]^{(q-g)\times g}$. □

3 Controllability

In this section we introduce, study and characterize the controllability of the system (1.3).

Definition 3.1 For $R(D) \in \mathcal{M}[D]^{g\times q}$, the system $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is called *locally controllable at* $t \in \mathbb{R}$ if, and only if, for every $w^1, w^2 \in \mathfrak{B}_R^{\ker}(t)$ and every $t_0 \in (-\infty, t) \cap \mathrm{dom}\ w^1$ there exist $t_1 \in \mathrm{dom}\ w^2 \cap (t, \infty)$ and $w \in \mathfrak{B}_R^{\ker}(t)$ such that

$$w(t) = \begin{cases} w^1(t), & t \in (-\infty, t_0] \cap \mathrm{dom}\ w^1 \\ w^2(t), & t \in [t_1, \infty) \cap \mathrm{dom}\ w^2. \end{cases}$$

$R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is called *controllable* if, and only if, it is locally controllable almost everywhere. □

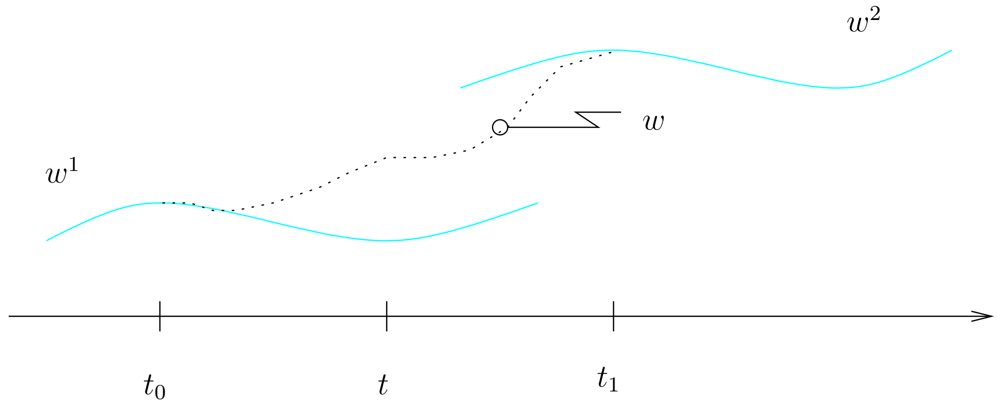

Figure 2: Local controllability at t

Remark 3.2

(i) Due to the linearity of the system $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, the trajectory w^2 in Definition 3.1 may be replaced, without restriction of generality, by $w^2 = 0$.

(ii) Loosely speaking, controllability means that any two trajectories $w^1, w^2 \in \mathfrak{B}_R^{\mathrm{ker}}(t)$ can be connected by another trajectory $w \in \mathfrak{B}_R^{\mathrm{ker}}(t)$ so that in finite time w^1 moves via w into w^2. A similar notion of controllability via trajectories was introduced in [10] for time-invariant Rosenbrock systems with $E = I$ of the form (1.10). For time-invariant state space systems of the form (1.6), the concept of controllability coincides with the one introduced in [21, Sect. 5.2]. $\square$

We are now in a position to prove the main theorem of this section which characterizes controllability in algebraic terms. Recall that $R(D)$ is called *right invertible* if, and only if, there exists some $R^{\#}(D) \in \mathcal{M}[D]^{q\times g}$ such that $R(D)R^{\#}(D) = I_g$.

Theorem 3.3 Let $R(D) \in \mathcal{M}[D]^{g\times q}$. Then the system $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is controllable if, and only if, $R(D)$ is right invertible.

Proof: Suppose that $R(D)$ is factorized as in (2.2) and let $\mathbb{T} = \mathbb{T}(R, U, V, r)$ denote the discrete set given in Remark 2.3. Then it remains to show that $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is locally controllable at $t \in \mathbb{R}\backslash\mathbb{T}$ if, and only if, $r(D)$ is a non-zero meromorphic function.

"$\Rightarrow$": Suppose that $\deg\ r(D) \geq 1$ and $t \in \mathbb{R}\backslash\mathbb{T}$. By [29, Ch. IV] there exists an open interval $\mathbb{I} \subset \mathbb{R}\backslash\mathbb{T}$ with $t \in \mathbb{I}$ and some non-zero real analytic solution $\varphi : \mathbb{I} \to \mathbb{R}$ which solves $r(\frac{\mathrm{d}}{\mathrm{d}t})\varphi = 0$. By the construction of $\mathbb{T}$ and letting e_g denote the g-th canonical basis vector in $\mathbb{R}^q$, it follows that

$$\hat{w}^1 \ := \ V(\tfrac{\mathrm{d}}{\mathrm{d}t})\,\varphi\, e_g \ \in \mathcal{C}^{\omega}(\mathbb{I}, \mathbb{R}^q)$$

and solves $R(\frac{\mathrm{d}}{\mathrm{d}t})\hat{w}^1 = 0$.

Seeking a contradiction, suppose that $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ were locally controllable at t. Let $t_0 \in$

$(-\infty, t) \cap \mathbb{I}$. Then there exist $t_1 \in (t, \infty)$ and $w \in \mathfrak{B}_R^{\ker}$ such that

$$w(t) = \begin{cases} w^1(t), & t \in (-\infty, t_0] \cap \operatorname{dom} w^1 \\ 0, & t \in [t_1, \infty). \end{cases} \tag{3.1}$$

Therefore,

$$\operatorname{diag}\{1, \dots, 1, r(\tfrac{\mathrm{d}}{\mathrm{d}t}), 0, \dots, 0\} V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1} w \;=\; 0 \qquad \forall\, t \in \operatorname{dom} w$$

which yields

$$V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1} w \;=:\; (0, \dots, 0, \varphi_g, \dots, \varphi_q)^T \in \mathcal{C}^\omega(\operatorname{dom} w, \mathbb{R}^q)$$

and $r(\frac{\mathrm{d}}{\mathrm{d}t})\varphi_g(t) = 0$ for all $t \in \operatorname{dom} w$. By (3.1) we have $\varphi_g(t) = 0$ for all $t \in [t_1, \infty)$, and since φ_g is real analytic, the identity property of real analytic functions gives $\varphi \equiv \varphi_g \equiv 0$, which is a contradiction.

