Edge colouring by total labellings

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Abstract

We introduce the concept of an edge-colouring total k-labelling. This is a labelling of the vertices and the edges of a graph G with labels $1, 2, \dots, k$ such that the weights of the edges define a proper edge colouring of G. Here the weight of an edge is the sum of its label and the labels of its two endvertices. We define $\chi'_t(G)$ to be the smallest integer k for which G has an edge-colouring total k-labelling. This parameter has natural upper and lower bounds in terms of the maximum degree Δ of G: $\lceil(\Delta + 1)/2\rceil \leq \chi'_t(G) \leq \Delta + 1$. We improve the upper bound by 1 for every graph and prove a general upper bound of $\chi'_t(G) \leq \Delta/2 + \mathcal{O}(\sqrt{\Delta \log \Delta})$. Moreover, we investigate some special classes of graphs.

Keywords Edge colouring; total labelling; irregularity strength; discrepancy
MSC Classification 05C15; 05C78; 05D40

1 Introduction

For a graph $G = (V(G), E(G))$ an *edge-colouring total k-labelling* is a function $f : V(G) \cup E(G) \to \{1, 2, \dots, k\}$ such that the weights of the edges defined by

$$w(uv) := f(u) + f(uv) + f(v)$$

form a proper edge colouring. The smallest integer k for which there exists an edge-colouring total k-labelling is denoted by $\chi'_t(G)$.

A related concept which has recently received a lot of attention was proposed by Karoński, Łuczak and Thomason [16]. They conjectured that the edges of every graph G with no K_2 component can be labeled with labels $1, 2, 3$ such that the sums of the edge labels incident to the vertices of G define a proper vertex colouring. Addario-Berry, Dalal and Reed [2] recently proved that the labels $1, 2, \dots, 16$ are always sufficient, i.e. every

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graph with no K_2 component has a *vertex-colouring edge* 16-*labelling* (cf. also [1, 3]). A total version of vertex-colouring labellings was discussed by Przybyło and Woźniak who proved [19] by similar methods as in [2] that every graph has a *vertex-colouring total* 11-*labelling* and conjecture that 2 labels are enough.

The vertex-colouring edge labellings can be considered a relaxation of the well-known *irregularity strength* of graphs [10, 4, 18, 14] where the label sums for all vertices are required to be different. Similarly, the edge-colouring total labellings which we study here can be considered a relaxation of *edge-irregular total labellings* introduced by Bača, Jendrol', Miller, and Ryan [6], where the weights of all edges are required to be different. The *total edge irregularity strength* $\mathrm{tes}(G)$ is defined as the smallest integer k for which a graph G has an edge-irregular total k-labelling. A simple lower bound is

$$\mathrm{tes}(G) \geq \max\left\{ \left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\}.$$

and Ivančo and Jendrol' [15] conjectured that this bound is attained for all graphs except K_5. Brandt, Miškuf, and Rautenbach [8, 9] recently proved that this is true for graphs whose size is at least 111000 times their maximum degree.

Let us return to the edge-colouring total k-labellings and the corresponding graph parameter $\chi'_t(G)$ which has natural upper and lower bounds in terms of the maximum degree Δ of G. Obviously,

$$\chi'_t(G) \leq \Delta + 1$$

by Vizing's Theorem [20], since a proper edge colouring together with a constant labelling of the vertices defines an edge-colouring total labelling of G. Furthermore, since the possible weights of the edges incident with a vertex v of maximum degree Δ in an edge-colouring k-labelling f are $f(v) + \{2, 3, \ldots, 2k\}$, we get a lower bound of

$$\chi'_t(G) \geq \left\lceil \frac{\Delta+1}{2} \right\rceil.$$

The following is our main result whose proof we postpone to Section 3.

Theorem 1.1 *If G is a graph of maximum degree Δ, then*

$$\chi'_t(G) \leq \left\lfloor \frac{1}{2}\left(\Delta + \left\lfloor \sqrt{2\Delta(1+\ln(2\Delta^2 - 2\Delta + 2))} \right\rfloor \right) \right\rfloor + 1 = \Delta/2 + \mathcal{O}(\sqrt{\Delta \log \Delta}).$$

Before we proceed to Section 2 where we study $\chi'_t(G)$ for some special graphs, we show how to reduce the upper bound by one for every graph and relate $\chi'_t(G)$ to the chromatic index. The next result already illustrates our general approach which is to combine edge colouring methods with suitable partitions of the vertex set.

