Cyclic Sums, Network Sharing and Restricted Edge Cuts in Graphs with Long Cycles

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Abstract

We study graphs $G = (V, E)$ containing a long cycle which for given integers a_1, $a_2, ..., a_k \in \mathbb{N}$ have an edge cut whose removal results in k components with vertex sets $V_1, V_2, ..., V_k$ such that $|V_i| \geq a_i$ for $1 \leq i \leq k$. Our results closely relate to problems and recent research in network sharing and network reliability.

Keywords: restricted edge connectivity; arbitrarily vertex decomposable graph; network reliability; network sharing

2000 Mathematics Subject Classification: 05A17; 05C40

1 Introduction

The problem we study in the present paper receives motivation from at least two sources: *network sharing* and *network reliability*.

For a given graph $G = (V, E)$ of order n one of the problems considered in the context of network sharing is whether for every $k \in \mathbb{N}$ and every choice of integers $a_1, a_2, ..., a_k \in \mathbb{N}$ with $n = a_1 + a_2 + ... + a_k$, the vertex set V of G can be partitioned into k sets $V = V_1 \cup V_2 \cup ... \cup V_k$ such that $|V_i| = a_i$ and the subgraph $G[V_i]$ induced in G by the set V_i is connected for all $1 \leq i \leq k$. Graphs having this property were called *arbitrarily vertex decomposable* ($\mathcal{AVD}$).

Trees which are $\mathcal{AVD}$ have been studied in some detail. No tree of maximum degree at least five is $\mathcal{AVD}$ [2, 10] and while it is NP-complete to decide the $\mathcal{AVD}$ property for general graphs (cf. [1]), the $\mathcal{AVD}$ trees homeomorphic to $K_{1,3}$ or $K_{1,4}$ can be recognized in polynomial time [1, 2]. Since graphs with a Hamiltonian path are clearly $\mathcal{AVD}$, Ore type conditions implying a graphs to be $\mathcal{AVD}$ have been studied [13]. $\mathcal{AVD}$ graphs in which almost all vertices lie in a unique and dominating cycle were studied in [4, 11].

The second source of motivation is related to the notion of restricted egde connectivity which was proposed as a natural measure of network fault-tolerance or reliability [5, 6, 8]. The central problem considered in this context for a given connected graph $G = (V, E)$ and some integer $a \in \mathbb{N}$ concerns the existence and minimum cardinality of edge cuts $S \subseteq E$ whose removal from G results in a graph $G - S = (V, E \setminus S)$ all components of which are of order at least a. If such a cut S exists the corresponding graph is called λ_a*-connected* and if $|S|$ is small the corresponding network can be considered vulnerable because the removal of few edge can separate large parts. λ_a-connected graphs and the sizes of the corresponding edge cuts have received notable attention [3, 9, 14, 15, 16, 17].

Being $\mathcal{AVD}$ is clearly an extremely restrictive property. A main reason for this is that the number of parts k in the desired partitions is arbitrary. Therefore, it seems a natural idea to study graphs which are arbitrarily vertex decomposable into a bounded number of parts which corresponds to sharing a network among a bounded number of parties.

For a minimal edge cut S whose removal from a connected graph G results in a graph all components of which are at least of some given order, the graph $G-S$ will always have exactly two components. Here it seems natural to consider the existence and minimum cardinality of edges cuts whose removal creates a given number of components which are all at least of some given order. Graphs which have such a cut of small cardinality can easily be split into many large parts.

These last two observations motivate to study graphs $G=(V,E)$ which for given integers $a_1, a_2, ..., a_k \in \mathbb{N}$ have an edge cut S whose removal results in k components with vertex sets $V_1, V_2, ..., V_k$ such that $|V_i| \geq a_i$ for $1 \leq i \leq k$. There are beautiful theorems due to Győri [7] and Lovász [12] which imply that k-connectivity forces the existence of such an edge cut provided the obvious necessary condition that the order of G is at least $a_1 + a_2 + ... + a_k$. We call graphs which have such an edge cut $\lambda_{a_1,a_2,...,a_k}$-connected and study conditions which imply this property for graphs which contain a long cycle. The structure of these graphs is similar to the graphs studied in [4, 11]. Our main tools are results about cyclic sums (Theorems 2.1 and 2.5) which we feel to be interesting on their own right.

2 Results

In our first result we consider the following question: *Given n positive integers arranged in a cycle; which values can we realize as the sum of cyclically consecutive integers?* We give a best-possible condition implying that all values between 1 and the sum of all integers are realizable up to some specified error as such a cyclic sum.

