

# Time-varying linear systems: relative degree and normal form

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## Abstract

We define the relative degree of time-varying linear systems, show that it coincides with Isidori's and with Liberzon/Morse/Sontag's definition if the system is understood as a time-invariant nonlinear system, characterize it in terms of the system data and their derivatives, derive a normal form with respect to a time-varying linear coordinate transformation, and finally characterize the zero dynamics.

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## 1 Introduction

The concept of relative degree goes back to single-input single-output linear systems described in the frequency domain by a transfer function  $p(s)/q(s)$  where the relative degree is defined by  $r = \deg q - \deg p$ ;  $p, q$  denote polynomials with real coefficients. To derive a characterization in the time domain, take any realization of  $p(s)/q(s)$ , say

$$\left. \begin{aligned} \dot{x}(t) &= Ax(t) + bu(t) \\ y(t) &= cx(t) \end{aligned} \right\} \quad (1.1)$$

with  $A \in \mathbb{R}^{n \times n}$ ,  $b, c^T \in \mathbb{R}^n$ . Then  $p(s)/q(s) = c(sI - A)^{-1}b = \sum_{k=0}^{\infty} cA^k b s^{-(k+1)}$ , and it is easy to see that  $r = \deg q - \deg p$  if, and only if,

$$\forall k = 0, \dots, r-2 : cA^k b = 0, \quad cA^{r-1} b \neq 0. \quad (1.2)$$

Isidori [3, p. 137] generalized the concept of relative degree to single-input single-output time-invariant nonlinear systems, affine in the control, of the form

$$\left. \begin{aligned} \dot{x} &= f(x) + g(x)u(t) \\ y(t) &= h(x(t)), \end{aligned} \right\} \quad (1.3)$$

with  $f, g \in \mathcal{C}^\ell(\mathbb{R}^n, \mathbb{R}^n)$ ,  $h \in \mathcal{C}^\ell(\mathbb{R}^n, \mathbb{R})$ , and  $\ell \in \mathbb{N}$ : the system (1.3) has relative degree  $r \in \{1, \dots, \ell\}$  at  $x^0 \in \mathbb{R}^n$  if, and only if, there exists an open neighbourhood  $\mathcal{U}$  of  $x^0$ , such that,

$$\forall x \in \mathcal{U} \quad \forall k \in \{0, \dots, r-2\} : L_g L_f^k h(x) = 0, \quad L_g L_f^{r-1} h(x^0) \neq 0, \quad (1.4)$$

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where  $L_f \lambda = \frac{\partial \lambda}{\partial x} f$  denotes the derivative of a function  $\lambda$  along a vector field  $f$ ; see, for example, [3, Sect. 1.2].

The importance of the relative degree is that it leads to a normal form [3, Sec. 4.1]: if (1.3) has relative degree  $r$  at  $x^0$ , then there exists a diffeomorphism  $\Phi$ , defined in a neighbourhood of  $x^0$ , which transforms (1.3) under  $(\xi, \eta) = \Phi(x)$ ,  $\xi = (y, \dots, y^{(r-1)})^T$ , to

$$\left. \begin{aligned} y^{(r)} &= L_f^r h(\Phi^{-1}(\xi, \eta)) + L_g L_f^{(r-1)} h(\Phi^{-1}(\xi, \eta)) u(t) \\ \dot{\eta} &= q(\xi, \eta) \\ y(t) &= \xi_1(t), \end{aligned} \right\} \quad (1.5)$$

for some  $q \in \mathcal{C}^\ell(\mathbb{R}^n, \mathbb{R}^{n-r})$ . This form gives immediately that, for  $x(t_0) = x^0$ , “the relative degree  $r$  is exactly equal to the number of times one has to differentiate the output  $y(t)$  at time  $t = t_0$  in order to have the value  $u(t_0)$  of the input explicitly appearing” [3, p. 139]. Moreover,  $u$  enters only in a single differential equation in (1.5) directly and it is possible to read off the zero dynamics, see [3, Sec. 4.3] and Section 3 of the present paper.

The purpose of the present note is to generalize the concept of relative degree to time-varying linear systems of the form

$$\left. \begin{aligned} \dot{x} &= A(t)x + B(t)u(t) \\ y(t) &= C(t)x(t), \end{aligned} \right\} \quad (1.6)$$

with  $A \in \mathcal{C}^\ell(\mathbb{R}, \mathbb{R}^{n \times n})$ ,  $B, C^T \in \mathcal{C}^\ell(\mathbb{R}, \mathbb{R}^{n \times m})$ ,  $\ell \in \mathbb{N}$ , and to derive a time-varying linear coordinate transformation which takes (1.6) to a normal form. To this end, we generalize the concept of relative degree to time-varying nonlinear systems first.

The paper is organized as follows. In Section 2, we present a definition of relative degree for time-varying nonlinear systems. It is shown that this definition coincides, if the system is time-invariant with Isidori’s definition ([3, p. 220] respectively with the definition by Liberzon et al. [4, Def. 2]; furthermore, if the system is linear time-varying and viewed as a time-invariant nonlinear system, the definition coincides again with Isidori’s definition. Our main result is a normal form for time-varying linear systems given in Section 3. In Section 4, we parameterize the zero dynamics of time-varying linear systems and character their stability properties. We have relegated a refined version of Doležal’s Theorem to the Appendix, which is used in the antecedent proofs.

We close this introduction with remarks on notation. Throughout,  $\mathbb{R}_{\geq 0} := [0, \infty)$  and  $\| \cdot \|$  is the Euclidean inner product or the induced norm on  $\mathbb{R}^n$ ; if  $M_1, M_2 \in \mathbb{R}^{n \times n}$  are symmetric, then the notion  $M_1 \geq M_2$  means  $x^T M_1 x \geq x^T M_2 x$  for all  $x \in \mathbb{R}^n$ ;  $\text{Gl}_n(\mathbb{R})$  denotes the general linear group of invertible matrices  $A \in \mathbb{R}^{n \times n}$ ;  $\mathcal{C}^\ell(U, W)$  is the vector space of  $\ell$ -times differentiable functions  $f : U \rightarrow W$ ,  $U$  and  $W$  are open sets; and  $\mathcal{L}^\infty(U, W)$  the set of essentially bounded functions  $f : U \rightarrow W$ .

## 2 Relative degree: definition and characterizations

For time-invariant nonlinear multi-input multi-output systems, affine in the control, of the form

$$\left. \begin{aligned} \dot{x} &= f(x) + g(x)u(t) \\ y(t) &= h(x(t)), \end{aligned} \right\} \quad (2.1)$$

with  $f \in \mathcal{C}^\ell(\mathbb{R}^n, \mathbb{R}^n)$ ,  $g \in \mathcal{C}^\ell(\mathbb{R}^n, \mathbb{R}^{n \times m})$ ,  $h \in \mathcal{C}^\ell(\mathbb{R}^n, \mathbb{R}^m)$ ,  $\ell \in \mathbb{N}$ , the strict relative degree is defined as follows.

**Definition 2.1** (Isidori ([3, p. 220]))

Let  $\mathcal{U} \subset \mathbb{R}^n$  be open and  $r \in \{1, \dots, \ell\}$ . The time-invariant nonlinear system (2.1) has (*strict*) *relative degree*  $r$  on  $\mathcal{U}$  if, and only if,

- (i)  $\forall \xi \in \mathcal{U} \forall k \in \{0, \dots, r-2\} : L_g L_f^k h(\xi) = 0_{m \times m}$ ,
- (ii)  $\forall \xi \in \mathcal{U} : L_g L_f^{r-1} h(\xi) \in \text{Gl}_m(\mathbb{R})$ .

This definition is due to Isidori ([3, p. 220]) who defines it more general for a vector relative degree; we consider only the “strict” relative degree, that is (ii). The original definition in [3, p. 220] defines the relative degree at a point  $\xi^0 \in \mathbb{R}^n$  and some neighbourhood; however, this is equivalent to Definition 2.1, the latter is technically easier to deal with in the following. In the single-input single-output case, the notion of “strict” is redundant and Isidori showed that “the relative degree  $r$  is exactly equal to the number of times one has to differentiate the output  $y(t)$  at time  $t = t_0$  in order to have the value  $u(t_0)$  of the input explicitly appearing” [3, p. 139]. This latter characterization was formalized by Liberzon et al. [4] and related to output-input stability. We extend their notion of functions  $H_k$  to time-varying nonlinear systems of the form

$$\left. \begin{aligned} \dot{x} &= F(t, x, u(t)) \\ y(t) &= H(t, x(t)) \end{aligned} \right\} \quad (2.2)$$

where

$$F \in \mathcal{C}^\ell(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n), \quad H \in \mathcal{C}^\ell(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^m), \quad \ell \in \mathbb{N},$$

and define recursively, for  $k = 0, 1, 2, \dots, \ell - 1$ , the functions  $H_0(t, x) := H(t, x)$ ,

$$\left. \begin{aligned} H_{k+1} : \mathbb{R} \times \mathbb{R}^n \times (\mathbb{R}^m)^{k+1} &\rightarrow \mathbb{R}^m \\ (t, x, u_0, \dots, u_k) &\mapsto \frac{\partial H_k}{\partial t} + \frac{\partial H_k}{\partial x} F(t, x, u_0) + \sum_{j=0}^{k-1} \frac{\partial H_k}{\partial u_j} u_{j+1}. \end{aligned} \right\} \quad (2.3)$$

This allows to express the  $k$ -th derivative of  $y(t)$  in terms of  $t$ ,  $x(t)$  and  $u(t), \dots, u^{(k-1)}(t)$ :

$$\forall t \in \mathbb{R} \quad \forall k \in \{1, \dots, \ell - 1\} : y^{(k)}(t) = H_k(t, x(t), u(t), \dots, u^{(k-1)}(t)).$$

**Definition 2.2** (Generalization of Liberzon et al. [4, Def. 2])

Let  $\mathcal{T} \subset \mathbb{R}$  and  $\mathcal{U} \subset \mathbb{R}^n$  be open sets, and  $r \in \mathbb{N}$ . Then a system (2.2) is said to have (*strict and uniform*) *relative degree*  $r \in \mathbb{N}$  on  $\mathcal{T} \times \mathcal{U}$  if, and only if,

- (i)  $\forall (t, x) \in \mathcal{T} \times \mathcal{U} \quad \forall k \in \{1, \dots, r-1\} \forall i \in \{0, \dots, k-1\} : \frac{\partial H_k}{\partial u_i}(t, x, u_0, \dots, u_{k-1}) = 0_{m \times m}$ ;
- (ii)  $\forall (t, x, u_0) \in \mathcal{T} \times \mathcal{U} \times \mathbb{R}^m : \frac{\partial H_r}{\partial u_0}(t, x, u_0, \dots, u_{r-1}) \in \text{Gl}_m(\mathbb{R})$ .