Theorem 1.2 *If G is a graph of maximum degree Δ, then $\chi'_t(G) \leq \Delta$*

Proof: Let $c : E(G) \to \{1, 2, \ldots, \Delta + 1\}$ be a proper edge colouring of G which exists by Vizing's Theorem [20]. Since the subgraph containing the edges coloured Δ and $\Delta + 1$ consists of paths and even cycles, it is bipartite. Fix a bipartition $A \cup B$ of $V(G)$ such that all edges with colours Δ and $\Delta + 1$ have one endvertex in A and the other endvertex in B.

Assign to all vertices of A the label 1 and to all vertices of B the label Δ. Assign label $c(e)$ to all edges between vertices of A and label $c(e) + 1$ to all edges between vertices of B. Finally, determine the labels of the edges in the bipartite graph spanned by the edges between A and B by a proper Δ-edge colouring c'.

The edges joining vertices of A receive weights between 3 and $(\Delta - 1) + 1 + 1 = \Delta + 1$, the edges joining A to B receive weights between $\Delta + 2$ and $2\Delta + 1$, and the edges joining vertices of B receive weights between $2\Delta + 2$ and 3Δ. Since these weights form proper edge colourings inside and between the sets, they form a proper edge colouring of the entire graph. $\square$

The upper bound $\chi'_t(G) \leq \Delta$ can only be tight for small values of Δ. From Theorem 1.1 follows that for $\Delta \geq 19$ we have $\chi'_t(G) < \Delta$, and, in fact, with a more refined reasoning along the same lines the threshold can be reduced to $\Delta \geq 14$. We are not aware of any graph with $\Delta > 3$ and $\chi'_t(G) = \Delta$.

Next we show that an edge-colouring total k-labelling gives rise to a proper edge colouring with $2k - 1$ colours. Conversely, this means that for every type II graph (i.e. $\chi'(G) = \Delta(G) + 1$) we have $\chi'_t(G) > \frac{\Delta(G)+1}{2}$.

Lemma 1.3 *If $\chi'_t(G) = k$ for a graph G, then $\chi'(G) \leq 2k - 1$.*

Proof: Consider an edge-colouring total k-labelling f of G. Note that for $l \leq k + 1$ the edges of weights l and $l + 2k - 1$ cannot have a common endvertex and therefore form a matching. Thus we can decompose the edge set into $2k - 1$ matchings: $k - 1$ matchings with the edges of weight l and $l + 2k - 1$ for $3 \leq l \leq k + 1$, and k matchings with the edges of weight l for $k + 2 \leq l \leq 2k + 1$. $\square$

2 Special classes of graphs

If G is a graph of maximum degree $\Delta = 1$, then $\chi'_t(G) = 1$. If $\Delta = 2$, then $\chi'_t(G) = 2$ by Theorem 1.2. Similarly, if $\Delta(G) = 3$, then $2 \leq \chi'_t(G) \leq 3$. In our first result we charaterize cubic graphs with $\chi'_t(G) = 2$.

Theorem 2.1 *A cubic graph G satisfies $\chi'_t(G) = 2$ if and only if its vertex set can be partitioned into two parts A and B that induce perfect matchings.*

Proof: Let f be an edge-colouring total 2-labelling of a cubic graph G. For every vertex $v \in V(G)$ the three edges incident with v must receive the weights $3, 4, 5$, if $f(v) = 1$, and the weights $4, 5, 6$, if $f(v) = 2$. The edges of weight 3 and weight 6 join two vertices with the same label.

If $f(v) = 1$, then the other endvertex of the edge of weight 5 incident with v has label 2. So there are at least as many vertices with label 2 as with label 1. Conversely, for $f(v) = 2$ the edge of weight 4 incident to v has its other endvertex labelled 1. So there are at least as many vertices labelled 1 as with label 2. Together, there are equally many vertices labelled 1 and 2 and the edges of weights 4 and 5 form a 2-regular graph joining vertices of label 1 to vertices of label 2. Therefore, G has the indicated structure.