Theorem 2.1 *Let $p \in \mathbb{N}$, $r \in \mathbb{N}_0$ and $x_0, x_1, ..., x_{p-1} \in \mathbb{N}$. For $y \in \mathbb{N}$ let $N_y = \{i \mid 0 \leq i \leq p-1, x_i = y\}$ and $n_y = |N_y|$.*

If

$$\sum_{y \leq r+1} y n_y \geq 1 + \sum_{y \geq r+2} (y-r-2) n_y,$$

then for all $X \in \{1, 2, ..., x_0 + x_1 + ... + x_{p-1}\}$ there are indices $0 \leq i, j \leq p-1$ such that

$$X \leq x_i + x_{i+1} + ... + x_{i+j} \leq X + r,$$

where the indices of the x_i's are taken modulo p.

Proof: We call a term of the form $x_i + x_{i+1} + ... + x_{i+j}$ a cyclic sum. Since $\sum_{y \le r+1} y n_y \ge 1$, some integer between 1 and $1 + r$ is a cyclic sum.

Now let $X \in \{2 + r, 3 + r, ..., x_0 + x_1 + ... + x_{p-1}\}$. We will prove that some integer between X and $X + r$ is a cyclic sum. For every $i \in \bigcup_{y \le r+1} N_y$ let $f(i) \in \{0, 1, ..., p - 1\}$ be such that

$$\begin{aligned} x_i + x_{i+1} + ... + x_{f(i)-1} &\le X - 1 \\ \text{and } x_i + x_{i+1} + ... + x_{f(i)} &\ge X. \end{aligned}$$

Clearly, $f(i)$ is well-defined for every $i \in \bigcup_{y \le r+1} N_y$.

If $x_i + x_{i+1} + ... + x_{f(i)} \le X + r$, then it is a cyclic sum between X and $X + r$. Hence we may assume that $x_i + x_{i+1} + ... + x_{f(i)} \ge X + r + 1$ which implies that

$$\begin{aligned} x_{f(i)} &= \big(x_i + x_{i+1} + ... + x_{f(i)}\big) - \big(x_i + x_{i+1} + ... + x_{f(i)-1}\big) \\ &\ge (X + r + 1) - (X - 1) = r + 2 \end{aligned}$$

and hence $f(i) \in \bigcup_{y \ge r+2} N_y$ for every $i \in \bigcup_{y \le r+1} N_y$, i.e.

$$f : \bigcup_{y \le r+1} N_y \to \bigcup_{y \ge r+2} N_y.$$

If there are $i_1, i_2, ..., i_q \in \bigcup_{y \le r+1} N_y$ and $j \in N_z$ for some $z \ge r + 2$ with cyclic order $i_1, i_2, ..., i_q, j$ and $f(i_1) = f(i_2) = ... = f(i_q) = j$, then

$$\begin{aligned} X &\le (X + r + 1) - x_{i_q} \\ &\le (x_{i_q} + x_{i_q+1} + ... + x_j) - x_{i_q} \\ &= x_{i_q+1} + x_{i_q+2} + ... + x_j \\ &\le (x_{i_1} + x_{i_1+1} + ... + x_{i_q} + x_{i_q+1} + ... + x_j) - (x_{i_1} + x_{i_2} + ... + x_{i_q}) \\ &= (x_{i_1} + x_{i_1+1} + ... + x_{j-1}) + z - (x_{i_1} + x_{i_2} + ... + x_{i_q}) \\ &\le (X - 1) + z - (x_{i_1} + x_{i_2} + ... + x_{i_q}). \end{aligned}$$

If $x_{i_1} + x_{i_2} + ... + x_{i_q} \ge z - r - 1$, then $x_{i_q+1} + x_{i_q+2} + ... + x_j$ is a cylic sum between X and $X + r$. Hence $x_{i_1} + x_{i_2} + ... + x_{i_q} \le z - r - 2$, i.e. for every $j \in N_z$ with $z \ge r + 2$ the sum of the x_i over the preimages i of j under f is at most $z - r - 2$. This implies the contradiction

$$\sum_{y \le r+1} y n_y \le \sum_{y \ge r+2} (y - r - 2) n_y$$

and the proof is complete. □

The choice $x_0 = x_1 = ... = x_{p-1} = r+2$ clearly implies that the condition given in Theorem 2.1 is best-possible.

If we want all possible values to be realized exactly as a cyclic sum, the condition from Theorem 2.1 can actually be simplified as follows.

Corollary 2.2 *If $p, x_0, x_1, ..., x_{p-1} \in \mathbb{N}$ and*

$$x_0 + x_1 + ... + x_{p-1} \leq 2p - 1,$$

then for all $X \in \{1, 2, ..., x_0 + x_1 + ... + x_{p-1}\}$ there are indices $0 \leq i, j \leq p - 1$ such that

$$X = x_i + x_{i+1} + ... + x_{i+j},$$

where the indices of the x_i's are taken modulo p.

