On $\mathcal{F}$-independence in Graphs

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Abstract. Let $\mathcal{F}$ be a set of graphs and for a graph G let $\alpha_{\mathcal{F}}(G)$ and $\alpha^*_{\mathcal{F}}(G)$ denote the maximum order of an induced subgraph of G which does not contain a graph in $\mathcal{F}$ as a subgraph and which does not contain a graph in $\mathcal{F}$ as an induced subgraph, respectively. Lower bounds on $\alpha_{\mathcal{F}}(G)$ and $\alpha^*_{\mathcal{F}}(G)$ and algorithms realizing them are presented.

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1 Introduction

We consider finite, undirected and simple graphs G with vertex set $V(G)$ and edge set $E(G)$ and refer to [5] for undefined notation.

As a generalization of the well-studied concept of independent sets [8] in graphs Peter Mihok [9] proposed the following problem: *For two given graphs F and G, what is the maximum order of an induced subgraph of G that either does not contain F as a subgraph or does not contain F as an induced subgraph?*

The purpose of the present paper is to formalize the independence concept corresponding to this problem and to initiate its study. Therefore, for a graph G and a set $\mathcal{M}$ of graphs we denoted by $f(G, \mathcal{M})$ the maximum order $|S|$ of a subgraph $G[S]$ of G induced by $S \subseteq V(G)$ such that $G[S]$ belongs to $\mathcal{M}$. Choosing $\mathcal{M}$ appropriately allows to capture Mihok's independence problem. More precisely, let $\mathcal{F}$ be a set of graphs and for a graph G let $\alpha_{\mathcal{F}}(G)$ and $\alpha^*_{\mathcal{F}}(G)$ denote the maximum order of an induced subgraph of G which does not contain a graph in $\mathcal{F}$ as a subgraph and which does not contain a graph in $\mathcal{F}$ as an induced subgraph, respectively. Clearly, if we define $\mathcal{M}_{\mathcal{F}}$ as the set of all graphs which do not contain a graph in $\mathcal{F}$ as a subgraph and $\mathcal{M}^*_{\mathcal{F}}$ as the set of all graphs which do not contain a graph in $\mathcal{F}$ as an induced subgraph, then $\alpha_{\mathcal{F}}(G) = f(G, \mathcal{M}_{\mathcal{F}})$ and $\alpha^*_{\mathcal{F}}(G) = f(G, \mathcal{M}^*_{\mathcal{F}})$. If $\mathcal{F} = \{F\}$, then we write $\alpha_F(G)$ and $\alpha^*_F(G)$ for short.

Several well-known graph parameters are special cases of these notions as shown in the following result which collects some obvious basic observations.

Proposition 1 *Let G be a graph.*

(i) $\alpha_{K_2}(G)$ *equals the independence number* $\alpha(G)$ *of* G*.*

(ii) $\alpha_{\bar{K}_2}(G)$ *equals the clique number of* G*.*

(iii) $\alpha_{P_3}(G)$ *equals the dissociation number of* G *[2].*

(iv) $\alpha_{K_r}(G) = \alpha^*_{K_r}(G)$*.*

(v) $\alpha_{\bar{K}_r}(G) = \min\{|V(G)|, r-1\}$*.*

(vi) $\alpha^*_{\bar{K}_r}(G) = \max\{|S| \mid S \subseteq V(G), \alpha(G[S]) \leq r-1\}$*.*

(vii) $\alpha^*_{\mathcal{F}}(G) = \alpha^*_{\{\bar{F} \mid F \in \mathcal{F}\}}(\bar{G})$*.*

Our next result is a lower bound on $f(G, \mathcal{M})$ provided the set $\mathcal{M}$ has some natural properties.

Theorem 2 *Let $\mathcal{M}$ be a set of graphs and let G be a graph.*

(i) If $\mathcal{M}$ is closed under taking induced subgraphs, then

$$f(G, \mathcal{M}) \geq \sum_{S: S \subseteq V(G), G[S] \in \mathcal{M}} \binom{|V(G)|}{|S|}^{-1}$$

(ii) If $\mathcal{M}$ is closed under taking induced subgraphs and under forming the union of graphs, then

$$f(G, \mathcal{M}) \geq \sum_{S: S \subseteq V(G), G[S] \in \mathcal{M}, G[S] \text{ is connected}} \binom{|N_G[S]|}{|S|}^{-1}$$

where $N_G[S] = \cup_{u \in S} N_G[u]$.