The notion ‘strict’ refers to the multivariable case where we do not allow for a relative degree vector  $(r_1, \dots, r_m) \in \mathbb{N}^m$  with different entries, see [3, Sect. 5.1], but assume that the matrices  $\frac{\partial}{\partial u_i} H_k$  are either ‘strictly’ zero or invertible, globally on their domain.

**Proposition 2.3** Let  $\mathcal{U} \subset \mathbb{R}^n$  be an open set,  $r, \ell \in \mathbb{N}$  with  $r \leq \ell$ , and consider the time-invariant nonlinear system (2.1), affine in the control. Then (2.1) has relative degree  $r$  on  $\mathcal{U}$  in the sense of Definition 2.1 if, and only if, (2.1) has relative degree  $r$  on  $\mathbb{R} \times \mathcal{U}$  in the sense of Definition 2.2.

**Proof:** The following analysis is considered for  $(t, x) \in \mathbb{R} \times \mathcal{U}$  only; for notational convenience, we omit to repeat this and also suppress the arguments of the functions for simplicity.

“ $\Rightarrow$ ”: Suppose that (2.1) has relative degree  $r$  in the sense of Definition 2.1. We first show by induction on  $k$  that the following holds:

$$\forall k \in \{1, \dots, r-1\} \quad \forall i \in \{1, \dots, k-1\} : \quad H_k = L_f^k h, \quad \frac{\partial H_k}{\partial u_i} = 0_{m \times m}. \quad (2.4)$$

We have, in view of (i) in Definition 2.1,

$$H_1 = \frac{\partial H_0}{\partial x} (f + g u_0) + \frac{\partial H_0}{\partial u_0} u_1 = L_f h + L_g h u_0 = L_f h,$$

and so  $\frac{\partial H_1}{\partial u_0} = \frac{\partial}{\partial u_0} L_f h = 0_{m \times m}$ . If (2.4) holds for all  $k \leq r-2$ , then, in view of (i) in Definition 2.1,

$$H_{k+1} = \frac{\partial H_k}{\partial x} (f + g u_0) + \sum_{j=0}^{k-1} \frac{\partial H_k}{\partial u_j} u_{j+1} = L_f^{k+1} h + L_g L_f^k h u_0 = L_f^{k+1} h,$$

and so, for all  $i \in \{1, \dots, k\}$ ,  $\frac{\partial H_{k+1}}{\partial u_i} = \frac{\partial}{\partial u_i} L_f^{k+1} h = 0_{m \times m}$ . This proves (2.4), and therefore (i) of Definition 2.2 holds.

To prove (ii) in Definition 2.2, note that, in view of (2.4),

$$H_r = \frac{\partial H_{r-1}}{\partial x} (f + g u_0) + \sum_{j=0}^{r-2} \frac{\partial H_{r-1}}{\partial u_j} u_{j+1} = L_f^r h + L_g L_f^{r-1} h u_0,$$

and thus, invoking (ii) in Definition 2.1,  $\frac{\partial H_r}{\partial u_0} = L_g L_f^{r-1} h \in \text{Gl}_m(\mathbb{R})$ . This proves (ii) in Definition 2.2.

“ $\Leftarrow$ ”: Suppose that (2.1) has relative degree  $r$  in the sense of Definition 2.2. We show first by induction on  $k$  that the following holds:

$$\forall k \in \{0, 1, \dots, r-2\} : \quad L_g L_f^k h = 0_{m \times m}, \quad H_{k+1} = L_f^{k+1} h. \quad (2.5)$$

We have

$$H_1 = \frac{\partial H_0}{\partial x} (f + g u_0) + \frac{\partial H_0}{\partial u_0} u_1 = \frac{\partial h}{\partial x} f + \frac{\partial h}{\partial x} g u_0 = L_f h + L_g h u_0,$$

and so, in view of (i) in Definition 2.2,  $0_{m \times m} = \frac{\partial H_1}{\partial u_0} = L_g h$ , which gives  $H_1 = L_f h$ . If (2.5) holds for all  $k \leq r-3$ , then, again in view of (i) in Definition 2.2,

$$H_{k+2} = \frac{\partial H_{k+1}}{\partial x} (f + g u_0) + \sum_{j=0}^k \frac{\partial H_{k+1}}{\partial u_j} u_{j+1} = \frac{\partial}{\partial x} L_f^{k+1} h f + \frac{\partial}{\partial x} L_f^{k+1} h g u_0 = L_f^{k+2} h + L_g L_f^{k+1} h u_0,$$

and furthermore  $0_{m \times m} = \frac{\partial H_{k+2}}{\partial u_0} = L_g L_f^{k+1} h$ , which yields  $H_{k+2} = L_f^{k+2} h$ . This proves (2.5).

Finally, by (2.5), (i) in Definition 2.1 follows. Applying (2.5) again gives

$$H_r = \frac{\partial H_{r-1}}{\partial x} (f + g u_0) + \sum_{j=0}^{r-2} \frac{\partial H_{r-1}}{\partial u_j} u_{j+1} = L_f^r h + L_g L_f^{r-1} h u_0,$$

and thus, invoking (ii) in Definition 2.2,  $\frac{\partial H_r}{\partial u_0} = L_g L_f^{r-1} h \in \text{Gl}_m(\mathbb{R})$ . This proves (ii) in Definition 2.1 and completes the proof of the proposition.  $\square$

**Remark 2.4**

- (i) It follows from the proof of Proposition 2.3 that the relative degree of the time-invariant system (2.1) does not depend on  $t$ : if its relative degree is defined on  $\mathcal{T} \times \mathcal{U}$  for some open set  $\mathcal{T} \subset \mathbb{R}$ , then it is defined on  $\mathbb{R} \times \mathcal{U}$ .
- (ii) It also follows from Proposition 2.3 and [4, Prop. 2] that for single-input single-output time-invariant systems (2.1), Definition 2.2 coincides with the definition of the relative degree given by Liberzon et al. [4, Def. 2].

The characterization of the relative degree for time-varying linear systems (1.6) in terms of the matrix-functions  $A, B, C$  and their derivatives relies crucially on the following right operator, a notational convention, which allows for a neat presentation.

**Definition 2.5** For  $\ell \in \mathbb{N}$ ,  $A \in \mathcal{C}^\ell(\mathbb{R}, \mathbb{R}^{n \times n})$ , and  $C \in \mathcal{C}^\ell(\mathbb{R}, \mathbb{R}^{m \times n})$  set

$$\begin{aligned} \forall t \in \mathbb{R} : \quad & \left( \frac{d}{dt} + A(t) \right)_R (C(t)) := \dot{C}(t) + C(t)A(t), \\ \forall t \in \mathbb{R} \quad \forall k \in \{1, \dots, \ell\} : \quad & \left( \frac{d}{dt} + A(t) \right)_R^k (C(t)) := \left( \frac{d}{dt} + A(t) \right)_R \left( \left( \frac{d}{dt} + A(t) \right)_R^{k-1} (C(t)) \right). \end{aligned}$$

The sub-script  $R$  indicates that  $A$  acts on  $C$  by multiplication from the right.

**Theorem 2.6** Let  $r, \ell \in \mathbb{N}$  with  $r \leq \ell$ , and  $\mathcal{T} \subset \mathbb{R}$  be an open set. Then the time-varying linear system (1.6) has relative degree  $r$  on  $\mathcal{T} \times \mathbb{R}^n$  if, and only if,  $(A, B, C)$  satisfy

$$\left. \begin{aligned} \forall t \in \mathcal{T} \quad \forall k = 0, \dots, r-2 : \quad & \left( \frac{d}{dt} + A(t) \right)_R^k (C(t)) B(t) = 0_{m \times m} \\ \forall t \in \mathcal{T} : \quad & \left( \left( \frac{d}{dt} + A(t) \right)_R^{r-1} (C(t)) B(t) \right) \in \text{Gl}_m(\mathbb{R}). \end{aligned} \right\} \quad (2.6)$$

The proof of Theorem 2.6 depends on the following technicality.