Conversely, if G has the indicated structure, then $|A| = |B|$. We assign label 1 to the vertices and edges in A and label 2 to the vertices and edges in B. Labelling the edges of the 2-regular bipartite graph between A and B by 1 and 2 according to a proper 2-edge colouring results is an edge-colouring total 2-labelling. □

It is an easy observation that the lower bound is tight for forests.

Theorem 2.2 *If F is a forest of maximum degree Δ, then $\chi'_t(F) = \lceil \frac{\Delta+1}{2} \rceil$.*

Proof: We prove the stronger statement that an edge-colouring total labelling exists using only two vertex labels 1 and $k = \lceil \frac{\Delta+1}{2} \rceil$. Obviously, it suffices to prove the statement for the tree components.

We proceed by induction on the number of vertices n. The statement is true for $n \leq 2$ so assume $n \geq 3$. Let vw be an edge such that v has degree at least 2 and all neighbours of v except possibly w are leaves. Note that such an edge vw exists. Delete all neighbours of v except w to obtain a tree T', which by induction has the required total labelling. Now label the deleted vertices with 1 and k such that at most $\frac{d(v)+1}{2}$ of the neighbours of v (including the already labelled vertex w) have the same label. Now the remaining edges can be easily labelled such that all edges incident with v have different weights. □

Next, we consider edge-colouring total labellings of complete graphs. In a graph G with a given edge colouring a *rainbow (perfect) matching* is a (perfect) matching, where all edges are of different colour. We need a lemma on rainbow matchings in the proof of our next result.

Lemma 2.3 *(a) Every complete bipartite graph $K_{k,k}$ has a proper k-edge colouring with a rainbow perfect matching if k is odd, and a rainbow matching of cardinality $k-1$ if k is even.*

(b) Every complete graph K_{2k} of even order has a proper $(2k-1)$-edge colouring with a rainbow perfect matching unless $k = 2$.

Proof:

(a) Let $u_1, \ldots, u_k$ and $w_1, \ldots, w_k$ be the vertices on both sides of the bipartition. Define a proper edge colouring of G by assigning the colour $\ell \in \{1, \ldots, k\}$ to the edge $u_i w_j$, if $j - i \equiv \ell \mod k$. Now let a and b be the largest even and odd integer $< \frac{k}{2} + 1$, respectively. Choose a maching M consisting of the edges $u_i w_{a+1-i}$ for $1 \leq i \leq a$ and $u_{a+i} w_{a+b+1-i}$ for $1 \leq i \leq b$. This is a rainbow matching of cardinality $a + b = k - 1$, if k is even and a rainbow perfect matching, if k is odd.

(b) Let $u_0, u_1, \ldots, u_{2k-1}$ be the vertices of K_{2k}. First assume that k is odd. Take as the first colour class of edges the perfect matching M_0 consisting of the edges $u_i u_{2k-i-1}$ for $0 \leq i \leq k-1$. The remaining colour classes are obtained as follows: Embed the vertices of K_{2k} in the plane such that $u_1, u_2, \ldots, u_{2k-1}$ form the vertices of a regular $(2k-1)$-gon with center u_0. Rotating M_0 by an angle of $\frac{2\pi}{2k-1}$ a total number of $2k-2$ times defines $2k-2$ further perfect matchings (cf. Figure 1). Since the geometric lengths of all edges in one matching are different, this defines a proper edge-colouring of K_{2k} for which the matching $u_0 u_{2k-1}$, $u_1 u_2, \ldots, u_{2k-3} u_{2k-2}$ is a rainbow perfect matching.

Next, assume that k is even. Here we choose as the first colour class of edges the perfect matching M_0 consisting of the edges $u_i u_{2k-i-1}$ for $0 \leq i < \frac{k}{4}$, $u_i u_{2k-i-2}$ for $\frac{k}{4} \leq i < \frac{3}{4}k - 1$, $u_i u_{2k-i-3}$ for $\frac{3}{4}k - 1 \leq i < k-1$, and the additional edge $u_i u_{i+\frac{k}{2}}$ for $i = \lfloor \frac{5}{4}k \rfloor - 1$. Again, the remaining $2k-2$ colour classes are obtained by embedding the vertices of K_{2k} and rotating M_0 as before (cf. Figure 1). Again the matching $u_0 u_{2k-1}$, $u_1 u_2, \ldots, u_{2k-3} u_{2k-2}$ is a rainbow perfect matching.