"$\Leftarrow$": Let $t \in \mathbb{R}\backslash\mathbb{T}$, $r(D)$ meromorphic and non-zero, and $w^1 \in \mathfrak{B}_R^{\ker}(t)$.
Then there exists some open interval $\mathbb{I} := (\tau_0, \tau_1) \subset (\mathbb{R}\backslash\mathbb{T}) \cap \operatorname{dom} w^1$ with $t \in \mathbb{I}$ such that

$$w^1 \;=:\; V(\tfrac{\mathrm{d}}{\mathrm{d}t})\ (0, \dots, 0, \varphi_{g+1}, \dots, \varphi_q)^T \in \mathcal{C}^\infty(\mathbb{I}, \mathbb{R}^q).$$

Choose $\delta \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ such that

$$\delta(t) = \begin{cases} 1, & t \le \tau_0 \\ 0, & t \ge \tau_1. \end{cases}$$

Then

$$w \;:=\; V(\tfrac{\mathrm{d}}{\mathrm{d}t})\ \delta\ \ (0, \dots, 0, \varphi_{g+1}, \dots, \varphi_q)^T \in \mathcal{C}^\infty(\mathbb{I}, \mathbb{R}^q)$$

satisfies $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ and

$$w(t) = \begin{cases} w^1(t), & t \le \tau_0 \\ 0, & t \ge \tau_1. \end{cases}$$

This completes the proof. $\square$

For time-invariant systems (1.3), Theorem 3.3 is derived differently in [21, Th. 5.2.10].

Remark 3.4 For time-varying systems (1.6) or (1.8), it is well known that controllability of the system yields that it can be controlled in arbitrary short time. The proof of Theorem 3.3, in particular the choice of (τ_0, τ_1) and δ, shows that this is also valid for systems $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$: If $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is controllable , then $t_0 < t$ and $t_1 > t$ in Definition 3.1 can be replaced by any arbitrary close to t. $\square$

In the following remark we recall the classical concept of controllability for time-varying state space systems and clarify the set of admissible input functions.

Remark 3.5 Controllability for state space systems (1.6) means (see for example [28, Def. 3.1.6]), that for any $x^0, x^1 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$, there exist $t_1 > t_0$ and a continuous function $u : [t_0, t_1] \to \mathbb{R}^m$ such that

$$x(t) \;=\; (Lu)(t) \;:=\; \Phi(t,t_0)x^0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau\,, \qquad t \in [t_0,t_1]$$

satisfies $x(t_1) = x^1$. Here Φ denotes the transition matrix of the homogeneous system $\dot{x} = Ax$. Using the fact that the set of $\mathcal{C}^\infty$–functions with support in $[t_0,t_1]$ lies dense, with respect to the $\mathcal{L}^1$–norm, in the set of piecewise continuous functions with support included in $[t_0,t_1]$, it follows from a straightforward modification of the proof of Lemma A2 in [13] that, for all $t \in (t_0,t_1)$,

$$\left\{(Lu)(t)\middle|u \in \mathcal{C}^\infty((t_0,t_1),\mathbb{R}^m)\right\} \;=\; \left\{(Lu)(t)\middle|u : [t_0,t_1] \to \mathbb{R}^m \text{ piecew. cont. with } \operatorname{supp} u \subset [t_0,t_1]\right\}.$$

Therefore, although in the original definition u is required to be continuous, we may choose, without any restriction of generality, $u \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^m)$ with supp $u \subset [t_0,t_1]$. □

In the following Proposition 3.6, it is shown how controllability encompasses other definitions of controllability, well established in the literature.

Proposition 3.6 Consider a time-varying Rosenbrock system of the form (1.8) with corresponding $R(D)$ as defined in (1.11), suppose that $R(D)$ has full row rank. Then the following conditions are equivalent:

(i) $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is controllable.

(ii) $[P(D),\ -Q(D)]$ is right invertible.

(iii) $[P(\frac{\mathrm{d}}{\mathrm{d}t}),\ -Q(\frac{\mathrm{d}}{\mathrm{d}t})]w = 0$ is controllable.

(iv) (1.8) is controllable in the sense defined in [13].

(v) If $R(D)$ represents a time-invariant Rosenbrock system (1.8), then (1.8) is controllable in the sense defined in [10].

(vi) If $R(D)$ represents a state space system (1.6) with corresponding $R(D)$ as defined in (1.10), then (1.6) is controllable in the classical sense as, for example given in [28, Def. 3.1.6].

Proof: The equivalences '(i) ⇔ (ii) ⇔ (iii)' follow from Theorem 3.3 and simple algebraic manipulations; '(ii)⇔ (iv)' follows from [13, Th. 6.4]. '(ii) ⇔ (v)' follows from [10, Cor. 7.3]. It remains to prove that the classical concept of controllability as given in Remark 3.5 is encompassed in the behavioural setup. It is easy to see that (iii) implies (vi) and we omit the proof. To prove the converse, suppose that (vi) holds. Then for given

$$(x^i,u^i) \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^n) \times \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^m) \quad \text{such that} \quad \tfrac{\mathrm{d}}{\mathrm{d}t}x^i(t) = A(t)x^i(t) + B(t)u^i(t), \quad i = 1,2$$

and given $t_0 \in \mathbb{R}$, we need to find

$$(x,u) \;\in\; \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^n) \times \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^m), \quad \text{so that} \quad \tfrac{\mathrm{d}}{\mathrm{d}t}x(t) \;=\; A(t)x(t) + B(t)u(t),$$

and $t_1 > t_0$ such that

$$(x(t),u(t)) \;=\; \begin{cases} \left(x^1(t),u^1(t)\right), & \text{for all } t \leq t_0 \\ \left(x^2(t),u^2(t)\right), & \text{for all } t \geq t_1. \end{cases} \tag{3.2}$$

Let $\bar{x}^1 = x^1(t_0)$ and, for arbitrary but fixed $t_1 > t_0$ let $\bar{x}^2 = x^2(t_1)$. Then by (vi) we may choose $\widehat{u} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ with supp $\widehat{u} \subset [t_0, t_1]$ such that

$$x(t) \;=\; \Phi(t,t_0)\bar{x}_1 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)\widehat{u}(\tau)d\tau \qquad \text{satisfies} \quad x(t_2) = \bar{x}^2.$$

Define, for all $t \in \mathbb{R}$,

$$u(t) = \begin{cases} u^1(t), & \text{for all } t \le t_0 \\ \widehat{u}(t), & \text{for all } t \in (t_0, t_1) \\ u^2(t), & \text{for all } t \ge t_1 \end{cases} \qquad \text{and} \quad x(t) = \Phi(t,t_0)\bar{x}_1 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau.$$

Then (x, u) satisfies $\dot{x} = Ax + Bu$ and (3.2). The function u is in general not infinitely many times differentiable at t_0 or at t_1, but applying Remark 3.5, one may replace $\widehat{u}$ so that $u \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^m)$. This completes the proof. $\square$

Next we study, for $R(D) \in \mathcal{M}[D]^{g\times q}$, the relationship between the local kernel representation $\mathfrak{B}_R^{\ker}(t)$ of the system $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ and the *local image representation* at $t \in \mathbb{R}$, i.e. for some $M(D) \in \mathcal{M}[D]^{q\times m}$ the real vector space

$$\mathfrak{B}_M^{\mathrm{im}}(t) := \left\{ w \in \mathcal{C}_t^\infty(\mathbb{R}^q) \,\middle|\, \exists\, l \in \mathcal{C}_t^\infty(\mathbb{R}^m) \;\; \forall\, t \in \operatorname{dom} w \cap \operatorname{dom} l \,:\, w(t) = M(\tfrac{\mathrm{d}}{\mathrm{d}t})l(t) \right\}.$$

Proposition 3.7 For $R(D) \in \mathcal{M}[D]^{g\times q}$ we have that $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is controllable if, and only if, there exist $m \in \mathbb{N}$ and $M(D) \in \mathcal{M}[D]^{q\times m}$ such that $\mathfrak{B}_R^{\ker}(t) \;=\; \mathfrak{B}_M^{\mathrm{im}}(t)$ for almost all $t \in \mathbb{R}$.