**Lemma 2.7** Let  $r, \ell \in \mathbb{N}$  with  $r \leq \ell$ . Then the functions  $H_k$  defined in (2.3) and applied to the linear time-varying system (1.6) satisfy, for all  $k \in \{0, \dots, \ell-1\}$  and all  $(t, x, u_0, \dots, u_k) \in \mathbb{R} \times \mathbb{R}^n \times (\mathbb{R}^m)^{k+1}$ ,

$$\begin{aligned} & H_{k+1}(t, x, u_0, \dots, u_k) \\ &= \left( \frac{d}{dt} + A(t) \right)_R^{k+1} (C(t)) x + \sum_{j=0}^k \sum_{i=j}^k \binom{i}{j} \left( \frac{d}{dt} \right)^{i-j} \left[ \left( \frac{d}{dt} + A(t) \right)_R^{k-i} (C(t)) B(t) \right] u_j. \end{aligned} \quad (2.7)$$

**Proof:** Applying the definition of  $H_k$  to

$$F(t, x, u) := A(t)x + B(t)u \quad \text{and} \quad H(t, x) := C(t)x \quad \text{for} \quad (t, x, u) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m,$$

we have

$$\begin{aligned} H_1(t, x, u_0) &= \frac{\partial}{\partial t} C(t)x + \frac{\partial}{\partial x} (C(t)x) [A(t)x + B(t)u_0] \\ &= \left( \frac{d}{dt} + A(t) \right)_R (C(t))x + C(t)B(t)u_0 \\ &= \left( \frac{d}{dt} + A(t) \right)_R^1 (C(t))x + \sum_{j=0}^0 \sum_{i=j}^0 \binom{i}{j} \left( \frac{d}{dt} \right)^{i-j} \left[ \left( \frac{d}{dt} + A(t) \right)_R^{0-i} (C(t)) B(t) \right] u_j, \end{aligned}$$

which shows (2.7) for  $k = 0$ . To prove (2.7) by induction over  $k \in \{0, \dots, \ell - 1\}$ , assume that (2.7) holds for all  $k \in \{0, \dots, \ell - 2\}$ . Then, suppressing the arguments of  $H_k$  and  $A, B, C$  for simplicity, we calculate

$$\begin{aligned}
H_{k+1} &= \frac{\partial}{\partial t} H_k + \frac{\partial}{\partial x} H_k [Ax + B u_0] + \sum_{l=0}^{k-1} \frac{\partial}{\partial u_l} H_k u_{l+1} \\
&= \frac{\partial}{\partial t} \left[ \left( \frac{d}{dt} + A \right)_R^k (C) x + \sum_{j=0}^{k-1} \sum_{i=j}^{k-1} \binom{i}{j} \left( \frac{d}{dt} \right)^{i-j} \left[ \left( \frac{d}{dt} + A \right)_R^{k-1-i} (C) B \right] u_j \right] \\
&\quad + \frac{\partial}{\partial x} \left[ \left( \frac{d}{dt} + A \right)_R^k (C) x \right] [Ax + B u_0] \\
&\quad + \sum_{l=0}^{k-1} \frac{\partial}{\partial u_l} \left[ \sum_{j=0}^{k-1} \sum_{i=j}^{k-1} \binom{i}{j} \left( \frac{d}{dt} \right)^{i-j} \left[ \left( \frac{d}{dt} + A \right)_R^{k-1-i} (C) B \right] u_j \right] u_{l+1} \\
&= \frac{d}{dt} \left( \left( \frac{d}{dt} + A \right)_R^k (C) \right) x + \sum_{j=0}^{k-1} \sum_{i=j}^{k-1} \binom{i}{j} \left( \frac{d}{dt} \right)^{i-j+1} \left[ \left( \frac{d}{dt} + A \right)_R^{k-1-i} (C) B \right] u_j \\
&\quad + \left( \frac{d}{dt} + A \right)_R^k (C) A x + \left( \frac{d}{dt} + A \right)_R^k (C) B u_0 \\
&\quad + \sum_{l=0}^{k-1} \sum_{i=l}^{k-1} \binom{i}{l} \left( \frac{d}{dt} \right)^{i-l} \left[ \left( \frac{d}{dt} + A \right)_R^{k-1-i} (C) B \right] u_{l+1} \\
&= \left[ \frac{d}{dt} \left( \left( \frac{d}{dt} + A \right)_R^k (C) \right) + \left( \frac{d}{dt} + A \right)_R^k (C) A \right] x + \left( \frac{d}{dt} + A \right)_R^k (C) B u_0 \\
&\quad + \sum_{j=0}^{k-1} \sum_{i=j}^{k-1} \binom{i}{j} \left( \frac{d}{dt} \right)^{i-j+1} \left[ \left( \frac{d}{dt} + A \right)_R^{k-1-i} (C) B \right] u_j \\
&\quad + \sum_{j=1}^k \sum_{i=j-1}^{k-1} \binom{i}{j-1} \left( \frac{d}{dt} \right)^{i-j+1} \left[ \left( \frac{d}{dt} + A \right)_R^{k-1-i} (C) B \right] u_j \\
&= \left( \frac{d}{dt} + A \right)_R^{k+1} (C) x + \left( \frac{d}{dt} + A \right)_R^k (C) B u_0 \\
&\quad + \sum_{i=0}^{k-1} \binom{i}{0} \left( \frac{d}{dt} \right)^{i+1} \left[ \left( \frac{d}{dt} + A \right)_R^{k-1-i} (C) B \right] u_0 \\
&\quad + \sum_{j=1}^{k-1} \left( \sum_{i=j}^{k-1} \left[ \binom{i}{j} + \binom{i}{j-1} \right] \left( \frac{d}{dt} \right)^{i-j+1} \left[ \left( \frac{d}{dt} + A \right)_R^{k-1-i} (C) B \right] u_j \right. \\
&\quad \quad \left. + \binom{j-1}{j-1} \left( \frac{d}{dt} \right)^{j-1-j+1} \left[ \left( \frac{d}{dt} + A \right)_R^{k-1-j+1} (C) B \right] u_j \right) \\
&\quad + \sum_{i=k-1}^{k-1} \binom{i}{k-1} \left( \frac{d}{dt} \right)^{i-k+1} \left[ \left( \frac{d}{dt} + A \right)_R^{k-1-i} (C) B \right] u_k \\
&= \left( \frac{d}{dt} + A \right)_R^{k+1} (C) x + \underbrace{\sum_{i=-1}^{k-1} \binom{i}{0} \left( \frac{d}{dt} \right)^{i+1} \left[ \left( \frac{d}{dt} + A \right)_R^{k-1-i} (C) B \right] u_0}_{= \sum_{i=0}^k \binom{i}{0} \left( \frac{d}{dt} \right)^{i-0} \left[ \left( \frac{d}{dt} + A \right)_R^{k-i} (C) B \right] u_0} \\
&\quad + \sum_{j=1}^{k-1} \left( \sum_{i=j}^{k-1} \binom{i+1}{j} \left( \frac{d}{dt} \right)^{i+1-j} \left[ \left( \frac{d}{dt} + A \right)_R^{k-(i+1)} (C) B \right] u_j \right. \\
&\quad \quad \left. + \left( \frac{d}{dt} + A \right)_R^{k-j} (C) B u_j \right) \\
&\quad + \left( \frac{d}{dt} + A \right)_R^0 (C) B u_k
\end{aligned}$$

$$\begin{aligned}
&= \left( \frac{d}{dt} + A \right)_R^{k+1} (C) x + \sum_{i=0}^k \binom{i}{0} \left( \frac{d}{dt} \right)^{i-0} \left[ \left( \frac{d}{dt} + A \right)_R^{k-i} (C) B \right] u_0 \\
&\quad + \sum_{j=1}^{k-1} \left( \sum_{i=j+1}^k \binom{i}{j} \left( \frac{d}{dt} \right)^{i-j} \left[ \left( \frac{d}{dt} + A \right)_R^{k-i} (C) B \right] u_j \right. \\
&\quad \quad \left. + \binom{j}{j} \left( \frac{d}{dt} \right)^{j-j} \left( \frac{d}{dt} + A \right)_R^{k-j} (C) B u_j \right) \\
&\quad + \left( \frac{d}{dt} + A \right)_R^0 (C) B u_k \\
&= \left( \frac{d}{dt} + A \right)_R^{k+1} (C) x + \sum_{i=0}^k \binom{i}{0} \left( \frac{d}{dt} \right)^{i-0} \left[ \left( \frac{d}{dt} + A \right)_R^{k-i} (C) B \right] u_0 \\
&\quad + \sum_{j=1}^{k-1} \sum_{i=j}^k \binom{i}{j} \left( \frac{d}{dt} \right)^{i-j} \left[ \left( \frac{d}{dt} + A \right)_R^{k-i} (C) B \right] u_j \\
&\quad + \binom{k}{k} \left( \frac{d}{dt} \right)^{k-k} \left( \frac{d}{dt} + A \right)_R^{k-k} (C) B u_k \\
&= \left( \frac{d}{dt} + A \right)_R^{k+1} (C) x + \sum_{j=0}^k \sum_{i=j}^k \binom{i}{j} \left( \frac{d}{dt} \right)^{i-j} \left[ \left( \frac{d}{dt} + A \right)_R^{k-i} (C) B \right] u_j.
\end{aligned}$$

This shows (2.7) for  $k + 1$  and therefore the proof of the lemma is complete.  $\square$

**Proof of Theorem 2.6:**

“ $\Rightarrow$ ”: Suppose that Definition 2.2 holds. We show the first condition in (2.6) by induction over  $k \in \{1, \dots, r - 2\}$  (omitting the arguments of the operators). For  $N = k = 0$  we have, by (2.7),

$$H_1 = \left( \frac{d}{dt} + A \right)_R (C) x + CB u_0,$$

and so, by Definition 2.2,  $0_{m \times m} = \frac{\partial H_1}{\partial u_0} = CB$ . Suppose that the first condition in (2.6) holds for all  $k = 0, \dots, N$ , where  $N \leq r - 3$ . Then (2.7) yields,

$$H_{N+2} = \left( \frac{d}{dt} + A \right)_R^{N+2} (C) x + \left( \frac{d}{dt} + A \right)_R^{N+1} (C) B u_0,$$

and so, invoking (i) of Definition 2.2,

$$0_{m \times m} = \frac{\partial H_{N+2}}{\partial u_0} = \left( \frac{d}{dt} + A \right)_R^{N+1} (C) B.$$

This proves the first condition in (2.6).

To see the second condition in (2.6), note that (2.7) yields

$$H_r = \left( \frac{d}{dt} + A \right)_R^r (C) x + \left( \frac{d}{dt} + A \right)_R^{r-1} (C) B u_0,$$

and so, by (ii) of Definition 2.2,

$$\left( \frac{d}{dt} + A \right)_R^{r-1} (C) B = \frac{\partial H_r}{\partial u_0} \in \text{Gl}_m(\mathbb{R}),$$

This completes the proof of (2.6).

