Tracking control with prescribed transient behaviour for systems of known relative degree*

Achim Ilchmann†, Eugene P. Ryan‡, Philip Townsend‡

†*Institute of Mathematics, Technical University Ilmenau, Weimarer Straße 25, 98693 Ilmenau, DE,* ilchmann@mathematik.tu-ilmenau.de

‡*Department of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, UK,* epr@maths.bath.ac.uk, p.townsend@bath.ac.uk

Abstract

Tracking of a reference signal (assumed bounded with essentially bounded derivative) is considered for multi-input, multi-output linear systems satisfying the following structural assumptions: (i) arbitrary – but known – relative degree, (ii) the "high-frequency gain" matrix is sign definite – but of unknown sign, (iii) exponentially stable zero dynamics. The first control objective is tracking, by the output y, with prescribed accuracy: given $\lambda > 0$ (arbitrarily small), determine a feedback strategy which ensures that, for every reference signal r, the tracking error $e = y - r$ is ultimately bounded by λ (that is, $\|e(t)\| < \lambda$ for all t sufficiently large). The second objective is guaranteed output transient performance: the evolution of the tracking error should be contained in a prescribed performance funnel $\mathcal{F}_\varphi$ (determined by a function φ). Both objectives are achieved by a filter in conjunction with a feedback function of the filter states, the tracking error and a gain parameter. The latter is generated via a feedback function of the tracking error and the funnel parameter φ. Moreover, the feedback system is robust to nonlinear perturbations bounded by some continuous function of the output. The feedback structure essentially exploits an intrinsic high-gain property of the system/filter interconnection by ensuring that, if $(t, e(t))$ approaches the funnel boundary, then the gain attains values sufficiently large to preclude boundary contact.

Key words: Output feedback, transient behaviour, tracking, relative degree

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1 Introduction

1.1 Class of systems

We consider the class of nonlinearly-perturbed (perturbation p), m-input ($u(t) \in \mathbb{R}^m$), m-output ($y(t) \in \mathbb{R}^m$) linear systems of the form

$$\left.\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) + p(t, x(t)), \quad x(0) = x^0 \in \mathbb{R}^n \\ y(t) &= C\,x(t) \in \mathbb{R}^m \end{aligned}\right\} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $p\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are such that the following hold.

Assumption A1: (*strict relative degree and sign-definite high-frequency gain*)
For some known $\rho \in \mathbb{N}$, $CA^iB = 0$ for $i = 1, ..., \rho - 2$ and $CA^{\rho-1}B$ is either strictly positive definite or strictly negative definite.

Assumption A2: (*minimum-phase*)

$$\det \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} \neq 0 \qquad \text{for all } s \in \mathbb{C} \text{ with } \operatorname{Re} s \geq 0.$$

Assumption A3: (*nonlinear perturbation*)
The perturbation $p\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function with the property that, for some continuous $\mu\colon \mathbb{R}^m \to \mathbb{R}_+$,

$$\|p(t,x)\| \leq \mu(Cx) \quad \forall\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

Remark 1

(i) If the transfer function $s \mapsto C(sI - A)^{-1}B = \sum_{i=0}^{\infty} CA^iBs^{-i+1}$ is non-trivial (not identically zero), then there exists $\rho \in \mathbb{N}$ such that $CA^iB = 0$ for $i = 1, ..., \rho - 2$ and $CA^{\rho-1}B \neq 0$. Assumption A1 requires that ρ be known, and $CA^{\rho-1}B$ to be not only invertible (i.e. *strict* relative degree) but also either strictly positive definite or strictly negative definite. The sign-definite assumption is redundant in the single-input, single-output case.

(ii) Coppel (Coppel, 1974, Th. 10) has shown (see also (Ilchmann, 1993, Prop. 2.1.2)) that Assumption A2 is equivalent to: (A, B) is stabilizable, (C, A) is detectable, and the transfer function $s \mapsto C(sI - A)^{-1}B$ has no zeros in the closed right half complex plane $\{\lambda \in \mathbb{C} |\ \operatorname{Re}(\lambda) \geq 0\}$.
Note that the minimum phase assumption implies that the unperturbed ($p \equiv 0$) system has exponentially stable zero dynamics, see, for example, (Isidori, 1995, Sec. 5.1).

(iii) Even in the absence of a nonlinear perturbation p, the results of the paper are new. We encompass perturbations for added generality and remark that perturbations satisfying A3 can be incorporated with relative ease in the analysis.

Linear systems satisfying Assumptions A1 and A2 are, at least in the single-input single-output case, typical of the class of systems underlying the area of high-gain adaptive control, as studied in (Morse, 1983), (Byrnes and Willems, 1984), and (Mareels, 1984) for example. Most early results pertain to systems of relative degree one. More recently, systems of higher relative degree have been investigated: in Section 3.2, we compare some of these investigations with the approach adopted here.

1.2 Control objectives and the performance funnel

The first control objective is approximate tracking, by the output y, of reference signals r of class $\mathcal{R} := W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$, i.e. the space of locally absolutely continuous bounded functions with bounded derivative, endowed with norm $\|r\|_{1,\infty} := \|r\|_\infty + \|\dot{r}\|_\infty$. In particular, for arbitrary $\lambda > 0$, we seek an output feedback strategy which ensures that, for every $r \in \mathcal{R}$, the closed-loop system has bounded solution and the tracking error $e(t) = y(t) - r(t)$ is ultimately bounded by λ (that is, $\|e(t)\| \le \lambda$ for all t sufficiently large). The second control objective is prescribed transient behaviour of the tracking error signal. We capture both objectives in the concept of a performance funnel

$$\mathcal{F}_\varphi := \left\{ (t,e) \in \mathbb{R}_+ \times \mathbb{R}^m \,\middle|\, \varphi(t)\,\|e\| < 1 \right\}$$

associated with a function φ (the reciprocal of which determines the funnel boundary) belonging to

$$\mathcal{B} := \left\{ \varphi \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}) \,\middle|\, \varphi(0) = 0,\ \varphi(s) > 0\ \forall\ s > 0,\ \liminf_{s\to\infty} \varphi(s) > 0 \right\}.$$

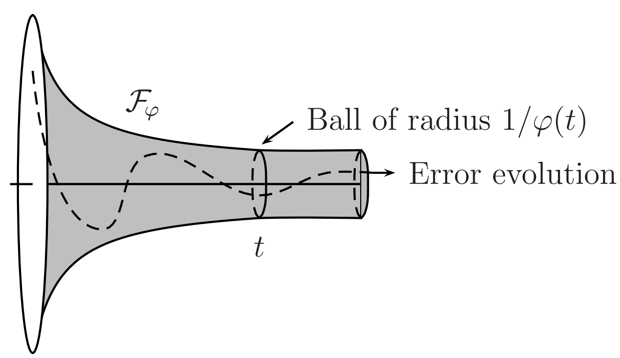


Fig. 1. Prescribed performance funnel $\mathcal{F}_\varphi$.

The aim is an output feedback strategy ensuring that, for every reference signal $r \in \mathcal{R}$, the tracking error $e = y - r$ evolves within the funnel $\mathcal{F}_\varphi$. For example, if $\liminf_{t\to\infty} \varphi(t) > 1/\lambda$, then evolution within the funnel ensures that the first control objective is achieved. If φ is chosen as the function $t \mapsto \min\{t/T, 1\}/\lambda$, then evolution within the funnel ensures that the prescribed tracking accuracy $\lambda > 0$ is achieved within the prescribed time $T > 0$.

