Diffusion limits for discrete velocity models in a thin gap

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Abstract

We consider discrete kinetic models of a gas moving in a small gap with thickness h. Under an appropriate scaling of the variables, we introduce a formal series expansion to study the limit $h \to 0$. As a necessary condition for the validity of the expansion we derive a nonlinear diffusion equation.

Key words: Discrete velocity models, diffusion limit, thin gap.

MSC: 82C40, 41A60.

1 Introduction

The mesoscopic description of a gas flow is based on the dynamics of a large system of particles in phase space – i.e. on the space composed of the spatial and the velocity variables of the particles. In full dimensionality, this is a six-dimensional space. Solving the evolution equation (e.g. of the type of the Boltzmann equation) is a very complex problem and even on modern supercomputers solvable only under very restrictive assumptions. Therefore there have been many attempts to derive simplified – macroscopic – equations. Such equations are usually (formally) obtained by introducing some appropriate scaling of the variables x (space), v (velocity) and t (time). A necessary condition for the validity of these simplified systems is that the microscopic length scale (mean free path) is much less than the characteristic macroscopic dimension.

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An interesting situation appears if one space scale has microscopic order, while the other ones are macroscopic. This is the case when particles are moving along a very thin channel. One may expect – under an appropriate scaling – a diffusion equation as the appropriate macroscopic description. The proper diffusion constant contains information about the particle-particle and the particle-surface interactions. A simplified situation is obtained in the case of a Knudsen gas (no particle collisions). If particles are diffusely reflected at the channel walls, then a diffusion limit can be obtained [2, 6, 9, 4]. First results including collisions have been obtained in [3].

The objective of the paper is to derive diffusion limits for discrete velocity models which give rise to model Boltzmann equations (see [11]). Our approach is a formal one: We introduce a series expansion of the solution in terms of a small variable ε (which governs the diffusion limit) and derive necessary conditions under which such an expansion is valid. The result is in lowest order a diffusion equation.

The plan of the paper is as follows. Below we give two examples of diffusion limits in simplified – linear – situations: collisionless flow (Knudsen gas) and scattering particles field (Lorenz gas). In Section 2 a general nonlinear collision interaction model is introduced. We formulate the assumptions we require from the model and give some properties of a linearized system. This collision model is a starting point of our investigation of a diffusion limit. Section 3 presents the main result of the paper. We derive the necessary conditions for the existence of the diffusion limit and present the diffusion equation it satisfies. Section 4 is an illustration to the developed theory. We give two examples of the discrete velocity models: 12-velocity model and generalized Broadwell model, for which we derive the diffusion limits.

1.1 Diffusion limits

Consider an N-velocity model

$$\mathcal{V} = \{(v_i, w_i), i = 1, \dots, N\} \tag{1.1}$$

which is symmetric in the sense that whenever $(v, w) \in \mathcal{V}$, then all $(\pm v, \pm w) \in \mathcal{V}$. We assume N to be an even number, $N = 2d$. We denote $\mathbf{v} := (v_1, \dots, v_N)^T$ and $\mathbf{w} := (w_1, \dots, w_N)^T$. Suppose there is given a collision interaction model on $\mathcal{V}$ related to a collision operator Q. We investigate the kinetic equation

$$(\partial_t + \mathbf{v} \cdot \partial_x + \mathbf{w} \cdot \partial_y)\mathbf{f} = \gamma \cdot Q[\mathbf{f}]$$

in a slab $\mathcal{S}_h := \mathbb{R} \times [-h, h]$ under diffuse reflection at the boundaries $y = \pm h$. To formulate these boundary conditions, suppose that there is a positive normalized equilibrium solution $\mathbf{m}$ of Q, i.e. a function satisfying $Q[\mathbf{m}] = 0$ and the symmetry conditions $m(v,w) = m(\pm v, \pm w)$. Denote by $\mathbf{m}_\pm$ the restrictions of $\mathbf{m}$ to the sets $\mathcal{V} \cap \{w > 0\}$ resp. $\mathcal{V} \cap \{w < 0\}$. If $\mathbf{f}$ is a function on $\mathcal{S}_h \times \mathcal{V}$ then the fluxes $\mathbf{f}^{in}(\pm h)$ into $\mathcal{S}_h$ are given by the restrictions of $\mathbf{f}(y = \pm h)$ to $\mathcal{V} \cap \{w > 0\}$ (at $y = -h$) resp. $\mathcal{V} \cap \{w < 0\}$ (at $y = h$). The boundary conditions now read

$$\mathbf{f}^{in}(\pm h) = c_\pm \cdot \mathbf{m}_\mp \ ,$$

where the quantities $c_\pm = c_\pm(x)$ have to be chosen such that the flux through the boundaries vanishes.

It is well-known, that under the scaling

$$x' = \varepsilon x, \quad t' = \varepsilon^2 t, \quad \varepsilon > 0 \tag{1.2}$$

one may obtain a diffusion equation in the limit $\varepsilon \to 0$. We present shortly some quick heuristic arguments for different settings based on intuitive stochastic arguments.

1.1.1 Knudsen flows

Here, we neglect the collision operator, i.e. we set $\gamma = 0$. The limit $\varepsilon \to 0$ has been treated e.g. in [2, 6, 4]. The solution of the (now linear) initial boundary value problem may be simulated by the motion of a test particle which moves in the interior of the slab with uniform velocity and which is scattered at the boundary via the reflection law

$$(v, w) \longrightarrow (v', w')$$

corresponding to the probability density

$$B(v', w') = \frac{2|w'|}{\overline{|w|}} \cdot m(v', w')$$

with the mean value

$$\overline{|w|} = \sum_{j=1}^{N} |w_j| m(v_j, w_j).$$

The motion of a test particle is now given as a sum of independent random increments Δx between two contacts of the wall. The time τ between is a random variable with mean value

$$\overline{\tau} = \frac{2h}{\overline{|w|}} \ .$$

Δx has expectation 0 and variance

$$\overline{(\Delta x)^2} = \frac{4h^2}{\overline{|w|}} \cdot \overline{v^2/w} \ .$$

For t large, the number n of contacts with the wall is approximately

$$n \approx \frac{t}{\overline{\tau}}$$

and the variance of the x-position is about

$$\overline{x^2(t)} \approx n \cdot \overline{(\Delta x)^2} \ .$$

Inserting the scaling (1.2) we find

$$\overline{(x')^2(t')} \underset{\varepsilon \to 0}{\longrightarrow} 2ht' \cdot \overline{v^2/w} \ .$$

The linear in time increase of the variance is modeled by the diffusion equation

$$\partial_t \phi = h \cdot \overline{v^2/w} \cdot \partial_x^2 \phi.$$

Such a scaling is appropriate in the case of finite variances. For infinite variance, the scaling has to be modified in order to obtain a diffusion limit [9].

1.1.2 A Lorenz gas

Now we consider the motion of a test particle in a field of randomly distributed scattering particles with density ψ. The test particle hits at exponentially distributed random times τ (given by density $\psi \cdot \exp(-\psi t)$) a field particle and is scattered according to the probability distribution m. (In particular, the scattering result is independent of the velocity (v, w) which is a considerable simplification!) The corresponding linear transport equation reads

$$(\partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}}) f(t, \mathbf{x}, \mathbf{v}) = m(\mathbf{v}) \cdot \psi \int f(t, \mathbf{x}, \mathbf{v}') \, d\mathbf{v}' - \psi f(t, \mathbf{x}, \mathbf{v}').$$

Suppose the particle moves in the gap $-h \leq y \leq h$ and is scattered corresponding to the density $\mathbf{m}$ when hitting the wall. We assume h to be very small and thus suppress the possibility that the particle hits twice the scattering medium between two contacts with the wall. Calculating the mean time and the variance of the x-displacement between two contacts with the wall one easily finds

$$\begin{aligned} \overline{\tau} &= 2h \cdot \frac{1}{\overline{|w|}} - 2\psi h^2 \cdot \frac{\overline{1/|w|}}{\overline{|w|}} + \mathrm{O}(h^3), \\ \overline{(\Delta x)^2} &= 4h^2 \cdot \frac{\overline{v^2/|w|}}{\overline{|w|}} - \frac{16\psi h^3}{3} \cdot \frac{\overline{v^2/w^2}}{\overline{|w|}} + \mathrm{O}(h^4). \end{aligned}$$

Following the arguments above and the scaling, this leads to a diffusion equation with diffusion coefficient

$$h \cdot \overline{v^2/w} + \psi h^2 \left(\frac{\overline{1/w}}{\overline{w}} - \frac{4}{3} \right) \cdot \overline{v^2/w^2} + \mathrm{O}(h^3).$$

2 Collision interaction model

In this section we define a collision interaction model on the discrete velocity set $\mathcal{V}$ (see (1.1)). Velocity pairs $(\mathbf{z}_1, \mathbf{z}_2) \in \mathcal{V} \times \mathcal{V}$ may interchange due to collisions:

$$(\mathbf{z}_1, \mathbf{z}_2) \rightarrow (\mathbf{z}_1', \mathbf{z}_2')$$

with a rate $k(\mathbf{z}_1, \mathbf{z}_2 | \mathbf{z}_1', \mathbf{z}_2')$ satisfying the *microreversibility law*

$$k(\mathbf{z}_1, \mathbf{z}_2 | \mathbf{z}_1', \mathbf{z}_2') = k(\mathbf{z}_1', \mathbf{z}_2' | \mathbf{z}_1, \mathbf{z}_2).$$

We require conservation of momentum:

$$k(\mathbf{z}_1, \mathbf{z}_2 | \mathbf{z}_1', \mathbf{z}_2') > 0 \Rightarrow \mathbf{z}_1 + \mathbf{z}_2 = \mathbf{z}_1' + \mathbf{z}_2'.$$

The corresponding kinetic equations for a density function $\mathbf{f} = \mathbf{f}(t, \mathbf{x}, \mathbf{z})$ on some phase space $\Omega \times \mathcal{V}$ read with $\mathbf{z} = (v, w)$

$$(\partial_t + v\partial_x + w\partial_y)\mathbf{f}(\mathbf{z}) = \gamma \cdot \sum_{\mathbf{z}_1', \mathbf{z}_2' \in \mathcal{V}} k(\mathbf{z}_1, \mathbf{z}_2 | \mathbf{z}_1', \mathbf{z}_2') \mathbf{f}(\mathbf{z}_1') \mathbf{f}(\mathbf{z}_2') - \gamma \cdot \mathbf{f}(\mathbf{z}_1) \cdot \sum_{\mathbf{z}_2 \in \mathcal{V}} K(\mathbf{z}_1, \mathbf{z}_2) \cdot \mathbf{f}(\mathbf{z}_2),$$

where

$$K(\mathbf{z}_1, \mathbf{z}_2) = \sum_{\mathbf{z}_1', \mathbf{z}_2'} k(\mathbf{z}_1', \mathbf{z}_2' | \mathbf{z}_1, \mathbf{z}_2).$$

Some more properties of the collision model are required.