Proof: Suppose $R(D)$ is factorized as in (2.2) and let $\mathbb{T}$ denote the discrete set given in Remark 2.3. By Theorem 3.3 it remains to show that $r(D)$ is a non-zero meromorphic function if, and only if, $\mathfrak{B}_R^{\ker}(t) = \mathfrak{B}_M^{\mathrm{im}}(t)$ for all $t \in \mathbb{R}\backslash\mathbb{T}$.

"$\Rightarrow$": Set

$$M(D) \;:=\; V(D)\begin{bmatrix} 0_{g\times(q-g)} \\ I_{q-g} \end{bmatrix}.$$

Then $\mathfrak{B}_M^{\mathrm{im}}(t) \subset \mathfrak{B}_R^{\ker}(t)$ for all $t \in \mathbb{R}\backslash\mathbb{T}$ is immediate. If $w \in \mathfrak{B}_R^{\ker}(t)$ for $t \in \mathbb{R}\backslash\mathbb{T}$, then $r(D)$ being non-zero and meromorphic yields

$$\left[I_g \;\middle|\; 0_{g\times(q-g)}\right] V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1} w(t) = 0 \qquad \forall\, t \in \operatorname{dom} w \cap (\mathbb{R} \subset \mathbb{T}),$$

and so there exists $l \in \mathcal{C}_t^\infty(\mathbb{R}^m)$ such that

$$V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1} w \;=\; \begin{bmatrix} 0_{g\times m} \\ I_m \end{bmatrix} l.$$

"$\Leftarrow$": Let $t \in \mathbb{R} \subset \mathbb{T}$ and choose an open interval $\mathbb{I} \subset (\mathbb{R}\backslash\mathbb{T})$ with $t \in \mathbb{I}$. Seeking a contradiction, by Theorem 3.3 one may assume that $\deg p(D) \ge 1$. Comparing the gth components of the identical vector spaces

$$\left\{ w \in \mathcal{C}^\infty(\mathbb{I}, \mathbb{R}^q) \,\middle|\, \left[\begin{array}{cc|c} I_{g-1} & & \\ & r(\tfrac{\mathrm{d}}{\mathrm{d}t}) & 0_{g\times(q-g)} \end{array}\right] V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1} w = 0 \right\}$$

and $\left\{ w \in \mathcal{C}^\infty(\mathbb{I},\mathbb{R}^q) \middle| \ \exists\, l \in \mathcal{C}^\infty(\mathbb{I},\mathbb{R}^m) \ : \ w = M(\tfrac{\mathrm{d}}{\mathrm{d}t})l \right\}$ yields that

$$\begin{aligned}\dim\ &\left\{ \left(V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1}w(t)\right)_g \middle|\ w \in \mathcal{C}^\infty(\mathbb{I},\mathbb{R}^q) \ \wedge\ r(\tfrac{\mathrm{d}}{\mathrm{d}t})\left(V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1}w(t)\right)_g = 0 \right\} \\ &= \qquad \dim\ \left\{ \left(V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1}M(\tfrac{\mathrm{d}}{\mathrm{d}t})l(t)\right)_g \middle|\ l \in \mathcal{C}^\infty(\mathbb{I},\mathbb{R}^m) \right\}.\end{aligned}$$

However, the former has finite dimension $\deg r(D) \geq 1$, while the latter is zero dimensional or has infinite dimension. This is a contradiction.
This completes the proof of the proposition. □

Proposition 3.7 is known for time-invariant systems, see [21, Theorem 6.6.1]; however the different proof presented here might also be of interest in the time-invariant case.

Note that the family of linear sub-spaces of a linear space may be partially ordered by inclusion, and thus constitutes a lattice with respect to $+$ and $\cap$. Hence the following definition is well-defined.

Definition 3.8 Let $R(D) \in \mathcal{M}[D]^{g\times q}$ and $t \in \mathbb{R}$. Any vector space $\mathfrak{B}_R(t)$ with $\mathfrak{B}_R(t) \subset \mathfrak{B}_R^{\ker}(t)$ is called *local sub-behaviour* of $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ at $t \in \mathbb{R}$.
A sub-vector space $\mathfrak{B}_R^{\mathrm{c}}(t)$ of $\mathfrak{B}_R^{\ker}(t)$ is called *locally controllable* at $t \in \mathbb{R}$ if, and only if, for every $w^1, w^2 \in \mathfrak{B}_R^{\mathrm{c}}(t)$ and every $t_0 \in (-\infty, t) \cap \mathrm{dom}\ w^1$ there exist $t_1 \in \mathrm{dom}\ w^2 \cap (t,\infty)$ and $w \in \mathfrak{B}_R^{\mathrm{c}}(t)$ such that

$$w(t) = \begin{cases} w^1(t), & t \in (-\infty, t_0] \cap \mathrm{dom}\ w^1 \\ w^2(t), & t \in [t_1, \infty) \cap \mathrm{dom}\ w^2. \end{cases}$$

$\mathfrak{B}_R^{\mathrm{c}}$ is called *controllable* if, and only if, it is locally controllable almost everywhere.
$\mathfrak{B}_R^{\mathrm{contr}}(t) \subset \mathfrak{B}_R^{\ker}$ is called the *largest controllable behaviour* of $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ at t if, and only if, every controllable behaviour $\mathfrak{B}_R^{\mathrm{c}}(t)$ of $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ satisfies $\mathfrak{B}_R^{\mathrm{c}}(t) \subset \mathfrak{B}_R^{\mathrm{contr}}(t)$ almost everywhere.
□

In the following Proposition 3.9 we show that the largest controllable behaviour is independent of the non-unique factorization (2.2).

Proposition 3.9 If $R(D) \in \mathcal{M}[D]^{g\times q}$ is factorized as in (2.2), then we have

$$\mathfrak{B}_R^{\mathrm{contr}}(t) \ = \ \left\{ w \in \mathfrak{B}_R^{\ker}(t) \middle|\ \left[I_g, 0_{g\times(q-g)}\right]\ V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1}w = 0 \right\} \quad \text{for a.a. } t \in \mathbb{R}.$$

Proof: Since $\left[I_g, 0_{g\times(q-g)}\right]\ V(D)^{-1}$ is right invertible, it follows from Theorem 3.3 that

$$\mathfrak{B}^{\mathrm{c}}(t) \ := \ \left\{ w \in \mathfrak{B}_R^{\ker}(t) \middle|\ \ [I_g, 0]\ V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1}w = 0 \right\}$$

is a controllable behaviour almost everywhere. Therefore, we have to show that $\mathfrak{B}_R^{\mathrm{contr}}(t) \subset \mathfrak{B}^{\mathrm{c}}(t)$ almost everywhere. Let $\mathbb{T}$ denote the union of all zeros and poles of the meromorphic coefficients in all entries of $U(D)$, $U(D)^{-1}$, $V(D)$, $V(D)^{-1}$, $r(D)$, $\bar{U}(D)$, $\bar{U}(D)^{-1}$, $\bar{V}(D)$, $\bar{V}(D)^{-1}$, and $\bar{r}(D)$.

