Domination in Cubic Graphs of Large Girth

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Abstract We prove that connected cubic graphs of order n and girth g have domination number at most $0.32127n + O\left(\frac{n}{g}\right)$.

Keywords domination number; minimum degree; girth; cubic graph

The domination number $\gamma(G)$ of a (finite, undirected and simple) graph $G = (V, E)$ is one of the most well-studied graph parameters [4] and is defined as the minimum cardinality of a set $D \subseteq V$ of vertices such that every vertex in $V \setminus D$ has a neighbour in D.

Initially motivated by Reed's [10] disproved [6] conjecture that every connected cubic graph of order n has domination number at most $\lceil \frac{n}{3} \rceil$, several authors recently studied the domination number of cubic graphs of large girth where the girth is the length of a shortest cycle in G.

Kawarabayashi, Plummer and Saito [5] proved $\gamma(G) \leq \left(\frac{1}{3} + \frac{1}{9k+3}\right) n$ for every 2-edge connected cubic graph G of order n and girth at least $3k$ for some $k \in \mathbb{N}$ and Kostochka and Stodolsky [7] proved $\gamma(G) \leq \left(\frac{1}{3} + \frac{8}{3g^2}\right) n$ for every connected cubic graph G of order $n > 8$ and girth g. While these two bounds tend to $n/3$ for $g \to \infty$, Löwenstein and Rautenbach [8] recently showed that one actually gets below $n/3$ for sufficiently large girth by proving $\gamma(G) \leq \left(\frac{44}{135} + \frac{82}{135g}\right) n \approx 0.3259n + O\left(\frac{n}{g}\right)$ for every cubic graph of order n and girth $g \geq 5$. In the present paper we will slightly improve the constant in this upper bound.

Since for fixed d and g the numbers of cycles in random cubic graphs of fixed lengths $r < g$ are asymptotically distributed as independent Poisson variables [2] with mean $(d-1)^r/2r$, the number of cycles in a random cubic graph of length smaller than g is asymptotically almost surely bounded. Therefore, the asymptotic bounds for the domination number of random cubic graphs carry over to cubic graph of large girth and indicate how much more the above constant could be improved. Molloy and Reed [9] proved that the domination number γ of a random cubic graph of order n asymptotically almost surely satisfies $0.2636n \leq \gamma \leq 0.3126n$ and Duckworth and Wormald [3] improved the upper bound to $0.2794n$.

We immediately proceed to our main result.

Theorem 1 *If $G = (V, E)$ is a connected, cubic graph of order n, girth g and domination number γ, then*

$$\gamma \leq 0.32127n + O\left(\frac{n}{g}\right).$$

Proof: Let the graph $G = (V, E)$ be as in the statement of the theorem.

Clearly, we may assume $g \geq 7$. We will first prove the existence of a matching that covers all but $O\left(\frac{n}{g}\right)$ vertices. Therefore, for some set $S \subseteq V$ let $q_1(S)$ denote the number of components of $G[V \setminus S]$ of odd order that are joined to S by only one edge and let $q_2(S)$ denote the number of components of $G[V \setminus S]$ of odd order that are joined to S by at least 2 edges. Since G is cubic, all of the $q_2(G)$ odd components are joined to S by at least 3 edges and we have $q_2(G) \leq |S|$. Furthermore, every component of $G[V \setminus S]$ of odd order that is joined S by only one edge must contain a cycle which implies that it has order at least g and $q_1(S) \leq \frac{n}{g}$. Altogether, we obtain $q_1(S) + q_2(S) - |S| \leq \frac{n}{g}$ and by the Tutte-Berge formula, there is a matching M in G such that the set D_0 of vertices not incident to an edge in M satisfies $|D_0| \leq \frac{n}{g}$.