Proof: We only prove (ii) and leave the very similar proof of (i) to the reader. We choose a permutation $v_1, v_2, ..., v_n$ of the vertices of G uniformly at random. Let $S_0 = \emptyset$ and for $1 \leq i \leq n$ let $S_i = S_{i-1} \cup \{v_i\}$ if $G[S_{i-1} \cup \{v_i\}] \in \mathcal{M}$ and $S_i = S_{i-1}$ otherwise. Clearly, $f(G, \mathcal{M}) \geq |S_n|$ and $v_i \in S_n$ if and only if $v_i \in S_i$ and the component H_i of $G[S_i]$ containing v_i belongs to $\mathcal{M}$. Therefore, for a set $S \subseteq V(G)$ with $v_i \in S$ such that $G[S] \in \mathcal{M}$ and $G[S]$ is connected, a lower bound for the probability that $H_i = G[S]$ is the probability that in the chosen permutation the vertices $S \setminus \{v_i\}$ preceed v_i while v_i preceeds the vertices in $N_G[S] \setminus S$ which equals $\frac{1}{|S|}\binom{|N_G[S]|}{|S|}^{-1}$. Therefore, by linearity of expectation

$$\begin{aligned}
f(G,\mathcal{M}) &\geq \mathbf{E}(|S_n|) = \sum_{i=1}^{n} \mathbf{P}(v_i \in S_n) \\
&\geq \sum_{i=1}^{n} \sum_{S: v_i \in S \subseteq V(G), G[S] \in \mathcal{M}, G[S] \text{ is connected}} \frac{1}{|S|}\binom{|N_G[S]|}{|S|}^{-1} \\
&= \sum_{S: S \subseteq V(G), G[S] \in \mathcal{M}, G[S] \text{ is connected}} \sum_{i: v_i \in S} \frac{1}{|S|}\binom{|N_G[S]|}{|S|}^{-1} \\
&= \sum_{S: S \subseteq V(G), G[S] \in \mathcal{M}, G[S] \text{ is connected}} \binom{|N_G[S]|}{|S|}^{-1}
\end{aligned}$$

and the proof is complete. $\square$

Corollary 3 *Let G be a graph.*

(i) $\alpha(G) \geq \sum_{u \in V(G)} \frac{1}{1+d_G(v)}$ **(Caro [3], Wei [11])**.

(ii) The dissociation number satisfies

$$\alpha_{P_3}(G) \geq \sum_{u \in V(G)} \frac{1}{1+d_G(v)} + \sum_{uv \in E(G)} \frac{2}{|N_G[u] \cup N_G[v]| \, (|N_G[u] \cup N_G[v]| - 1)}.$$

Proof: Note that $\mathcal{M}_{\{K_2\}} = \left\{\bar{K}_r \mid r \in \mathbb{N}\right\}$ and $\mathcal{M}_{\{P_3\}} = \mathcal{M}_{\{K_2\}} \cup \left\{K_2 \cup \bar{K}_r \mid r \in \mathbb{N}\right\}$. Both statements follow immediately from Theorem 2(ii) and the observation that the only connected graph in $\mathcal{M}_{\{K_2\}}$ is K_1 and the only connected graphs in $\mathcal{M}_{\{P_3\}}$ are K_1 and K_2. $\square$

The famous bound due to Caro [3] and Wei [11] from Corollary 3 has yet another generalization in this context.

Theorem 4 *If G is a graph and $r \in \mathbb{N}$, then* $\alpha_{K_{r+1}}(G) \geq \sum\limits_{v \in V(G)} \frac{1}{1+d_G(v)-\alpha_{K_r}(G[N_G(v)])}$.

Proof: We mimic a proof from [1]. For every vertex $v \in V(G)$ let the set $X_v \subseteq N_G(v)$ be such that $|X_v| = d_G(v) - \alpha_{K_r}(G[N_G(v)])$ and $G[N_G(v) \setminus X_v]$ does not contain K_r as a subgraph. Let $v_1, v_2, ..., v_n$ be a permutation of the vertices of G chosen uniformly at random and let $v_i \in S$ if and only if $X_{v_i} \cap \{v_1, v_2, ..., v_i\} = \emptyset$, i.e. v_i is the first vertex of $\{v_i\} \cup X_{v_i}$ that appears within the permutation. Clearly, $G[S]$ does not contain K_{r+1} as a subgraph and

$$\alpha_{K_{r+1}}(G) \geq \mathbf{E}(|S|) = \sum_{v \in V(G)} \mathbf{P}(v \in S) = \sum_{v \in V(G)} \frac{1}{1+d_G(v)-\alpha_{K_r}(G[N_G(v)])}. \square$$

The next result relies on methods proposed in [7].

Theorem 5 *If G is a graph with vertex set $\{v_1, v_2, ..., v_n\}$ and $r \in \mathbb{N}$, then*

$$\alpha_{K_{1,r}}(G) = \max \sum_{v_i \in V(G)} p_i \sum_{Y: Y \subseteq N_G(v_i), |Y| < r} \left(\prod_{v_j \in Y} p_j \prod_{v_k \in N_G(v_i) \setminus Y} (1 - p_j) \right),$$

where the maximum is taken over all $(p_1, p_2, ...p_n) \in [0,1]^n$.