“ $\Leftarrow$ ”: Suppose that (2.6) holds. Then (2.7) yields

$$\forall k = 0, \dots, r - 2 : H_{k+1} = \left( \frac{d}{dt} + A \right)_R^{k+1} (C) x,$$

and thus (i) in Definition 2.2 follows. Finally, the second statement in (2.6) together with (2.7) gives

$$\frac{\partial H_r}{\partial u_0} = \frac{\partial}{\partial u_0} \left( \left( \frac{d}{dt} + A \right)_R^r (C) x + \left( \frac{d}{dt} + A \right)_R^{r-1} (C) B u_0 \right) = \left( \frac{d}{dt} + A \right)_R^{r-1} (C) B \in \text{Gl}_m(\mathbb{R}).$$

This proves (ii) of Definition 2.2 and completes the proof of the theorem.  $\square$

**Remark 2.8**

- (i) It follows from the proof of Theorem 2.6 that the relative degree of the time-varying linear system (1.6) does not depend on  $x$ : if its relative degree is defined on some  $\mathcal{T} \times U$ , where  $U \subset \mathbb{R}^n$  is open, then it is defined on  $\mathcal{T} \times \mathbb{R}^n$ . We therefore omit, at most places in the following, the second component in  $\mathcal{T} \times \mathbb{R}^n$ .
- (ii) If  $A, B, C$  are real analytic matrices and the linear system (1.6) has relative degree  $r$  on  $\mathcal{T} \times \mathbb{R}^n$  for some open  $\mathcal{T} \subset \mathbb{R}$ , then the Identity Theorem for analytic functions implies that (1.6) has relative degree  $r$  on  $(\mathbb{R} \setminus D) \times \mathbb{R}^n$ , where  $D$  denotes a discrete set.
- (iii) If the linear system (1.6) is time-invariant, then Theorem 2.6 yields that (1.6) has relative degree  $r \in \mathbb{N}$  on  $\mathbb{R} \times \mathbb{R}^n$  if, and only if,

$$CA^k B = 0_{m \times m} \text{ for all } k = 0, \dots, r-2 \quad \text{and} \quad CA^{r-1} B \in \text{Gl}_m(\mathbb{R}).$$

This is the well known characterization of strict relative degree, see [3, Rem. 4.1.2] for single-input single-output systems.

Instead of defining the relative degree of the linear time-varying system (1.6) as in Definition 2.2, we may consider the equivalent description of (1.6) as a time-invariant nonlinear system and determine the relative degree according to Definition 2.1. In the following we will show that both definitions coincide. Introducing an additional variable  $z$  with initial condition  $z(0) = 0$ , (1.6) is equivalent to

$$\left. \begin{aligned} \frac{d}{dt} \begin{pmatrix} x \\ z \end{pmatrix} &= \begin{pmatrix} A(z)x \\ 1 \end{pmatrix} + \begin{pmatrix} B(z) \\ 0 \end{pmatrix} u(t) \\ y(t) &= C(z(t))x(t). \end{aligned} \right\} \quad (2.8)$$

**Proposition 2.9** Let  $\mathcal{T} \subset \mathbb{R}$  and  $\mathcal{U} \subset \mathbb{R}^n$  be open sets,  $r, \ell \in \mathbb{N}$  with  $r \leq \ell$ . The time-varying linear system (1.6) has relative degree  $r$  on  $\mathcal{T} \times \mathcal{U}$  in the sense of Definition 2.2 if, and only if, the equivalent nonlinear time-invariant system (2.8) has relative degree  $r$  on  $\mathcal{U} \times \mathcal{T}$  in the sense of Definition 2.1.

**Proof:** Writing

$$f(x, z) = \begin{pmatrix} A(z)x \\ 1 \end{pmatrix}, \quad g(x, z) = \begin{pmatrix} B(z) \\ 0 \end{pmatrix}, \quad h(x, z) = C(z)x,$$

we show by induction over  $N \in \{0, \dots, \ell\}$  that

$$\forall (x, z) \in \mathcal{U} \times \mathcal{T} \quad \forall k \in \{0, \dots, \ell\} : L_f^k h(x, z) = \left( \frac{d}{dz} + A(z) \right)_R^k (C(z)) x. \quad (2.9)$$



For  $k = N = 0$ , we obviously have  $L_f^0 h(x, z) = C(z)x$ . Suppose that (2.9) holds for all  $k \in \{0, \dots, N\}$  for some  $N \in \{0, \dots, \ell - 1\}$ . Then

$$\begin{aligned}
L_f^{N+1} h(x, z) &= L_f \left( \left( \frac{d}{dz} + A \right)_R^N (C) x \right) \\
&= \left( \frac{\partial}{\partial x} \left( \left( \frac{d}{dz} + A \right)_R^N (C) x \right), \frac{\partial}{\partial z} \left( \left( \frac{d}{dz} + A \right)_R^N (C) x \right) \right) \begin{pmatrix} Ax \\ 1 \end{pmatrix} \\
&= \left( \frac{d}{dz} + A \right)_R^N (C) Ax + \frac{\partial}{\partial z} \left( \left( \frac{d}{dz} + A \right)_R^N (C) x \right) \\
&= \left( \frac{d}{dz} + A \right)_R^{N+1} (C) x.
\end{aligned}$$

This completes the proof of (2.9) and gives, for all  $(x, z) \in \mathcal{U} \times \mathcal{T}$  and all  $k \in \{0, \dots, \ell\}$ ,

$$\begin{aligned}
L_g L_f^k h(x, z) &= \left( \frac{\partial}{\partial x} \left( \left( \frac{d}{dz} + A(z) \right)_R^k (C(z)) x \right), \frac{\partial}{\partial z} \left( \left( \frac{d}{dz} + A(z) \right)_R^k (C(z)) x \right) \right) \begin{pmatrix} B(z) \\ 0 \end{pmatrix} \\
&= \left( \frac{d}{dz} + A(z) \right)_R^k (C(z)) B(z). \tag{2.10}
\end{aligned}$$

Finally, Theorem 2.6 is a consequence of (2.9) and (2.10).  $\square$

### 3 Normal form

In this section, we derive a normal form for time-varying linear systems (1.6). Theorem 2.6 may already indicate that the matrix function  $\left( \frac{d}{dt} + A(\cdot) \right)_R^k (C(\cdot))$ ,  $k = 0, \dots, r - 1$  are candidates for a new basis, however, this potential basis needs to be completed. We introduce the following matrix functions which will serve to derive a time-varying linear transformation.

Let  $r, \ell \in \mathbb{N}$  with  $r \leq \ell$ . Consider the system (1.6) and define, for  $r \in \mathbb{N}$  and all  $t \in \mathbb{R}$ ,

$$\begin{aligned}
\mathcal{C}(t) &:= \begin{bmatrix} C(t) \\ \left( \frac{d}{dt} + A(t) \right)_R (C(t)) \\ \vdots \\ \left( \frac{d}{dt} + A(t) \right)_R^{r-1} (C(t)) \end{bmatrix} \in \mathbb{R}^{rm \times n} \\
\mathcal{B}(t) &:= \left[ B(t), \left( \frac{d}{dt} - A(t) \right) (B(t)), \dots, \left( \frac{d}{dt} - A(t) \right)^{r-1} (B(t)) \right] \in \mathbb{R}^{n \times rm} \\
\Gamma(t) &:= \left( \frac{d}{dt} + A(t) \right)_R^{r-1} (C(t)) B(t) \in \mathbb{R}^{m \times m}.
\end{aligned}$$

The following proposition presents two more characterizations for (1.6) having relative degree  $r$ . They are rather technical but essential to design the coordinate transformation for the normal form.

**Proposition 3.1** Let  $r, \ell \in \mathbb{N}$  with  $r \leq \ell$  and  $\mathcal{T} \subset \mathbb{R}$  be an open set. Then the following conditions are equivalent:

- (i) The system (1.6) has relative degree  $r$  on  $\mathcal{T}$ .
- (ii) (a)  $\forall t \in \mathcal{T} \quad \forall (i, j) \in \{(i, j) \in \mathbb{N}_0^2 \mid 0 \leq i + j \leq r - 2\}$  :  
 $\left( \frac{d}{dt} + A(t) \right)_R^i (C(t)) \left( \frac{d}{dt} - A(t) \right)^j (B(t)) = 0_{m \times m}$ ;

(b)  $\forall t \in \mathcal{T} \quad \forall (i, j) \in \{(i, j) \in \mathbb{N}_0^2 \mid i + j = r - 1\} :$

$$\left(\frac{d}{dt} + A(t)\right)_R^i (C(t)) \left(\frac{d}{dt} - A(t)\right)_R^j (B(t)) \in \text{Gl}_m(\mathbb{R}).$$

(iii)

$$\forall t \in \mathcal{T} : \quad \mathcal{C}(t)\mathcal{B}(t) = \begin{bmatrix} 0 & & (-1)^{r-1}\Gamma(t) \\ & \dots & \\ \Gamma(t) & & * \end{bmatrix} \in \text{Gl}_{rm}(\mathbb{R}).$$

The proof of Proposition 3.1 depends crucially on the following technical lemma.