The paper is structured as follows. In Section 2, we describe the control strategy and present the main result in Theorem 2. A discussion, including intuition and literature review, is presented in Section 3. An example is given in Section 4, which illustrates the control strategy by numerical simulations. For purposes of exposition, all proofs are deferred to Section 5.

2 The control

Let Assumptions A1 and A2 hold, with relative degree $\rho \geq 2$; the relative degree 1 case will be treated separately.

2.1 Filter

Introduce the filter

$$\begin{aligned} \dot{\xi}_i(t) &= -\xi_i(t) + \xi_{i+1}, & \xi_i(0) &= \xi_i^0 \in \mathbb{R}^m, \quad i = 1, \ldots, \rho - 2 \\ \dot{\xi}_{\rho-1}(t) &= -\xi_{\rho-1}(t) + u(t), & \xi_{\rho-1}(0) &= \xi_{\rho-1}^0 \in \mathbb{R}^m, \end{aligned}$$

which, on writing

$$\xi(t) = \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \\ \vdots \\ \xi_{\rho-2}(t) \\ \xi_{\rho-1}(t) \end{pmatrix}, \qquad F = \begin{bmatrix} -I & I & 0 \cdots & 0 & 0 \\ 0 & -I & I \cdots & 0 & 0 \\ 0 & 0 & -I \cdots & 0 & 0 \\ \vdots & \vdots & \vdots \ddots & \vdots & \vdots \\ 0 & 0 & 0 \cdots & -I & I \\ 0 & 0 & 0 \cdots & 0 & -I \end{bmatrix}, \qquad G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix},$$

may be expressed as

$$\left.\begin{aligned} \dot{\xi}(t) &= F\xi(t) + Gu(t), & \xi(0) &= \xi^0 \in \mathbb{R}^{(\rho-1)m}, \\ \xi_1(t) &= H\xi(t), & H &:= \begin{bmatrix} I_m \vdots 0 \vdots \cdots \vdots 0 \end{bmatrix}. \end{aligned}\right\} \tag{2}$$

2.2 Feedback

Let $\nu\colon \mathbb{R} \to \mathbb{R}$ be any C^∞ function with the properties

$$\limsup_{k\to\infty} \nu(k) = +\infty \quad \text{and} \quad \liminf_{k\to\infty} \nu(k) = -\infty\,. \tag{3}$$

Introduce the projections

$$\pi_i \colon \mathbb{R}^{(\rho-1)m} \to \mathbb{R}^{im}, \quad \xi = (\xi_1, \ldots, \xi_{\rho-1}) \mapsto (\xi_1, \ldots, \xi_i), \qquad i = 1, \ldots, \rho - 1,$$

and define the C^∞ function

$$\gamma_1 \colon \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m, \qquad (k, e) \mapsto \gamma_1(k, e) := -\nu(k) e\,, \tag{4}$$

with derivative (Jacobian matrix function) $D\gamma_1$. Next, for $i = 2, \ldots, \rho$, define the C^∞ function $\gamma_i : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{(i-1)m} \to \mathbb{R}^m$ by the recursion

$$\begin{aligned} \gamma_i(k, e, \pi_{i-1}\xi) &:= \gamma_{i-1}(k, e, \pi_{i-2}\xi) \\ &\quad + \|D\gamma_{i-1}(k, e, \pi_{i-2}\xi)\|^2 \, k^4 \, (1 + \|\pi_{i-1}\xi\|^2) \Big(\xi_{i-1} + \gamma_{i-1}(k, e, \pi_{i-2}\xi)\Big)\,, \end{aligned} \tag{5}$$

wherein we adopt the notational convention $\gamma_1(k, e, \pi_0\xi) := \gamma_1(k, e)$.

For arbitrary $r \in \mathcal{R}$, the control strategy is given by

$$\left.\begin{aligned} u(t) &= -\gamma_\rho\Big(k(t), Cx(t) - r(t), \xi(t)\Big), \\ k(t) &= \frac{1}{1 - \Big(\varphi(t)\|Cx(t) - r(t)\|\Big)^2}\,. \end{aligned}\right\} \tag{6}$$

2.3 Closed-loop system

The conjunction of (1), (2) and (6) defines the closed-loop initial-value problem

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + p(t, x(t)) - B\gamma_\rho(k(t), Cx(t) - r(t), \xi(t)), \; x(0) = x^0 \\ \dot{\xi}(t) &= F\xi(t) - G\gamma_\rho(k(t), Cx(t) - r(t), \xi(t)), \qquad \xi(0) = \xi^0 \\ k(t) &= \frac{1}{1 - (\varphi(t)\|Cx(t) - r(t)\|)^2}\,. \end{aligned}\right\} \tag{7}$$

Noting the potential singularity in the function k, some care must be exercised in defining the concept of a solution of (7): a function $(x, \xi) \colon [0, \omega) \to \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$, with $0 < \omega \le \infty$, is deemed a *solution* of (7) if, and only if, it is absolutely continuous, with $(x(0), \xi(0)) = (x^0, \xi^0)$, satisfies the differential equations in (7) for almost all $t \in [0, \omega)$ and $\varphi(t)\,\|Cx(t) - r(t)\| < 1$ for all $t \in [0, \omega)$. A solution is *maximal* if, and only if, it has no proper right extension that is also a solution. Observe that the tracking objective is achieved if it can be shown that a solution exists, and that every solution can be extended to a (maximal) solution on $\mathbb{R}_+$.

Firstly, we consider systems of relative degree $\rho \geq 2$.

Theorem 2 *Let A, B and C be such that Assumptions A1 and A2 hold with $\rho \geq 2$; let p be such that A3 holds. Let $\mathcal{F}_\varphi$ be a performance funnel associated with $\varphi \in \mathcal{B}$. For every $r \in \mathcal{R}$ and $(x^0, \xi^0) \in \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$, application of the feedback* (6) *in conjunction with the filter* (2) *to system* (1) *yields the initial-value problem* (7) *which has a solution and every solution can be extended to a maximal solution. Every maximal solution $(x, \xi)\colon [0, \omega) \to \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$ has the properties:*

(i) $\omega = \infty$;
(ii) all variables (x, ξ), k and u are bounded;
(iii) the tracking error evolves within the funnel $\mathcal{F}_\varphi$ and is bounded away from the funnel boundary, i.e. there exists $\varepsilon > 0$ such that, for all $t \geq 0$, $\varphi(t)\,\|Cx(t) - r(t)\| \leq 1 - \varepsilon$.