(a) Symmetry conditions: Suppose the velocity set is symmetric with respect to the x- and y-axes:

$$(v, w) \in \mathcal{V} \Rightarrow (\sigma_1 v, \sigma_2 w) \in \mathcal{V} \quad \text{for all} \quad \sigma_1, \sigma_2 \in \{-1, 1\}.$$

We assume that the collision kernel reflects this symmetry in the following sense:
Let T be one of the reflection operators

$$T_x(v, w) = (-v, w), \quad T_y(v, w) = (v, -w), \quad T_{xy}(v, w) = (-v, -w).$$

Then we require

Assumption 2.1 *For all* $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}'_1, \mathbf{z}'_2 \in \mathcal{V}$

$$k(T\mathbf{z}_1, T\mathbf{z}_2 | T\mathbf{z}'_1, T\mathbf{z}'_2) = k(\mathbf{z}_1, \mathbf{z}_2 | \mathbf{z}'_1, \mathbf{z}'_2).$$

(b) The linearized system: Suppose $\mathbf{m}$ is an equilibrium solution of the collision operator satisfying the symmetry condition

$$m(v, w) = m(\sigma_1 v, \sigma_2 w) \quad \text{for all} \quad \sigma_1, \sigma_2 \in \{-1, 1\} \tag{2.1}$$

and denote by $\tilde{M}$ the collision operator linearized around $\mathbf{m}$. We introduce in $\mathcal{V}$ a numbering such that

$$(v_{d+i}, w_{d+i}) = (v_i, -w_i) \quad \forall i \in \{1, \dots, d\}.$$

This implicitly requires that $w_i \neq 0$ for $i = 1, \dots, N$. Then, due to Assumption 2.1, $\tilde{M}$ has the form

$$\tilde{M} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{B} & \tilde{A} \end{pmatrix}. \tag{2.2}$$

The matrix $\tilde{M}$ is in general not symmetric. However, if we denote

$$D := diag(m_i, i = 1, \dots, N),$$

then a symmetry property of the original collision operator is expressed by

Assumption 2.2 $\tilde{M}$ *satisfies*

$$\tilde{M} D = D \tilde{M}^{\mathrm{T}}$$

or, equivalently,

$$D^{-1/2} \tilde{M} D^{1/2} \quad \textit{is symmetric.}$$

This assumption reflects a typical property of the classical linearized Boltzmann collision operator, see [10, Sect. 7.1].

If we define the matrix W as follows

$$W = \mathrm{diag}(w_i, i = 1, \dots, N) = \begin{pmatrix} W_d & \\ & -W_d \end{pmatrix}$$

then the matrix $M = W^{-1} \tilde{M}$ has the form

$$M = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}. \tag{2.3}$$

Boundary value problems with discrete structures as that of (2.2) or (2.3) have been analyzed in [5, 8]. The Jordan structure of M plays a prominent role in our investigations. It turns out that

i) $-\lambda$ is an eigenvalue of M if λ is,

ii) the algebraic multiplicity of the eigenvalue 0 is larger than the geometric one.

A conclusion of the latter property is that the nullspace of M consists of Jordan blocks of dimension larger than one. In accordance to the properties of the classical linearized Boltzmann collision operator we require

Assumption 2.3 *The matrix M is similar to a matrix L in a Jordan normal form*

$$L = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & \Lambda & & \\ & & & -\Lambda & \\ & & & & J \end{pmatrix} \tag{2.4}$$

with $\Lambda = diag(\lambda_1, \ldots, \lambda_k)$, $\lambda_i > 0$ and J denoting Jordan blocks:

$$J = \begin{pmatrix} J_2 & & \\ & \ddots & \\ & & J_2 \end{pmatrix} \quad with\ J_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

where $k \geq 0$ and $l \geq 1$ is the number of J_2 blocks in J.

If $k = 0$ then there are no blocks Λ and $-\Lambda$. In the case that the only invariants of the collision operator are mass, the two momenta and kinetic energy, one can show that there are two blocks J_2.

3 Diffusion limit for a general model

For a given density vector $\mathbf{f} \in \mathbb{R}^{\mathcal{V}}$ we define a (quadratic) collision operator $Q[\mathbf{f}] = (Q_i[\mathbf{f}])_{i=1}^N$ of the form

$$Q_i[\mathbf{f}] = \sum_{j,k,l} A_{ij}^{kl}(f_k f_l - f_i f_j)$$

with A_{ij}^{kl} such that Q reflects the symmetry of the grid in the sense of Assumption 2.1. (More complex models including e.g. ternary collisions like the hexagonal models as

described in [1] are easily included, since the basic equations to be solved are *linearized* kinetic equations.)
We introduce some notation which will be used throughout the paper

$$\begin{aligned}
\mathbb{1}_{\mathrm{d}} &:= (1,\dots,1)^{\mathrm{T}} \in \mathbb{R}^d, \\
\mathbb{1} &:= \begin{pmatrix} \mathbb{1}_{\mathrm{d}} \\ \mathbb{1}_{\mathrm{d}} \end{pmatrix}, \\
\mathbf{v} &:= (v_1,\dots,v_N)^{\mathrm{T}}, \\
\mathbf{w} &:= (w_1,\dots,w_N)^{\mathrm{T}}, \\
V &:= \begin{pmatrix} V_1 & 0 \\ 0 & V_1 \end{pmatrix}, \quad V_1 = diag(v_1,\dots,v_d), \\
W &:= \begin{pmatrix} W_1 & 0 \\ 0 & -W_1 \end{pmatrix}, \quad W_1 = diag(w_1,\dots,w_d).
\end{aligned}$$

The starting point of our investigations is the kinetic equation for density functions $\mathbf{f} = \mathbf{f}(t,x,y)$

$$(\partial_t + v_i\partial_x + w_i\partial_y) f_i = Q_i[\mathbf{f}].$$

Inserting the scaling (1.2) we get the following equation

$$\varepsilon^2\, \partial_t \mathbf{f}^\varepsilon + \varepsilon\, V \cdot \partial_x \mathbf{f}^\varepsilon + W \cdot \partial_y \mathbf{f}^\varepsilon = Q[\mathbf{f}^\varepsilon]. \tag{3.1}$$

We are interested in solutions of (3.1) as $\varepsilon \to 0$. Hence we consider the reduced system

$$W \cdot \partial_y \mathbf{f} = Q[\mathbf{f}], \tag{3.2}$$

where $\mathbf{f}$ depends only on y.

Suppose

$$\mathbf{m}^{(0)} = \begin{pmatrix} \mathbf{m}_d^{(0)} \\ \mathbf{m}_d^{(0)} \end{pmatrix}, \quad \mathbf{m}_d^{(0)} = (m_1,\dots,m_d)$$

is an equilibrium solution of Q with nonnegative coefficients and the first moments

$$\langle \mathbf{v}, \mathbf{m}^{(0)} \rangle = \langle \mathbf{w}, \mathbf{m}^{(0)} \rangle = 0, \tag{3.3}$$

normalized such that $\langle \mathbf{m}^{(0)}, \mathbb{1} \rangle = 1$. In the sequel we always assume the symmetry condition for $\mathbf{m}^{(0)}$ given by (2.1).

For a given function $\mathbf{f} : [-h,h] \to \mathbb{R}^N$ we denote the vectors of components pointing into the slab $[-h,h]$ by

$$\begin{aligned}
\mathbf{f}^{in}(-h) &= (f_1(-h),\dots,f_d(-h))^{\mathrm{T}}, \\
\mathbf{f}^{in}(h) &= (f_{d+1}(h),\dots,f_{2d}(h))^{\mathrm{T}}.
\end{aligned}$$

We consider the equation (3.2) under the *diffuse boundary conditions*

$$\mathbf{f}^{in}(\pm h) \parallel \mathbf{m}_d^{(0)}, \tag{3.4}$$

$$\langle \mathbf{f}(\pm h), \mathbf{w} \rangle = 0, \tag{3.5}$$

which we will denote for brevity as $\mathbf{f}_+ = \mathcal{B}\mathbf{f}_-$. The equation (3.5) states that the flux at the boundaries in y-direction is zero. These conditions describe complete diffuse reflection at the boundaries.

In order to study the linearized version of (3.2) we need the following assumption.

Assumption 3.1 *The problem*

$$W \cdot \partial_y \mathbf{f} = Q[\mathbf{f}], \quad \mathbf{f}_+ = \mathcal{B}\mathbf{f}_-$$

admits at least one solution.

Suppose a solution $\mathbf{f} = \mathbf{f}(y)$ of (3.2) is of the form

$$\mathbf{f} = \varphi \mathbf{m}^{(0)} + \varepsilon \mathbf{g}$$

with a small parameter $\varepsilon > 0$. Then linearization of Q yields an equation for $\mathbf{g}$ of the form

$$W \cdot \partial_y \mathbf{g} = \varphi \tilde{M} \mathbf{g}.$$

with the matrix $\tilde{M}$ satisfying the symmetry condition of Assumption 2.2.

In the following we denote a vector $\mathbf{u} = (u_i)_{i=1}^N = (\mathbf{u}_1, \mathbf{u}_2)^{\mathrm{T}}$ as *even*, if $\mathbf{u}_1 = \mathbf{u}_2$, and *odd*, if $\mathbf{u}_1 = -\mathbf{u}_2$.

Due to the physical conservation laws $\tilde{M}$ is physically reasonable if $\tilde{M}^{\mathrm{T}}$ has the invariants $\mathbb{1}$, $\mathbf{v}$ and $(0.5(v_i^2 + w_i^2); i = 1, \ldots, N)^{\mathrm{T}}$ (even vectors; conservation of mass, x-momentum and energy) and $\mathbf{w}$ (odd vector; conservation of y-momentum).
Hence we have

$$\mathbb{1},\, \mathbf{v},\, \mathbf{w} \in ker \tilde{M}^{\mathrm{T}}. \tag{3.6}$$

and it follows from Assumption 2.2 that

$$D\mathbb{1},\, D\mathbf{v},\, D\mathbf{w} \in ker \tilde{M}.$$

We define a matrix M:

$$M := W^{-1} \cdot \tilde{M} = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}$$

and suppose that the linearized reduced equation satisfies:

Assumption 3.2 *The homogeneous problem*

$$\partial_y \mathbf{f} = \varphi \cdot M\mathbf{f}, \quad \mathbf{f}_+ = \mathcal{B}\mathbf{f}_-$$

has a one-dimensional solution space spanned by the equilibrium $\mathbf{m}^{(0)}$.

To study the inhomogeneous problem we need to define for functions $\mathbf{f}, \mathbf{g} : [-h, h] \to \mathbb{R}^N$ the scalar product

$$\langle \mathbf{f}, \mathbf{g} \rangle_h := \frac{1}{4hd} \int_{-h}^{h} \langle \mathbf{f}(y), \mathbf{g}(y) \rangle \, dy.$$

The following lemma is a straightforward consequence of the above assumption and the definition of the no-flux condition (3.5).