**Lemma 3.2** The linear time-varying system (1.6) satisfies, for all  $i, j \in \mathbb{N}_0$  with  $i + j \leq \ell$  and all  $t \in \mathbb{R}$ ,

$$\begin{aligned} & \left(\frac{d}{dt} + A(t)\right)_R^i (C(t)) \cdot \left(\frac{d}{dt} - A(t)\right)^j (B(t)) \\ &= \frac{d}{dt} \left[ \left(\frac{d}{dt} + A(t)\right)_R^i (C(t)) \cdot \left(\frac{d}{dt} - A(t)\right)^{j-1} (B(t)) \right] \\ & \quad - \left(\frac{d}{dt} + A(t)\right)_R^{i+1} (C(t)) \cdot \left(\frac{d}{dt} - A(t)\right)^{j-1} (B(t)), \quad (3.1) \end{aligned}$$

and

$$\begin{aligned} & \left(\frac{d}{dt} + A(t)\right)_R^i (C(t)) \left(\frac{d}{dt} - A(t)\right)^j (B(t)) = \\ & \quad \sum_{\mu=0}^j (-1)^\mu \binom{j}{\mu} \left(\frac{d}{dt}\right)^{j-\mu} \left[ \left(\frac{d}{dt} + A(t)\right)_R^{i+\mu} (C(t)) B(t) \right]. \quad (3.2) \end{aligned}$$

**Proof:** The equality (3.1) follows from the calculation

$$\begin{aligned} & \left(\frac{d}{dt} + A\right)_R^i (C) \cdot \left(\frac{d}{dt} - A\right)^j (B) \\ &= \left(\frac{d}{dt} + A\right)_R^i (C) \cdot \left(\frac{d}{dt} - A\right) \left( \left(\frac{d}{dt} - A\right)^{j-1} (B) \right) \\ &= \left(\frac{d}{dt} + A\right)_R^i (C) \cdot \left[ \frac{d}{dt} \left( \left(\frac{d}{dt} - A\right)^{j-1} (B) \right) - A \left( \left(\frac{d}{dt} - A\right)^{j-1} (B) \right) \right] \\ &= \left(\frac{d}{dt} + A\right)_R^i (C) \cdot \left( \frac{d}{dt} \left( \left(\frac{d}{dt} - A\right)^{j-1} (B) \right) - A \left( \left(\frac{d}{dt} - A\right)^{j-1} (B) \right) \right) \\ & \quad + \frac{d}{dt} \left[ \left(\frac{d}{dt} + A\right)_R^i (C) \cdot \left( \left(\frac{d}{dt} - A\right)^{j-1} (B) \right) \right] \\ & \quad - \frac{d}{dt} \left( \left(\frac{d}{dt} + A\right)_R^i (C) \right) \cdot \left( \left(\frac{d}{dt} - A\right)^{j-1} (B) \right) - \left( \left(\frac{d}{dt} + A\right)_R^i (C) \right) \cdot \frac{d}{dt} \left( \left(\frac{d}{dt} - A\right)^{j-1} (B) \right) \\ &= \frac{d}{dt} \left[ \left(\frac{d}{dt} + A\right)_R^i (C) \cdot \left( \left(\frac{d}{dt} - A\right)^{j-1} (B) \right) \right] \\ & \quad - \underbrace{\left[ \frac{d}{dt} \left( \left(\frac{d}{dt} + A\right)_R^i (C) \right) + \left(\frac{d}{dt} + A\right)_R^i (C) A \right]}_{= \left(\frac{d}{dt} + A\right)_R \left( \left(\frac{d}{dt} + A\right)_R^i (C) \right)} \cdot \left(\frac{d}{dt} - A\right)^{j-1} (B) \\ &= \frac{d}{dt} \left[ \left(\frac{d}{dt} + A\right)_R^i (C) \cdot \left( \left(\frac{d}{dt} - A\right)^{j-1} (B) \right) \right] - \left(\frac{d}{dt} + A\right)_R^{i+1} (C) \cdot \left(\frac{d}{dt} - A\right)^{j-1} (B). \end{aligned}$$

We prove (3.2) by fixing  $i \in \mathbb{N}_0$  and induction over  $j = 0, \dots, \ell - i$ . For  $j = 0$ , (3.2) is obvious. Suppose that (3.2) holds for  $j \leq \ell - i - 1$ . Then, invoking (3.1), it follows that

$$\begin{aligned}
& \left(\frac{d}{dt} + A\right)_R^i (C) \cdot \left(\frac{d}{dt} - A\right)^{j+1} (B) \\
&= \frac{d}{dt} \left[ \sum_{\mu=0}^j (-1)^\mu \binom{j}{\mu} \left(\frac{d}{dt}\right)^{j-\mu} \left[ \left(\frac{d}{dt} + A\right)_R^{i+\mu} (C) B \right] \right] \\
&\quad - \sum_{\mu=0}^j (-1)^\mu \binom{j}{\mu} \left(\frac{d}{dt}\right)^{j-\mu} \left[ \left(\frac{d}{dt} + A\right)_R^{i+1+\mu} (C) B \right] \\
&= \sum_{\mu=0}^j (-1)^\mu \binom{j}{\mu} \left(\frac{d}{dt}\right)^{j+1-\mu} \left[ \left(\frac{d}{dt} + A\right)_R^{i+\mu} (C) B \right] \\
&\quad - \sum_{\mu=1}^{j+1} (-1)^{\mu-1} \binom{j}{\mu-1} \left(\frac{d}{dt}\right)^{j-\mu+1} \left[ \left(\frac{d}{dt} + A\right)_R^{i+1+\mu-1} (C) B \right] \\
&= \binom{j}{0} \left(\frac{d}{dt}\right)^{j+1} \left[ \left(\frac{d}{dt} + A\right)_R^{i+0} (C) B \right] \\
&\quad + \sum_{\mu=1}^j \left[ (-1)^\mu \binom{j}{\mu} - (-1)^{\mu-1} \binom{j}{\mu-1} \right] \left(\frac{d}{dt}\right)^{j+1-\mu} \left[ \left(\frac{d}{dt} + A\right)_R^{i+\mu} (C) B \right] \\
&\quad - (-1)^{j+1-1} \binom{j}{j+1-1} \left(\frac{d}{dt}\right)^{j-j-1+1} \left[ \left(\frac{d}{dt} + A\right)_R^{i+j+1} (C) B \right] \\
&= (-1)^0 \binom{j}{0} \left(\frac{d}{dt}\right)^{j+1-0} \left[ \left(\frac{d}{dt} + A\right)_R^{i+0} (C) B \right] \\
&\quad + \sum_{\mu=1}^j (-1)^\mu \binom{j+1}{\mu} \left(\frac{d}{dt}\right)^{j+1-\mu} \left[ \left(\frac{d}{dt} + A\right)_R^{i+\mu} (C) B \right] \\
&\quad + (-1)^{j+1} \binom{j+1}{j+1} \left(\frac{d}{dt}\right)^{j+1-(j+1)} \left[ \left(\frac{d}{dt} + A\right)_R^{i+j+1} (C) B \right] \\
&= \sum_{\mu=0}^{j+1} (-1)^\mu \binom{j+1}{\mu} \left(\frac{d}{dt}\right)^{j+1-\mu} \left[ \left(\frac{d}{dt} + A\right)_R^{i+\mu} (C) B \right].
\end{aligned}$$

This completes the proof of the lemma.  $\square$

**Proof of Proposition 3.1:** The equivalence “(i)  $\Leftrightarrow$  (ii)” follows from (3.2) and (2.6), and the equivalence “(ii)  $\Leftrightarrow$  (iii)” follows from (3.2).  $\square$

The following corollary is a direct consequence of Proposition 3.1.

**Corollary 3.3** If the linear time-varying system (1.6) has relative degree  $r \in \mathbb{N}$  on some open set  $\mathcal{T} \subset \mathbb{R}$ , then the following hold:

- (i)  $\forall t \in \mathcal{T} : \text{rk } \mathcal{C}(t) = rm$  and  $\text{rk } \mathcal{B}(t) = rm$  ;

- (ii) the two sets of matrix functions  $C(\cdot), \left(\frac{d}{dt} + A(t)\right)_R (C(\cdot)), \dots, \left(\frac{d}{dt} + A(t)\right)_R^{r-1} (C(\cdot))$  and  $B(\cdot), \left(\frac{d}{dt} - A(\cdot)\right) (B(\cdot)), \dots, \left(\frac{d}{dt} - A(\cdot)\right)^{r-1} (B(\cdot))$  are both linearly independent over  $\mathcal{T}$ ;
- (iii)  $rm \leq n$ .

We are now in a position to design a time-varying linear basis transformation which will lead to a normal form.

**Remark 3.4** Let  $r, \ell \in \mathbb{N}$  with  $r \leq \ell$  and  $\mathcal{T} \subset \mathbb{R}$  be an open set. Suppose that the time-varying linear system (1.6) has relative degree  $r \in \mathbb{N}$  on  $\mathcal{T}$ . By Corollary 3.3, the rows in  $\mathcal{C}$  qualify as new basis but the basis needs to be completed. By Theorem 5.1, we may choose  $T = [t_1, \dots, t_n] \in \mathcal{C}^1(\mathcal{T}, \text{Gl}_n(\mathbb{R}))$  such that

$$\begin{bmatrix} \mathcal{C} \\ 0 \end{bmatrix} T = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{C}^1(\mathcal{T}, \mathbb{R}^{n \times n}) \quad \text{with } F \in \mathcal{C}^1(\mathcal{T}, \text{Gl}_{rm}(\mathbb{R})).$$

Defining

$$\mathcal{V} := [t_{rm+1}, \dots, t_n] \in \mathcal{C}^1(\mathcal{T}, \mathbb{R}^{n \times (n-rm)})$$

we have

$$\forall t \in \mathcal{T} : \text{im } \mathcal{V}(t) = \ker \mathcal{C}(t) \quad \text{and} \quad \text{rk } \mathcal{V}(t)^T \mathcal{V}(t) = n - rm, \quad (3.3)$$

and writing

$$U := \begin{bmatrix} \mathcal{C} \\ \mathcal{N} \end{bmatrix} \in \mathcal{C}^1(\mathcal{T}, \mathbb{R}^{n \times n}) \quad \text{and} \quad \mathcal{N} := (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T [I - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \mathcal{C}] \in \mathcal{C}^1(\mathcal{T}, \mathbb{R}^{(n-rm) \times n}),$$

it follows from

$$\begin{bmatrix} \mathcal{C} \\ \mathcal{N} \end{bmatrix} [\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}, \mathcal{V}] = I_n,$$

that  $U \in \mathcal{C}^1(\mathcal{T}, \text{Gl}_n(\mathbb{R}))$  with inverse

$$U^{-1} = [\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}, \mathcal{V}] \in \mathcal{C}^1(\mathcal{T}, \text{Gl}_n(\mathbb{R})).$$

We are now in a position to derive the main result of this note, that is a normal form of the time-varying linear system (1.6).