Secondly, we consider the case wherein the triple (A, B, C) defines a minimum-phase system of relative degree $\rho = 1$. In this case, a filter is not necessary and the controller simplifies to

$$u(t) = \nu(k(t))(Cx(t) - r(t)), \quad k(t) = \frac{1}{1 - \big(\varphi(t)\|Cx(t) - r(t)\|\big)^2}. \tag{8}$$

The closed-loop initial-value problem then becomes

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + p(t, x(t)) + B\nu(k(t))(Cx(t) - r(t)), \quad x(0) = x^0 \\ k(t) &= \frac{1}{1 - (\varphi(t)\|Cx(t) - r(t)\|)^2}. \end{aligned}\right\} \tag{9}$$

Theorem 3 *Let A, B and C be such that Assumptions A1 and A2 hold with $\rho = 1$; let p be such that A3 holds. Let $\mathcal{F}_\varphi$ be a performance funnel associated with $\varphi \in \mathcal{B}$. For every $r \in \mathcal{R}$ and $x^0 \in \mathbb{R}^n$, the initial-value problem* (9) *has a solution and every solution can be extended to a maximal solution. Every maximal solution $x\colon [0, \omega) \to \mathbb{R}^n$ has the properties:*

(i) $\omega = \infty$;
(ii) x, k and u are bounded;
(iii) there exists $\varepsilon > 0$ such that, for all $t \geq 0$, $\varphi(t)\,\|Cx(t) - r(t)\| \leq 1 - \varepsilon$.

Remark 4

(i) A simple example of a function satisfying (3) is $\nu : k \mapsto k\,\cos k$. The rôle of the function ν is similar to the concept of a "Nussbaum" function in adaptive control. Note, however, that the requisite properties (3) are less restrictive than (a) the "Nussbaum property"

$$\limsup_{k\to\infty} \frac{1}{k}\int_0^k \nu(\kappa)\,\mathrm{d}\kappa = \infty, \qquad \liminf_{k\to\infty} \frac{1}{k}\int_0^k \nu(\kappa)\,\mathrm{d}\kappa = -\infty,$$

as required in (Ye, 1999), for example, or (b) the stronger "scaling invariant Nussbaum property", as required in (Jiang *et al.*, 2004), for example.

(ii) In the specific case of a system of relative degree $\rho = 2$ in Theorem 2, writing $e(t) = Cx(t) - r(t)$ and omitting the argument t for simplicity, the control strategy takes the explicit form

$$\begin{aligned} u &= \nu(k)e - \Big[(\nu'(k)\|e\|)^2 + (\nu(k))^2\Big]\, k^4\, [1 + \|\xi\|^2]\theta \\ k &= [1 - \varphi^2\|e\|^2]^{-1}, \quad \theta = \xi - \nu(k)e\,, \quad \dot{\xi} = -\xi + u, \quad \xi(0) = \xi^0. \end{aligned}$$

(iii) If $CA^{\rho-1}B$ is known to be positive (respectively, negative) definite, the need for the function ν, with properties (3), in (4) or (8) is obviated and it may be replaced by $k \mapsto \nu(k) = -k$, ($k \mapsto \nu(k) = k$), respectively. The proofs of Theorem 2 and 3 are readily modified to confirm this claim. In the case of sign-definite $CA^{\rho-1}B$ of known sign, the result of Theorem 3 is proved in (Ilchmann *et al.*, 2002): the general case of Theorem 3, wherein $CA^{\rho-1}B$ is of unknown sign, is new.

3 Discussion

3.1 Intuition

The intuition behind the filter (2) and feedback control strategy (6) is as follows. With $\xi^0 = 0$, the transfer function from u to ξ_1 is given by

$$H(sI - F)^{-1}G = (s+1)^{1-\rho} I\,.$$

Therefore, with reference to Figure 2 below, the transfer function from the signal ξ_1 to the output y is given by

$$(s+1)^{\rho-1}\, C(sI-A)^{-1}B = C[I+A]^{\rho-1}(sI-A)^{-1}B = C(sI-A)^{-1}[I+A]^{\rho-1}B,$$

which has the minimum phase property and is of relative degree one (see Lemma 5 below): thus, the triple $\big(A, [I + A]^{\rho-1}B, C\big)$ defines a minimum-phase system of strict relative degree one with high-frequency gain $CA^{\rho-1}B$.

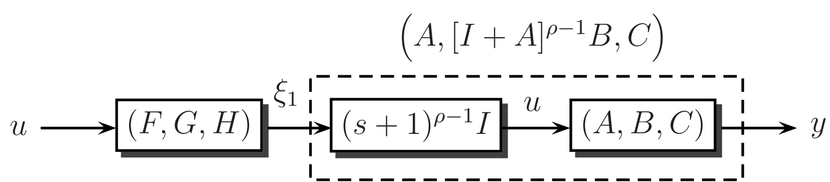


Fig. 2.

Lemma 5 *Let* (1) *be such that Assumptions A1 and A2 hold with* $\rho \geq 2$ *and assume* $p = 0$. *Then the following hold.*

(i) *The triple* $(A, [I + A]^{\rho-1}B, C)$ *has the minimum phase property.*

(ii) *There exist* $K \in \mathbb{R}^{n\times m(\rho-1)}$ *and invertible* $U \in \mathbb{R}^{n\times n}$ *such that, under the coordinate change*

$$\begin{pmatrix} y(t) \\ z(t) \\ \xi(t) \end{pmatrix} = L \begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix}, \quad \begin{pmatrix} y^0 \\ z^0 \\ \xi^0 \end{pmatrix} = L \begin{pmatrix} x^0 \\ \xi^0 \end{pmatrix}, \quad L := \begin{bmatrix} U & -UK \\ 0 & I \end{bmatrix}, \tag{10}$$

the conjunction of system (1) *and filter* (2) *is represented by*

$$\left.\begin{aligned} \dot y(t) &= A_1 y(t) + A_2 z(t) + CA^{\rho-1}B\ \xi_1(t), && y(0) = y^0 \in \mathbb{R}^m \\ \dot z(t) &= A_3 y(t) + A_4 z(t), && z(0) = z^0 \in \mathbb{R}^{n-m} \\ \dot\xi(t) &= F\xi(t) + Gu(t), && \xi(0) = \xi^0 \in \mathbb{R}^{(\rho-1)m} \end{aligned}\right\} \tag{11}$$

where $A_4 \in \mathbb{R}^{(n-m)\times(n-m)}$ *has spectrum in open left half complex plane.*

If $(x, \xi) : [0, \omega) \to \mathbb{R}^n \times \mathbb{R}^{m(\rho-1)}$ is a maximal solution of the nonlinearly-perturbed closed-loop system (7), then, in view of Lemma 5 and writing

$$y(t) = Cx(t), \qquad e(t) = y(t) - r(t), \quad e^0 = y^0 - r(0)\,, \tag{12}$$

there exists an invertible linear coordinate transformation L (with associated invertible submatrix U) which takes (7) into the equivalent form

$$\left.\begin{aligned} \dot e(t) &= A_1 e(t) + A_2 z(t) + CA^{\rho-1}B\xi_1(t) + f_1(t), && e(0) = e^0 \\ \dot z(t) &= A_3 e(t) + A_4 z(t) + f_2(t), && z(0) = z^0 \\ \dot\xi(t) &= F\xi(t) - G\gamma_\rho(k(t), e(t), \xi(t)), && \xi(0) = \xi^0 \\ k(t) &= 1/\big(1 - (\varphi(t)\|e(t)\|)^2\big), \end{aligned}\right\} \tag{13}$$

where the functions f_1 and f_2 are given by

$$\left.\begin{aligned} f_1(t) &:= A_1 r(t) + [I_m \,\vdots\, 0\,]\, U\, p(t, x(t)) - \dot r(t), \\ f_2(t) &:= A_3 r(t) + [0 \,\vdots\, I_{n-m}]\, U\, p(t, x(t)). \end{aligned}\right\} \tag{14}$$

Since $(\varphi(t)\|e(t)\|)^2 < 1$, properties of $\varphi \in \mathcal{B}$ yield boundedness of the function e which, together with boundedness of r, implies boundedness of y. By boundedness of r, essential boundedness of $\dot r$ and Assumption A3, we may now conclude that f_1 is essentially bounded and f_2 is bounded. Now observe that, since A_4 is Hurwitz and f_2 is bounded, the second of the differential equations in (13) implies that z is bounded. We record these observations in the following.