Lemma 3.1 *If the inhomogeneous problem*

$$\partial_y \mathbf{f} = \varphi \cdot M\mathbf{f} + \mathbf{g}, \quad \mathbf{f}_+ = \mathcal{B}\mathbf{f}_- \tag{3.7}$$

for a given function $\mathbf{g}$ *is solvable then* $\mathbf{g}$ *satisfies:*

$$\langle \mathbf{g}, \mathbf{w} \rangle_h = 0. \tag{3.8}$$

Proof. Let $\mathbf{f}$ be a solution of (3.7). Then we have the following equality

$$\partial_y \langle \mathbf{f}, \mathbf{w} \rangle = \varphi \langle M\mathbf{f}, \mathbf{w} \rangle + \langle \mathbf{g}, \mathbf{w} \rangle.$$

Using (3.6) for the first term of the right-hand side we get

$$\varphi \langle M\mathbf{f}, \mathbf{w} \rangle = \varphi \langle \mathbf{f}, M^{\mathrm{T}}\mathbf{w} \rangle = \varphi \langle \mathbf{f}, \tilde{M}^{\mathrm{T}} W^{-1}\mathbf{w} \rangle = \varphi \langle \mathbf{f}, \tilde{M}^{\mathrm{T}} \mathbb{1} \rangle = 0.$$

Hence solving the differential equation we obtain for $y = h$:

$$\varphi(\langle \mathbf{f}(h), \mathbf{w} \rangle - \langle \mathbf{f}(-h), \mathbf{w} \rangle) = \int_{-h}^{h} \langle \mathbf{g}(y), \mathbf{w} \rangle \, dy.$$

This gives the desired result since from the no-flux condition (3.5) the left-hand side of this equality is zero. □

In what follows the condition given by equation (3.8) will be called *the compatibility condition*. As we shall show later on it is also the sufficient condition for solvability of (3.7) under suitably chosen assumptions.

Series expansion

We introduce the following formal series expansion

$$\mathbf{f}^{\varepsilon}(t,x,y) = \varphi(t,x)\cdot \mathbf{m}^{(0)} + \varepsilon\,\mathbf{f}^1(t,x,y) + \varepsilon^2\,\mathbf{f}^2(t,x,y) + \mathrm{O}(\varepsilon^3),$$

where we take $\mathbf{f}^i$ such that

$$\langle \mathbf{f}^i, \mathbf{m}^{(0)}\rangle_h = 0, \quad i = 1,2. \tag{3.9}$$

Inserting $\mathbf{f}^{\varepsilon}(t,x,y)$ into (3.1) and comparing respective powers of ε we have formally

$$\varepsilon^1: \quad W\partial_y \mathbf{f}^1 - \varphi\cdot\tilde{M}\mathbf{f}^1 = -\partial_x\varphi\cdot V\mathbf{m}^{(0)}, \tag{3.10}$$

$$\varepsilon^2: \quad W\partial_y \mathbf{f}^2 - \varphi\cdot\tilde{M}\mathbf{f}^2 = -\partial_t\varphi\cdot \mathbf{m}^{(0)} - V\partial_x\mathbf{f}^1 + Q(\mathbf{f}^1), \tag{3.11}$$

where the terms of higher order in ε are neglected. The equation with ε^0 is omitted because it is identity for $\varphi(t,x)\cdot\mathbf{m}^{(0)}$. The right hand side of (3.10) satisfies the compatibility condition since

$$\langle W^{-1}V\mathbf{m}^{(0)}, \mathbf{w}\rangle = \langle V\mathbf{m}^{(0)}, \mathbb{1}\rangle = \langle \mathbf{m}^{(0)}, \mathbf{v}\rangle = 0,$$

where we used (3.3) for the last equality. The compatibility condition applied to the equation (3.11) reduces to the following equality:

$$\langle \partial_t\varphi\cdot\mathbf{m}^{(0)} + V\cdot\partial_x\mathbf{f}^1, \mathbb{1}\rangle_h = 0. \tag{3.12}$$

We choose the basis of $\mathbb{R}^N$ denoted by

$$\{\tilde{\mathbf{b}}_i\} := \{\tilde{\mathbf{b}}_0^{\pm}, \tilde{\mathbf{b}}_1^{+}, \ldots, \tilde{\mathbf{b}}_k^{+}, \tilde{\mathbf{b}}_1^{-}, \ldots, \tilde{\mathbf{b}}_k^{-}, \tilde{\mathbf{b}}_{k+1}^{\pm}, \ldots, \tilde{\mathbf{b}}_{k+l}^{\pm}\}$$

such that $\{\tilde{\mathbf{b}}_0^{\pm}\}$, $\{\tilde{\mathbf{b}}_{k+1}^{+}, \ldots, \tilde{\mathbf{b}}_{k+l}^{+}\}$ and $\{\tilde{\mathbf{b}}_1^{+}, \ldots, \tilde{\mathbf{b}}_k^{+}\}$, $\{\tilde{\mathbf{b}}_1^{-}, \ldots, \tilde{\mathbf{b}}_k^{-}\}$ are eigenvectors of M corresponding to the eigenvalues 0 ($l{+}2$ times) and $\lambda_1, \ldots, \lambda_k$, $-\lambda_1, \ldots, -\lambda_k$ respectively. Moreover $\{\tilde{\mathbf{b}}_{k+1}^{-}, \ldots, \tilde{\mathbf{b}}_{k+l}^{-}\}$ are vectors satisfying $M\tilde{\mathbf{b}}_{k+i}^{-} = \tilde{\mathbf{b}}_{k+i}^{+}$ for $i = 1, \ldots, l$. We take also $\tilde{\mathbf{b}}_0^{+} = D\mathbb{1}$, $\tilde{\mathbf{b}}_0^{-} = \beta\cdot D\mathbf{w}$ and $\tilde{\mathbf{b}}_{k+1}^{+} = D\mathbf{v}$, where the constant β is taken such that $\langle\tilde{\mathbf{b}}_0^{-}, \mathbf{w}\rangle = 1$. We use as a convention the notation $\mathbf{b}_i^{\pm}$ in place of two consecutive vectors $\mathbf{b}_i^{+}, \mathbf{b}_i^{-}$. Obviously, the matrix M takes the form (2.4) in the new basis.

To simplify the calculation of $\mathbf{f}^1$ we transform the basis $\{\tilde{\mathbf{b}}_i\}$ to obtain the new basis $\{\mathbf{b}_i\}$. The new basis will consist of only even and odd vectors. We do this changing only for the vectors corresponding to the nonzero eigenvalues:

$$\mathbf{b}_i^{+} = \tilde{\mathbf{b}}_i^{+} + \tilde{\mathbf{b}}_i^{-}, \qquad \mathbf{b}_i^{-} = \tilde{\mathbf{b}}_i^{+} - \tilde{\mathbf{b}}_i^{-}, \qquad \text{for } i = 1, \ldots, k. \tag{3.13}$$

Then the basis vectors take the following form

$$\mathbf{b}_0^+ = \begin{pmatrix} \widehat{\mathbf{b}}_0^+ \\ \widehat{\mathbf{b}}_0^+ \end{pmatrix}, \qquad \mathbf{b}_0^- = \begin{pmatrix} \widehat{\mathbf{b}}_0^- \\ -\widehat{\mathbf{b}}_0^- \end{pmatrix},$$
$$\vdots \qquad\qquad \vdots$$
$$\mathbf{b}_{d'}^+ = \begin{pmatrix} \widehat{\mathbf{b}}_{d'}^+ \\ \widehat{\mathbf{b}}_{d'}^+ \end{pmatrix}, \qquad \mathbf{b}_{d'}^- = \begin{pmatrix} \widehat{\mathbf{b}}_{d'}^- \\ -\widehat{\mathbf{b}}_{d'}^- \end{pmatrix},$$

where $d' = d - 1$. Obviously $\{\mathbf{b}_i^+\}$ and $\{\mathbf{b}_i^-\}$ are bases of "even" and "odd" parts of $\mathbb{R}^N$ respectively. The matrix M in the new basis changes to

$$L = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \bar{\Lambda} & \\ & & & J \end{pmatrix},$$

where

$$\bar{\Lambda} = \begin{pmatrix} 0 & \lambda_1 & & & \\ \lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \lambda_k \\ & & & \lambda_k & 0 \end{pmatrix}.$$

One can easily check that the following lemma holds.

Lemma 3.2 *Both systems of vectors $\{\widehat{\mathbf{b}}_i^+\}$ and $\{\widehat{\mathbf{b}}_i^-\}$ form bases of $\mathbb{R}^d$.*

We denote by

$$\tilde{T} := (\tau_{ij})_{0 \le i,j \le d'}$$

the matrix of change of basis in $\mathbb{R}^d$, defined by

$$\widehat{\mathbf{b}}_i^- = \sum_{j=0}^{d'} \tau_{ij} \, \widehat{\mathbf{b}}_j^+.$$

We will also use the following notation

$$\begin{gathered} T := (\tau_{ij})_{1 \le i,j \le d'}, \quad T_0 := (\tau_{01}, \dots, \tau_{0d'})^{\mathrm{T}}, \quad T_i := (\tau_{1i}, \dots, \tau_{d'i})^{\mathrm{T}}, \\ \boldsymbol{\sigma} := (\sigma_1, \dots, \sigma_{d'})^{\mathrm{T}}, \quad S := \boldsymbol{\sigma} \cdot T_0^{\mathrm{T}}, \end{gathered} \tag{3.14}$$

where $\sigma_i = \langle \mathbf{b}_i^-, \mathbf{w} \rangle$ and use the following assumption.

Assumption 3.3 *The matrix $T - S$ is regular.*

Since the right hand side of equation (3.10) is independent of y, we can restrict our analysis of the problem (3.7) to the case, where the function $\mathbf{g}$ does not depend on y either. Hence we can write $\mathbf{g}$ in our basis $\{\mathbf{b}_i\}$ as

$$\mathbf{g} = \sum_{i=0}^{d'} r_i^{\pm} \mathbf{b}_i^{\pm},$$

where $r_i^{\pm} = r_i^{\pm}(t, x)$. Assume also that $\mathbf{f}^1$ has the following form:

$$\mathbf{f}^1(y) = \sum_{i=0}^{d'} \mu_i^{\pm}(y) \mathbf{b}_i^{\pm}, \tag{3.15}$$

where, for brevity, we omit the dependence of $\mathbf{f}^1$ and $\mu_i^{\pm}$ on t and x. The problem (3.7) in this case reads as follows

$$\partial_y \boldsymbol{\mu} = \varphi L \boldsymbol{\mu} + \mathbf{r}, \tag{3.16}$$

where $\boldsymbol{\mu} = (\mu_0^{\pm}, \ldots, \mu_{d'}^{\pm})^{\mathrm{T}}$ and $\mathbf{r} = (r_0^{\pm}, \ldots, r_{d'}^{\pm})^{\mathrm{T}}$, with boundary conditions suitably transformed for $\boldsymbol{\mu}$.

Lemma 3.3 *The boundary conditions for the system (3.16) are given by*

$$\begin{aligned} \boldsymbol{\mu}^+(-h) + (T - S)^{\mathrm{T}} \boldsymbol{\mu}^-(-h) &= 0, \\ \boldsymbol{\mu}^+(h) - (T - S)^{\mathrm{T}} \boldsymbol{\mu}^-(h) &= 0, \end{aligned} \tag{3.17}$$

$$\mu_0^-(\pm h) + \langle \boldsymbol{\sigma}, \boldsymbol{\mu}^-(\pm h) \rangle = 0, \tag{3.18}$$

where $\boldsymbol{\mu}^+ = (\mu_1^+, \ldots, \mu_{d'}^+)^{\mathrm{T}}$ and $\boldsymbol{\mu}^- = (\mu_1^-, \ldots, \mu_{d'}^-)^{\mathrm{T}}$.