□

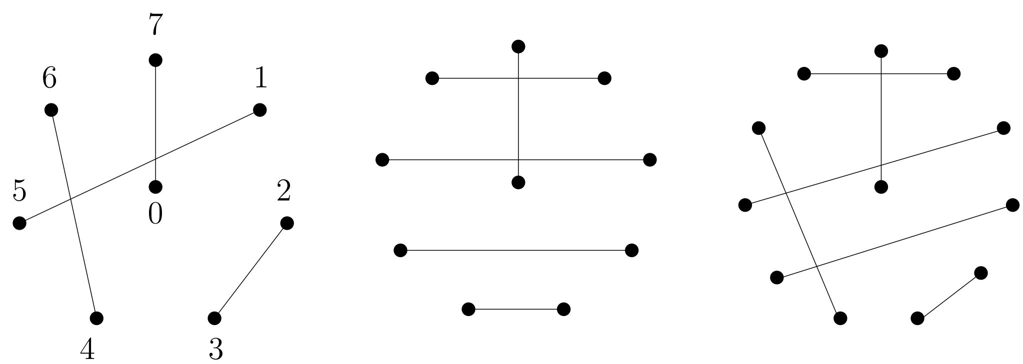

Figure 1

In a graph with a total labelling we denote the set of vertices with label i by V_i.

Theorem 2.4 *If $n \not\equiv 2 \mod 4$, then $\chi'_t(K_n) = \lceil \frac{n}{2} \rceil$ and if $n \equiv 2 \mod 4$, then $\chi'_t(K_n) \leq \frac{n}{2} + 1$.*

Proof: In view of the lower bound it suffices to describe suitable labellings of the complete graph.

First assume that $n \equiv 0 \mod 4$. Label half the vertices by 1 and the other half by $k = \frac{n}{2}$. Determine a proper edge colouring of the edges in V_1 with labels $1, \ldots, k-1$, a proper edge colouring of the edges in V_k with labels $2, \ldots, k$, and a proper edge colouring of the edges joining V_1 to V_k with labels $1, \ldots, k$. It is now easy to verify, that this is an edge-colouring total k-labelling. Note that this also implies the result for $n \equiv 3 \mod 4$ and $n \equiv 2 \mod 4$ by considering edge-colouring total labellings of complete graphs of order $n+1$ and $n+2$, respectively.

Therefore, only the case $n \equiv 1 \mod 4$ remains.

Label $\frac{n-1}{2} = k-1$ vertices by label 1 and k, respectively, and the remaining vertex v by $\frac{k+1}{2}$. Let $u_{\frac{k+1}{2}+2}, \ldots, u_{\frac{k+1}{2}+k}$ denote the vertices of V_1 and $w_{\frac{k+1}{2}+k+2}, \ldots, w_{\frac{k+1}{2}+2k}$ be the vertices of V_k. Label the edges from v to u_i with $i - \frac{k+1}{2} - 1$ and the edges v to w_j with $j - \frac{k+1}{2} - k$. Note that each vertex u_i and w_j is joined to v by an edge of weight i and j, respectively.

It remains to show that we can find an edge labelling of the edges not incident with v, such that the labels form a proper edge colouring of the remaining graph and the weight of each edge is different from the indices of its endvertices. The edges inside V_1 will obtain the weights $3, \ldots, k+1$, inside V_k the weights $2k+2, \ldots, 3k$, and the edges between V_1 and V_k will obtain weights $k+2, \ldots, 2k+1$.

By Lemma 2.3 (b) we know that the complete graph K_{k-1} induced by V_1 has a proper $(k-2)$-edge colouring c which has a rainbow perfect matching. Let $\{2, \ldots, k-1\}$ be the colours of the colouring and let $\{\frac{k-1}{2}+1, \ldots, k-1\}$ be the colours occurring in the rainbow perfect matching M. Assign the indices in such a way that the vertex u_i of index i is incident with the edge of colour $i-2$ in the rainbow matching for $\frac{k+1}{2} + 2 \leq i \leq k+1$. Finally, recolour the edges of the rainbow perfect matching M with colour 1 and take the colours of this new colouring c' as the labels of the edges inside V_1. Note that this edge labelling has the desired property that u_i is not joined to a vertex in V_1 by an edge of weight i. Along the same line of argument we obtain a labelling of the edges inside V_k with labels $\{2, \ldots, k\}$ such that each vertex w_j is not joined to a vertex in V_k by an edge of weight j.