Then $\mathbb{T}$ is a discrete set. Let $w \in \mathfrak{B}_R^{\text{contr}}(t)$ for $t \in \mathbb{R}\backslash\mathbb{T}$. Choose an open interval $\mathbb{I} \subset \mathbb{T}$ with $t \in \mathbb{R}\subset\mathbb{T}$. Then

$$V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1}w \;=:\; (0,\ldots,0,\varphi_g,\ldots,\varphi_q)^T \in \mathcal{C}^\infty(\mathbb{I},\mathbb{R}^q) \qquad \text{and} \quad r(\tfrac{\mathrm{d}}{\mathrm{d}t})\varphi_g = 0.$$

The function φ_g, as a solution of a linear ordinary differential equation with real analytic coefficients on $\mathbb{I}$, is real analytic on $\mathbb{I}$ itself. Therefore, the normal form (2.2) and the identity property of analytic function yields $\varphi_g \equiv 0$. This proves $w(t) = V(\tfrac{\mathrm{d}}{\mathrm{d}t})\,(0,\ldots,0,\varphi_{g+1},\ldots,\varphi_q)^T \in \mathfrak{B}_R^{\mathrm{c}}(t)$. If $R(D)$ is factorized as in (2.4), then by Remark 2.5 one concludes that

$$[I_g,\ 0]\,\bar{V}(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1}w \;=\; [I_g,\ 0]\begin{bmatrix} W_1(\tfrac{\mathrm{d}}{\mathrm{d}t}) & 0 \\ W_3(\tfrac{\mathrm{d}}{\mathrm{d}t}) & W_4(\tfrac{\mathrm{d}}{\mathrm{d}t})\end{bmatrix} V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1}w \;=\; \left[W_1(\tfrac{\mathrm{d}}{\mathrm{d}t}),\ 0\right] V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1}w,$$

and the result follows since $W_1(D)$ is unimodular. This completes the proof. □

4 Autonomous behaviour

In this section we show that the local behaviour (in the sense almost everywhere) can be decomposed into the direct sum of the controllability subspace and an autonomous subspace.

Definition 4.1 For $R(D) \in \mathcal{M}[D]^{g\times q}$, the system $R(\tfrac{\mathrm{d}}{\mathrm{d}t})w = 0$ is called *locally autonomous at* $t \in \mathbb{R}$ if, and only if, for any $w^1, w^2 \in \mathfrak{B}_R^{\ker}(t)$ with $w_1 \equiv w_2$ on some open interval $\mathbb{I} \subset \operatorname{dom}\ w^1 \cap \operatorname{dom}\ w^2$ with $t \in \mathbb{I}$ it follows that $w_1 \equiv w_2$ on $\operatorname{dom}\ w^1 \cap \operatorname{dom}\ w^2$.
The system $R(\tfrac{\mathrm{d}}{\mathrm{d}t})w = 0$ is called *autonomous* if, and only if, it is autonomous almost everywhere.
A real vector space $\mathfrak{B}_R^{\text{aut}}(t) \subset \mathfrak{B}_R^{\ker}(t)$ is called *autonomous* if, and only if, for any $w^1, w^2 \in \mathfrak{B}_R^{\text{aut}}(t)$ with $w_1 \equiv w_2$ on some open interval $\mathbb{I} \subset \operatorname{dom}\ w^1 \cap \operatorname{dom}\ w^2$ with $t \in \mathbb{I}$ it follows that $w_1 \equiv w_2$ on $\operatorname{dom}\ w^1 \cap \operatorname{dom}\ w^2$.
A set $\mathfrak{B}_R^{\text{aut}}$, associated to $t \mapsto \mathfrak{B}_R^{\text{aut}}(t) \subset \mathfrak{B}_R^{\ker}(t)$, is called *autonomous behaviour* if, and only if, $\mathfrak{B}_R^{\text{aut}}(t)$ is autonomous for almost all $t \in \mathbb{R}$. □

The above definition is a generalization of autonomous sub-behaviour of time-invariant systems as, for example, defined in [21, p. 67].

Proposition 4.2 Consider the system $R(\tfrac{\mathrm{d}}{\mathrm{d}t})w = 0$ with $R(D) \in \mathcal{M}[D]^{g\times q}$ and factorization (2.2). For any autonomous behaviour $\mathfrak{B}_R^{\text{aut}}$, the following properties hold.

(i) $\quad \mathfrak{B}_R^{\text{aut}}(t) \;\cap\; \mathfrak{B}_R^{\text{contr}}(t) \;=\; \{0\} \quad$ for almost all $t \in \mathbb{R}$.

(ii) If $w \in \mathfrak{B}_R^{\text{aut}}(t)$, then

$$\begin{bmatrix} I_{g-1} & & \\ & r(\tfrac{\mathrm{d}}{\mathrm{d}t}) & \\ & & I_{q-g}\end{bmatrix} V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1}w \;=0.$$

(iii)

$$\left\{ w \in \mathcal{C}_t^\infty(\mathbb{R}^q) \,\middle|\, \begin{bmatrix} I_{g-1} & & \\ & r(\frac{\mathrm{d}}{\mathrm{d}t}) & \\ & & I_{q-g} \end{bmatrix} V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1} w \; = 0 \right\}$$

is an autonomous behaviour for a.a. $t \in \mathbb{R}$.

Proof:
(i): If $w \in \mathfrak{B}_R^{\mathrm{aut}}$ and $w \not\equiv 0$, then it cannot belong to the controllable behaviour, otherwise Definition 4.1 would be violated.
(ii): By (i) and Proposition 3.9, any $w \in \mathfrak{B}_R^{\mathrm{aut}}(t)$ satisfies $[0_{(q-g)\times g}, I_{q-g}]V(\frac{\mathrm{d}}{\mathrm{d}t})^{-1}w = 0$. Hence (ii) follows from (2.2).
(iii): Let $\mathbb{T}$ denote the discrete set given in Remark 2.3 and $t \in \mathbb{R}\backslash\mathbb{T}$. If $w \in \mathfrak{B}_R^{\ker}(t)$ and satisfies

$$\begin{bmatrix} I_{g-1} & & \\ & r(\frac{\mathrm{d}}{\mathrm{d}t}) & \\ & & I_{q-g} \end{bmatrix} V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1} w \; = \; 0\,,$$

then (2.2) yields that w is of the form

$$w = V(\tfrac{\mathrm{d}}{\mathrm{d}t}) \; (0, \ldots, 0, \varphi_g, 0, \ldots, 0)^T\,,$$

for some $\varphi_g \in \mathcal{C}_t^\infty(\mathbb{R})$ with $r(\frac{\mathrm{d}}{\mathrm{d}t})\varphi_g = 0$. Since r has real analytic coefficients, the solution is real analytic, too, and the identity property of real analytic functions ensures local uniqueness of w as in Definition 4.1. This completes the proof. $\square$

Note that the autonomous behaviour in Proposition 4.2 (iii) is not uniquely defined, it depends on the factorization (2.2); this holds already true for time-invariant systems, see [21, Ex. 5.6]. However, the dimension of of this autonomous behaviour is unique; this follows from the fact that $r(D)$ is unique up to similarity, and the latter preserves the degree, see Proposition 2.2. For time-invariant systems (1.3), the results of Proposition 4.2 can be found in [21, Sect. 5.2].