Proof. From (3.15) and the definition of σ_i the no-flux condition (3.5) can be written as

$$0 = \langle \mathbf{f}^1(\pm h), \mathbf{w} \rangle = \sum_{i=0}^{d'} \mu_i^{\pm}(\pm h) \langle \mathbf{b}_i^{\pm}, \mathbf{w} \rangle = \sum_{i=0}^{d'} \mu_i^-(\pm h) \sigma_i, \tag{3.19}$$

where we used also $\langle \mathbf{b}_i^+, \mathbf{w} \rangle = 0$. This gives the equation (3.18) since $\langle \mathbf{b}_0^-, \mathbf{w} \rangle = 1$.

Using the definition of vectors $\mathbf{b}_i^+$ and $\mathbf{b}_i^-$ we can now calculate the fluxes $\mathbf{f}^{in}(\pm h)$ into the slab as follows

$$\begin{aligned} \mathbf{f}^{in}(\pm h) &= \sum_{i=0}^{d'} \mu_i^+(\pm h) \widehat{\mathbf{b}}_i^+ \mp \sum_{i=0}^{d'} \mu_i^-(\pm h) \widehat{\mathbf{b}}_i^- \\ &= \sum_{i=0}^{d'} \mu_i^+(\pm h) \widehat{\mathbf{b}}_i^+ \mp \sum_{i=0}^{d'} \mu_i^-(\pm h) \sum_{j=0}^{N} \tau_{ij} \widehat{\mathbf{b}}_j^+ \\ &= \sum_{i=0}^{d'} \Big[\mu_i^+(\pm h) \mp \sum_{j=0}^{d'} \mu_j^-(\pm h) \tau_{ji} \Big] \widehat{\mathbf{b}}_i^+. \end{aligned}$$

The boundary condition (3.4) is now equivalent to the following system

$$\left.\begin{aligned} \mu_i^+(-h) + \sum_{j=0}^{d'} \mu_j^-(-h)\tau_{ji} = 0 \\ \mu_i^+(h) - \sum_{j=0}^{d'} \mu_j^-(h)\tau_{ji} = 0 \end{aligned}\right\} \quad i = 1, \ldots, d',$$

which using the formulation of the no-flux condition as in (3.19) can be written as

$$\left.\begin{aligned} \mu_i^+(-h) + \sum_{j=1}^{d'} \mu_j^-(-h)(\tau_{ji} - \sigma_j \tau_{0i}) = 0 \\ \mu_i^+(h) - \sum_{j=1}^{d'} \mu_j^-(h)(\tau_{ji} - \sigma_j \tau_{0i}) = 0 \end{aligned}\right\} \quad i = 1, \ldots, d'.$$

This gives the equations (3.17). □

The compatibility condition (3.8) rewritten for the problem (3.16) has the following form

$$\sum_{i=0}^{d'} \sigma_i r_i^- = 0. \tag{3.20}$$

The following lemma gives the solution to the problem (3.16).

Lemma 3.4 *Let* $\mathbf{r}$ *satisfy the compatibility condition (3.20). Moreover assume*

$$h \ll 1. \tag{3.21}$$

Then there exists a one-parameter family of solutions of the problem (3.16) with the boundary conditions (3.17), (3.18) that is given by

$$\begin{aligned}
&\mu_0^+(y) = y r_0^+ + c_0^+, \\
&\mu_0^-(y) = (y+h) r_0^- + h\langle \boldsymbol{\sigma}, \mathbf{r}^- \rangle \\
&\qquad\qquad -h\langle \boldsymbol{\sigma}_{1..k}, (T-S)_{\mathrm{I}}^{-\mathrm{T}} \mathbf{r}_{1..k}^+ \rangle - h\langle \boldsymbol{\sigma}_{k+1..d'}, \mathcal{R} \rangle, \\
&\boldsymbol{\mu}_{1..k}^+(y) = \tfrac{1}{2}(y^2 - h^2)\varphi \Lambda \mathbf{r}_{1..k}^- + y \mathbf{r}_{1..k}^+ + hy\varphi\Lambda (T-S)_{\mathrm{I}}^{-\mathrm{T}} \mathbf{r}_{1..k}^+ \\
&\qquad\qquad +h(T-S)_{\mathrm{I,III}}^{\mathrm{T}} \mathbf{r}^- + h^2 \varphi (T-S)_{\mathrm{I}}^{\mathrm{T}} \Lambda (T-S)_{\mathrm{I,III}}^{\mathrm{T}} \mathbf{r}^-, \\
&\boldsymbol{\mu}_{1..k}^-(y) = \tfrac{1}{2} y^2 \varphi \Lambda \mathbf{r}_{1..k}^+ + y \mathbf{r}_{1..k}^- + h(T-S)_{\mathrm{I}}^{-\mathrm{T}} \mathbf{r}_{1..k}^+ \\
&\qquad\qquad +hy\varphi\Lambda(T-S)_{\mathrm{I,III}}^{\mathrm{T}} \mathbf{r}^-, \\
&\boldsymbol{\mu}_{k+1..d'}^+(y) = \tfrac{1}{2}(y^2 - h^2)\varphi \mathbf{r}_{k+1..d'}^- + y \mathbf{r}_{k+1..d'}^+ + hy\varphi\mathcal{R} \\
&\qquad\qquad +h(T-S)_{\mathrm{II,IV}}^{\mathrm{T}} \mathbf{r}^- + h^2\varphi(T-S)_{\mathrm{II}}^{\mathrm{T}} \Lambda (T-S)_{\mathrm{I,III}}^{\mathrm{T}} \mathbf{r}^-, \\
&\boldsymbol{\mu}_{k+1..d'}^-(y) = y \mathbf{r}_{k+1..d'}^- + h\mathcal{R},
\end{aligned} \tag{3.22}$$

with

$$\mathcal{R} = (T-S)_{\mathrm{IV}}^{-\mathrm{T}} \left(\mathbf{r}_{k+1..d'}^- - (T-S)_{\mathrm{II}}^{\mathrm{T}} (T-S)_{\mathrm{I}}^{-\mathrm{T}} \mathbf{r}_{1..k}^+ \right),$$

where the constant c_0^+ *is a parameter of the family and we use a representation of the matrix* $T-S$*:*

$$T-S = \begin{pmatrix} (T-S)_{\mathrm{I}} & (T-S)_{\mathrm{II}} \\ (T-S)_{\mathrm{III}} & (T-S)_{\mathrm{IV}} \end{pmatrix}.$$

Proof. It is easy to find a general solution to the system (3.16) of the following form

$$\begin{aligned}
&\mu_0^+(y) = yr_0^+ + c_0^+,\\
&\mu_0^-(y) = (y+h)r_0^- + c_0^-,\\
&\left.\begin{aligned}\mu_i^+(y) &= c_i^+\cosh(y\lambda_i\varphi) + c_i^-\sinh(y\lambda_i\varphi) - \tfrac{r_i^-}{\lambda_i\varphi},\\ \mu_i^-(y) &= c_i^-\cosh(y\lambda_i\varphi) + c_i^+\sinh(y\lambda_i\varphi) - \tfrac{r_i^+}{\lambda_i\varphi},\end{aligned}\right\}\ i=1,\dots,k,\\
&\left.\begin{aligned}\mu_i^+(y) &= \tfrac{1}{2}y^2\varphi r_i^- + yr_i^+ + y\varphi c_i^- + c_i^+,\\ \mu_i^-(y) &= yr_i^- + c_i^-,\end{aligned}\right\}\ i=k+1,\dots,d'.
\end{aligned} \tag{3.23}$$

To calculate the constants in (3.23) we apply the boundary conditions (3.17) formulated in an equivalent form

$$\left.\begin{aligned}
\mu_i^+(h) + \mu_i^+(-h) - \sum_{i=1}^{d'}\big(\mu_i^-(h) - \mu_i^-(-h)\big)(\tau_{ji} - \sigma_j\tau_{0i}) = 0\\
\mu_i^+(h) - \mu_i^+(-h) - \sum_{i=1}^{d'}\big(\mu_i^-(h) + \mu_i^-(-h)\big)(\tau_{ji} - \sigma_j\tau_{0i}) = 0
\end{aligned}\right\}\ i=1,\dots,d', \tag{3.24}$$

which is obtained by adding and subtracting the respective equations in (3.17). Substituting in (3.24) $\mu_i^\pm(\pm h)$ calculated as in (3.23) we get two systems of equations for $\mathbf{c}^+ := (c_1^+,\dots,c_{d'}^+)^{\mathrm{T}}$ and $\mathbf{c}^- := (c_1^-,\dots,c_{d'}^-)^{\mathrm{T}}$:

$$\begin{aligned} P^+\mathbf{c}^+_{1..k} &= \mathbf{u}^+_{1..k},\\ P^-\mathbf{c}^- &= \mathbf{u}^-, \end{aligned} \tag{3.25}$$

with

$$\begin{gathered}
P^+ = D_{ch} - (T-S)_{\mathrm{I}}^{\mathrm{T}}D_{sh},\\
P^- = \begin{pmatrix} D_{sh} - (T-S)_{\mathrm{I}}^{\mathrm{T}}D_{ch} & -(T-S)_{\mathrm{III}}^{\mathrm{T}} \\ -(T-S)_{\mathrm{II}}^{\mathrm{T}}D_{ch} & h\varphi\, Id - (T-S)_{\mathrm{IV}}^{\mathrm{T}} \end{pmatrix},\\
\mathbf{u}^+ = \tilde{\mathbf{r}}^-_{1..k} + h(T-S)_{\mathrm{III}}^{\mathrm{T}}\mathbf{r}^-_{k+1..d'},\\
\mathbf{u}^- = \begin{pmatrix} -(T-S)_{\mathrm{I}}^{\mathrm{T}}\tilde{\mathbf{r}}^+_{1..k} \\ -(T-S)_{\mathrm{II}}^{\mathrm{T}}\tilde{\mathbf{r}}^+_{1..k} - h\mathbf{r}^-_{k+1..d'} \end{pmatrix},\\
\mathbf{c}^+_{k+1..d'} = (T-S)_{\mathrm{II}}^{\mathrm{T}}D_{sh}\mathbf{c}^+_{1..k} + \left(-\tfrac{1}{2}h^2\varphi\, Id + h(T-S)_{\mathrm{IV}}^{\mathrm{T}}\right)\mathbf{r}^-_{k+1..d'},
\end{gathered} \tag{3.26}$$

where

$$\begin{gathered} D_{ch} = diag(\cosh(h\lambda_i\varphi)_{i=1,\dots,k}), \quad D_{sh} = diag(\sinh(h\lambda_i\varphi)_{i=1,\dots,k}),\\ \tilde{\mathbf{r}}^-_{1..k} = (\tfrac{r_1^\pm}{\lambda_1\varphi},\dots,\tfrac{r_k^\pm}{\lambda_k\varphi})^{\mathrm{T}}. \end{gathered}$$