Since there are at most $3|D_0| \leq \frac{3n}{g}$ edges between D_0 and $V \setminus D_0$, the graph $G[V \setminus D_0] - M$ consists of cycles and at most $\frac{3n}{2g}$ paths. Since all the cycles are of length at least g, we can decompose them into paths of lengths between $\frac{g}{4}$ and $\frac{g}{4} + \frac{g}{3 \cdot 4} = \frac{g}{3}$. This implies that there is a collection of vertex disjoint paths $\mathcal{P}$ with $|\mathcal{P}| \leq \frac{n}{g/4} + \frac{3n}{2g} = \frac{11n}{2g}$ which are all of lengths at most $\frac{g}{3}$ and contain all vertices in $V \setminus D_0$.

Since $\frac{g}{3} < \frac{g-2}{2}$, these paths are induced and no two of these paths are joined by more than one edge.

Let H denote the graph with vertex set $V \setminus D_0$ whose edges are the edges of the paths in $\mathcal{P}$ together with the matching M. Note that M is a perfect matching of H. We will describe a probabilistic procedure for constructing a small dominating set D_1 of H in five phases.

Phase 1

We select a random subset $\mathcal{P}_0$ of $\mathcal{P}$ by assigning each $P \in \mathcal{P}$ to $\mathcal{P}_0$ independently at random with probability p for some $0 \leq p \leq 1$ to be specified later. Let $\mathcal{P}_1 = \mathcal{P} \setminus \mathcal{P}_0$.

Phase 2

For every path $P : x_1x_2...x_l \in \mathcal{P}_0$ we choose independently at random a parity $j \in \{0, 1, 2\}$ each with probability $\frac{1}{3}$ and set

$$D(P) = \{x_i \mid 1 \leq i \leq l, i \equiv j \pmod 3\}.$$

Phase 3

For every path $P : x_1x_2...x_l \in \mathcal{P}_1$ we will determine a set $D(P) \subseteq \{x_1, x_2, ..., x_l\}$ by the following procedure:

(1) We set $D(P) := \emptyset$ and $i := 1$.

(2) We query whether x_i is adjacent in H to a vertex in $\bigcup_{Q \in \mathcal{P}_0} D(Q)$.

- If the answer to the query is 'yes' and $i \leq l-1$, then we set $i := i+1$ and go to (2).
- If the answer to the query is 'yes' and $i = l$, then we terminate.
- If the answer to the query is 'no' and $i \leq l-3$, then we set $D(P) := D(P) \cup \{x_i\}$, $i := i+3$ and go to (2).
- If the answer to the query is 'no' and $i \geq l-2$, then we set $D(P) := D(P) \cup \left\{x_{\min\{i+1,l\}}\right\}$ and terminate.

Phase 4

For every edge $uv \in M$ such that $u \in D(P)$, $v \in D(Q)$ with $P, Q \in \mathcal{P}_0$ we delete at most one vertex, say u, from $D(P) \cup D(Q)$, if the neighbour(s) of u on P are adjacent in H to a vertex in $\bigcup_{R \in \mathcal{P}_0} D(R) \setminus D(P)$. (Note that in H the vertex u has no neighbour on a path in $\mathcal{P}_1$.)

Phase 5

Let D' denote the set of endvertices of the paths in $\mathcal{P}_0$ and let $D_1 = D' \cup \bigcup_{P \in \mathcal{P}} D(P)$. This terminates the last phase.

It is obvious from the construction that the set D_1 is a dominating set of H. We will now estimate the expected value of $|D_1|$. Therefore, let $n_1 = n - |D_0|$.

The expected number of vertices added to $\bigcup_{P \in \mathcal{P}_0} D(P)$ in the Phase 2 is $\frac{pn_1}{3}$, because the probability that a path in $\mathcal{P}$ is in $\mathcal{P}_0$ is p and subject to this the probability of a vertex on P to belong to $D(P)$ is $\frac{1}{3}$.

Now we proceed to the Phase 3. For some path $P : x_1 x_2 ... x_l$ in $\mathcal{P}_1$ let q denote the total number of queries and let q_y denote the number of queries with answer 'yes' during the construction of $D(P)$. Note that while the answers to previous queries influence for which vertex we ask the next query, the answer to every query is independent of the answers to previous queries, because no two paths are joined by more than one edge and the random choices for different paths are independent.