Lemma 6 *Let Assumptions A1 and A2 hold with $\rho \geq 2$. Let p be such that Assumption A3 holds. Let $\mathcal{F}_\varphi$ be a performance funnel associated with $\varphi \in \mathcal{B}$. Let $r \in \mathcal{R}$ and $(x^0, \xi^0) \in \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$. If $(x, \xi)\colon [0, \omega) \to \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$ is a maximal solution of* (7)*, then the functions y, z and e, given by* (10) *and* (12)*, are bounded. Furthermore, the functions f_1 and f_2, given by* (14)*, are, respectively, essentially bounded and bounded.*

From existing results on relative degree one systems, see (Ilchmann *et al.*, 2002), and momentarily regarding ξ_1 as an independent input variable, it is known that, in the case wherein $CA^{\rho-1}B$ is positive definite, the choice $\xi_1 = -ke$ achieves the control objective for system defined by the first two of equations (13); Theorem 3 extends this to the case of sign definite $CA^{\rho-1}B$ of unknown sign, asserting that the choice $\xi_1 = -\gamma_1(k, e) = \nu(k)\, e$ achieves the control objective for the latter system of relative degree one. However (with $\rho \geq 2$), ξ_1 is not an independent input but instead is generated via the filter. The essence of the strategy is a procedure which "backsteps" through the filter variables to arrive at an input u which assures boundedness of the signals $\theta_i = \xi_i + \gamma_i(k, e, \pi_{i-1}\xi)$, $i = 1, ..., \rho - 1$ (and, in particular, yields boundedness of the "mismatch" $\theta_1 = \xi_1 - \nu(k)e$). More precisely, we show the following.

Lemma 7 *Let the hypotheses of Lemma 6 hold. If $(x, \xi)\colon [0, \omega) \to \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$ is a maximal solution of* (7)*, then the signal*

$$\theta = (\theta_1, \ldots, \theta_{\rho-1})\colon [0, \omega) \to \mathbb{R}^{(\rho-1)m}$$

defined, componentwise, by

$$\theta_i(t) = \xi_i(t) + \gamma_i(k(t), Cx(t) - r(t), \pi_{i-1}\xi(t))\,, \qquad i = 1, \ldots, \rho - 1, \tag{15}$$

is bounded.

3.2 Literature review

The present paper is in the spirit of the adaptive results in (Ye, 1999) and the non-adaptive results in (Ilchmann *et al.*, 2002). The paper (Ye, 1999) restricts the class of systems (1) satisfying Assumptions A1 and A2 to the single-input, single-output case; the control objective is (continuous) adaptive λ-tracking with non-decreasing gain; transient behaviour is not addressed, however nonlinear perturbations as in Assumption A3 are allowed. The filter and the "backstepping" construction of the feedback strategy in the present paper is akin to that of (Ye, 1999). The approach of (Ilchmann *et al.*, 2002) restricts the class of systems (1) (satisfying Assumptions A1 and A2) to those of relative degree one with sign-definite high-frequency gain CB of known sign. For this restricted class, the funnel control objective is achieved. The control law is a special case of (6): the associated gain k in (9) is not monotone (non-decreasing) – which contrasts with typical high-gain adaptive control schemes; $k(t)$ becomes large only when the distance between the output and the funnel boundary becomes small which, in conjunction with the underlying high-gain properties of the system class, precludes boundary contact.

The paper (Miller and Davison, 1991) considers the class of systems (1) satisfying Assumption A1 and A2 restricted to the single-input, single-output case with high-frequency gain of known sign. Therein, a controller is introduced which guarantees the "error to be less than an (arbitrarily small) prespecified constant after an (arbitrarily small) prespecified period of time, with an (arbitrarily small) prespecified upper bound on the amount of overshoot." However, the controller is adaptive with non-decreasing gain k, invokes a piecewise-constant switching strategy, and is less flexible in its scope for shaping transient behaviour (in particular, an *a priori* bound on the initial data is required).

Jiang *et al.* (Jiang *et al.*, 2004) consider a large class of non-linear systems which are single-input, single-output, have known relative degree and zero dynamics which are stable in an appropriate sense. The emphasis therein lies on the nonlinear nature of the system class. The filter used in the present paper and the attendant backstepping procedure resemble the methodology of (Jiang *et al.*, 2004). However, the controller in the latter incorporates a non-decreasing adaptive gain and achieves output stabilization – neither tracking nor transient behaviour is addressed.

In (Khalil and Saberi, 1987) adaptive stabilization of the output is achieved for a class of systems (1) satisfying Assumption A2 and a strengthened Assumption A1. The adaptive strategy is based on a high-gain compensator and is piecewise constant: transient behaviour is not considered.

Finally, we remark that, for clarity of exposition, we have not chosen the most general presentation. The matrix F in the filter (2) could have arbitrary negative eigenvalues on the diagonal; inherent conservatism in the functions γ_i for the feedback law could be improved if tighter estimates are used in the analysis; the design of k may allow for different measures of the distance to the funnel boundary. These features relate to issues of controller synthesis: we view the contribution of the paper as analytical in nature - addressing the question of existence of controllers which guarantee performance under weak hypothesis.

4 Example

We illustrate the controller strategy (6) for the single-input, single-output, relative degree two system with nonlinear perturbations modelling a pendulum (with input force u):

$$\ddot{y}(t) + a\sin y(t) = b\,u(t)\,, \tag{16}$$

with unknown real parameters a and $b \neq 0$. Equation (16) is equivalent to (1) with $x(t) = (y(t), \dot{y}(t))^T$,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ b \end{bmatrix}, \quad C = [1 \vdots 0], \quad p(t, x(t)) = a\sin y(t),\ t \geq 0.$$

The funnel is specified by the smooth function

$$t \mapsto \varphi(t) = \begin{cases} 10(1 - (0.1\,t - 1)^2), & 0 \leq t < 10 \\ 10, & t \geq 10, \end{cases} \tag{17}$$