Finally, we need to label the edges in the bipartite graph spanned by the (V_1, V_k)-edges. This graph is isomorphic to $K_{k-1,k-1}$, where $k-1 \equiv 0 \mod 2$. By Lemma 2.3 (a) this graph has a proper $(k-1)$-edge colouring using the colours $\{1, \ldots, k-1\}$ with a rainbow matching M of cardinality $k-2$ that avoids the colour $\frac{k-1}{2}$. Assign the indices in such a way that u_i is incident with the edge of M of weight $i-k-1$ for $k \leq i \leq \frac{k+1}{2}+k$, and w_j is incident with the edge of M of weight $j-k-1$ for $\frac{k+1}{2}+k+2 \leq j \leq 2k$. Moreover, let w_{2k+1} be the vertex in V_k that is not incident with an edge of M. Now recolour the edges of M with colour k to obtain a new colouring, which we use as the labelling of the (V_1, V_k) edges. By the construction it is easy to verify that the result is an edge-colouring total k-edge labelling. □

We conclude this section with some further results concerning the case $n \equiv 2 \mod 4$ which might eventually allow to determine for which $n \equiv 2 \mod 4$, $\chi'_t(K_n) = \frac{\Delta+1}{2}$ holds, and for which $\chi'_t(K_n) = \frac{\Delta+1}{2} + 1$. We can show that the second equality holds for $6 \leq n \leq 22$. At the same time our result describes the distribution of the labels in some detail if the first equality holds.

Lemma 2.5 *Let K_n be a complete graph with $k = \chi'_t(K_n) = \frac{n}{2}$. If V_i denotes the set of vertices labelled i in an edge-colouring total k-labelling of K_n, then the cardinality of each set V_i is even, $|V_i| = |V_{k-i+1}|$ and $|V_i| \leq |V_1| = |V_k|$ for $i = 1, \ldots, k$. The edges of weight*

$k+2$ *have label* 1 *and the edges of weight* $2k+1$ *have label* k. *Moreover, if* $n \equiv 2 \mod 4$ *then* $6 \leq |V_{\frac{k+1}{2}}| \equiv 2 \mod 4$.

Proof: Since $k = \frac{\Delta+1}{2}$, each vertex $v \in V_i$ is incident with an edge of weight $i+\ell$ for $2 \leq \ell \leq 2k$. For $2 \leq i \leq k$ the edges of weight $i+2$ form a matching between V_i and V_1 and hence $|V_i| \leq |V_1|$. Similarly, each vertex $v \in V_j$ is incident with an edge of weight $j+2k$ and for $1 \leq j \leq k-1$ these edges form a matching between V_j and V_k, implying $|V_j| \leq |V_k|$. Since the inequalities hold for $i=k$ and $j=1$, we obtain $|V_1| = |V_k|$.

Next we show that each of the sets V_i has even cardinality. This is true for V_1, since the edges of weight 3 form a perfect matching between the vertices in V_1. Now consider the vertex set $U_i = V_1 \cup V_2 \cup \ldots \cup V_i$. Since the edges of weight $i+2$ form a perfect matching of U_i, and, by induction, U_{i-1} has even cardinality, the set V_i has even cardinality as well.

For $i \leq \frac{k+1}{2}$ we prove by induction over i that the edges of weight $2k+1$ incident to a vertex in V_i have their other endvertex in V_{k-i+1}, and the edges of weight $k+2$ incident to a vertex in V_{k-i+1} have their other endvertex in V_i. In particular, $|V_i| = |V_{k-i+1}|$ and the edges of weight $k+2$ and $2k+1$ have weight 1 and k, respectively.