### Theorem 3.5

Let  $r, \ell \in \mathbb{N}$  with  $r \leq \ell$  and  $\mathcal{T} \subset \mathbb{R}$  be an open set. Suppose the time-varying linear system (1.6) has relative degree  $r$  on  $\mathcal{T}$  and choose  $U \in \mathcal{C}^1(\mathcal{T}, \text{Gl}_n(\mathbb{R}))$ ,  $\mathcal{V} \in \mathcal{C}^1(\mathcal{T}, \mathbb{R}^{n \times (n-r)})$ , and  $\mathcal{N} \in \mathcal{C}^1(\mathcal{T}, \mathbb{R}^{(n-rm) \times n})$  as in Remark 3.4. Then the coordinate transformation

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} := Ux, \quad \xi(t) = (y(t)^T, \dots, y^{(r-1)}(t)^T)^T \in \mathbb{R}^{rm}, \quad \eta(t) \in \mathbb{R}^{n-rm}$$

converts (1.6) on  $\mathcal{T}$  into

$$\left. \begin{aligned} \frac{d}{dt} \begin{pmatrix} \xi \\ \eta \end{pmatrix} &= \hat{A}(t) \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \hat{B}(t) u(t) \\ y(t) &= \hat{C}(t) \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} \end{aligned} \right\} \quad (3.4)$$

where

$$\left. \begin{aligned} \hat{A}(t) &= \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & I & 0 \\ R_1(t) & R_2(t) & \dots & R_r(t) & S(t) \\ P(t) & 0 & \dots & 0 & Q(t) \end{bmatrix}, & \hat{B}(t) &= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \Gamma \\ 0 \end{bmatrix}, \\ \hat{C}(t) &= [I \ 0 \ \dots \ 0], \end{aligned} \right\} \quad (3.5)$$

and

$$\Gamma = \left(\frac{d}{dt} + A\right)_R^{r-1} (C)B, \quad (3.6)$$

$$[R_1, \dots, R_r, S] = [0_{m \times (rm-m)}, I_m] \left[ \left(\frac{d}{dt} + A\right)_R^r (C)\mathcal{B}(\mathcal{CB})^{-1}, \left(\frac{d}{dt} + A\right)_R^r (C)\mathcal{V} \right] \quad (3.7)$$

$$= \left[ \left(\frac{d}{dt} + A\right)_R^r (C)\mathcal{B}(\mathcal{CB})^{-1}, \left(\frac{d}{dt} + A\right)_R^r (C)\mathcal{V} \right], \quad (3.8)$$

$$Q = \left(\frac{d}{dt} + A\right)_R (\mathcal{N})\mathcal{V} \quad (3.9)$$

$$= -(\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \left[ \left(\frac{d}{dt} - A\right)\mathcal{V} - B\Gamma^{-1} \left(\frac{d}{dt} + A\right)_R^r (C)\mathcal{V} \right], \quad (3.10)$$

$$P = \left(\frac{d}{dt} + A\right)_R (\mathcal{N})\mathcal{B}(\mathcal{CB})^{-1} \begin{bmatrix} I_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (3.11)$$

$$= (-1)^{r-1} (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T [\mathcal{B}(\mathcal{CB})^{-1} \mathcal{C} - I] \left(\frac{d}{dt} - A\right)^r (B)\Gamma^{-1}. \quad (3.12)$$

**Proof:** The special form of  $U$  and  $U^{-1}$  gives immediately  $\hat{B} = UB$ ,  $\hat{C}U = C$  with the special structure as shown in (3.5) and  $\Gamma$  as in (3.6); furthermore,

$$\hat{A} = [UA + \dot{U}]U^{-1} = \left(\frac{d}{dt} + A\right)_R (U)U^{-1}. \quad (3.13)$$

In view of Remark 3.4, it remains to show (3.8)-(3.12) and that

$$\begin{bmatrix} \left(\frac{d}{dt} + A\right)_R (C) \\ \vdots \\ \left(\frac{d}{dt} + A\right)_R^{r-1} (C) \\ \left(\frac{d}{dt} + A\right)_R^r (C) \\ \left(\frac{d}{dt} + A\right)_R (\mathcal{N}) \end{bmatrix} = \begin{bmatrix} 0 & I & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & 0 & I & 0 \\ R_1 & R_2 & \dots & R_r & S \\ P_1 & P_2 & \dots & P_r & Q \end{bmatrix} \begin{bmatrix} \mathcal{C} \\ \mathcal{N} \end{bmatrix} \quad \text{with } P_2 = \dots = P_r = 0. \quad (3.14)$$

We first show (3.14). Since equality of the upper blocks in (3.14) is immediate, it remains to show that

$$\left(\frac{d}{dt} + A\right)_R (\mathcal{N})\mathcal{B}(\mathcal{CB})^{-1} = [P_1, 0, \dots, 0]. \quad (3.15)$$

Writing

$$\mathcal{CB} = [\eta_1, \dots, \eta_r], \quad (\mathcal{CB})^{-1} = \begin{bmatrix} \psi^1 \\ \vdots \\ \psi^r \end{bmatrix}, \quad \mathcal{B} = [\beta_1, \dots, \beta_r]$$

we have

$$\begin{aligned}
\mathcal{B}(\mathcal{CB})^{-1}\mathcal{C}\left(\frac{d}{dt} - A\right)(\mathcal{B}) &= \mathcal{B}(\mathcal{CB})^{-1}\mathcal{C}\left[\beta_2, \dots, \beta_r, \left(\frac{d}{dt} - A\right)^r(B)\right] \\
&= [\beta_1, \dots, \beta_r] \begin{bmatrix} \psi^1 \\ \vdots \\ \psi^r \end{bmatrix} \left[\eta_2, \dots, \eta_r, \mathcal{B}(\mathcal{CB})^{-1}\mathcal{C}\left(\frac{d}{dt} - A\right)^r(B)\right] \\
&= [\beta_1, \dots, \beta_r] \begin{bmatrix} 0 & 0 & \dots & 0 & * \\ I & 0 & \dots & 0 & * \\ 0 & I & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \\ 0 & \dots & 0 & I & * \end{bmatrix} \\
&= [\beta_2, \dots, \beta_r, \mathcal{B}(\mathcal{CB})^{-1}\mathcal{C}\left(\frac{d}{dt} - A\right)^r(B)] ,
\end{aligned}$$

and thus

$$\begin{aligned}
&[\mathcal{B}(\mathcal{CB})^{-1}\mathcal{C} - I_n] \left(\frac{d}{dt} - A\right)(\mathcal{B}) \\
&= [\beta_2, \dots, \beta_r, (\mathcal{CB})^{-1}\mathcal{C}\left(\frac{d}{dt} - A\right)^r(B)] - [\beta_2, \dots, \beta_r, \left(\frac{d}{dt} - A\right)^r(B)] \\
&= [0, \dots, 0, [\mathcal{B}(\mathcal{CB})^{-1}\mathcal{C} - I_n] \left(\frac{d}{dt} - A\right)^r(B)] , \quad (3.16)
\end{aligned}$$

which, by invoking

$$\begin{aligned}
&\left(\frac{d}{dt} + A\right)_R(\mathcal{N}) \\
&= \frac{d}{dt} \left( (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \right) (I - \mathcal{B}(\mathcal{CB})^{-1}\mathcal{C}) + (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T (A - \left(\frac{d}{dt} + A\right)_R(\mathcal{B}(\mathcal{CB})^{-1}\mathcal{C})) , \quad (3.17)
\end{aligned}$$

implies

$$\begin{aligned}
\left(\frac{d}{dt} + A\right)_R(\mathcal{N})\mathcal{B} &= \left[\frac{d}{dt} \left( (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \right) (I - \mathcal{B}(\mathcal{CB})^{-1}\mathcal{C}) \right. \\
&\quad \left. + (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T (A - \left(\frac{d}{dt} + A\right)_R(\mathcal{B}(\mathcal{CB})^{-1}\mathcal{C})) \right] \mathcal{B} \\
&= (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T [A - \left(\frac{d}{dt} + A\right)_R(\mathcal{B}(\mathcal{CB})^{-1}\mathcal{C})] \mathcal{B} \\
&= (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T [\mathcal{B}(\mathcal{CB})^{-1}\mathcal{C} - I_n] \left(\frac{d}{dt} - A\right)(\mathcal{B}) \\
&= (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T [0, \dots, 0, [\mathcal{B}(\mathcal{CB})^{-1}\mathcal{C} - I_n] \left(\frac{d}{dt} - A\right)^r(B)] . \quad (3.18)
\end{aligned}$$

Finally,

$$(\mathcal{CB})^{-1} = \begin{bmatrix} * & & \Gamma^{-1} \\ & \ddots & \\ (-1)^{r-1} \Gamma^{-1} & & 0 \end{bmatrix} ,$$

applied to (3.18) yields (3.15), whence (3.14).

Next we prove (3.8). By (3.13),

$$[R_1, \dots, R_r, S] = [0_{m \times (rm-m)}, I_m] \left(\frac{d}{dt} + A\right)_R(\mathcal{C}) [\mathcal{B}(\mathcal{CB})^{-1}, \mathcal{V}] ,$$

and (3.8) follows from the definition of  $\mathcal{C}$ .