Proof: For $y \in \mathbb{N}$ let $N_y = \{i \mid 0 \leq i \leq p-1, x_i = y\}$ and $n_y = |N_y|$. The condition $x_0 + x_1 + ... + x_{p-1} \leq 2p - 1$ is easily seen to be equivalent to the condition $n_1 \geq 1 + \sum_{y \geq 2}(y-2)n_y$ and the result follows from Theorem 2.1 for $r = 0$. $\square$

From Theorem 2.1 we can derive a sufficient condition for a graph of large enough order containing a cycle long enough to be $\lambda_{a,b}$-connected. Note that graphs corresponding to the example given immediately after the proof of Theorem 2.1 show that the following result is best-possible.

Corollary 2.3 *Let $a, b, p \in \mathbb{N}$ and $r \in \mathbb{N}_0$ with $p \geq 3$ and $a \leq b$. Let $G = (V, E)$ be a connected graph of order $n \geq a + b + r$ which contains a cycle C of length p. Let $G - E(C)$ contain exactly n_i components of order i for $i \in \mathbb{N}$.*

If $\sum_{y \leq r+1} yn_y \geq 1 + \sum_{y \geq r+2}(y - r - 2)n_y$, then G is $\lambda_{a,b}$-connected.

Proof: By Theorem 2.1, the graph G is $\lambda_{a',n-a'}$-connected for some $a \leq a' \leq a + r$. Since $n - a' \geq n - a - r \geq b$, the desired result follows. $\square$

Similarly, we can derive a graph-theoretic consequence from Corollary 2.2.

Corollary 2.4 *Let $a, b, p \in \mathbb{N}$ with $p \geq 3$ and $a + b \leq 2p - 1$. If $G = (V, E)$ is a connected graph of order $n \geq a + b$ which contains a cycle of order p, then G is $\lambda_{a,b}$-connected.*

Proof: Clearly, the graph G has a spanning subgraph G' with a unique cycle C of order p. If $p > a + b$, then G is obviously $\lambda_{a,b}$-connected. Hence we may assume that $p \leq a + b$. By iteratively deleting endvertices from G', we obtain a connected subgraph G'' of order exactly $a + b$ which contains C. Corollary 2.2 implies that G'' is $\lambda_{a,b}$-connected. Therefore, also G is $\lambda_{a,b}$-connected. $\square$

Now we consider the problem to split a graph with a long cycle into more than two large parts. As before, the main tool is a result about cyclic sums. While Theorem 2.1 was best-possible, we were not able to obtain a similarly strong result in this situation.

Theorem 2.5 *Let $k, p \in \mathbb{N}$, $r \in \mathbb{N}_0$ and $x_0, x_1, ..., x_{p-1} \in \mathbb{N}$. For $y \in \mathbb{N}$ let $N_y = \{i \mid 0 \leq i \leq p-1, x_i = y\}$ and $n_y = |N_y|$.*

If

$$\sum_{y\le r+1} yn_y \ge 1 + k\sum_{y\ge r+2}(y-1)n_y,$$

then for all $S_1, S_2, ..., S_k \in \mathbb{N}$ *with*

$$1 \le S_1 < S_2 < ... < S_k \le x_0 + x_1 + ... + x_{p-1}$$

there exist indices $0 \le i_0, i_1, i_2, ..., i_k \le p-1$ *such that*

$$S_j \le x_{i_0} + x_{i_0+1} + ... + x_{i_0+i_j} \le S_j + r$$

for all $1 \le j \le k$*, where the indices of the* x_i*'s are taken modulo* p*.*

Proof: Let $k, p, x_0, x_1, ..., x_{p-1}, N_y, n_y$ be as in the statement of the result. Furthermore, let

$$\sum_{y\le r+1} yn_y \ge 1 + k\sum_{y\ge r+2}(y-1)n_y.$$

Let $S_1, S_2, ..., S_k \in \mathbb{N}$ be such that $1 \le S_1 < S_2 < ... < S_k \le x_0 + x_1 + ... + x_{p-1}$.