For time-invariant systems (1.9), it is well-known that an autonomous behaviour is not uniquely defined by $R(D)$, but depends on the factorization, see [21, Rem. 5.2.15]. However, the sum of an autonomous behaviour and the controllable behaviour is indeed uniquely defined. In the following we generalize this result to time-varying systems.

Theorem 4.3 Consider the system $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ with $R(D) \in \mathcal{M}[D]^{g\times q}$, factorizations (2.2), (2.4), and define, for all $t \in \mathbb{R}$,

$$\begin{aligned}
\mathfrak{B}_R^{\mathrm{contr}}(t) &= \left\{ w \in \mathfrak{B}_R^{\ker}(t) \middle| \quad [I_g, \quad 0_{g\times(q-g)}]\, V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1} w = 0 \right\},\\
\mathfrak{B}_R^{\mathrm{aut}}(t) &= \left\{ w \in \mathfrak{B}_R^{\ker}(t) \,\middle|\, \begin{bmatrix} I_{g-1} & & \\ & r(\frac{\mathrm{d}}{\mathrm{d}t}) & \\ & & I_{q-g} \end{bmatrix} V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1} w \; = 0 \right\},\\
\overline{\mathfrak{B}}_R^{\mathrm{aut}}(t) &= \left\{ w \in \mathfrak{B}_R^{\ker}(t) \,\middle|\, \begin{bmatrix} I_{g-1} & & \\ & \bar{r}(\frac{\mathrm{d}}{\mathrm{d}t}) & \\ & & I_{q-g} \end{bmatrix} \bar{V}(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1} w \; = 0 \right\},
\end{aligned}$$

where the latter is defined with respect to (2.4). Then

$$\mathfrak{B}_R^{\ker}(t) \;=\; \mathfrak{B}_R^{\mathrm{aut}}(t) \oplus \mathfrak{B}_R^{\mathrm{contr}}(t) \;=\; \overline{\mathfrak{B}}_R^{\mathrm{aut}}(t) \oplus \mathfrak{B}_R^{\mathrm{contr}}(t) \quad \text{for almost all } t \in \mathbb{R}. \tag{4.1}$$

Proof: Let $\mathbb{T}$ denote the union of all zeros and poles of the meromorphic coefficients in all entries of $U(D), U(D)^{-1}, V(D), V(D)^{-1}, r(D)$ and $\bar{U}(D), \bar{U}(D)^{-1}, \bar{V}(D), \bar{V}(D)^{-1}, \bar{r}(D)$. $\mathbb{T}$ is a discrete set. In the following we consider $t \in \mathbb{R}\backslash\mathbb{T}$ and an open interval $\mathbb{I} \subset \mathbb{T}$ with $t \in \mathbb{I}$. We proceed in several steps.
Step 1: By Proposition 4.2 (i) the sums in (4.1) are direct sums.

Step 2: The inclusion

$$\mathfrak{B}_R^{\ker}(t) \;\supset\; \mathfrak{B}_R^{\mathrm{aut}}(t) \oplus \mathfrak{B}_R^{\mathrm{contr}}(t)$$

follows from the definition of $\mathfrak{B}_R^{\mathrm{aut}}(t)$ and $\mathfrak{B}_R^{\mathrm{contr}}(t)$.

Step 3: We show

$$\mathfrak{B}_R^{\ker}(t) \;\subset\; \mathfrak{B}_R^{\mathrm{aut}}(t) \oplus \mathfrak{B}_R^{\mathrm{contr}}(t)\,. \tag{4.2}$$

Let $w \in \mathfrak{B}_R^{\ker}(t)$ and set

$$(\varphi_1, \dots, \varphi_q)^T \;:=\; V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1} w \in \mathcal{C}^\infty(\mathbb{I}; \mathbb{R}^q)\,.$$

Then

$$\begin{bmatrix} I_{g-1} & & \\ & r(\tfrac{\mathrm{d}}{\mathrm{d}t}) & \\ & & I_{q-g} \end{bmatrix} V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1} w \;=\; 0\,,$$

and hence

$$(\varphi_1, \dots, \varphi_q)^T \;=\; (0, \dots, 0, \varphi_g, 0, \dots, 0)^T \quad \text{with } r(\tfrac{\mathrm{d}}{\mathrm{d}t})\varphi_g = 0.$$

Finally

$$\begin{aligned} w_1 \;&:=\; V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1}\big(0, \dots, 0, \varphi_g, 0, \dots, 0\big)^T \in \mathfrak{B}_R^{\mathrm{aut}}(t), \\ w_2 \;&:=\; V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1}\,(0, \dots, 0, \varphi_{g+1}, \dots, \varphi_q)^T \in \mathfrak{B}_R^{\mathrm{contr}}(t), \end{aligned}$$

yields $w_1 + w_2 = w$, whence (4.2).

Step 4: We show

$$\mathfrak{B}_R^{\mathrm{aut}}(t) \oplus \mathfrak{B}_R^{\mathrm{contr}}(t) \;\subset\; \overline{\mathfrak{B}}_R^{\mathrm{aut}}(t) \oplus \mathfrak{B}_R^{\mathrm{contr}}(t)\,.$$

Let $w_1 \in \mathfrak{B}_R^{\mathrm{aut}}(t)$ and $w_2 \in \mathfrak{B}_R^{\mathrm{contr}}(t)$. Then

$$\begin{aligned} (0, \dots, 0, \varphi_g, 0, \dots, 0)^T \;&:=\; V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1} w_1 \in \mathcal{C}^\infty(\mathbb{I}; \mathbb{R}^q) \quad \text{with } r(\tfrac{\mathrm{d}}{\mathrm{d}t})\varphi_g = 0, \\ (0, \dots, 0, \varphi_{g+1}, \dots, \varphi_q)^T \;&:=\; V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1} w_2 \in \mathcal{C}^\infty(\mathbb{I}; \mathbb{R}^q) \end{aligned}$$