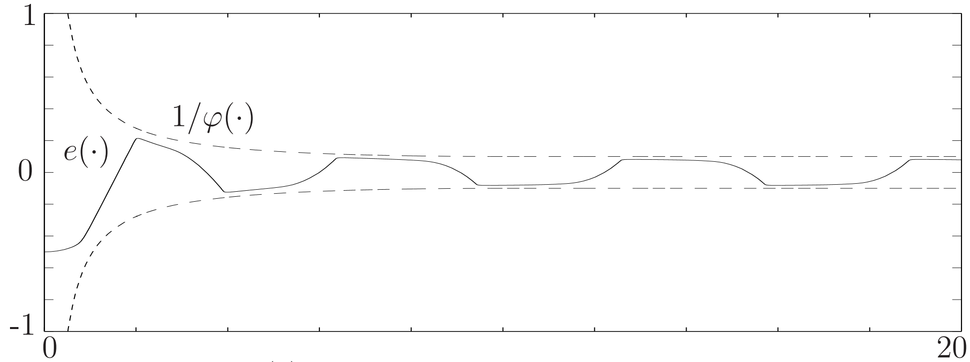
which assures a tracking accuracy $|e(t)| < 0.1$ for all $t \geq 10$. If non-zero b is of unknown sign, then, choosing $\nu : k \mapsto k\cos k$, writing $e(t) = y(t) - r(t)$ and suppressing the argument t for simplicity, the control strategy is

$$\left.\begin{aligned} u &= (k\cos k)e \\ &\qquad -\Big[\xi - (k\cos k)e\Big]\Big[(\cos k - k\sin k)^2\, e^2 + k^2\cos^2 k\Big]\, k^4\,[1 + \xi^2] \\ k &= [1 - \varphi^2 e^2]^{-1} \\ \dot{\xi} &= -\xi + u, \qquad\qquad \xi(0) = 0. \end{aligned}\right\} \tag{18}$$

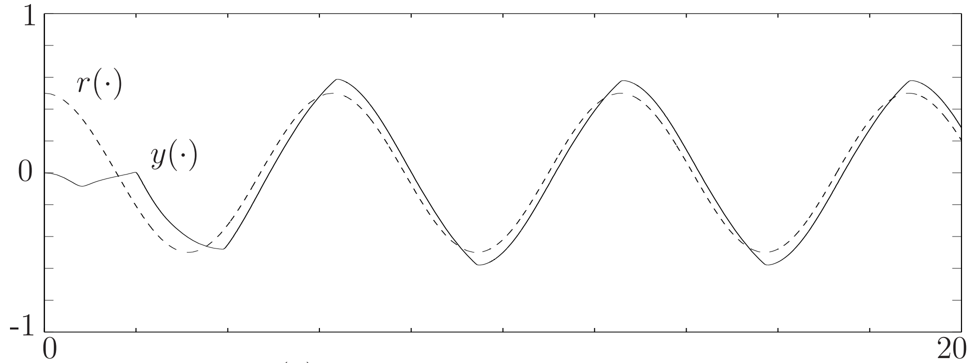
Adopting the values $a = \frac{1}{2}$, $b = 1$, initial data $(y(0), \dot{y}(0)) = (0, 0)$ and reference signal $t \mapsto r(t) = \frac{1}{2}\cos t$, the behaviour of the closed-loop system (16)-(18) over the time interval $[0, 20]$ is depicted in Figure 3. The "peaks" in the control action occur whenever the tracking error is close to the boundary of the funnel. However, if $b \neq 0$ is known *a priori* to be positive, then the peaking behaviour is considerably mollified by choosing the function $\nu\colon k \mapsto -k$ in place of $k \mapsto k\cos k$ in which case the strategy is

$$\left.\begin{aligned} u &= -ke - [\xi + ke]\Big[e^2 + k^2\Big]\, k^4\,[1 + \xi^2] \\ k &= [1 - \varphi^2 e^2]^{-1} \\ \dot{\xi} &= -\xi + u, \qquad\qquad \xi(0) = 0. \end{aligned}\right\} \tag{19}$$

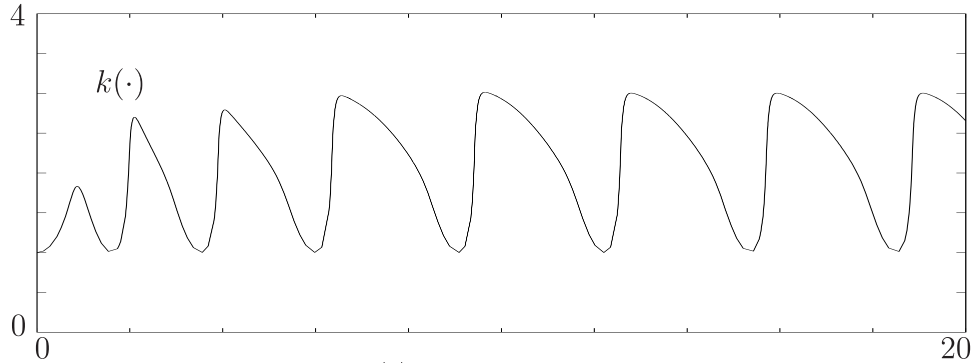
For the same parameter values and initial data as above, the behaviour (16), under control (19), is shown in Figure 4.