We will solve the systems (3.25) under the assumption (3.21) with $\mathrm{O}(h^3)$ and $\mathrm{O}(h^2)$ approximations for calculating the vectors $\mathbf{c}^+$ and $\mathbf{c}^-$ respectively. Thus we approximate $\cosh(x) = 1 + \frac{1}{2}x^2 + \mathrm{O}(x^4)$ to calculate $\mathbf{c}^+$, $\cosh(x) = 1 + \mathrm{O}(x^2)$ to calculate $\mathbf{c}^-$ and $\sinh(x) = x + \mathrm{O}(x^3)$ for both unknown vectors. In this case (3.26) can be rewritten as follows

$$\begin{aligned} P^+ &= Id - h\varphi(T-S)_{\mathrm{I}}^{\mathrm{T}}\Lambda + \tfrac{1}{2}h^2\varphi^2\Lambda^2, \\ P^- &= \begin{pmatrix} -h\varphi\Lambda - (T-S)_{\mathrm{I}}^{\mathrm{T}} & -(T-S)_{\mathrm{III}}^{\mathrm{T}} \\ -(T-S)_{\mathrm{II}}^{\mathrm{T}} & h\varphi\, Id - (T-S)_{\mathrm{IV}}^{\mathrm{T}} \end{pmatrix}, \\ \mathbf{c}^+_{k+1..d'} &= h\varphi(T-S)_{\mathrm{II}}^{\mathrm{T}}\Lambda\mathbf{c}^+_{1..k} + \left(-\tfrac{1}{2}h^2\varphi\, Id + h(T-S)_{\mathrm{IV}}^{\mathrm{T}}\right)\mathbf{r}^-_{k+1..d'}, \end{aligned}$$

with the vectors $\mathbf{u}^+$ and $\mathbf{u}^-$ defined as before. Both systems can be easily solved if we note that under $\mathrm{O}(h)$-approximation the solutions are $\mathbf{c}^+_{1..k} = \tilde{\mathbf{r}}^-_{1..k}$ and $\mathbf{c}^- = \binom{\tilde{\mathbf{r}}^+_{1..k}}{0}$. Hence we make the substitution:

$$\mathbf{c}^+_{1..k} := \tilde{\mathbf{r}}^-_{1..k} + \boldsymbol{\xi}^+, \quad \mathbf{c}^- := \begin{pmatrix} \tilde{\mathbf{r}}^+_{1..k} + \boldsymbol{\xi}^- \\ \boldsymbol{\zeta}^- \end{pmatrix}.$$

Omitting in calculations the terms of order $\mathrm{O}(h^3)$ and $\mathrm{O}(h^2)$ respectively we obtain the following formulas

$$\begin{aligned} \mathbf{c}^+_{1..k} &= \tilde{\mathbf{r}}^-_{1..k} + h(T-S)_{\mathrm{I,III}}^{\mathrm{T}}\mathbf{r}^- + h^2\varphi(T-S)_{\mathrm{I}}^{\mathrm{T}}\Lambda(T-S)_{\mathrm{I,III}}^{\mathrm{T}}\mathbf{r}^- - \tfrac{1}{2}h^2\varphi\Lambda\mathbf{r}^-_{1..k}, \\ \mathbf{c}^+_{k+1..d'} &= h(T-S)_{\mathrm{II,IV}}^{\mathrm{T}}\mathbf{r}^- + h^2\varphi(T-S)_{\mathrm{II}}^{\mathrm{T}}\Lambda(T-S)_{\mathrm{I,III}}^{\mathrm{T}}\mathbf{r}^- - \tfrac{1}{2}h^2\varphi\mathbf{r}^-_{k+1..d'}, \\ \mathbf{c}^- &= \begin{pmatrix} \tilde{\mathbf{r}}^+_{1..k} + h(T-S)_{\mathrm{I}}^{\mathrm{T}}\mathbf{r}^+_{1..k} \\ h\mathcal{R} \end{pmatrix}. \end{aligned}$$

Substituting this result into (3.23), using the approximations: $\cosh(x) = 1+\frac{1}{2}x^2+\mathrm{O}(x^4)$, $\sinh(x) = x + \mathrm{O}(x^3)$ and omitting the terms $\mathrm{O}(h^3)$ in $\boldsymbol{\mu}^+(y)$ and $\mathrm{O}(h^2)$ in $\boldsymbol{\mu}^-(y)$ we obtain the formulas (3.22) for $\boldsymbol{\mu}^\pm(y)$.

It remains to calculate c_0^-. Using the no-flux condition (3.18) and again omitting the terms of order $\mathrm{O}(h^2)$ we get two formulas for the constant

$$\begin{aligned} c_0^- &= h\langle\boldsymbol{\sigma}, \mathbf{r}^-\rangle - \theta, \\ c_0^- &= -2hr_0^- - h\langle\boldsymbol{\sigma}, \mathbf{r}^-\rangle - \theta, \end{aligned}$$

where

$$\theta = h\langle\boldsymbol{\sigma}_{1..k}, (T-S)_{\mathrm{I}}^{-\mathrm{T}}\mathbf{r}^+_{1..k}\rangle + h\langle\boldsymbol{\sigma}_{k+1..d'}, \mathcal{R}\rangle$$

The formulas coincide if we apply the compatibility condition $r_0^- = -\langle\boldsymbol{\sigma}, \mathbf{r}^-\rangle$ to the second one. This ends the proof since we get the formula (3.22) for $\mu_0^-(y)$. □

Using the fact that the right-hand side of equation (3.10) is an even vector and thus multiplied by W^{-1} becomes an odd one, we can write

$$\mathbf{g} = -W^{-1}\partial_x\varphi\,\mathbf{b}_{k+1}^{+} = \sum_{i=0}^{d'}(-\alpha_i\partial_x\varphi)\mathbf{b}_i^{-}.$$

Then only "odd" components of the vector $\mathbf{r}$ in equation (3.16) are nonzero. Hence

$$\mathbf{r} = (0, r_0^-, \ldots, 0, r_{d'}^-)^{\mathrm{T}}, \tag{3.27}$$

where

$$r_i^- = -\alpha_i\partial_x\varphi.$$

Hence using Lemma 3.4 with (3.27) we can write $\mathbf{f}^1$ as in (3.15) with $\mu_i^{\pm}(y)$ given by

$$\begin{aligned}
\mu_0^+(y) &= c_0^+,\\
\mu_0^-(y) &= -\left[(y+h)\alpha_0 + h\left(\langle\boldsymbol{\sigma},\boldsymbol{\alpha}\rangle - \langle\boldsymbol{\sigma}_{k+1..d'}, (T-S)_{\mathrm{IV}}^{-\mathrm{T}}\boldsymbol{\alpha}_{k+1..d'}\rangle\right)\right]\partial_x\varphi,\\
\boldsymbol{\mu}_{1..k}^+(y) &= -\left[\tfrac{1}{2}(y^2-h^2)\varphi\Lambda\boldsymbol{\alpha}_{1..k} + h(T-S)_{\mathrm{I,III}}^{\mathrm{T}}\boldsymbol{\alpha}\right.\\
&\qquad \left. + h^2\varphi(T-S)_{\mathrm{I}}^{\mathrm{T}}\Lambda(T-S)_{\mathrm{I,III}}^{\mathrm{T}}\boldsymbol{\alpha}\right]\partial_x\varphi,\\
\boldsymbol{\mu}_{1..k}^-(y) &= -\left[y\boldsymbol{\alpha}_{1..k} + hy\varphi\Lambda(T-S)_{\mathrm{I,III}}^{\mathrm{T}}\boldsymbol{\alpha}\right]\partial_x\varphi,\\
\boldsymbol{\mu}_{k+1..d'}^+(y) &= -\left[\tfrac{1}{2}(y^2-h^2)\varphi\boldsymbol{\alpha}_{k+1..d'} + hy\varphi(T-S)_{\mathrm{IV}}^{-\mathrm{T}}\boldsymbol{\alpha}_{k+1..d'}\right.\\
&\qquad \left. + h(T-S)_{\mathrm{II,IV}}^{\mathrm{T}}\boldsymbol{\alpha} + h^2\varphi(T-S)_{\mathrm{II}}^{\mathrm{T}}\Lambda(T-S)_{\mathrm{I,III}}^{\mathrm{T}}\boldsymbol{\alpha}\right]\partial_x\varphi,\\
\boldsymbol{\mu}_{k+1..d'}^-(y) &= -\left[y\boldsymbol{\alpha}_{k+1..d'} + h(T-S)_{\mathrm{IV}}^{-\mathrm{T}}\boldsymbol{\alpha}_{k+1..d'}\right]\partial_x\varphi,
\end{aligned}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{d'})^{\mathrm{T}}$. The constant c_0^+ can be specified using the condition (3.9), but we shall not do this due to the reason described below.

Diffusion equation

The necessary and sufficient condition for solvability of equation (3.11) for $\mathbf{f}^2$ is the equality (3.12). It is obvious that only the coefficients of the even basis vectors in $\mathbf{f}^1$ contribute to (3.12) since $\mathbf{v}$ is even vector. But note that the term $c_0^+\mathbf{b}_0^+$ does not contribute due to the fact that $\langle\mathbf{v}, \mathbb{1}\rangle = 0$. The term in $\boldsymbol{\mu}_{k+1..d'}^+(y)$, which is odd with respect to y-variable, does not contribute either. Thus denoting

$$\varrho_i := \langle V\mathbf{b}_i^+, \mathbb{1}\rangle, \quad i = 1, \ldots, d',$$

we obtain the main theorem of this section.

Theorem 3.1 *Let $h \ll 1$. Then the equation (3.11) is solvable for $\mathbf{f}^2$ if and only if φ satisfies the diffusion equation*

$$\bar{m}\partial_t\varphi = \partial_x\left((c_1h + c_2h^2\varphi)\partial_x\varphi\right), \tag{3.28}$$

where

$$c_1 = \langle \boldsymbol{\varrho}, (T-S)^{\mathrm{T}} \boldsymbol{\alpha} \rangle,$$
$$c_2 = -\tfrac{1}{3}\langle \Lambda \boldsymbol{\alpha}_{1..k}, \boldsymbol{\varrho}_{1..k}\rangle - \tfrac{1}{3}\langle \boldsymbol{\alpha}_{k+1..d'}, \boldsymbol{\varrho}_{k+1..d'}\rangle + \langle (T-S)^{\mathrm{T}}_{\mathrm{I,II}} \Lambda (T-S)^{\mathrm{T}}_{\mathrm{I,III}} \boldsymbol{\alpha}, \boldsymbol{\varrho}\rangle.$$

and $\boldsymbol{\varrho} = (\varrho_1, \dots, \varrho_{d'})^{\mathrm{T}}$.

4 Examples

4.1 12-velocity model

The velocities in the model $(v_i, w_i)_{i=1,\dots,12}$ are defined as components of the velocity vectors $\mathbf{v} = (v_1, \dots, v_{12})^{\mathrm{T}}$ and $\mathbf{w} = (w_1, \dots, w_{12})^{\mathrm{T}}$ in x and y directions respectively:

$$\mathbf{v} = \begin{pmatrix} \hat{\mathbf{v}} \\ \hat{\mathbf{v}} \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} \hat{\mathbf{w}} \\ -\hat{\mathbf{w}} \end{pmatrix},$$

where

$$\hat{\mathbf{v}} = \begin{pmatrix} 1 \\ 1 \\ 3 \\ -1 \\ -1 \\ -3 \end{pmatrix}, \quad \hat{\mathbf{w}} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 3 \\ 1 \\ 1 \end{pmatrix}.$$

The velocity grid and the numbering of velocities in the 12-velocity model are demonstrated in Fig. 1.