For contradiction, we assume that indices $0 \le i_0, i_1, i_2, ..., i_k \le p-1$ with

$$S_j \le x_{i_0} + x_{i_0+1} + ... + x_{i_0+i_j} \le S_j + r$$

for all $1 \le j \le k$ do not exist. For every $i \in \bigcup_{y\le r+1} N_y$ let $l(i) \in \{1, 2, ..., k\}$ be minimum such that there is no index $0 \le j \le p-1$ with

$$S_{l(i)} \le x_i + x_{i+1} + ... + x_{i+j} \le S_{l(i)} + r.$$

Furthermore, let $f(i) \in \{0, 1, ..., p-1\}$ be such that

$$\begin{aligned} x_i + x_{i+1} + ... + x_{f(i)-1} &\le S_{l(i)} - 1 \\ \text{and } x_i + x_{i+1} + ... + x_{f(i)} &\ge S_{l(i)}. \end{aligned}$$

Clearly, $l(i)$ and $f(i)$ are well-defined for every $i \in \bigcup_{y\le r+1} N_y$ and $x_i + x_{i+1} + ... + x_{f(i)} \ge S_{l(i)} + r + 1$ which implies that $f(i) \in \bigcup_{y\ge r+2} N_y$.

If there are $i_1, i_2, ..., i_q \in N_1$, $l \in \{1, 2, ..., k\}$ and $j \in N_z$ for some $z \ge 2$ with cyclic order $i_1, i_2, ..., i_q, j$, $l(i_1) = l(i_2) = ... = l(i_q) = l$ and $f(i_1) = f(i_2) = ... = f(i_q) = j$, then

$$\begin{aligned} S_l &\le (S_l + r + 1) - x_{i_q} \\ &\le (x_{i_q} + x_{i_q+1} + ... + x_j) - x_{i_q} \\ &= x_{i_q+1} + x_{i_q+2} + ... + x_j \\ &\le (x_{i_1} + x_{i_1+1} + ... + x_{i_q} + x_{i_q+1} + ... + x_j) - (x_{i_1} + x_{i_2} + ... + x_{i_q}) \\ &= (x_{i_1} + x_{i_1+1} + ... + x_{j-1}) + z - (x_{i_1} + x_{i_2} + ... + x_{i_q}) \\ &\le (S_l - 1) + z - (x_{i_1} + x_{i_2} + ... + x_{i_q}) \end{aligned}$$

which implies $(x_{i_1} + x_{i_2} + ... + x_{i_q}) \leq z - 1$. (*Note that we cannot conclude an upper bound of $z - r - 2$ as in the proof of Theorem 2.1 because $x_{i_q+1} + x_{i_q+2} + ... + x_j \leq X + r$ would not imply a contradiction.*)

Therefore for every tupel $(l, j) \in \{1, 2, ..., k\} \times N_z$ for some $z \geq 2$ the sum of the x_i over all i with $(l(i), f(i)) = (l, j)$ is at most $z - 1$. This implies the contradiction

$$\sum_{y \leq r+1} y n_y \leq k \sum_{y \geq r+2} (y - 1) n_y$$

and the proof is complete. □

Again, we derive a result about exact realizations.

Corollary 2.6 *Let $k, p \in \mathbb{N}$ and $x_0, x_1, ..., x_{p-1} \in \mathbb{N}$.*

If

$$x_0 + x_1 + ... + x_{p-1} < \frac{k+2}{k+1} p,$$

then for all $S_1, S_2, ..., S_k \in \mathbb{N}$ with

$$1 \leq S_1 < S_2 < ... < S_k \leq x_0 + x_1 + ... + x_{p-1}$$

there exist indices $0 \leq i_0, i_1, i_2, ..., i_k \leq p - 1$ such that

$$S_j = x_{i_0} + x_{i_0+1} + ... + x_{i_0+i_j}$$

for all $1 \leq j \leq k$, where the indices of the x_i's are taken modulo p.

Proof: Since the average value of the x_i is less than $\frac{k+2}{k+1}$, there are more than $(k+1)y - (k+2)$ different x_i's equal to 1 for every x_j equal to $y \geq 2$. Since $(k+1)y - (k+2) \geq k(y-1)$ for $y \geq 2$, the result follows from Theorem 2.5 for $r = 0$. □

We close with a corollary for graphs containing a long cycle.

Corollary 2.7 *Let $k, p, a_1, a_2, ..., a_k \in \mathbb{N}$ with $k, p \geq 2$ and $a_1 + a_2 + ... + a_k < \frac{k+2}{k+1} p$. If $G = (V, E)$ is a connected graph of order $n \geq a_1 + a_2 + ... + a_k$ which contains a cycle of order p, then G is $\lambda_{a_1, a_2, ..., a_k}$-connected.*

Numerous questions motivated by our results are obvious and we just pose two: *What about $\lambda_{a_1, a_2, ..., a_k}$-connected graphs which are neither highly connected nor have long cycles or other nicely structures subgraphs along which the desired components can be cut? What is a best-possible version of Theorem 2.5?*

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