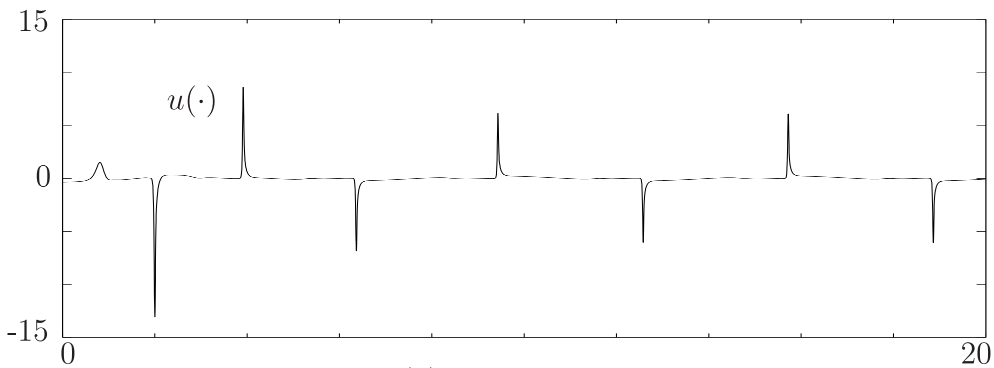
(a) The funnel and tracking error e



(b) The reference r and output y

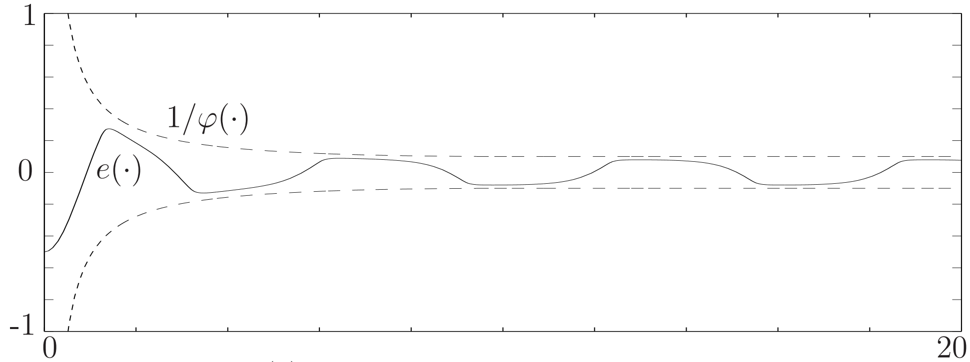


(c) The function k

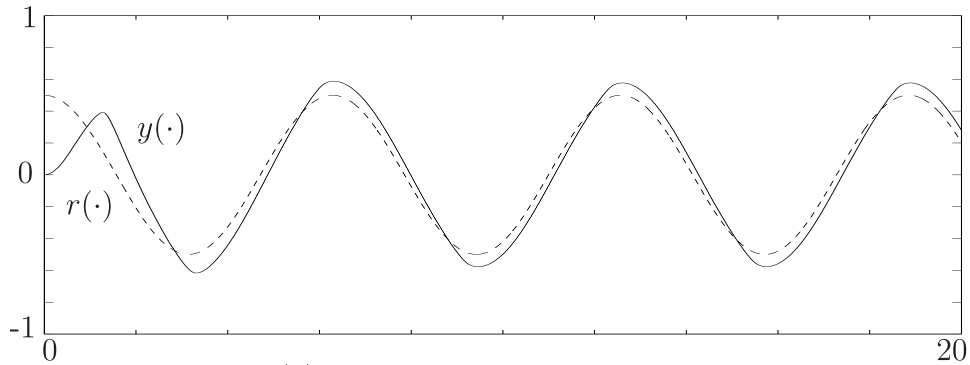


(d) The control u

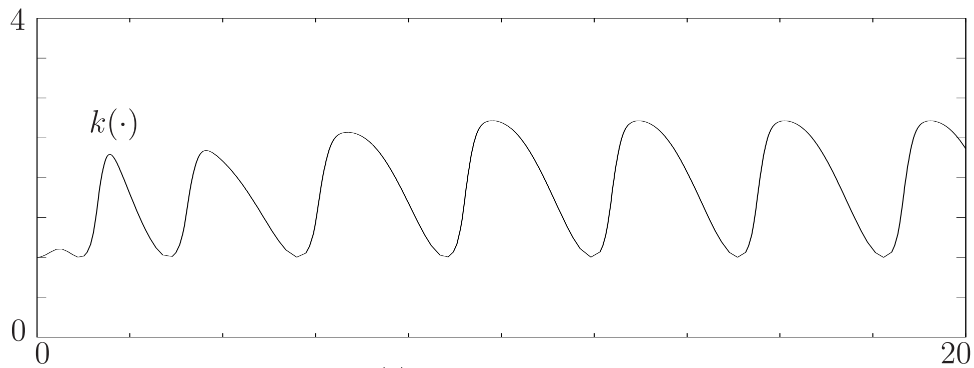
Fig. 3. Unknown sign $b \neq 0$: control (18) applied to the nonlinear pendulum (16).



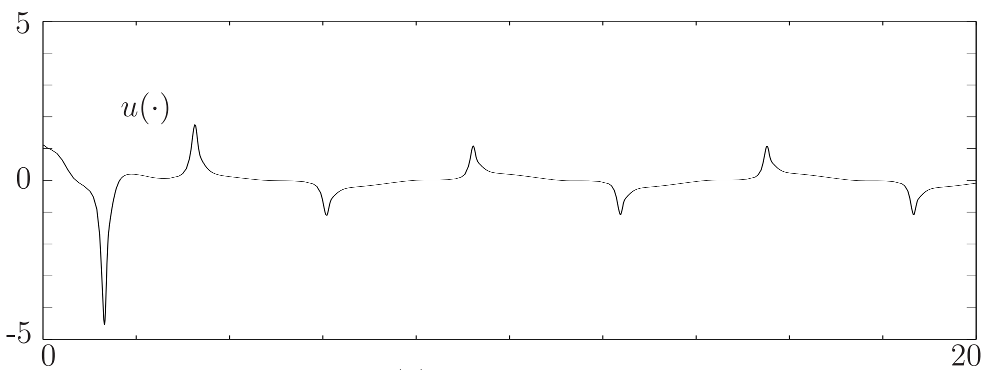
(a) The funnel and tracking error e



(b) The reference r and output y



(c) The function k



(d) The control u

Fig. 4. Known sign $b > 0$: control (19) applied to the nonlinear pendulum (16).

5 Proofs

5.1 Proof of Lemma 5

Note that

$$K := \Big[[I+A]^{\rho-2}B \,\vdots\, [I+A]^{\rho-3}B \,\vdots\, \ldots \,\vdots\, [I+A]B \,\vdots\, B\Big] \in \mathbb{R}^{n\times(\rho-1)m}$$

is such that

$$AK - KF = \Big[[I+A]^{\rho-1}B \,\vdots\, 0 \,\vdots\, \ldots \,\vdots\, 0\Big], \quad KG = B \quad \text{and} \quad CK = 0.$$

The coordinate transformation

$$\zeta(t) = x(t) - K\xi(t),$$

together with (1) (with $p = 0$) and (2), yields

$$\left.\begin{aligned} \dot{\zeta}(t) &= A\,\zeta(t) + [I+A]^{\rho-1}\,B\,\xi_1(t) \\ y(t) &= C\,\zeta(t). \end{aligned}\right\} \tag{20}$$

Since $C[I+A]^{\rho-1}B = CA^{\rho-1}B$ is invertible by Assumption A1, we have $\mathbb{R}^n = \text{ im } [I+A]^{\rho-1}\,B \oplus \text{ ker } C$, and thus there exists an invertible $U \in \mathbb{R}^{n\times n}$ so that, under the change of coordinates

$$\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = U\,\zeta(t),$$

we may express (20) as

$$\left.\begin{aligned} \dot{y}(t) &= A_1\,y(t) + A_2\,z(t) + CA^{\rho-1}\,B\,\xi_1(t) \\ \dot{z}(t) &= A_3\,y(t) + A_4\,z(t). \end{aligned}\right\} \tag{21}$$

Therefore, the coordinate transformation matrix

$$L = \begin{bmatrix} U & -UK \\ 0 & I \end{bmatrix}$$

takes (1) and (2) into form (11). It remains to show that A_4 has spectrum in open left half complex plane. Writing

$$M_1(s) = \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} \quad \text{and} \quad M_2(s) = \begin{bmatrix} sI - A & 0 & B \\ 0 & sI - F & -G \\ C & 0 & 0 \end{bmatrix},$$

we have

$$M_3(s) := \begin{bmatrix} I & K & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} M_2(s) \begin{bmatrix} I & K & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} sI - A & AK - KF & 0 \\ 0 & sI - F & -G \\ C & 0 & 0 \end{bmatrix}.$$

In view of the particular structure of F, G and $AK - KF$, it is readily verified that

$$|\det M_3(s)| = |\det M_4(s)|, \qquad \text{where } M_4(s) = \begin{bmatrix} sI - A & [I + A]^{\rho-1}B \\ C & 0 \end{bmatrix}.$$

Moreover,

$$M_5(s) := \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} M_4(s) \begin{bmatrix} U^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} sI - A_1 & -A_2 & CA^{\rho-1}B \\ -A_3 & sI - A_4 & 0 \\ I & 0 & 0 \end{bmatrix}.$$

We may now conclude that, for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq 0$,

$$\begin{aligned} |\det(CA^{\rho-1}B)\, \det(sI - A_4)| &= |\det M_5(s)| = |\det M_4(s)| \\ &= |\det M_3(s)| = |\det M_2(s)| = |\det(sI - F)\, \det M_1(s)| \neq 0, \end{aligned}$$

whence Assertions (i) and (ii). □

5.2 *Proof of Lemma 7*

Assume that $(x, \xi)\colon [0, \omega) \to \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$ is a maximal solution of (7). Write $y(t) = Cx(t)$ and $e(t) = y(t) - r(t)$ for all $t \in [0, \omega)$. By Lemma 5, there exists an invertible linear transformation L under which the closed-loop system (7) may be expressed in the form (13), wherein, by Lemma 6, e and z are bounded and the functions f_1 and f_2, given by (14), are, respectively, essentially bounded and bounded. By the first of equations (13), we may infer the existence of $c_1 > 0$ such that

$$\|\dot e(t)\| \le c_1\Big(1 + \|\xi_1(t)\|\Big) \quad \text{for a.a. } t \in [0,\omega).$$