An equilibrium state is given by an even vector

$$\mathbf{m}^{(0)} = \begin{pmatrix} \hat{\mathbf{m}}^{(0)} \\ \hat{\mathbf{m}}^{(0)} \end{pmatrix}, \quad \hat{\mathbf{m}}^{(0)} = \frac{1}{4(2a+b)} \begin{pmatrix} a \\ b \\ a \\ a \\ b \\ a \end{pmatrix},$$

where $a \geq 0$ and $b > 0$. The vector $\mathbf{m}^{(0)}$ is normalized such that $\langle \mathbf{m}^{(0)}, \mathbb{1} \rangle = 1$. Obviously it satisfies the conditions (3.3).

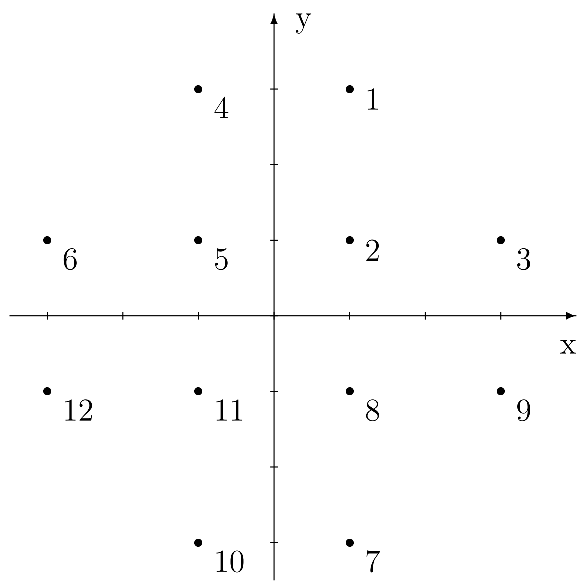
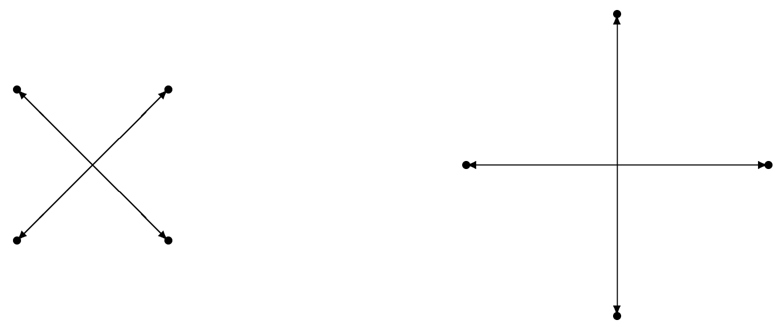


Figure 1: Velocity grid and numbering of velocities.

Let $\mathbf{f} = (f_1, \ldots, f_{12})^{\mathrm{T}}$ and $Q[\mathbf{f}] = (Q_1(\mathbf{f}), \ldots, Q_{12}(\mathbf{f}))^{\mathrm{T}}$ with the components defined as follows

$$
\begin{aligned}
Q_1(\mathbf{f}) &= \gamma_1(f_2f_4 - f_1f_5) + \gamma_2(f_3f_5 - f_1f_8),\\
Q_2(\mathbf{f}) &= \gamma_1(f_1f_5 - f_2f_4 + f_3f_8 - f_2f_9 + f_5f_8 - f_2f_{11}) +\\
&\quad +\gamma_2(f_9f_{11} - f_2f_7 + f_4f_{11} - f_2f_6),\\
Q_3(\mathbf{f}) &= \gamma_1(f_2f_9 - f_3f_5) + \gamma_2(f_1f_8 - f_3f_5),\\
Q_4(\mathbf{f}) &= \gamma_1(f_1f_5 - f_2f_4) + \gamma_2(f_2f_6 - f_4f_{11}),\\
Q_5(\mathbf{f}) &= \gamma_1(f_2f_4 - f_1f_5 + f_2f_{11} - f_5f_8 + f_6f_{11} - f_5f_{12}) +\\
&\quad +\gamma_2(f_1f_8 - f_3f_5 + f_8f_{12} - f_5f_{10}),\\
Q_6(\mathbf{f}) &= \gamma_1(f_5f_{12} - f_6f_{11}) + \gamma_2(f_4f_{11} - f_2f_6),\\
Q_7(\mathbf{f}) &= \gamma_1(f_8f_{10} - f_7f_{11}) + \gamma_2(f_9f_{11} - f_2f_7),\\
Q_8(\mathbf{f}) &= \gamma_1(f_2f_9 - f_3f_8 + f_2f_{11} - f_5f_8 + f_7f_{11} - f_8f_{10}) +\\
&\quad +\gamma_2(f_3f_5 - f_1f_8 + f_5f_{10} - f_8f_{12}),\\
Q_9(\mathbf{f}) &= \gamma_1(f_3f_8 - f_2f_9) + \gamma_2(f_2f_7 - f_9f_{11}),\\
Q_{10}(\mathbf{f}) &= \gamma_1(f_7f_{11} - f_8f_{10}) + \gamma_2(f_8f_{12} - f_5f_{10}),\\
Q_{11}(\mathbf{f}) &= \gamma_1(f_5f_8 - f_2f_{11} + f_5f_{12} - f_6f_{11} + f_8f_{10} - f_7f_{11}) +\\
&\quad +\gamma_2(f_2f_6 - f_4f_{11} + f_2f_7 - f_9f_{11}),\\
Q_{12}(\mathbf{f}) &= \gamma_1(f_6f_{11} - f_5f_{12}) + \gamma_2(f_5f_{10} - f_8f_{12}),
\end{aligned}
$$

where $\gamma_1, \gamma_2 > 0$ are the collision rates for the two types of collisions respectively (see Fig. 2).



a) Type 1 b) Type 2

Figure 2: Collisions types schemes.

The only collision invariants are given by the vectors

$$\mathbb{1}, \quad \mathbf{v}, \quad \mathbf{w}, \quad \mathbf{e},$$

where $\mathbf{e} = (v_1^2 + w_1^2, \ldots, v_{12}^2 + w_{12}^2)^{\mathrm{T}}$.

Linearization of Q around $\mathbf{m}^{(0)}$ yields the matrix $\tilde{M}$ of the following form

$$\tilde{M} = \gamma_1 \tilde{M}_1 + \gamma_2 \tilde{M}_2, \quad \text{with } \tilde{M}_i = \frac{1}{4(2a+b)} \begin{pmatrix} \tilde{A}_i & \tilde{B}_i \\ \tilde{B}_i & \tilde{A}_i \end{pmatrix}, \ i = 1, 2,$$

where

$$\tilde{A}_1 = \begin{pmatrix} -b & a & 0 & b & -a & 0 \\ b & -2a-b & b & -b & a+b & 0 \\ 0 & a & -b & 0 & 0 & 0 \\ b & -a & 0 & -b & a & 0 \\ -b & a+b & 0 & b & -2a-b & b \\ 0 & 0 & 0 & 0 & a & -b \end{pmatrix},$$

$$\tilde{B}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a+b & -b & 0 & -b & 0 \\ 0 & -a & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -b & 0 & 0 & a+b & -b \\ 0 & 0 & 0 & 0 & -a & b \end{pmatrix},$$

$$\tilde{A}_2 = \begin{pmatrix} -b & 0 & b & 0 & a & 0 \\ 0 & -2a & 0 & b & 0 & -b \\ b & 0 & -b & 0 & -a & 0 \\ 0 & a & 0 & -b & 0 & b \\ b & 0 & -b & 0 & -2a & 0 \\ 0 & -a & 0 & b & 0 & -b \end{pmatrix},$$

$$\tilde{B}_2 = \begin{pmatrix} 0 & -a & 0 & 0 & 0 & 0 \\ -b & 0 & b & 0 & 2a & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a & 0 \\ 0 & 2a & 0 & -b & 0 & b \\ 0 & 0 & 0 & 0 & a & 0 \end{pmatrix}.$$

Now let $M = W^{-1} \cdot \tilde{M}$. It is easy to see that the matrix M has the form

$$M = \frac{1}{4(2a+b)} \begin{pmatrix} A & B \\ -B & -A \end{pmatrix},$$

where

$$A = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & B_2 \\ B_2 & B_1 \end{pmatrix},$$

and the matrices A_1, A_2, B_1, B_2 are defined as follows

$$A_1 = \begin{pmatrix} -\frac{1}{3}b(\gamma_1+\gamma_2) & \frac{1}{3}a\gamma_1 & \frac{1}{3}b\gamma_2 \\ b\gamma_1 & -2a(\gamma_1+\gamma_2) - b\gamma_1 & b\gamma_1 \\ b\gamma_2 & a\gamma_1 & -b(\gamma_1+\gamma_2) \end{pmatrix},$$

$$A_2 = \begin{pmatrix} \frac{1}{3}b\gamma_1 & \frac{1}{3}a(\gamma_2-\gamma_1) & 0 \\ b(\gamma_2-\gamma_1) & (a+b)\gamma_1 & -b\gamma_2 \\ 0 & -a\gamma_2 & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 & -\frac{1}{3}a\gamma_2 & 0 \\ -b\gamma_2 & (a+b)\gamma_1 & b(\gamma_2-\gamma_1) \\ 0 & a(\gamma_2-\gamma_1) & b\gamma_1 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2a\gamma_2 - b\gamma_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix M has the Jordan form (2.4) with two Jordan blocks J_2 and the matrix $\Lambda = diag(\lambda_1, \lambda_2, \lambda_3)$ where

$$\lambda_1 = \frac{b\sqrt{2\gamma_2(9\gamma_1 + 8\gamma_2)}}{12(2a+b)}, \quad \lambda_{2,3} = \frac{1}{12(2a+b)}\sqrt{c_1 \pm \sqrt{c_1^2 - 72c_2}},$$

and

$$c_1 = 12a(3a+4b)\gamma_1(\gamma_1+2\gamma_2) + b^2(2\gamma_1^2 + 11\gamma_1\gamma_2 + 8\gamma_2^2),$$
$$c_2 = b^2(36a^2+48ab+b^2)\gamma_1^2\gamma_2(\gamma_1+\gamma_2).$$