Proof: Let $p_i \in [0,1]$ for $1 \leq i \leq n$. We consider a random subset X of $V(G)$ formed by choosing every vertex v_i independently with probability p_i. If $S = \{v \in X \mid d_{G[X]}(v) < r\}$, then

$$\alpha_{K_{1,r}}(G) \quad \geq \quad \mathbf{E}(S) = \sum_{v_i \in V(G)} p_i \sum_{Y: Y \subseteq N_G(v_i), |Y| < r} \left(\prod_{v_j \in Y} p_j \prod_{v_k \in N_G(v_i) \setminus Y} (1 - p_j) \right).$$

Conversely, if $S \subseteq$ is such that $\alpha_{K_{1,r}}(G) = |S|$ and $G[S]$ has maximum degree less than r, then setting $p_i^* = 1$ for all $v_i \in S$ and $p_i^* = 0$ for all $v_i \not\in S$ yields

$$\alpha_{K_{1,r}}(G) = \mathbf{E}(S) = \sum_{v_i \in V(G)} p_i^* \sum_{Y: Y \subseteq N_G(v_i), |Y| < r} \left(\prod_{v_j \in Y} p_j^* \prod_{v_k \in N_G(v_i) \setminus Y} (1 - p_j^*) \right)$$

which completes the proof. $\square$

It is trivial that for several specific choices of $\mathcal{M}$ and $\mathcal{F}$ the decision problems associated with $f(G, \mathcal{M})$, $\alpha_{\mathcal{F}}(G)$ and $\alpha^*_{\mathcal{F}}(G)$ are NP-complete. In view of Mihok's original problem, we consider the case that $\mathcal{F}$ consists of just one graph in more detail.

Theorem 6 *If F is a graph containing at least one edge, then the following problems are NP-complete problem.*

(i) For a given graph G and $k \in \mathbb{N}$, decide whether $\alpha_F(G) \geq k$.

(ii) For a given graph G and $k \in \mathbb{N}$, decide whether $\alpha^*_F(G) \geq k$.

Proof: Let uv be an arbitrary edge of F. For a graph G let the graph G' arise as follows: For every edge xy of G add a copy F_{xy} of F and identify the copy of the edge uv in F_{xy} with xy (in any orientation).

It is obvious that for every set $T \subseteq V(G')$ of minimum cardinality such that $G'[V(G') \setminus T]$ does not contain F as a subgraph (or induced subgraph), T must intersect every copy F_{xy} of F in G'. Since deleting either x or y from F_{xy} clearly deletes this copy of F, we can assume that $T \subseteq V(G)$ and that $T \cap \{x, y\} \neq \emptyset$ for all $xy \in E(G)$. Hence T is exactly a vertex cover of G. This implies $\alpha(G) = \alpha_F(G') = \alpha^*_F(G')$ and the desired statement follows from the NP-completeness of the corresponding decision problem for the independence number [6]. □

Note that in view Proposition 1(vii), the decision problem "$\alpha^*_{\mathcal{F}}(G) \geq k$?" remains NP-complete even if F is edge-less.

Tuza [10] observed the following nice relation between the independence number and the domination number $\gamma(G)$ of a graph G [7]:

$$\alpha(G) = \max\{\gamma(H) \mid H \text{ is an induced subgraph of } G\}.$$

We close with a generalization of this equality. For a set $\mathcal{F}$ of graphs and a graph G let $\gamma_{\mathcal{F}}(G)$ ($\gamma^*_{\mathcal{F}}(G)$) denote the minimum cardinality $|D|$ of a set $D \subseteq V(G)$ such that for every vertex $u \in V(G) \setminus D$ there is a graph $F \in \mathcal{F}$ and a set $D' \subseteq D$ with $|D'| = |V(F)| - 1$ such that $G[D' \cup \{u\}]$ contains a graph in $\mathcal{F}$ as a(n induced) subgraph.

Theorem 7 *If $\mathcal{F}$ is a set of graphs and let G is a graph G, then*

$$\begin{aligned} \alpha_{\mathcal{F}}(G) &= \max\{\gamma_{\mathcal{F}}(H) \mid H \textit{ is an induced subgraph of } G\} \\ \alpha^*_{\mathcal{F}}(G) &= \max\{\gamma^*_{\mathcal{F}}(H) \mid H \textit{ is an induced subgraph of } G\}. \end{aligned}$$

Proof: We only prove the first equality and leave the very similar proof of the second equality to the reader.

If $S \subseteq V(G)$ is such that $|S| = \alpha_{\mathcal{F}}(G)$ and $G[S]$ does not contain a graph in $\mathcal{F}$ as a subgraph, then $\gamma_{\mathcal{F}}(G[S]) = |S| \geq \alpha_{\mathcal{F}}(G)$.

Conversely, if $G[S]$ is an induced subgraph of G for which $\gamma_{\mathcal{F}}(G[S])$ is maximum, then let $S' \subseteq S$ be of maximum cardinality such that $G[S']$ does not contain a graph in $\mathcal{F}$ as a subgraph. We obtain $\gamma_{\mathcal{F}}(G[S]) \leq |S'| = \alpha_{\mathcal{F}}(G[S]) \leq \alpha_{\mathcal{F}}(G)$ and the proof is complete. □

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