

# Domination in Bipartite Graphs

Jochen Harant and Dieter Rautenbach

Institut für Mathematik, TU Ilmenau, Postfach 100565, D-98684 Ilmenau,  
Germany, email: {jochen.harant,dieter.rautenbach}@tu-ilmenau.de

**Abstract** We prove that the domination number of a graph of order  $n$  and minimum degree at least 2 that does not contain cycles of lengths 4, 5, 7, 10 or 13 is at most  $\frac{3}{8}n$ . Furthermore, we derive upper bounds on the domination number of bipartite graphs of given minimum degree.

**Keywords** domination number; cycle length; bipartite; probabilistic method

## 1 Introduction

The domination number  $\gamma(G)$  of a (finite, undirected and simple) graph  $G = (V, E)$  is the minimum cardinality of a set  $D \subseteq V$  of vertices such that every vertex in  $V \setminus D$  has a neighbour in  $D$ . This parameter is one of the most well-studied in graph theory and the two volume monograph [9, 10] provides an impressive account of the research related to this concept.

Fundamental results about the domination number  $\gamma(G)$  are upper bounds in terms of the order  $n$  and the minimum degree  $\delta$  of the graph  $G$ . Ore [14] proved that  $\gamma(G) \leq \frac{n}{2}$  provided  $\delta \geq 1$ . For  $\delta \geq 2$  and all but 7 exceptional graphs Blank [3] and McCuaig and Shepherd [13] proved

$$\gamma(G) \leq \frac{2n}{5}. \quad (1)$$

In [17] Reed proved that  $\gamma(G) \leq \frac{3}{8}n$  for  $\delta \geq 3$ .

Bounds which are interesting for large minimum degree were obtained by Alon and Spencer [1], Aronov [2] and Payan [15] who proved (see also Caro and Roditty [5, 6])

$$\gamma(G) \leq \left( \frac{1 + \ln(\delta + 1)}{\delta + 1} \right) n. \quad (2)$$

While all these bounds hold without restricting the structure of the graph, there are several partly quite recent results [4, 11, 12, 16, 18, 19] that involve conditions on the girth of the graph, i.e. the length of a shortest cycle.

In the present paper we consider the domination number of graphs of given minimum degree under different cycle conditions related to bipartite graphs. We prove a best-possible bound on the domination number of graphs of minimum degree 2 that do not contain cycles of lengths 4, 5, 7, 10 or 13 and bounds on the domination number of bipartite graphs of given minimum degree.

## 2 Results

Graphs as in Figure 1 show that the bound (1) [3, 13] actually remains best-possible for bipartite graphs. Therefore, it makes sense to forbid cycles of length 4. Since we are eventually interested in the domination number of bipartite graphs, we will also forbid some odd cycle lengths. Cycles of length 3 and long odd cycles can be dominated by (roughly) one third of their vertices and do not pose a problem. Therefore, it suffices to forbid some small odd cycle length at least 5. Up to the assumption on cycles of length 10 these comments motivate the hypothesis of the following result.

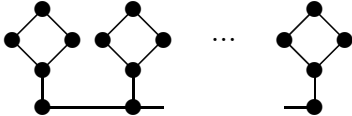


Figure 1

**Theorem 1** *If  $G$  is a graph of order  $n$ , minimum degree at least 2 and domination number  $\gamma$  that does not contain cycles of lengths 4, 5, 7, 10 or 13, then  $\gamma \leq \frac{3}{8}n$ .*

*Proof:* For contradiction, we assume that  $G = (V, E)$  is a counterexample of minimum sum of order  $n$  and size. Let  $n$  and  $\gamma$  be as in the statement of the theorem. Since  $n$  and  $\gamma$  are linear with respect to the components of  $G$ , the graph  $G$  is connected. Furthermore, the set of vertices of degree at least 3 is independent.

It is easy to check the theorem for cycles and hence we can assume that  $G$  has at least one vertex of degree at least 3.

A path between vertices of degree at least 3 with  $a$  internal vertices which are all of degree 2 is called an  $a$ -path. Similarly, a cycle containing a vertex of degree at least 3 and  $a$  further vertices which are all of degree 2 is called an  $a$ -loop. See Figure 2 for an illustration.

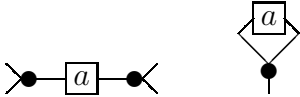


Figure 2

In what follows we will consider several times a set  $V_0 \subseteq V$  of vertices with the property that  $G[V \setminus V_0]$  has no vertex of degree less than 2. Note that  $G[V \setminus V_0]$  satisfies the assumptions of the theorem. We will always use the following notation  $n_0 = |V_0|$ ,  $n_1 = n - n_0$ ,  $G_0 = G[V_0]$ ,  $G_1 = G[V \setminus V_0]$ ,  $\gamma_0 = \gamma(G_0)$  and  $\gamma_1 = \gamma(G_1)$ . Note that  $\gamma \leq \gamma_0 + \gamma_1$  because the union of a dominating set of  $G_0$  and a dominating set of  $G_1$  is a dominating set of  $G$ . Instead of a

dominating set of  $G_0$ , we will sometimes consider a set  $D_0 \subseteq V$  such that every vertex in  $V_0$  is either in  $D_0$  or adjacent to a vertex in  $D_0$ . Clearly,  $\gamma \leq |D_0| + \gamma_1$ .