If we query the corresponding adjacency for some vertex, then the path Q containing its neighbour outside of P lies in $\mathcal{P}_0$ with probability p and subject to this the neighbour lies in $D(Q)$ with probability $1/3$. Therefore, the probability of a positive answer to an individual query is $p/3$ and the expected values for q_y and q satisfy $E[q_y] = \frac{pE[q]}{3}$.

Since with the exception of queries within distance $O(1)$ of the end of P, for every positive anser to a query the index i is incremented by 1 and for every negative anser to a query the index i is incremented by 3, we have $q = \frac{l+2q_y}{3} + O(1)$. Therefore, $E[q] = \frac{l+2E[q_y]}{3} + O(1)$ which together with $E[q_y] = \frac{pE[q]}{3}$ implies that $E[q_y] = \frac{p}{9-2p} l + O(1)$.

Since for every query with a negative answer, one vertex is added to $D(P)$ we have

$$E[|D(P)|] = E[q - q_y] = E[q] - E[q_y] = (1 - p/3)E[q_y] = \frac{3-p}{9-2p} l + O(1).$$

Finally, since every of the $O\left(\frac{n}{g}\right)$ paths in $\mathcal{P}$ belongs to $\mathcal{P}_1$ with probability $(1-p)$, we obtain, by linearity of expectation, that

$$E\left[\left|\bigcup_{P\in\mathcal{P}_1} D(P)\right|\right] = \frac{3-p}{9-2p}(1-p)n_1 + O\left(\frac{n}{g}\right) = \frac{(1-p)n_1}{3} + \frac{p(p-1)n_1}{3(9-2p)} + O\left(\frac{n}{g}\right).$$

We proceed to Phase 4. The expected number of edges among the total $\frac{n_1}{2}$ edges in M which join two paths in $\mathcal{P}_0$ and for which we remove one vertex from $\bigcup_{P\in\mathcal{P}_0} D(P)$ in the fourth phase equals

$$\frac{n_1}{2} p^2 \frac{1}{9}\left(2\left(\frac{p}{3}\right)^2 - \left(\frac{p}{3}\right)^4\right).$$

(The two paths containing the endpoints of an edge uv in M lie in $\mathcal{P}_0$ with probability p^2 and the two vertices u and v lie in $\bigcup_{P\in\mathcal{P}_0} D(P)$ as constructed in the second phase with probability $\frac{1}{9}$. The term $\left(2\left(\frac{p}{3}\right)^2 - \left(\frac{p}{3}\right)^4\right)$ is the probability that the neighbour(s) of u and v are still adjacent in H to a vertex in $\bigcup_{R\in\mathcal{P}_0} D(R)$ after removing either u or v.)

Putting everything together, we obtain

$$\begin{aligned} E[|D_1|] &= \frac{pn_1}{3} + \frac{(1-p)n_1}{3} + \frac{p(p-1)n_1}{3(9-2p)} + O\left(\frac{n}{g}\right) - \frac{p^2 n_1}{18}\left(2\left(\frac{p}{3}\right)^2 - \left(\frac{p}{3}\right)^4\right) \\ &= \frac{n_1}{3} - n_1\left(\frac{p(1-p)}{3(9-2p)} + \frac{p^2}{18}\left(2\left(\frac{p}{3}\right)^2 - \left(\frac{p}{3}\right)^4\right)\right) + O\left(\frac{n}{g}\right). \end{aligned}$$

Over the interval $[0,1]$ the function

$$f(p) = \frac{p(1-p)}{3(9-2p)} + \frac{p^2}{18}\left(2\left(\frac{p}{3}\right)^2 - \left(\frac{p}{3}\right)^4\right)$$

has maximum value $f(0.74379) > 0.012117$. Since $D_0 \cup D_1$ is a dominating set of G, we obtain $\gamma \leq |D_0| + (n - |D_0|)\left(\frac{1}{3} - 0.012117\right) + O\left(\frac{n}{g}\right) = 0.32127n + O\left(\frac{n}{g}\right)$ and the proof is complete. $\Box$

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