Since $w := w_1 + w_2 \in \mathfrak{B}_R^{\ker}(t)$, it follows from (2.4) that

$$\bar{V}(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1} w = (0, \dots, 0, \bar{\varphi}_g, \dots, \bar{\varphi}_q)^T \in \mathcal{C}^\infty(\mathbb{I}; \mathbb{R}^q) \quad \text{with } r(\tfrac{\mathrm{d}}{\mathrm{d}t})\bar{\varphi}_g = 0.$$

Finally, setting

$$\begin{aligned} \bar{w}_1 \;&:=\; \bar{V}(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1}\big(0, \dots, 0, \bar{\varphi}_g, 0, \dots, 0\big)^T \in \overline{\mathfrak{B}}_R^{\mathrm{aut}}, \\ \bar{w}_2 \;&:=\; \bar{V}(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1}\,(0, \dots, 0, \bar{\varphi}_{g+1}, \dots, \bar{\varphi}_q)^T \in \overline{\mathfrak{B}}_R^{\mathrm{contr}}, \end{aligned}$$

shows $w = \bar{w}_1 + \bar{w}_2 \in \overline{\mathfrak{B}}_R^{\text{aut}}(t) \oplus \mathfrak{B}_R^{\text{contr}}(t)$.

Step 5: The inclusion

$$\mathfrak{B}_R^{\text{aut}}(t) \oplus \mathfrak{B}_R^{\text{contr}}(t) \ \supset \ \overline{\mathfrak{B}}_R^{\text{aut}}(t) \oplus \mathfrak{B}_R^{\text{contr}}(t)\,.$$

follows by symmetry as in Step 4. This completes the proof of the theorem. $\square$

Example 4.4 Revisiting the example (1.2), we now can show that this system is locally controllable almost everywhere.

Without loss of generality, we may assume that the coordinate system for the Lagrange multipliers is such that $F_0 = [F_1\,0]$ with nonsingular $F_1 \in \mathbb{R}^{2\times 2}$ and if we partition

$$-K_0 = \begin{bmatrix} K_{11}(t) & K_{12}(t) \\ K_{21}(t) & K_{22}(t) \end{bmatrix},\ M_0 = \begin{bmatrix} M_{11}(t) & M_{12}(t) \\ M_{21}(t) & M_{22}(t) \end{bmatrix},\ -D_0 = \begin{bmatrix} D_{11}(t) & D_{12}(t) \\ D_{21}(t) & D_{22}(t) \end{bmatrix},\ S_0 = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix},$$

with $K_{11}(t), M_{11}(t), D_{11}(t), S_1 \in \mathbb{R}^{2\times 2}$ and all other formats accordingly, then the system (1.2), for $t \in \mathbb{I}$, may be written as

$$\begin{bmatrix} I_2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & M_{11}(t) & M_{12}(t) & 0 \\ 0 & 0 & M_{21}(t) & M_{22}(t) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & I_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ K_{11}(t) & K_{12}(t) & D_{11}(t) & D_{12}(t) & F_1^T \\ K_{21}(t) & K_{22}(t) & D_{21}(t) & D_{22}(t) & 0 \\ F_1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ S_1 \\ S_2 \\ 0 \end{bmatrix} u.$$

Since F_1 is non-singular, and S_1, S_2 are constant matrices of full row rank, it follows that $x_1 = 0$ and $\dot{x}_1 = 0$, whence $x_3 = 0$. Therefore, (1.2) is equivalent to

$$\begin{bmatrix} D & -1 & 0_{2\times 1} & 0 \\ -K_{12}(t) & M_{12}(t) & -F_1 & S_1 \\ -K_{22}(t) & M_{22}(t) & 0 & S_2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \\ x_5 \\ u \end{bmatrix} = 0\,,$$

with corresponding correponding right invertible matrix $R(D)$. By Theorem 3.3, the system (1.2) is locally controllable almost everywhere on $\mathbb{I}$.

5 Observability

In this section, we study how one behaviour can be observed from another. Essential for this are the concepts of adjoints of matrices over $\mathcal{M}[D]$ and the adjoint of a kernel representation $\mathfrak{B}_R^{\text{ker}}$.

Definition 5.1 The *adjoint* for matrices over $\mathcal{M}[D]$ is defined as

$$\cdot^{\mathrm{ad}} : \mathcal{M}^{n\times m}[D] \to \mathcal{M}^{m\times n}[D], \qquad \sum_{i=0}^{k} P_i D^i \mapsto \Big(\sum_{i=0}^{k} P_i D^i\Big)^{\mathrm{ad}} := \sum_{i=0}^{k}(-1)^i D^i P_i^T.$$

Proposition 5.2 The adjoint is an anti-isomorphism, i.e., it is surjective, injective, and satisfies, for arbitrary matrices $P(D), Q(D)$ over $\mathcal{M}[D]$ with appropriate formats,

$$[P(D)+Q(D)]^{\mathrm{ad}} = P(D)^{\mathrm{ad}} + Q(D)^{\mathrm{ad}}, \tag{5.1}$$

$$[P(D)\cdot Q(D)]^{\mathrm{ad}} = Q(D)^{\mathrm{ad}} \cdot P(D)^{\mathrm{ad}}. \tag{5.2}$$

Proof: Surjectivity, injectivity, and addition are straightforward. It remains to prove the anti-multiplication rule (5.2). This is well known in the scalar case, see for example [18, p. 25]. To prove the matrix case, denote the entries of $P(D) \in \mathcal{M}^{n\times m}[D]$, $Q(D) \in \mathcal{M}^{m\times l}[D]$ by $p_{ij}(D)$, $q_{ij}(D)$, respectively. Then

$$P(D)^{\mathrm{ad}} = \Big(p_{ji}(D)^{\mathrm{ad}}\Big)_{1\le i\le n,\, 1\le j\le m}, \qquad Q(D)^{\mathrm{ad}} = \Big(q_{ji}(D)^{\mathrm{ad}}\Big)_{1\le i\le m,\, 1\le j\le l}$$

and applying this to

$$\big(P(D)\cdot Q(D)\big)_{ij} = \sum_{\lambda=1}^{k} p_{i\lambda}(D) q_{\lambda j}(D)$$

and using the anti-multiplication rule (5.2) for scalar polynomials yields the result. This completes the proof. □

Definition 5.3 Let $R(D) \in \mathcal{M}[D]^{g\times q}$ and $t \in \mathbb{R}$. The *local adjoint* of the kernel representation $\mathfrak{B}_R^{\ker}(t)$ of the system $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is the image representation

$$\big\{\tilde{w} \in \mathcal{C}_t^\infty(\mathbb{R}^q)\big| \quad \exists\, l \in \mathcal{C}_t^\infty(\mathbb{R}^g)\ \forall\, \tau \in \operatorname{dom} w \cap \operatorname{dom} l \ :\ \tilde{w}(\tau) = R(\tfrac{\mathrm{d}}{\mathrm{d}\tau})^{\mathrm{ad}} l(\tau)\big\}\ . \tag{5.3}$$

□

Certainly, the projection onto the first component of the kernel representation

$$\left\{ (\tilde{w}, l) \in \mathcal{C}_t^\infty(\mathbb{R}^q) \times \mathcal{C}_t^\infty(\mathbb{R}^g) \;\middle|\; \forall\, \tau \in \operatorname{dom}\tilde{w} \cap \operatorname{dom} l \ :\ \big[I_q, R(\tfrac{\mathrm{d}}{\mathrm{d}\tau})^{\mathrm{ad}}\big] \begin{pmatrix} \tilde{w}(\tau) \\ l(\tau) \end{pmatrix} = 0 \right\}$$

yields the image representation (5.3).