The statement is true for $i=1$, so assume that it is true for all indices $< i$. Let vw be the edge of weight $2k+1$ that is incident to $v \in V_i$. Since the label of vw is at most k, the vertex w has label $s \geq k-i+1$. If $s > k-i+1$, then by induction the other endvertex v of the edge of weight $2k+1$ incident to w has label $t = k-s+1 < i$, contradicting $v \in V_i$. Analogously, for the vertices of V_{k-i+1} the other endvertex of the incident edge of weight $k+2$ lies in V_i. This completes the induction. If $n \equiv 2 \mod 4$, then $6 \leq |V_{\frac{k+1}{2}}| \equiv 2 \mod 4$, because of the parity conditions and since $V_{\frac{k+1}{2}}$ has two disjoint perfect matchings consisting of the edges of weight $k+2$ and $2k+1$. □

Lemma 2.6 *Every edge-colouring total* $(2p+1)$*-labelling of* K_{4p+2} *for* $p \geq 1$ *uses at least* 5 *different vertex labels.*

Proof: For contradiction, we assume the existence of an edge-colouring total $(2p+1)$-labelling using less than 5 different vertex labels. By Lemma 2.5, this implies that it has exactly 3 label classes V_1, V_{p+1}, and V_{2p+1}. Moreover $|V_{2p+1}| = |V_1| \geq |V_{p+1}| \geq 6$. We know that all edges with weights $3, \ldots, p+2$ have both endvertices in V_1 and for each such weight value these edges form a perfect matching in V_1. Furthermore, all edges of weight $p+3, \ldots, 2p+2$ incident with a vertex in V_{p+1} have the other endvertex in V_1, and, finally, there is a perfect matching between V_{2p+1} and V_1 of edges of weight $2p+3$.

Let n_1 be the number of vertices in V_1 and n_{p+1} the number of vertices in V_{p+1}. Since we have $n_1 = \frac{n-n_{p+1}}{2}$ for $n = 4p+2$, there are exactly $n_1/2$ edges in V_1 of weight w for each $3 \leq w \leq p+2$ and $\frac{n_1-n_{p+1}}{2}$ edges in V_1 of weight w for each $p+3 \leq w \leq 2p+2$ and no edges of weight $\geq 2p+3$. Altogether, there are at most

$$p\left(\frac{n_1}{2}\right) + p\left(\frac{n_1 - n_{p+1}}{2}\right) = p\left(n_1 - \frac{n_{p+1}}{2}\right)$$

edges in V_1. Since $2n_1 + n_{p+1} = n = 4p + 2$ we get $p = \frac{1}{2}(n_1 + \frac{n_{p+1}}{2} - 1)$ and

$$\binom{n_1}{2} \leq \binom{n_1}{2} - \frac{1}{2}n_{p+1}^2 + \frac{1}{2}(n_1 - n_1 + 1)n_{p+1},$$

which is a contradiction since $n_{p+1} \geq 6 > 1$. □

3 The general upper bound

Our goal in this section is to prove Theorem 1.1. In order to clarify our approach, we present a number of intermediate results, some of which we think to be interesting on their own right. The first is a consequence of Vizing's Adjacency Lemma [21] (see also [13]). A graph $G = (V, E)$ of maximum degree Δ is called *critical* if $\chi'(G) = \Delta+1$ but $\chi'(G-e) = \Delta$ for all $e \in E$.

Lemma 3.1 (Vizing's Adjacency Lemma [21]) *Let $G = (V, E)$ be a critical graph with maximum degree Δ and $\chi'(G) = \Delta + 1$. If $uv \in E$ then u is adjacent to at least* $\max\{2, \Delta - d_G(v) + 1\}$ *many vertices of maximum degree.*

Proposition 3.2 *Every graph G with maximum degree Δ has a proper $(\Delta+1)$-edge colouring such that no edge of colour $\Delta + 1$ is incident with a vertex of degree less than Δ.*

Proof: We apply induction on $m := |E(G)|$. If G has a proper Δ-edge colouring, then the statement is vacuously true. Note that this already implies the result for $m \leq 2$. Therefore, we assume now that $m \geq 3$ and that $\chi'(G) = \Delta + 1$.

It follows from Lemma 3.1 applied to a critical subgraph of G and a vertex u of maximum degree, that a neighbour w of u has maximum degree as well. By induction, $G - uw$ has a proper $(\Delta + 1)$-edge colouring such that no edge of colour $\Delta + 1$ is incident to a vertex of degree less than Δ. Therefore, assigning the colour $\Delta + 1$ to the edge uw yields the desired colouring. □

The construction in the next result relies on a suitable partition of the vertex set.