Transforming the eigenvectors of M according to (3.13) we get the basis $\{\mathbf{b}_0^+, \mathbf{b}_0^-, \ldots, \mathbf{b}_5^+, \mathbf{b}_5^-\}$ with

$$\widehat{\mathbf{b}}_0^+ = \tfrac{1}{4(2a+b)}\begin{pmatrix} a \\ b \\ a \\ a \\ b \\ a \end{pmatrix}, \qquad \widehat{\mathbf{b}}_0^- = \tfrac{1}{4(10a+b)}\begin{pmatrix} 3a \\ b \\ a \\ 3a \\ b \\ a \end{pmatrix},$$

$$\widehat{\mathbf{b}}_1^+ = \begin{pmatrix} \frac{4}{3}b\gamma_2(\lambda_1 - \frac{4}{3}b\gamma_2) \\ 2b(3\gamma_1 - 4\gamma_2)(\lambda_1 - \frac{4}{3}b\gamma_2) \\ -2b(3\gamma_1 + 2\gamma_2)(\lambda_1 - \frac{4}{3}b\gamma_2) \\ \frac{4}{3}b\gamma_2(\lambda_1 - \frac{4}{3}b\gamma_2) \\ 2b(3\gamma_1 - 4\gamma_2)(\lambda_1 - \frac{4}{3}b\gamma_2) \\ -2b(3\gamma_1 + 2\gamma_2)(\lambda_1 - \frac{4}{3}b\gamma_2) \end{pmatrix},$$

$$\widehat{\mathbf{b}}_1^- = \begin{pmatrix} -\frac{2}{3}b\gamma_2(\frac{1}{3}b(9\gamma_1+8\gamma_2) - 2\lambda_1) \\ 0 \\ 2b\gamma_2(\frac{1}{3}b(9\gamma_1+8\gamma_2) - 2\lambda_1) \\ -\frac{2}{3}b\gamma_2(\frac{1}{3}b(9\gamma_1+8\gamma_2) - 2\lambda_1) \\ 0 \\ 2b\gamma_2(\frac{1}{3}b(9\gamma_1+8\gamma_2) - 2\lambda_1) \end{pmatrix},$$

$$\widehat{\mathbf{b}}_{2,3}^+ = \begin{pmatrix} 6\lambda^3 - 8\gamma_1(6a^2(\gamma_1+2\gamma_2) + ab(7\gamma_1+20\gamma_2) + b^2\gamma_2)\lambda \\ -8b(6a+7b)\gamma_1(\gamma_1-4\gamma_2)\lambda \\ -18\lambda^3 + 8\gamma_1(18a^2(\gamma_1+2\gamma_2) + 9ab(3\gamma_1+4\gamma_2) + b^2(\gamma_1-\gamma_2))\lambda \\ -6\lambda^3 + 8\gamma_1(6a^2(\gamma_1+2\gamma_2) + ab(7\gamma_1+20\gamma_2) + b^2\gamma_2)\lambda \\ 8b(6a+7b)\gamma_1(\gamma_1-4\gamma_2)\lambda \\ 18\lambda^3 - 8\gamma_1(18a^2(\gamma_1+2\gamma_2) + 9ab(3\gamma_1+4\gamma_2) + b^2(\gamma_1-\gamma_2))\lambda \end{pmatrix},$$

$$\widehat{\mathbf{b}}_{2,3}^- = \begin{pmatrix} -4b(\gamma_1+2\gamma_2)\lambda^2 + 8b(36a^2+48ab+b^2)\gamma_1^2\gamma_2 \\ 12b(\gamma_1-4\gamma_2)\lambda^2 \\ 24b\gamma_2\lambda^2 - 8b(36a^2+48ab+b^2)\gamma_1^2\gamma_2 \\ 4b(\gamma_1+2\gamma_2)\lambda^2 - 8b(36a^2+48ab+b^2)\gamma_1^2\gamma_2 \\ -12b(\gamma_1-4\gamma_2)\lambda^2 \\ -24b\gamma_2\lambda^2 + 8b(36a^2+48ab+b^2)\gamma_1^2\gamma_2 \end{pmatrix},$$

$$\widehat{\mathbf{b}}_4^+ = \tfrac{1}{4(2a+b)}\begin{pmatrix} a \\ b \\ 3a \\ -a \\ -b \\ -3a \end{pmatrix}, \qquad \widehat{\mathbf{b}}_4^- = \begin{pmatrix} -\frac{a(6a+7b)}{4b^2\gamma_1} \\ -\frac{6a+b}{4b\gamma_1} \\ -\frac{a(6a+7b)}{4b^2\gamma_1} \\ \frac{a(6a+7b)}{4b^2\gamma_1} \\ \frac{6a+b}{4b\gamma_1} \\ \frac{a(6a+7b)}{4b^2\gamma_1} \end{pmatrix},$$

$$\widehat{\mathbf{b}}_5^+ = \tfrac{1}{4(2a+b)}\begin{pmatrix} -b \\ 10b \\ -b \\ -b \\ 10b \\ -b \end{pmatrix}, \qquad \widehat{\mathbf{b}}_5^- = \begin{pmatrix} \frac{3(a+b)\gamma_1+2(b-2a)\gamma_2}{(10a+b)\gamma_1\gamma_2} \\ -\frac{b(9\gamma_1+8\gamma_2)}{(10a+b)\gamma_1\gamma_2} \\ -\frac{9a\gamma_1-2(6a+b)\gamma_2}{(10a+b)\gamma_1\gamma_2} \\ \frac{3(a+b)\gamma_1+2(b-2a)\gamma_2}{(10a+b)\gamma_1\gamma_2} \\ -\frac{b(9\gamma_1+8\gamma_2)}{(10a+b)\gamma_1\gamma_2} \\ -\frac{9a\gamma_1-2(6a+b)\gamma_2}{(10a+b)\gamma_1\gamma_2} \end{pmatrix},$$

where $\lambda = \lambda_2, \lambda_3$ for the vectors $\widehat{\mathbf{b}}_2^{\pm}$ and $\widehat{\mathbf{b}}_3^{\pm}$ respectively.

We have for this base $\boldsymbol{\sigma} = (0,0,0,0,0)^{\mathrm{T}}$ (see (3.14)), which implies $S = 0$, and the matrix T is regular. Moreover one can easily deduce from the proof of Lemma 3.4 that Assumption 3.2 is satisfied for small h.

Finally from Theorem 3.1 we get that $\varphi = \varphi(t,x)$ satisfies the following diffusion equation for the 12-velocity model

$$\partial_t\varphi = \partial_x\left[\left(\Big(\frac{88}{3}a+1\Big)h + \frac{1}{27}(8a-1)\Big(4a(4\gamma_1-13\gamma_2)-9\gamma_1\Big)h^2\varphi\right)\partial_x\varphi\right].$$

where we take $b = \frac{1}{4} - 2a$ (due to the normalization of the equilibrium state $\mathbf{m}^{(0)}$) and then $a \in [0, \frac{1}{8})$ (to preserve positivity of b).

Remarks:

a) The coefficient of a linear part of the above diffusion equation can be expressed as a mean value of velocity (see Section 1.1.1) as follows

$$\sum_{i=1}^{12}\frac{v_i^2}{|w_i|}\mathbf{m}^{(0)}(v_i,w_i) = 4\Big(\frac{1}{3}a + \frac{1}{4} - 2a + 9a\Big) = \frac{88}{3}a + 1.$$

b) If we take $a = 0$ and $\gamma_1 = \gamma_2 = \frac{1}{2}$ we get

$$\partial_t\varphi = \partial_x\left(\Big(h + \frac{1}{6}h^2\varphi\Big)\partial_x\varphi\right),$$

which is the diffusion equation for the classical Broadwell model (see Section 4.2).

c) If we take $a = \frac{1}{12}$ (the case of constant maxwellian – hot gas) we get

$$\partial_t \varphi = \partial_x \left[\left(\frac{31}{9} h + \frac{1}{243} (23\gamma_1 + 13\gamma_2) h^2 \varphi \right) \partial_x \varphi \right].$$

4.2 Generalized Broadwell model

We consider a $2d$-velocity model with velocities regularly distributed over a circle:

$$\begin{aligned} v_i &= \cos\left(2\pi \frac{i - 1/2}{2d}\right), \\ w_i &= \sin\left(2\pi \frac{i - 1/2}{2d}\right) \end{aligned} \tag{4.1}$$

for $i = 1, \ldots, d$, and

$$v_{d+i} = v_i, \quad w_{d+i} = -w_i. \tag{4.2}$$

We observe that

$$v_i = -v_{d+1-i} \qquad \text{and} \tag{4.3}$$

$$w_i = w_{d+1-i} \quad \text{for} \quad i = 1, \ldots, d. \tag{4.4}$$

In the sequel we denote velocity vectors by greek letters, like $\xi = (v, w)^T$. The collision model is such that a pair $(\xi, -\xi)$ of opposite velocities may interact and form (equally distributed) another pair $(\eta, -\eta)$. The collision operator $Q[\mathbf{f}] = (Q_1(\mathbf{f}), \ldots, Q_{2d}(\mathbf{f}))^T$ reads

$$Q_i(\mathbf{f}) = \frac{1}{d} \sum_{j=1}^{d} f_j f_{\bar{j}} - f_i f_{\bar{i}} =: S(\mathbf{f}) - f_i f_{\bar{i}} \,, \tag{4.5}$$

where $\bar{j}$ is defined such that

$$j + \bar{j} = 2d + 1. \tag{4.6}$$

The slab problem for this model has been analyzed in [7], where it has been proven that the only steady solutions under diffuse reflection laws are the equilibrium solutions $\mathbf{m}^{(0)} = c \cdot \mathbb{1}$.

In the following we consider the equilibrium solution

$$\mathbf{m}^{(0)} = \frac{1}{2d} \cdot \mathbb{1}.$$

Linearization of Q around $\mathbf{m}^{(0)}$ yields the linearized operator

$$\tilde{L} = \frac{1}{2d^2} \cdot \mathbb{1} \cdot \mathbb{1}^T - \frac{1}{2d} \begin{pmatrix} I & \hat{I} \\ \hat{I} & I \end{pmatrix}, \tag{4.7}$$

where

$$\hat{I} = \begin{pmatrix} & & 1 \\ & \cdot & \\ 1 & & \end{pmatrix} = (\delta_{i,d+1-j})_{1\le i,j\le d} \ . \tag{4.8}$$

We denote by $\mathbf{e}^{(i)}$ the i-th canonical unit vector in $\mathbb{R}^d$ and construct the kernel $\ker(L)$ of L. *Odd* vectors $(\xi,-\xi)^T$ belong obviously to $\ker(L)$, if and only if $\xi = \hat{I}\xi$, i.e. if $\xi \in \mathrm{span}(\mathbf{e}^{(i)} + \mathbf{e}^{(d+1-i)}, i = 1,2,\ldots)$. *Even* vectors $(\xi,\xi)^T$ are in $\ker(L)$, if and only if $(I+\hat{I})\xi \parallel \mathbb{1}$, i.e. if for some $\alpha \in \mathbb{R}$, $\xi - \alpha\cdot\mathbb{1} \in \mathrm{span}(\mathbf{e}^{(i)} - \mathbf{e}^{(d+1-i)}, i = 1,2,\ldots)$. Thus this model does not satisfy all the assumptions of Section 3. However, we may use the ideas developed above to discuss the diffusion limit.

From now on we only consider the case of even d, $d = 2\delta$. (The case of odd d can be treated with slight modifications.)