The following definition is a straightforward generalization of observability for time-invariant systems in the behavioural set-up, see [21, Def. 5.3.2].

Definition 5.4 Let $[R_1(D), R_2(D)] \in \mathcal{M}[D]^{g\times(q_1+q_2)}$ and $t \in \mathbb{R}$. Then $w_2 \in \mathcal{C}_t^\infty(\mathbb{R}^{q_2})$ is called *locally observable at* $t \in \mathbb{R}$ *from* $w_1 \in \mathcal{C}_t^\infty(\mathbb{R}^{q_1})$ for $t \in \mathbb{R}$ if, and only if,

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ \tilde{w}_2 \end{bmatrix} \in \mathfrak{B}_{[R_1,R_2]}^{\ker}(t)$$

implies that

$$w_2(\tau) = \tilde{w}_2(\tau)\ \forall\, \tau \in \operatorname{dom} w_2 \cap \qquad \operatorname{dom} \tilde{w}_2.$$

□

An algebraic characterization of observability is given in the following theorem.

Theorem 5.5 Let $[R_1(D), R_2(D)] \in \mathcal{M}[D]^{g\times(q_1+q_2)}$. Then w_2 is locally observable almost everywhere from w_1 if, and only if, $R_2(D)$ is left invertible.

Proof: First note that in view of the linearity of the system, it remains to show that for almost all $t \in \mathbb{R}$ we have

$$\left[w_2 \in \mathfrak{B}_{R_2}^{\ker}(t) \implies w_2 = 0\right] \qquad \Longleftrightarrow \qquad R_2(D) \text{ is left invertible.}$$

"$\Leftarrow$" is immediate.
"$\Rightarrow$": Let $\mathbb{T}$ denote the discrete set of the union of all zeros and poles of the meromorphic coefficients in all entries of $U_2(D), U_2(D)^{-1}, V_2(D), V_2(D)^{-1}, r_2(D)$ which take $R_2(D)$ into a normal form (2.2).
Seeking a contradiction, suppose $R_2(D)$ is not left invertible and let $t \in \mathbb{T}$. Now either $\mathrm{rk}_{\mathcal{M}[D]} R_2(D) < q_2$ (in which case the normal form (2.2) applied to $R_2(D)$ yields the existence of some $w_2 \in \mathfrak{B}_{R_2}^{\ker}(t)$ with $w_2 \neq 0$) or, again by Theorem 2.2, there exist $r_2(D) \in \mathcal{M}[D]$ with $\deg r_2(D) \geq 1$ and unimodular $U_2(D) \in \mathcal{M}[D]^{g\times g}$, $V_2(D) \in \mathcal{M}[D]^{q_2\times q_2}$ such that

$$U_2(D)^{-1} R_2(D) V_2(D)^{-1} \;=\; \begin{bmatrix} I_{q_2-1} & 0_{(q_2-1)\times 1} \\ 0_{1\times(q_2-1)} & r_2(D) \\ 0_{(g-q_2)\times(q_2-1)} & 0 \end{bmatrix}. \tag{5.4}$$

By $\deg r_2(D) \geq 1$ there exists $\varphi \in \mathcal{C}_t^\infty(\mathbb{R}) \setminus \{0\}$ such that $r_2(\frac{\mathrm{d}}{\mathrm{d}t})\varphi = 0$. Therefore $w_2 := (0, \ldots, 0, \varphi)^T \in \mathfrak{B}_{R_2}^{\ker}(t)$, which is a contradiction. This completes the proof. $\square$

The following theorem relates the concepts of controllability and observability.

Theorem 5.6 For $[R_1(D), R_2(D)] \in \mathcal{M}[D]^{g\times(q_1+q_2)}$ the following two statements are equivalent:

(i) The system $[R_1(\frac{\mathrm{d}}{\mathrm{d}t}), R_2(\frac{\mathrm{d}}{\mathrm{d}t})] \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0$ is locally controllable almost everywhere;

(ii) l is locally observable almost everywhere from w with respect to the system
$\left[I_q \;,\; \begin{matrix} R_1(\frac{\mathrm{d}}{\mathrm{d}t})^{\mathrm{ad}} \\ R_2(\frac{\mathrm{d}}{\mathrm{d}t})^{\mathrm{ad}} \end{matrix} \right] \begin{pmatrix} w \\ l \end{pmatrix} \;=\; 0.$

$\square$

Proof: By Theorem 3.3, the statement (i) is equivalent to $[R_1(D), R_2(D)]$ being right invertible, which, by Proposition 5.2, is equivalent to $[R_1(D), R_2(D)]^{\mathrm{ad}} = \begin{bmatrix} R_1(D)^{\mathrm{ad}} \\ R_2(D)^{\mathrm{ad}} \end{bmatrix}$ being left invertible. The latter is, by invoking Proposition 5.5, equivalent to statement (ii) This completes the proof of the theorem. $\square$

In order to relate the classical concepts of observability known in the literature to observability as introduced above, we have to permute the columns in the presentation (1.10), (1.11) in the following proposition.

Proposition 5.7 For a time-varying Rosenbrock system of the form (1.8) represented in the form

$$R(D) = [R_1(D), R_2(D)]\ , \quad R_1(D) = \begin{bmatrix} -Q(D), & 0 \\ W(D), & -I_p \end{bmatrix}\ , \quad R_2(D) = \begin{bmatrix} P(D) \\ V(D) \end{bmatrix}\ ,$$

the following conditions are equivalent.

(i) w_2 is locally observable from w_1 almost everywhere w.r.t. the system $[R_1(\frac{\mathrm{d}}{\mathrm{d}t}), R_2(\frac{\mathrm{d}}{\mathrm{d}t})] \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0.$

(ii) $R_2(D)$ is left invertible.

(iii) $[R_1(D), R_2(D)]$ is observable in the sense defined in [13].

(iv) If $R(D)$ represents a time-invariant Rosenbrock system, then it is observable in the sense defined in [10].

(v) If $R(D)$ represents a state space system (1.6) in the form

$$R_1(D) = \begin{bmatrix} -B & 0 \\ -F & I_p \end{bmatrix}\ , \quad R_2(D) = \begin{bmatrix} DI_n - A \\ -C \end{bmatrix}\ ,$$

then it is observable in the classical sense, see for example [26].