By boundedness of φ, e and essential boundedness of $\dot\varphi$, there exists $c_2 > 0$ such that

$$\begin{aligned}|\dot k(t)| &= 2k^2(t)\Big|\varphi^2(t)\langle e(t), \dot e(t)\rangle + \varphi(t)\dot\varphi(t)\|e(t)\|^2\Big| \\ &\le c_2\, k^2(t)\,(1 + \|\xi_1(t)\|) \quad \text{for a.a. } t \in [0,\omega).\end{aligned}$$

Since $k(t) \ge 1$ for all $t \in [0,\omega)$, we may now conclude the existence of $c_3 > 0$ such that

$$\|(\dot k(t), \dot e(t))\|^2 \le c_3\, k^4(t)\,\Big(1 + \|\xi_1(t)\|^2\Big) \quad \text{for a.a. } t \in [0,\omega).$$

Then, for some constant $c_{4,1} > 0$, we have, by invoking (5),

$$\begin{aligned}\langle\theta_1(t), \dot\theta_1(t)\rangle &\le \Big\langle \theta_1(t), (-\xi_1(t) + \xi_2(t))\Big\rangle + \|\theta_1(t)\|\|D\gamma_1(k(t), e(t))\|\|(\dot k(t), \dot e(t))\| \\ &\le -\|\theta_1(t)\|^2 + \Big\langle \theta_1(t), (\xi_2(t) + \gamma_1(k(t), e(t)))\Big\rangle \\ &\qquad + \|\theta_1(t)\|\,\|D\gamma_1(k(t), e(t))\|\,\sqrt{c_3}\,k^2(t)\,\sqrt{1 + \|\xi_1(t)\|^2} \\ &\le c_{4,1} - \|\theta_1(t)\|^2 + \langle\theta_1(t), \xi_2(t)\rangle + \Big\langle\theta_1(t), \gamma_1(k(t), e(t))\Big\rangle \\ &\qquad + \|\theta_1(t)\|^2\,\|D\gamma_1(k(t), e(t))\|^2\,k^4(t)\,(1 + \|\xi_1(t)\|^2) \\ &= c_{4,1} - \|\theta_1(t)\|^2 + \Big\langle\theta_1(t), \xi_2(t) + \gamma_2(k(t), e(t), \xi_1(t))\Big\rangle \\ &= c_{4,1} - \|\theta_1(t)\|^2 + \langle\theta_1(t), \theta_2(t)\rangle \qquad \text{for a.a. } t \in [0,\omega).\end{aligned}$$

Analogous calculations yield the existence of constants $c_{4,2}, \ldots, c_{4,\rho-1} > 0$, such that

$$\langle\theta_i(t), \dot\theta_i(t)\rangle \le c_{4,i} - \|\theta_i(t)\|^2 + \langle\theta_i(t), \theta_{i+1}(t)\rangle \quad \text{a.a. } t \in [0,\omega),\ \ i = 2, \ldots, \rho-2$$

and

$$\langle\theta_{\rho-1}(t), \dot\theta_{\rho-1}(t)\rangle \le c_{4,\rho-1} - \|\theta_{\rho-1}(t)\|^2 \quad \text{for a.a. } t \in [0,\omega).$$

Writing $c_4 = c_{4,1} + \ldots + c_{4,\rho-1}$, we have

$$\begin{aligned}\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|\theta(t)\|^2 &\le c_4 - \|\theta(t)\|^2 + \langle\theta_1(t), \theta_2(t)\rangle + \cdots + \langle\theta_{\rho-2}(t), \theta_{\rho-1}(t)\rangle \\ &= c_4 - \langle\theta(t), P\theta(t)\rangle \qquad \text{for a.a. } t \in [0,\omega),\end{aligned}$$

where P is a positive-definite, symmetric, tridiagonal matrix with all diagonal entries equal to 1 and all sub- and superdiagonal entries equal to -1/2 (in fact, P is the symmetric part of F). By positivity of P, it follows that θ is bounded. This completes the proof of the lemma. □

Introducing the open set

$$\mathcal{D} := \left\{ (x,\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m} \times \mathbb{R} \,\middle|\, \left(\varphi(|\eta|)\,\|Cx - r(|\eta|)\|\right)^2 < 1 \right\},$$

and defining, on $\mathcal{D}$,

$$\gamma_\rho^* \colon (x,\xi,\eta) \mapsto \gamma_\rho \left(1/(1 - (\varphi(|\eta|)\,\|Cx - r(|\eta|)\|)^2)\,,\, Cx - r(|\eta|), \xi \right),$$

the initial-value problem (7) may be recast on $\mathcal{D}$ as

$$\left.\begin{aligned} &\dot{x}(t) = Ax(t) + p(t, x(t)) - B\gamma_\rho^*(x(t),\xi(t),\eta(t)) \\ &\dot{\xi}(t) = F\xi(t) - G\gamma_\rho^*(x(t),\xi(t),\eta(t)) \\ &\dot{\eta}(t) = 1 \\ &(x(0),\xi(0),\eta(0)) = (x^0,\xi^0,0) \in \mathcal{D}. \end{aligned}\right\} \tag{22}$$

The standard theory of ordinary differential equations now applies to conclude the existence of a solution $t \mapsto (x(t),\xi(t),\eta(t)) \in \mathcal{D}$ to (22) and, moreover, every solution can be extended to a maximal solution $(x,\xi,\eta)\colon [0,\omega) \to \mathcal{D}$. We will make use of the following fact in due course: if there exists a compact set $\mathcal{C} \subset \mathcal{D}$ such that $(x(t),\xi(t),\eta(t)) \in \mathcal{C}$ for all $t \in [0,\omega)$, then $\omega = \infty$. To see this, assume that such a set $\mathcal{C}$ exists and, seeking a contradiction, suppose that $\omega < \infty$. By Assumption A3 of p, continuity of γ_ρ^* and boundedness of (x,ξ,η), it follows from (22) that (x,ξ,η) is uniformly continuous on the bounded interval $[0,\omega)$. Therefore, the limit $(x^*,\xi^*,\omega) = \lim_{t\nearrow\omega}(x(t),\xi(t),\eta(t))$ exists and, by compactness, lies in $\mathcal{C} \subset \mathcal{D}$. By the existence theory, the initial-value problem (22), with initial data (x^*,ξ^*,ω) replacing $(x^0,\xi^0,0)$ has a solution: concatenation of this solution with (x,ξ,η) yields a proper right extension of the latter, contradicting its maximality.

Clearly, if $(x,\xi,\eta)\colon [0,\omega) \to \mathcal{D}$ is a solution of (22), then $(x,\xi)\colon [0,\omega) \to \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$ is a solution of (7); conversely, if $(x,\xi) : [0,\omega) \to \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$ is a solution of (7), then $(x,\xi,\eta) : [0,\omega) \to \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m} \times \mathbb{R}$, with component η given by $\eta(t) = t$, is a solution of (22). We may now conclude that, for each $(x^0,\xi^0) \in \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$, (7) has a solution and every solution can be maximally extended.