We show (3.9) and (3.10). Equality (3.9) follows immediately from the normal form (3.4) and (3.5). To see equality (3.9), note that (3.17) yields

$$\begin{aligned}
& \left(\frac{d}{dt} + A\right)_R (\mathcal{N})\mathcal{V} \\
&= \left[\frac{d}{dt} [(\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T] [I_n - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \mathcal{C}] - (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \frac{d}{dt} [\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \mathcal{C}] \right. \\
&\quad \left. + (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T A - (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T (\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \mathcal{C}) A\right] \mathcal{V} \\
&= \frac{d}{dt} [(\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T] \left[\mathcal{V} - \underbrace{\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \mathcal{C}\mathcal{V}}_{=0}\right] + (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T A \mathcal{V} \\
&\quad - (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \left[\frac{d}{dt} (\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \mathcal{C}) + (\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \mathcal{C}) A\right] \mathcal{V} \\
&= \underbrace{\frac{d}{dt} [(\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T] \mathcal{V} + (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T A \mathcal{V}}_{= \left(\frac{d}{dt} + A\right)_R ((\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}) \mathcal{V}} \\
&\quad - (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \left[\underbrace{\dot{\mathcal{B}}(\mathcal{C}\mathcal{B})^{-1} \mathcal{C}\mathcal{V}}_{=0} + \mathcal{B} \frac{d}{dt} ((\mathcal{C}\mathcal{B})^{-1}) \underbrace{\mathcal{C}\mathcal{V}}_{=0} + \mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \dot{\mathcal{C}}\mathcal{V} + (\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \mathcal{C}) A \mathcal{V}\right],
\end{aligned}$$

which gives

$$\left(\frac{d}{dt} + A\right)_R (\mathcal{N})\mathcal{V} = \left(\frac{d}{dt} + A\right)_R ((\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}) \mathcal{V} - (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \left(\frac{d}{dt} + A\right)_R (\mathcal{C})\mathcal{V}.$$

Invoking

$$\begin{aligned}
& \left(\frac{d}{dt} + A\right)_R ((\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T) \mathcal{V} \\
&= \frac{d}{dt} ((\mathcal{V}^T \mathcal{V})^{-1}) \mathcal{V}^T \mathcal{V} + (\mathcal{V}^T \mathcal{V})^{-1} \frac{d}{dt} (\mathcal{V}^T) \mathcal{V} + (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T A \mathcal{V} \\
&= -(\mathcal{V}^T \mathcal{V})^{-1} \frac{d}{dt} (\mathcal{V}^T \mathcal{V}) (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \mathcal{V} + (\mathcal{V}^T \mathcal{V})^{-1} \frac{d}{dt} (\mathcal{V}^T) \mathcal{V} + (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T A \mathcal{V} \\
&= -(\mathcal{V}^T \mathcal{V})^{-1} \frac{d}{dt} (\mathcal{V}^T) \mathcal{V} - (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \frac{d}{dt} (\mathcal{V}) + (\mathcal{V}^T \mathcal{V})^{-1} \frac{d}{dt} (\mathcal{V}^T) \mathcal{V} + (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T A \mathcal{V} \\
&= -(\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \left(\frac{d}{dt} - A\right) (\mathcal{V}),
\end{aligned}$$

we arrive at

$$\begin{aligned}
\left(\frac{d}{dt} + A\right)_R (\mathcal{N})\mathcal{V} &= -(\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \left(\frac{d}{dt} - A\right) (\mathcal{V}) \\
&\quad - (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \left(\frac{d}{dt} + A\right)_R (\mathcal{C})\mathcal{V} \\
&= -(\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \left(\frac{d}{dt} - A\right) (\mathcal{V}) \\
&\quad - (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \left(\frac{d}{dt} + A\right)_R^r (\mathcal{C})\mathcal{V} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
&= -(\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \left( \frac{d}{dt} - A \right) (\mathcal{V}) \\
&\quad - (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \mathcal{B} \begin{bmatrix} * & & \Gamma^{-1} \\ & \ddots & \\ (-1)^{r-1} \Gamma^{-1} & & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \left( \frac{d}{dt} + A \right)_R^r (C) \mathcal{V} \end{bmatrix} \\
&= -(\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \left( \frac{d}{dt} - A \right) (\mathcal{V}) \\
&\quad - (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \left[ B, \dots, \left( \frac{d}{dt} - A \right)^{r-1} (B) \right] \begin{bmatrix} \Gamma^{-1} \left( \frac{d}{dt} + A \right)_R^r (C) \mathcal{V} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= -(\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \left[ \left( \frac{d}{dt} - A \right) (\mathcal{V}) - B \Gamma^{-1} \left( \frac{d}{dt} + A \right)_R^r (C) \mathcal{V} \right].
\end{aligned}$$

This proves (3.10).

Finally, we show (3.11) and (3.12) for  $P := P_1$ . First, (3.14) together with Remark 3.4 yields

$$\left( \frac{d}{dt} + A \right)_R (\mathcal{N}) [\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}, \mathcal{V}] = [P, 0, \dots, 0] \quad (3.19)$$

and hence (3.11) follows.

To see (3.12), note that (3.18) gives

$$\begin{aligned}
&\left( \frac{d}{dt} + A \right)_R (\mathcal{N}) \mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \begin{bmatrix} I_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \left[ 0, \dots, 0, [\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \mathcal{C} - I_n] \left( \frac{d}{dt} - A \right)^r (B) \right] \begin{bmatrix} * \\ \vdots \\ * \\ (-1)^{r-1} \Gamma^{-1} \end{bmatrix}
\end{aligned}$$

which proves (3.12). This completes the proof of the theorem.  $\square$

As a direct consequence of Proposition 3.6 we obtain the following corollary.

**Corollary 3.6** Under the assumptions of Theorem 3.5 and using the same notation, the normal form (3.4) may be written as

$$\begin{aligned}
y^{(r)} &= \sum_{i=1}^r R_i(t) y^{(i-1)} + S(t) \eta + \Gamma(t) u(t) \\
\dot{\eta} &= Q(t) \eta + P(t) y,
\end{aligned}$$

and, if moreover (1.6) is time-invariant,

$$\begin{aligned}
y^{(r)} &= \sum_{i=1}^r R_i y^{(i-1)} + S \eta + C A^{r-1} B u(t) \\
\dot{\eta} &= \mathcal{N} A \mathcal{V} \eta + (-1)^{r-1} (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T C A^r B (C A^{r-1} B)^{-1} y.
\end{aligned}$$



## 4 Zero dynamics

In the time-invariant, possibly nonlinear, case one can read off the zero dynamics from the normal form and, if the zero dynamics are asymptotically stable, a high-gain output derivative feedback controller may stabilize the system; see [3, Sec. 4.2]. In the time-varying case, an analogue cannot be expected unless severe restrictions on the time-variation of  $A, B, C$  are imposed. The reason is that, in general, neither the coordinate transformation  $U$ , designed in Remark 3.4, nor its inverse will be bounded. However, by Theorem 5.1 – an extended version of Doležal’s Theorem – it is ensured that  $\mathcal{V}$  and its left inverse  $(\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T$  are both bounded matrix functions and therefore the stability properties of the zero dynamics are equivalent to the stability properties of  $\dot{\eta} = Q\eta$  (see (3.10)). To be precise, we first define the zero dynamics of a linear time-varying systems.

**Definition 4.1** For any  $r, \ell \in \mathbb{N}$  with  $r \leq \ell$  and  $\mathcal{T} \subset \mathbb{R}$  an open set, the *zero dynamics of system (1.6) on  $\mathcal{T}$*  are defined as the real vector space of trajectories

$$\mathcal{ZD}_{\mathcal{T}}(A, B, C) := \left\{ (x, u) \in \mathcal{C}^1(\mathcal{T}, \mathbb{R}^n) \times \mathcal{C}^1(\mathcal{T}, \mathbb{R}^m) \mid \begin{array}{l} (x, u) \text{ solves (1.6)} \\ \text{with } y = 0 \text{ on } \mathcal{T} \end{array} \right\}.$$

Also for time-varying systems, as known for time-invariant systems, the zero dynamics can be read off the normal form (3.4). This is shown in the following proposition. In fact, the zero dynamics can be parameterized.

**Proposition 4.2** Let  $r, \ell \in \mathbb{N}$  with  $r \leq \ell$  and  $\mathcal{T} \subset \mathbb{R}$  an open set. Then for any system (1.6) with relative degree  $r \leq \ell$  on  $\mathcal{T} \subset \mathbb{R}$  and normal form (3.4) the following holds.

$$\mathcal{ZD}_{\mathcal{T}}(A, B, C) = \left\{ (\mathcal{V}\eta, -\Gamma^{-1}S\eta) \in \mathcal{C}^1(\mathcal{T}, \mathbb{R}^n) \times \mathcal{C}^1(\mathcal{T}, \mathbb{R}^m) \mid \dot{\eta} = Q\eta \right\}. \quad (4.1)$$

**Proof:** Set

$$\mathcal{Z} = \left\{ (\mathcal{V}\eta, -\Gamma^{-1}S\eta) \in \mathcal{C}^1(\mathcal{T}, \mathbb{R}^n) \times \mathcal{C}^1(\mathcal{T}, \mathbb{R}^m) \mid \dot{\eta} = Q\eta \right\}.$$

“ $\subseteq$ ”: If  $(x, u) \in \mathcal{ZD}_{\mathcal{T}}(A, B, C)$ , then on  $\mathcal{T}$  we have that  $y = 0$  and so

$$\xi = (y^T, \dots, (y^{(r-1)})^T)^T = 0,$$

which yields, in view of (3.4),

$$x = \mathcal{V}\eta = U^{-1} \begin{pmatrix} 0 \\ \eta \end{pmatrix} \quad \text{and} \quad 0 = S\eta + \Gamma u,$$

and therefore,  $(x, u) \in \mathcal{Z}$ .

“ $\supseteq$ ”: If  $(\tilde{x}, \tilde{u}) = (\mathcal{V}\eta, -\Gamma^{-1}S\eta) \in \mathcal{Z}$ , then on  $\mathcal{T}$  we have that

$$\tilde{y} := C\tilde{x} = C\mathcal{V}\eta = 0,$$

and so

$$\tilde{\xi} = (\tilde{y}^T, \dots, (\tilde{y}^{(r-1)})^T)^T = 0,$$

and therefore  $\left( \begin{pmatrix} 0 \\ \eta \end{pmatrix}, \tilde{u} \right)$  solves the first equation in (3.4) with  $\tilde{y} = 0$ , and it follows that

$$(\tilde{x}, \tilde{u}) = \left( U^{-1} \begin{pmatrix} 0 \\ \eta \end{pmatrix}, \tilde{u} \right) = (\mathcal{V}\eta, \tilde{u}) \in \mathcal{ZD}_{\mathcal{T}}(A, B, C).$$

This completes the proof of the proposition.  $\square$

In the remainder of this section we study stability of the zero dynamics.