In $\mathbb{R}^d$ we define the vectors

$$\hat{\mathbf{b}}_i^+ = \mathbf{e}^{(i)} + \mathbf{e}^{(d+1-i)}, \quad \hat{\mathbf{b}}_i^- = \mathbf{e}^{(i)} - \mathbf{e}^{(d+1-i)}, \quad i = 1,\ldots,\delta. \tag{4.9}$$

We easily find

$$L\cdot\begin{pmatrix} \hat{\mathbf{b}}_i^- \\ \hat{\mathbf{b}}_i^- \end{pmatrix} = 0, \quad L\cdot\begin{pmatrix} \hat{\mathbf{b}}_i^+ \\ -\hat{\mathbf{b}}_i^+ \end{pmatrix} = 0, \quad L\cdot\begin{pmatrix} \hat{\mathbf{b}}_i^- \\ -\hat{\mathbf{b}}_i^- \end{pmatrix} = -\frac{1}{d\cdot w_i}\cdot\begin{pmatrix} \hat{\mathbf{b}}_i^- \\ \hat{\mathbf{b}}_i^- \end{pmatrix}. \tag{4.10}$$

Furthermore,

$$\begin{aligned} L\cdot\begin{pmatrix} \hat{\mathbf{b}}_i^+ \\ \hat{\mathbf{b}}_i^+ \end{pmatrix} &= \frac{2}{d^2}\cdot W^{-1}\mathbb{1} - \frac{1}{d}\cdot W^{-1}\begin{pmatrix} \hat{\mathbf{b}}_i^+ \\ \hat{\mathbf{b}}_i^+ \end{pmatrix} \\ &= \frac{2}{d^2}\cdot\sum_{j=1}^{\delta} w_j^{-1}\cdot\begin{pmatrix} \hat{\mathbf{b}}_j^+ \\ -\hat{\mathbf{b}}_j^+ \end{pmatrix} - \frac{1}{d\cdot w_i}\cdot\begin{pmatrix} \hat{\mathbf{b}}_i^+ \\ -\hat{\mathbf{b}}_i^+ \end{pmatrix}. \end{aligned} \tag{4.11}$$

In order to establish a complete basis, we are looking for even vectors $(\hat{\mathbf{c}},\hat{\mathbf{c}})^T$ which are mapped into multiples of the even vectors $(\hat{\mathbf{c}},-\hat{\mathbf{c}})^T$. Choosing the ansatz

$$\hat{\mathbf{c}} = \sum_{j=1}^{\delta} \alpha_j\cdot\hat{\mathbf{b}}_j^+$$

we find as a necessary and sufficient condition that $\alpha = (\alpha_1,\ldots,\alpha_\delta)^T$ is an eigenvector of the $\delta\times\delta$-matrix

$$M := W_\delta^{-1}(2/d^2\cdot\mathbb{1} - 1/d\cdot I). \tag{4.12}$$

M is similar to the symmetric nonpositive matrix $W_\delta^{-1/2}(2/d^2 \cdot \mathbb{1} - 1/d \cdot I)W_\delta^{-1/2}$ and thus has δ linearly independent eigenvectors $\alpha^{(i)}$. The kernel of M is spanned by $\alpha^{(1)} := \mathbb{1}_\delta$. The other vectors $\alpha^{(i)}$ have eigenvectors $-\lambda_i < 0$.

For given $\alpha^{(i)} = (\alpha_{ij})$ denote

$$\hat{\mathbf{c}}^{(i)} := \sum_{j=1}^{\delta} \alpha_{ij} \cdot \hat{\mathbf{b}}_j^+. \tag{4.13}$$

Then $\hat{\mathbf{c}}^{(1)} = \mathbb{1}$, and for $i > 1$,

$$L \cdot \begin{pmatrix} \hat{\mathbf{c}}^{(i)} \\ \hat{\mathbf{c}}^{(i)} \end{pmatrix} = -\lambda_i \cdot \begin{pmatrix} \hat{\mathbf{c}}^{(i)} \\ -\hat{\mathbf{c}}^{(i)} \end{pmatrix}. \tag{4.14}$$

Furthermore, from the definition of M follows for the fluxes (in y-direction)

$$\langle \mathbf{w}_d, \hat{\mathbf{c}}^{(i)} \rangle = 0. \tag{4.15}$$

We now define the basis

$$\mathbf{b}^{(j)} = \begin{cases} \mathbb{1} & \text{for} \quad j = 1, \\ \mathbf{w} & \text{for} \quad j = 2, \\ (\hat{\mathbf{c}}^{(i)}, -\hat{\mathbf{c}}^{(i)})^T & \text{for } j = 2i-1, \quad i = 2, \ldots, \delta, \\ (\hat{\mathbf{c}}^{(i)}, \hat{\mathbf{c}}^{(i)})^T & \text{for } j = 2i, \quad i = 2, \ldots, \delta, \\ (\hat{\mathbf{b}}_i^-, \hat{\mathbf{b}}_i^-)^T & \text{for } j = d + 2i - 1, \quad i = 1, \ldots, \delta, \\ (\hat{\mathbf{b}}_i^-, -\hat{\mathbf{b}}_i^-)^T & \text{for } j = d + 2i, \quad i = 1, \ldots, \delta. \end{cases} \tag{4.16}$$

Corresponding to this basis, L takes the Jordan normal structure

$$J = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & J_2 & & \\ & & & \ddots & \\ & & & & J_d \end{pmatrix} \tag{4.17}$$

with

$$J_j = \begin{cases} \begin{pmatrix} 0 & -\lambda_j \\ 0 & 0 \end{pmatrix} & \text{for} \quad j = 2, \ldots, \delta, \\ \begin{pmatrix} 0 & -1/(d \cdot w_{j-\delta}) \\ 0 & 0 \end{pmatrix} & \text{for} \quad j = \delta + 1, \ldots, d. \end{cases} \tag{4.18}$$

We turn to the calculation of $\mathbf{f}^1$. First, we notice the series representation for the inhomogeneous part

$$-\frac{1}{2d}\cdot W^{-1}V\mathbb{1} = -\sum_{i=1}^{\delta}\frac{v_i}{2dw_i}\cdot\mathbf{b}^{(d+2i)}. \tag{4.19}$$

Assuming an ansatz for $\mathbf{f}^1$,

$$\mathbf{f}^1 = \sum_{i=1}^{2d}\mu_i(y)\mathbf{b}^{(i)}, \tag{4.20}$$

we find from Lemma 3.4 that

$$\mu_i(y) = \mu_i(0) \quad \text{for } i=1 \text{ and } i=2j, \quad j=1,\ldots,\delta, \tag{4.21}$$

$$\mu_i(y) = \mu_i(0) - y\phi\lambda_j\mu_{i+1}(0) \quad \text{for } i=2j-1, \quad j=2,\ldots,\delta, \tag{4.22}$$

$$\mu_i(y) = \mu_i(0) - y\partial_x\phi\cdot\frac{v_j}{2dw_j} \quad \text{for } i=2j+d, \quad j=1,\ldots,\delta, \tag{4.23}$$

$$\begin{aligned}\mu_i(y) &= \mu_i(0) - 2y\phi\frac{1}{2dw_j}\mu_{i+1}(0)\\ &\quad + y^2\phi\partial_x\phi\frac{v_j}{(2dw_j)^2} \quad \text{for } i=2j+d-1, \quad j=1,\ldots,\delta.\end{aligned} \tag{4.24}$$

From the no-flux condition and (3.15) follows that $\mu_2 = 0$. The fluxes into the slab at $\pm h$ are

$$\begin{aligned}\mathbf{f}^{in}(-h) &= \mu_1(0)\cdot\mathbb{1} + \sum_{j=2}^{\delta}\left[\mu_{2j-1}(0) + (1+h\phi\lambda_j)\mu_{2j}(0)\right]\hat{\mathbf{c}}^{(j)}\\ &\quad + \sum_{j=1}^{\delta}\left[(1+2h\phi\frac{1}{2dw_j})\mu_{d+2j}(0) + \mu_{d+2j-1}(0) + h\partial_x\phi\frac{v_j}{2dw_j} + h^2\phi\partial_x\phi\frac{v_j}{(2dw_j)^2}\right]\hat{\mathbf{b}}_j^-\end{aligned}$$

$$\begin{aligned}\mathbf{f}^{in}(h) &= \mu_1(0)\cdot\mathbb{1} + \sum_{j=2}^{\delta}\left[-\mu_{2j-1}(0) + (1+h\phi\lambda_j)\mu_{2j}(0)\right]\hat{\mathbf{c}}^{(j)}\\ &\quad + \sum_{j=1}^{\delta}\left[-(1+2h\phi\frac{1}{2dw_j})\mu_{d+2j}(0) + \mu_{d+2j-1}(0) + h\partial_x\phi\frac{v_j}{2dw_j} + h^2\phi\partial_x\phi\frac{v_j}{(2dw_j)^2}\right]\hat{\mathbf{b}}_j^-\end{aligned}$$

We find the linear combinations

$$\mathbf{f}^{in}(h) - \mathbf{f}^{in}(-h) = -2\sum_{j=2}^{\delta}\mu_{2j-1}(0)\hat{\mathbf{c}}^{(j)} - 2\sum_{j=1}^{\delta}(1+2h\phi\frac{1}{2dw_j})\mu_{d+2j}(0)\hat{\mathbf{b}}_j^-$$

$$\mathbf{f}^{in}(h) + \mathbf{f}^{in}(-h) = 2\mu_1(0)\cdot\mathbb{1} + 2\sum_{j=2}^{\delta}(1+h\phi\lambda_j)\mu_{2j}(0)\hat{\mathbf{c}}^{(j)} + 2\sum_{j=1}^{\delta}\left[\mu_{d+2j-1}(0) + h\partial_x\phi\frac{v_j}{2dw_j} + h^2\phi\partial_x\phi\frac{v_j}{(2dw_j)^2}\right]\hat{\mathbf{b}}_j^-$$

From the conditions $\mathbf{f}^{in} \parallel \mathbb{1}$ and with $\mu_1(0) = 0$ follows

$$\mathbf{f}^1 = \sum_{j=1}^{\delta}\left[-h\partial_x\phi\frac{v_j}{2dw_j} - (h^2-y^2)\phi\partial_x\phi\frac{v_j}{(2dw_j)^2}\right]\begin{pmatrix}\hat{\mathbf{b}}_j^-\\ \hat{\mathbf{b}}_j^-\end{pmatrix} - y\partial_x\phi\cdot\sum_{j=1}^{\delta}\frac{v_j}{2dw_j}\begin{pmatrix}\hat{\mathbf{b}}_j^-\\ -\hat{\mathbf{b}}_j^-\end{pmatrix}$$

Applying the compatibility condition (3.12) and using

$$\left\langle V\cdot\begin{pmatrix}\hat{\mathbf{b}}_j^-\\ \hat{\mathbf{b}}_j^-\end{pmatrix},\mathbb{1}\right\rangle_h = \frac{4v_j}{2d}\,,\quad j=1,\ldots,\delta$$

now immediately yields the diffusion equation

$$\partial_t\phi = \partial_x\left[h\cdot\overline{v^2/w}\cdot\partial_x\phi + \frac{h^2}{3d}\cdot\overline{v^2/w^2}\cdot\phi\partial_x\phi\right],$$

where the overlines denote averages, i.e.

$$\overline{v^2/w} = \frac{1}{\delta}\sum_{j=1}^{\delta}\frac{v_j^2}{w_j}\,,\quad \overline{v^2/w^2} = \frac{1}{\delta}\sum_{j=1}^{\delta}\frac{v_j^2}{w_j^2}\,. \tag{4.25}$$

Remarks:

a) Classical Broadwell model: $d=2$, $|v_j| = |w_j| = 1/\sqrt{2}$ and thus

$$\partial_t\phi = \partial_x\left[\frac{h}{\sqrt{2}}\partial_x\phi + \frac{h^2}{6}\phi\partial_x\phi\right].$$

b) Limit $d\to\infty$: The averages (4.25) diverge, i.e. the diffusion constants are not finite. In this case, a different scaling has to be used to possibly obtain a diffusion limit. (See [9] for an analogous problem for the pure Knudsen gas in three dimensions.)

c) It is not necessary to choose the velocities equally distributed over the angles. The whole analysis holds whenever the symmetry conditions (a) of section 2 are valid.

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