Let $(x^0,\xi^0) \in \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$ be arbitrary and let (x,ξ) be a maximal solution of (7) with interval of existence $[0,\omega)$. Writing $y(t) = Cx(t)$, $e(t) = y(t) - r(t)$ for all $t \in [0,\omega)$ and invoking Lemma 5, there exists an invertible linear transformation L which takes (7) into the equivalent form (13)-(14). Introducing

$\theta_1 : [0,\omega) \to \mathbb{R}^m$ given by (15), viz.

$$\theta_1(t) = \xi_1(t) - \nu(k(t))e(t),$$

then, by the first of equations (13), we have

$$\dot{e}(t) = f_3(t) + \nu(k(t))\, CA^{\rho-1}B\, e(t) \quad \text{for a.a. } t \in [0,\omega), \tag{23}$$

with

$$f_3(t) := A_1 e(t) + A_2 z(t) + CA^{\rho-1}B\theta_1(t) + f_1(t).$$

By Lemmas 6 and 7, the functions y, z, e, and $\theta = (\theta_1, ..., \theta_{\rho-1})$, given by (15), are bounded which, together with essential boundedness of f_1, implies essential boundedness of f_3. Therefore, there exists $c_5 > 0$ such that

$$\langle e(t), \dot{e}(t)\rangle \le c_5 + \nu(k(t))\,\langle e(t), CA^{\rho-1}Be(t)\rangle \quad \text{for a.a. } t \in [0,\omega). \tag{24}$$

We are now in a position to prove boundedness of k. Writing

$$\beta_0 := \tfrac{1}{2}\left\| \left((CA^{\rho-1}B)^T + CA^{\rho-1}B \right)^{-1} \right\|^{-1} \quad \text{and} \quad \beta_1 := \|CA^{\rho-1}B\|,$$

and recalling that $CA^{\rho-1}B$ is either positive or negative definite, we have

$$\beta_0 \|e\|^2 \le |\langle e, CA^{\rho-1}Be\rangle| \le \beta_1 \|e\|^2 \quad \forall\, e \in \mathbb{R}^m.$$

Define the continuous function $\tilde{\nu} \colon \mathbb{R} \to \mathbb{R}$ as follows.

Case (a): If $CA^{\rho-1}B$ is positive definite, then set

$$\tilde{\nu}(k) := \begin{cases} -\beta_1 \nu(k), & \nu(k) \ge 0 \\ -\beta_0 \nu(k), & \nu(k) < 0\,. \end{cases}$$

Case (b): If $CA^{\rho-1}B$ is negative definite, then set

$$\tilde{\nu}(k) := \begin{cases} \beta_0 \nu(k), & \nu(k) \ge 0 \\ \beta_1 \nu(k), & \nu(k) < 0\,. \end{cases}$$

Therefore,

$$\nu(k)\langle e, CA^{\rho-1}Be\rangle \le -\tilde{\nu}(k)\|e\|^2 \quad \forall\, e \in \mathbb{R}^m\,,\ \forall\, k \ge 0\,,$$

which, together with boundedness of e, φ, essential boundedness of $\dot{\varphi}$ and (24), implies the existence of $c_6 > 0$ such that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi(t)\|e(t)\|\right)^2 &= 2\varphi(t)\dot{\varphi}(t)\|e(t)\|^2 + 2\varphi^2(t)\langle e(t), \dot{e}(t)\rangle \\ &\le c_6 - 2\,\varphi^2(t)\,\tilde{\nu}(k(t))\,\|e(t)\|^2 \quad \text{for a.a. } t \in [0,\omega). \end{aligned}$$

By properties (3) of ν, there exists a strictly increasing unbounded sequence (k_j) in $(1,\infty)$ such that $\tilde{\nu}(k_j) \to \infty$ as $j \to \infty$. Seeking a contradiction, suppose that k is unbounded. For each $j \in \mathbb{N}$, define

$$\begin{aligned}
\tau_j &:= \inf\{t \in [0,\omega) |\ k(t) = k_{j+1}\} \\
\sigma_j &:= \sup\{t \in [0,\tau_j] |\ \tilde{\nu}(k(t)) = \tilde{\nu}(k_j)\} \\
\tilde{\sigma}_j &:= \sup\{t \in [0,\tau_j] |\ k(t) = k_j\} \le \sigma_j \,.
\end{aligned}$$

Then, for all $j \in \mathbb{N}$ and all $t \in [\sigma_j, \tau_j]$, we have $k(t) \ge k_j$ and $\tilde{\nu}(k(t)) \ge \tilde{\nu}(k_j)$. Therefore,

$$(\varphi(t)\|e(t)\|)^2 \ge 1 - \frac{1}{k_j} \ge 1 - \frac{1}{k_1} =: c_7 > 0 \quad \forall\, t \in [\sigma_j, \tau_j] \ \ \forall\, j \in \mathbb{N}$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\varphi(t)\|e(t)\|)^2 \le c_6 - 2c_7\,\tilde{\nu}(k(t)) \quad \forall\, t \in [\sigma_j, \tau_j] \ \ \forall\, j \in \mathbb{N}.$$

Let $j^* \in \mathbb{N}$ be sufficiently large so that $c_6 - 2c_7\tilde{\nu}(k_{j^*}) < 0$. Then,

$$\Big(\varphi(\tau_{j^*})\|e(\tau_{j^*})\|\Big)^2 - \Big(\varphi(\sigma_{j^*})\|e(\sigma_{j^*})\|\Big)^2 < 0,$$

whence the contradiction

$$0 > \frac{1}{1 - \Big(\varphi(\tau_{j^*})\|e(\tau_{j^*})\|\Big)^2} - \frac{1}{1 - \Big(\varphi(\sigma_{j^*})\|e(\sigma_{j^*})\|\Big)^2} = k(\tau_{j^*}) - k(\sigma_{j^*}) \ge 0.$$

This proves boundedness of k.

Next we show boundedness of ξ, x and u.
Since k is bounded, there exists $\varepsilon > 0$ such that $\varphi(t)\|e(t)\| \le 1 - \varepsilon$ for all $t \in [0,\omega)$. By boundedness of y, z, θ and k, it follows from the recursive construction in (15) that, for $i = 1, ..., \rho - 1$, γ_i and ξ_i are bounded. Consequently x is bounded and, by (4) and (5), boundedness of γ_ρ (and hence of u) follows.

Finally, it remains to prove that $\omega = \infty$. Suppose that ω is finite. Let $c_8 > 0$ be such that $\|(x(t), \xi(t))\| \le c_8$ for all $t \in [0,\omega)$. Let

$$\mathcal{C} := \Big\{(x,\, \xi,\, \eta) \in \mathcal{D} \,\Big|\, \varphi(|\eta|)\,\|Cx - r(|\eta|)\| \le 1 - \varepsilon, \ \ \|(x,\xi)\| \le c_8, \ \ \eta \in [0,\omega]\Big\}.$$

Then $\mathcal{C}$ is a compact subset of $\mathcal{D}$ and contains the trajectory of the maximal solution $t \mapsto (x(t), \xi(t), t)$ of (22). Therefore, the supposition that ω is finite is false. This completes the proof of the theorem. □

5.4 Proof of Theorem 3

This is a straightforward modification of the proof of Theorem 2, essentially excising all vestiges of the filter equations. □

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