Theorem 3.3 *If G is a graph of maximum degree Δ whose vertex set has a partition $V(G) = A \cup B$ such that every vertex has at most $k - 1$ neighbours in A and at most $k - 1$ neighbours in B for some k with $k - 1 > \frac{\Delta}{2}$, then $\chi'_t(G) \leq k$.*

Proof: Let $V(G) = A \cup B$ be a partition as in the statement. Label the vertices of A with 1 and the vertices of B with k.

By Proposition 3.2, $G[A]$ has a proper k-edge colouring that avoids colour k at the vertices of degree $d_{G[A]}(v) < k - 1$. Similarly, $G[B]$ has a proper k-edge colouring that avoids colour 1 at the vertices of degree $d_{G[B]}(v) < k - 1$. We choose these edge colourings as the labellings of the edges in A and B, respectively. Let A' denote the set of vertices in

A incident with an edge labelled k and let B' denote the set of vertices in B incident with an edge labelled 1.

It remains to label the edges between A and B. Let $G(A,B)$ denote the bipartite spanning subgraph of G of maximum degree at most $k-1$ containing all edges between A and B. Considering a perfect matching in a bipartite $(k-1)$-regular supergraph of $G(A,B)$, it follows that $G(A,B)$ has a minimal matching M that saturates all vertices v with $d_{G(A,B)}(v) = k-1$. Note that by the minimality requirement, M does not contain an (A',B')-edge, since for each vertex in $u \in A' \cup B'$ we have $d_{G(A,B)}(u) \leq \Delta(G) - (k-1) < k-1$. We label the edges of M with one endvertex in A' with k and the remaining edges with 1. Now $G(A,B) - M$ has maximum degree $\leq k-2$ and hence has a proper $(k-2)$-edge colouring with colours $2, 3, \ldots, k-1$ which we use as the labelling for the edges. It is easy to verify that the edge weights defined by this total k-labelling form a proper edge colouring of G. □

Our next goal is to find a partition as in Theorem 3.3 for some k close to $\Delta/2$. We do this using the probabilistic method via a discrepancy argument: For a graph G we consider the *discrepancy* $\operatorname{disc}(G)$ defined as follows:

$$\operatorname{disc}(G) \;:=\; \min_{g:V(G)\to\{-1,1\}} \max_{u\in V(G)} \left| \sum_{v\in N_G(u)} g(u) \right|.$$

Note that $\operatorname{disc}(G)$ corresponds to the ordinary discrepancy of the hypergraph on the ground set $V(G)$ whose hyperedges are the neighbourhoods of vertices in G.

Together with Theorem 3.3 we obtain.

Corollary 3.4 *If G is a graph of maximum degree Δ, then*

$$\chi_t(G) \leq \frac{\Delta + \operatorname{disc}(G)}{2} + 1.$$

Proof: Let $g : V(G) \to \{-1,1\}$ be such that $\operatorname{disc}(G) = \max\limits_{u\in V(G)} \left|\sum_{v\in N_G(u)} g(u)\right|$. Let $A = g^{-1}(1)$ and $B = g^{-1}(-1)$. For $u \in V(G)$ let $d_A(u) = \{v \in N_G(u) \mid g(v) = 1\}$ and $d_B(u) = \{v \in N_G(u) \mid g(v) = -1\}$. Since $|d_A(u) - d_B(u)| \leq \operatorname{disc}(G)$ and $d_A(u) + d_B(u) \leq \Delta$, we have $\max\{d_A(u), d_B(u)\} \leq \frac{\Delta+\operatorname{disc}(G)}{2}$ for every $u \in V(G)$ and Theorem 3.3 implies the desired result. □

In order to bound the discrepancy we combine Chernoff's inequality with the Lovász Local Lemma.

Lemma 3.5 (Chernoff's inequality [11], see also [5]) *Let $X_1, \ldots, X_n$ be mutually independent random variables with $\mathbf{P}(X_i = 1) = \mathbf{P}(X_i = -1) = \frac{1}{2}$. Then for $S = X_1 + \ldots + X_n$ and $\delta > 0$ we get $\mathbf{P}(|S| > \delta) < 2\exp\left(\frac{-\delta^2}{2n}\right)$.*

