An eigenvalue conjecture of P.C. Müller (University of Wuppertal)

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1) Definitions

Let $M = M_n = (m_{ij}) \in \mathbb{R}^{(n,n)}$ be an $n \times n$ *matrix* with integer entries m_{ij} and $U = U_n$ denote the $n \times n$ *unit matrix*. The determinant $\det(\lambda U - M) = f_M(\lambda)$ is called the *characteristic polynomial* of M. Value λ_0 is an *eigenvalue* of M if and only if $f_M(\lambda_0) = 0$. Vector $v_0 = v(\lambda_0) \neq \vec{0}$ is an *eigenvector* belonging to λ_0 of M if and only if $Mv_0 = \lambda_0 v_0$. Matrices M and N are *similar* (briefly: $M \sim N$) if and only if there is an regular matrix R with $MR = RN$; that yields that both matrices M, N have the same eigenvalues (with the same multiplicities).

2) The conjecture

At the Elgersburg Workshop 2006, P.C. Müller posed the following *eigenvalue conjecture*: The eigenvalues of the matrix

$$A = A_n := (a_{ij}) \in \mathbb{R}^{(n,n)}$$

$$\text{and} \quad a_{ij} = \begin{cases} -i, & \text{for } j = n - i \\ i, & \text{for } j = n - i + 1 \\ 0 & \text{otherwise} \end{cases}$$

are for $n \in \mathbb{N}$: $\lambda_j = \lambda_j(A) = (-1)^{j-1} \cdot j, \quad j = 1, 2, \ldots, n$.

3) The Proof

Let $v_{ij} = (-1)^{n+1+i+j}\binom{j-1}{n-i}$, $a^*_{ij} = (-1)^{j-1} \cdot j \cdot \binom{n-j}{i-j}$. Matrices $V = (v_{ij})$ and $A^* = \left(a^*_{ij}\right)$ are triangular, $a^*_{jj} = (-1)^{j-1} \cdot j$, so these numbers are the eigenvalues of A^*.
Next we shall show

$$AV = VA^* \tag{1}$$

then the theorem follows.
(i) Matrix $B = A \cdot V = (b_{ik})$ with

$$\begin{aligned} b_{ik} &= \sum_{j=1}^{n} a_{ij} v_{jk} \\ &= (-1)^{i+k} \cdot i \cdot \left\{\binom{k-1}{i-1} + \binom{k-1}{i}\right\} \\ &= (-1)^{i+k} \cdot i \cdot \binom{k}{i} = (-1)^{i+k} \cdot k \cdot \binom{k-1}{i-1} \end{aligned} \tag{2}$$

is an upper triangular matrix.

(ii) Matrix $B^* = V \cdot A^* = (b^*_{ik})$ has entries

$$\begin{aligned} b^*_{ik} &= \sum_{j=1}^{n} v_{ij} \cdot a^*_{jk} \\ &= (-1)^{i+k} \cdot k \cdot \sum_{j=1}^{n} (-1)^{n+j} \binom{j-1}{n-i}\binom{n-k}{j-k}. \end{aligned} \tag{3}$$

A transformation $j \longrightarrow n+1-j$ gives for the sum of (3)

$$\sum_{j=1}^{n}(-1)^{n+j}\binom{j-1}{n-i}\binom{n-k}{j-k} = -\sum_{j=1}^{n}(-1)^{j}\binom{n-j}{n-i}\binom{n-k}{j-1}.$$

From [G,K,P, page 169] we take the identity

$$\sum_{j=1}^{n}\binom{n-j}{m}\binom{s}{j-l}(-1)^j = (-1)^{n+m}\binom{s-m-1}{n-m-l} \quad (l,m,n \geq 0).$$

Replace in this equation m by $(n-i)$, s by $(n-k)$ and l by 1 to obtain

$$\begin{aligned} -\sum_{j=1}^{n}\binom{n-j}{n-i}\binom{n-k}{j-1}(-1)^j &= (-1)^{i-1}\binom{i-1-k}{i-1} \\ &= \frac{(-1)^{i-1}\cdot(i-1-k)\cdot(i-2-k)\cdot\ \ldots\ \cdot(1-k)}{1\ \cdot\ 2\ \cdot\ldots\cdot\ (i-1)} \\ &= \frac{(k-1)\cdot(k-2)\cdot\ \ldots\ \cdot(k-(i-1))}{1\ \cdot\ 2\ \cdot\ldots\cdot\ (i-1)} \\ &= \binom{k-1}{i-1}. \end{aligned} \tag{4}$$

and from equations (3),(4) follow $b^*_{ik} = b_{ik}$.
This proves (1) and the eigenvalue conjecture of P.C. Müller. □

4) Remark concerning the motivation

We solve the problem for some small values of n, say, $n = 1, 2, 3$.
Given matrix $E'_n = \left(e'_{ij}\right)$ with

$$e'_{ij} = \begin{cases} 1, & \text{if } i+j = n+1 \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $A'_n = E'_n \cdot A_n = \left(a'_{ij}\right)$ is a lower triangular matrix with diagonal entries

$$a'_{jj} = n+1-j.$$

Therefore the eigenvalues of A'_n are $\lambda'_j = j$, $j = 1, 2, \ldots, n$.
Next calculate matrix V_n such that the j-th column of V_n represents an eigenvector to eigenvalue $\lambda'_j = j$ of matrix A'_n:

$$V_1 = (1),\ V_2 = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix},\ V_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix}.$$

Clearly, V_n is non-singular and the inverse matrices are:

$$V_1^{-1} = (1),\ V_2^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},\ V_3^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Now calculate $A_n^* = V_n^{-1} \cdot A_n \cdot V_n$:

$$A_1^* = (1),\ A_2^* = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix},\ A_3^* = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 1 & -2 & 3 \end{pmatrix}.$$

For $n = 1, 2, 3$ is $A_n \sim A_n^*$. From this matrices V_n and A_n^* follow immediately the matrices $V = (v_{ij})$ and $A^* = \left(a_{ij}^*\right)$ (see 3)).

5) Concluding remark

Matrix $A(x, y) = (a_{ij}(x, y))$, where

$$a_{ij}(x, y) = \begin{cases} i \cdot x, & \text{if } i + j = n \\ i \cdot y, & \text{if } i + j = n + 1 \\ 0 & \text{otherwise,} \end{cases}$$

has for $x, y \in \mathbb{R}$ and $|x| = y > 0$ the following eigenvalues

$$\begin{array}{lll} n & \text{odd}: & (-1)^{j-1} \cdot j \cdot y \quad \text{, and} \\ n & \text{even}: & (-1)^j \cdot j \cdot x. \end{array}$$

6) Reference

[G,K,P]: Graham, R.L.; D.E. Knuth and O. Patashnik: Concrete Mathematics, Addison-Wesley Publishing Company, ... New York ... 1989