**Definition 4.3** For any  $t_0 \in \mathbb{R}$  and  $\ell \in \mathbb{N}$  consider (1.6) on  $\mathcal{T} = (t_0, \infty)$ . Then the zero dynamics of (1.6) are called *asymptotically stable* if, and only if,

$$\forall (x, u) \in \mathcal{ZD}_{\mathcal{T}}(A, B, C) : \lim_{t \rightarrow \infty} x(t) = 0.$$

Although the time-varying coordinate transformation  $U$  which converts (1.6) to (3.4) may be unbounded or its inverse may be unbounded, we will show in the following theorem that – surprisingly – the zero dynamics of (1.6) are asymptotically stable if, and only if,  $\dot{\eta} = Q\eta$  is an asymptotically stable system.

**Theorem 4.4** Let  $r, \ell \in \mathbb{N}$  with  $r \leq \ell$ ,  $t_0 \in \mathbb{R}$ , and  $\mathcal{T} = (t_0, \infty)$ . Suppose the system (1.6) has relative degree  $r \leq \ell$  on  $\mathcal{T}$  and consider its normal form (3.4). Then the zero dynamics of (1.6) are asymptotically stable if, and only if,  $\dot{\eta} = Q\eta$  is an asymptotically stable system.

**Proof:** It follows from Theorem 5.1 that  $\mathcal{V}$  as defined in Remark 3.4 satisfies  $\mathcal{V} \in \mathcal{L}^\infty(\mathcal{T}, \mathbb{R}^{n \times (n-rm)})$  and  $(\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \in \mathcal{L}^\infty(\mathcal{T}, \mathbb{R}^{(n-rm) \times n})$ . Now the claim of the theorem is a consequence of the fact that if  $(x, u) \in \mathcal{ZD}_{\mathcal{T}}(A, B, C)$ , then Proposition 4.2 yields

$$x = \mathcal{V}\eta \quad \text{and} \quad \dot{\eta} = (Q\eta).$$

□

## 5 Appendix: Doležal's Theorem re-revisited

Doležal's Theorem [1], which states convenient representations of range and kernels of time-varying matrices, has found numerous applications in systems theory and has been generalized and improved in various directions [2, 5, 6, 7]. In the following we give a generalization of [5, Theorem 2] which is tailored for the needs of the present paper.

### Theorem 5.1

Let  $\ell \in \mathbb{N}$ ,  $M \in \mathcal{C}^\ell(\mathbb{R}_{\geq 0}, \mathbb{R}^{n \times n})$  and suppose that there exists  $r \in \{1, \dots, n\}$  such that, for all  $t \geq 0$ ,  $\text{rk } M(t) = r$ . Then there exists  $T \in \mathcal{C}^\ell(\mathbb{R}_{\geq 0}, \text{Gl}_n(\mathbb{R}))$  such that

$$\forall t \geq 0 : M(t)T(t) = [F(t), 0_{n \times (n-r)}], \quad (5.1)$$

$$\exists \beta > 0 \quad \forall t \geq 0 : \|T(t)\| \leq \beta, \quad (5.2)$$

$$\exists \varepsilon \in (0, 1) \quad \forall t \geq 0 : \varepsilon \leq T^T(t)T(t) \leq \frac{1}{\varepsilon}, \quad (5.3)$$

where, obviously,  $\text{rk } F(t) = r$  for all  $t \geq 0$ .

Moreover, for any  $m \in \{1, \dots, n\}$  and partition

$$T(t) = [X(t), V(t)], \quad X(t) \in \mathbb{R}^{n \times (n-m)}, \quad V(t) \in \mathbb{R}^{n \times m}, \quad (5.4)$$

we have

$$\exists \delta \in (0, 1) \quad \forall t \geq 0 : \delta \leq V^T(t)V(t) \leq \frac{1}{\delta}, \quad (5.5)$$

and  $V \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^{n \times m})$  with left inverse  $(V^T V)^{-1} V^T \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^{m \times n})$ .

We preface the proof with the following lemma.

**Lemma 5.2** Let  $\ell \in \mathbb{N}$  and  $P \in \mathcal{C}^\ell(\mathbb{R}_{\geq 0}, \mathbb{R}^{n \times n})$  such that, for all  $t \geq 0$ ,  $P(t) = P(t)^T > 0$ . Then the following statements are equivalent:

- (i)  $\exists \varepsilon \in (0, 1) \quad \forall t \geq 0 : \varepsilon \leq \det P(t) \quad \text{and} \quad \|P(t)\| \leq 1/\varepsilon$
- (ii)  $\exists \delta \in (0, 1) \quad \forall t \geq 0 : \delta \leq P(t) \leq 1/\delta$ ,
- (iii)  $\exists \delta \in (0, 1) \quad \forall t \geq 0 : \delta \leq P(t)^{-1} \leq 1/\delta$ .

**Proof:** Note that positivity of  $P(t)$  yields

$$\forall t \geq 0 \quad \exists U(t) \in \mathbb{R}^{n \times n} \quad \text{and} \quad p_1(t), \dots, p_n(t) > 0 :$$

$$U^T(t)U(t) = I_n \quad \text{and} \quad P(t) = U(t) \text{diag}(p_1(t), \dots, p_n(t)) U^T(t), \quad (5.6)$$

and therefore,

$$\forall t \geq 0 \quad : \quad \det P(t) = \prod_{i=1}^n p_i(t) \quad \text{and} \quad \|\text{diag}(p_1(t), \dots, p_n(t))\| = \|P(t)\|. \quad (5.7)$$

Hence we may conclude that

$$\begin{aligned} \text{(i)} \quad & \stackrel{(5.7)}{\iff} \exists \delta \in (0, 1) \quad \forall t \geq 0 \quad \forall i \in \{1, \dots, n\} : \delta \leq p_i(t) \leq 1/\delta \\ & \iff \exists \delta \in (0, 1) \quad \forall t \geq 0 : \delta \leq \text{diag}(p_1(t), \dots, p_n(t)) \leq 1/\delta \\ & \stackrel{(5.6)}{\iff} \text{(ii)} \\ & \iff \exists \delta \in (0, 1) \quad \forall t \geq 0 : \delta \leq \text{diag}(p_1(t), \dots, p_n(t)) \leq 1/\delta \\ & \iff \exists \delta \in (0, 1) \quad \forall t \geq 0 \quad \forall i \in \{1, \dots, n\} : \delta \leq p_i(t) \leq 1/\delta \\ & \iff \exists \delta \in (0, 1) \quad \forall t \geq 0 \quad \forall i \in \{1, \dots, n\} : \delta \leq p_i(t)^{-1} \leq 1/\delta \\ & \stackrel{(5.6)}{\iff} \text{(iii)}. \end{aligned}$$

This completes the proof of the lemma. □

**Proof of Theorem 5.1:** In [5, Theorem 2] it is shown that there exists  $T \in \mathcal{C}^\ell(\mathbb{R}_{\geq 0}, \mathbb{R}^{n \times n})$  such that

$$\exists \alpha \in (0, 1) \quad \forall t \geq 0 : M(t)T(t) = [F(t), 0_{n \times (n-r)}] \quad \text{and} \quad \alpha \leq |\det T(t)| \quad \text{and} \quad \|T(t)\| \leq 1/\alpha.$$

This yields (5.1) and (5.2), and furthermore

$$\exists \alpha \in (0, 1) \quad \forall t \geq 0 : \alpha^2 \leq \det T^T(t)T(t) \quad \text{and} \quad \|T^T(t)T(t)\| \leq 1/\alpha^2,$$

which implies, in view of Lemma 5.2, equation (5.3).

Let, for  $m \in \{1, \dots, n\}$ ,  $T$  be partitioned as in (5.4). The second inequality in (5.5) is a direct consequence of (5.2) and it remains to show the first inequality in (5.5). Seeking a contradiction, suppose that

$$\exists (\eta_i)_{i \in \mathbb{N}} \in (\mathcal{S}^{m-1})^{\mathbb{N}} \quad \exists (t_i)_{i \in \mathbb{N}} \in (\mathbb{R}_{\geq 0})^{\mathbb{N}} : \lim_{i \rightarrow \infty} \eta_i^T V^T(t_i) V(t_i) \eta_i = 0,$$

where  $\mathcal{S}^{m-1} := \{\eta \in \mathbb{R}^m \mid \|\eta\| = 1\}$ . Then

$$\begin{aligned} 0 &= \lim_{i \rightarrow \infty} \eta_i^T V^T(t_i) V(t_i) \eta_i \\ &= \lim_{i \rightarrow \infty} (0_{1 \times (n-m)}, \eta_i^T) \begin{bmatrix} X^T(t_i) X(t_i) & X^T(t_i) V(t_i) \\ V^T(t_i) X(t_i) & V^T(t_i) V(t_i) \end{bmatrix} \begin{pmatrix} 0 \\ \eta_i \end{pmatrix} \\ &= \lim_{i \rightarrow \infty} (0_{1 \times (n-m)}, \eta_i^T) T^T(t_i) T(t_i) \begin{pmatrix} 0_{n-m} \\ \eta_i \end{pmatrix}, \end{aligned}$$

and this contradicts the first inequality in (5.3).

Finally,  $V \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^{n \times m})$  is immediate from (5.2), and (5.5) together with Lemma 5.2 gives  $V^T V \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^{m \times m})$ , and therefore, again by Lemma 5.2,  $(V^T V)^{-1} V^T \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^{m \times n})$ . This completes the proof of the theorem.  $\square$

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