Proof: The equivalence '(i)⇔(ii)' follows from Theorem 5.5. The equivalences '(ii)⇔(iii)' and '(ii)⇔(iv)' follow from [13, Th. 6.5] and [10, Cor. 7.6], respectively. They all can be shown directly, but only for state space systems we prove '(i)⇔(v)' directly; it shows how observability in the classical sense and in the behavioural set-up are related. Note that in the case of time-varying state space systems and time-invariant Rosenbrock systems the set of critical points $\mathbb{T}$ is empty, the system is defined on the whole time axis.
Complete observability for time-varying state space systems of the form (1.6) means, see [26, Def. 9.7], that for any open interval $\mathbb{I} \subset \mathbb{R}$ we have

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} I_n - A(t) \\ -C(t) \end{bmatrix} z(t) = 0 \quad \forall\, t \in \mathbb{I} \qquad \Longrightarrow \qquad z(t) = 0 \quad \forall\, t \in \mathbb{I}. \tag{5.5}$$

(5.5) is equivalent to $R_2(D)$ being left invertible, and hence '(i)⇔(v)' follows from Theorem 5.5. This completes the proof of the theorem. □

Example 5.8 Revisiting example (1.2) and adding to (**??**) the output equation

$$y = \begin{bmatrix} 0 & 0 & I_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} x \tag{5.6}$$

corresponding to measuring the positions, we see that the resulting matrix

$$\begin{bmatrix} E(t)D - A(t) \\ C \end{bmatrix},$$

is left invertible if, and only if, the matrix

$$\begin{bmatrix} D & -1 & 0 \\ -K_{12}(t) & M_{12}(t)D & -F_1 \\ -K_{22}(t) & M_{22}(t)D & 0 \end{bmatrix}$$

is left invertible, which holds if, and only if, $K_{22}(t)$ is non-zero. The latter is typically the case in practice, since the stiffness matrix $K_0(t)$ is symmetric and positive definite. An application of Theorem 5.5 yields: x is locally observable from (u, y) at t with respect to the system (**??**), (5.6) if, and only if, $K_{22}(t)$ is non-zero.

6 Latent variables and elimination

In [21, Sect. 6.2], full and manifest behaviour is considered for time-invariant systems. We do not repeat these definitions for time-varying systems but show a time-varying version of the crucial Theorem 6.2.6 in [21].

Theorem 6.1 Let $[R(D), S(D)] \in \mathcal{M}[D]^{g\times(q+s)}$. Then there exists $R'(D) \in \mathcal{M}[D]^{g'\times q}$ such that

$$\mathfrak{B}^{\ker}_{R'}(t) \;=\; \left\{ w \in \mathcal{C}^\infty_t(\mathbb{R}^q) \,\middle|\, \forall \tau \in \operatorname{dom} w \cap \operatorname{dom} l \;:\; R(\tfrac{\mathrm{d}}{\mathrm{d}\tau})w(\tau) = S(\tfrac{\mathrm{d}}{\mathrm{d}\tau})l(\tau) \right\} \quad \text{for a.a. } t \in \mathbb{R}. \tag{6.1}$$

Proof: By Theorem 2.2, there exists some unimodular $U(D) \in \mathcal{M}[D]^{g\times g}$ such that

$$U(D)R(D) = \begin{bmatrix} R'(D) \\ R''(D) \end{bmatrix}, \quad U(D)S(D) = \begin{bmatrix} 0 \\ S''(D) \end{bmatrix},$$

where $R'(D) \in \mathcal{M}[D]^{g'\times q}$, $R''(D) \in \mathcal{M}[D]^{g''\times q}$, $S''(D) \in \mathcal{M}[D]^{g''\times s}$, and $\operatorname{rk}_{\mathcal{M}[D]} S''(D) = g''$. Applying Theorem 2.2 again, there exist $\mathcal{M}[D]$-unimodular matrices $U(D)$ and $V(D)$ of sizes g'' and s, and $r(d) \in \mathcal{M}[D]$ such that

$$S''(D) \;=\; U(D)^{-1} \left[\begin{array}{cc|c} I_{g''-1} & 0 & \\ 0 & r(D) & 0_{g''\times(q-g'')} \end{array} \right] V(D)^{-1}.$$

Choose $\mathbb{T}$ as the discrete set of the union of all zeros and poles of the meromorphic coefficients in all entries of $U(D), V(D), U(D)^{-1}, V(D)^{-1}, r(D)$. Let $\mathbb{I}$ be an open interval with $\mathbb{I} \subset \mathbb{R} \setminus \mathbb{T}$ and $t \in \mathbb{I}$.

Then

$$R(\tfrac{\mathrm{d}}{\mathrm{d}\tau})w(\tau) = S(\tfrac{\mathrm{d}}{\mathrm{d}\tau})l(\tau) \quad \Longleftrightarrow \quad \begin{bmatrix} R'(\tfrac{\mathrm{d}}{\mathrm{d}\tau}) & 0 \\ R''(\tfrac{\mathrm{d}}{\mathrm{d}\tau}) & S''(\tfrac{\mathrm{d}}{\mathrm{d}\tau}) \end{bmatrix} \begin{pmatrix} w(\tau) \\ l(\tau) \end{pmatrix} \qquad \forall\, \tau \in \mathbb{I}.$$

Hence the inclusion '$\supset$' in (6.1) is obvious. To show '$\subset$' in (6.1), let $w \in \mathfrak{B}^{\ker}_{R'}(t)$ for $t \in \mathbb{I}$. Let $\tilde{l}_{g''} \in \mathcal{C}^\infty(\mathbb{I}, \mathbb{R})$ denote the solution of

$$r(\tfrac{\mathrm{d}}{\mathrm{d}\tau})\tilde{l}_{g''}(\tau) = \left(U(\tfrac{\mathrm{d}}{\mathrm{d}\tau})S''(\tfrac{\mathrm{d}}{\mathrm{d}\tau})w(\tau)\right)_{g''} \qquad \text{on } \mathbb{I}.$$

This solution exists, see for example [29, Ch. IV]. Setting

$$l := V[0, \ldots, 0, \tilde{l}_{g''}]^T$$

yields

$$U(\tfrac{\mathrm{d}}{\mathrm{d}\tau})R''(\tfrac{\mathrm{d}}{\mathrm{d}\tau})w(\tfrac{\mathrm{d}}{\mathrm{d}\tau}) = \left[\begin{array}{cc|c} I_{g''-1} & 0 & \\ 0 & r(D) & 0_{g''\times(q-g'')} \end{array}\right] V(\tfrac{\mathrm{d}}{\mathrm{d}t})^{-1} l(\tau)$$

which is equivalent to $R(\frac{\mathrm{d}}{\mathrm{d}\tau})w(\tau) = S(\frac{\mathrm{d}}{\mathrm{d}\tau})l(\tau)$. This completes the proof of the theorem. □

7 Conclusions

We have introduced a general behavioural approach for linear systems with time-varying coefficients. We have characterized autonomous, controllable and observable behaviour and have generalized the results on time-varying ordinary differential equations and on time-invariant linear algebraic-differential equations. The results have been illustrated by several examples and it has been demonstrated that the approach also helps in the understanding of practical problems such as constrained multibody systems.

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