Lemma 3.6 (Lovász Local Lemma [12], see also [5]) *Let $A_1, A_2, \dots, A_n$ be events in an arbitrary probability space. Let $\mathbf{P}(A_i) \leq p$ and let A_i be mutually independent of all but at most $d \geq 2$ of the events A_j with $j \neq i$ for each $1 \leq i \leq n$. If $ep(d+1) \leq 1$, then $\mathbf{P}\left(\bigwedge_{i=1}^{n} \overline{A}_i\right) > 0$, i.e. with positive probability none of the events A_i occurs.*

Proposition 3.7 *If G is a graph of maximum degree Δ, then*

$$\operatorname{disc}(G) \leq \left\lfloor \sqrt{2\Delta(1+\ln(2\Delta^2 - 2\Delta + 2))} \right\rfloor .$$

Proof: We consider a random function $g : V(G) \to \{-1, 1\}$ where all values $g(v)$ are 1 independently at random with probability 1/2.

For some $\delta > 0$ and $u \in V(G)$ consider the event A_u: $\left|\sum_{v \in N_G(u)} g(u)\right| > \delta$. By Chernoff's inequality,

$$\mathbf{P}\left(A_u\right) \leq 2\exp\left(\frac{-\delta^2}{2d_G(u)}\right) \leq 2\exp\left(\frac{-\delta^2}{2\Delta}\right).$$

The events A_u and A_v are dependent only if there is a path of length exactly two between u and v. Therefore, A_u is independent of all but at most $\Delta(\Delta-1)$ many events A_v with $v \neq u$. For $\delta := \sqrt{2\Delta(1+\ln(2\Delta^2 - 2\Delta + 2))}$ we obtain

$$2\exp\left(1 - \frac{\delta^2}{2\Delta}\right)(\Delta(\Delta-1)+1) = 1$$

and the Lovász Local Lemma implies the existence of a function $g : V(G) \to \{-1, 1\}$ with $\left|\sum_{v \in N_G(u)} g(u)\right| \leq \delta$ for all $u \in V(G)$. $\square$

Proof of Theorem 1.1: The result follows immediately from Corollary 3.4 and Proposition 3.7. $\square$

4 Concluding remarks

The upper bound $\mathcal{O}(\sqrt{\Delta \log \Delta})$ for the discrepancy of a Δ-regular graph is not far from being best possible. This is due to the fact, that there are graphs with discrepancy $\Omega(\sqrt{\Delta})$. The Paley graphs for example form an infinite sequence of graphs with $\Delta = \frac{n-1}{2}$ and discrepancy $\Omega(\sqrt{\Delta})$ by a result of Lovász and Sós (see [17, Theorem 4.5]). Conversely, the Beck-Fiala Conjecture (see [17]) says that the vertices of every hypergraph where each vertex belongs to at most Δ hyperedges has discrepancy $\mathcal{O}(\sqrt{\Delta})$. If the Beck-Fiala Conjecture — or its restriction to Δ-regular, Δ-uniform hypergraphs — is true then we can improve the upper bound in Theorem 1.1 with the same reasoning to

$$\chi'_t(G) \leq \frac{\Delta+1}{2} + \mathcal{O}(\sqrt{\Delta}). \tag{1}$$

On the other hand, if, like in most of our explicit labellings, the typical total k-labellings use on the vertices almost only the labels 1 and k, then the reduced upper bound in the formula above is tight in view of the Paley graphs.

So the main open question in this context might be the following:

Problem 4.1 *Is there a constant K with*

$$\chi_t'(G) \leq \frac{\Delta+1}{2} + K \tag{2}$$

for all graphs G of maximum degree Δ?

Surely there are further options except (1) and (2). Indications could be obtained from an answer to the question whether all graphs G have an edge-colouring total $\chi_t'(G)$-labelling with only few vertex labels.

In view of the graphs where we know the exact value of $\chi_t'(G)$, the constant K must be at least 1. With $K = 1$ the bound (2) is attained with equality e.g. for cubic snarks and K_{4k+2} for $1 \leq k \leq 5$. For even Δ we are not aware of any graph with $\chi_t'(G) > \lceil\frac{\Delta+1}{2}\rceil$. One first question in this direction is whether $\chi_t'(G) = 3$ for all graphs with $\Delta = 4$. As a potential candidate for the general problem we checked the unique Paley graph on 17 vertices ($\Delta = 8$), which is at the same time the $(4,4)$-Ramsey graph, with a computer program, that came up with an edge-colouring total 5-labelling.

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