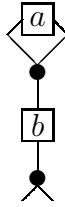
**Claim 1.** *There is no  $a$ -path with  $a \equiv 0 \pmod{3}$ .*

*Proof of Claim 1:* For contradiction, we assume the existence of such an  $a$ -path. If  $V_0$  is the set of internal vertices of the  $a$ -path, then  $\gamma_0 = \frac{a}{3}$ . By the choice of  $G$  as a minimum counterexample, we have  $\gamma_1 \leq \frac{3(n-a)}{8}$  which implies the contradiction  $\gamma \leq \frac{a}{3} + \frac{3(n-a)}{8} \leq \frac{3n}{8}$  and the proof of the claim is complete.  $\square$

**Claim 2.** *There is no  $a$ -loop with  $a \equiv 0 \pmod{3}$ .*

*Proof of Claim 2:* For contradiction, we assume the existence of such an  $a$ -loop containing the vertex  $u$  of degree at least 3.

If the degree of  $u$  is at least 4, we can choose  $V_0$  as the set of the  $a$  vertices of degree 2 of the  $a$ -loop and argue as in the proof of Claim 1. Hence we can assume that the degree of  $u$  is exactly 3 and that there is a  $b$ -path leading to another vertex  $v$  of degree at least 3 (cf. Figure 3).



**Figure 3**

If  $V_0$  consists of the  $a + 1$  vertices of the  $a$ -loop and the  $b$  internal vertices of the  $b$ -path, then  $\gamma_0 \leq \frac{a}{3} + \lceil \frac{b+1}{3} \rceil$ . Since  $G$  does not contain cycles of lengths 4, 7, 10 or 13, we have  $a \geq 15$  and  $b \geq 1$  which implies

$$\gamma_0 \leq \frac{a}{3} + \left\lceil \frac{b+1}{3} \right\rceil \leq \frac{a+b}{3} + 1 \leq \frac{3(a+b)}{8} + \frac{3}{8} = \frac{3n_0}{8}$$

and we obtain a similar contradiction as in the proof of Claim 1.  $\square$

**Claim 3.** *There is no vertex of degree at least 3 that lies on an  $a$ -loop and a  $b$ -path with  $a, b \equiv 1 \pmod{3}$ .*

*Proof of Claim 3:* For contradiction, we assume the existence of such a vertex  $u$ . Let the  $b$ -path lead to the vertex  $v$  of degree at least 3.

If the degree of  $u$  is 3, then let  $V_0$  consist of the  $a + 1$  vertices of the  $a$ -loop and the  $b$  internal vertices of the  $b$ -path (cf. first graph in Figure 4). Clearly,  $n_0 \equiv 0 \pmod{3}$ ,  $\gamma_0 = \frac{n_0}{3}$  and we obtain a similar contradiction as in the proof of Claim 1.

If the degree of  $u$  is at least 5, then let  $V_0$  consist of the  $a$  vertices of degree 2 of the  $a$ -loop and the  $b$  internal vertices of the  $b$ -path (cf. second graph in Figure 4). Now there is a set  $D_0 \subseteq V$  containing  $u$  such that  $|D_0| = \frac{a+b+1}{3}$  and every vertex in  $V_0$  is either in  $D_0$  or adjacent to a vertex in  $D_0$ . Since  $G$  does not contain cycles of lengths 5, we have  $a \geq 7$  and  $b \geq 1$  which implies  $|D_0| \leq \frac{3n_0}{8}$ . Clearly,  $\gamma \leq |D_0| + \gamma_1$  and we obtain a similar contradiction as in the proof of Claim 1.

Hence we can assume that the degree of  $u$  is exactly 4 and that there is a  $c$ -path leading to a vertex  $w$  of degree at least 3. Let  $V_0$  be a minimal set of vertices containing  $u$  for which  $G_1$  has no vertex of degree less than 2.

Either  $v \neq w$  (cf. third graph in Figure 4) or  $v = w$  and  $v$  is of degree at least 4 (cf. fourth graph in Figure 4) or  $v = w$ ,  $v$  is of degree 3 and there is a  $d$ -path leading to a vertex of degree at least 3 (cf. fifth graph in Figure 4). In all these three cases,  $G_0$  has a spanning subgraph that arises by attaching a path with  $b \geq 1$  vertices and another path with  $c' \geq 1$  vertices to one vertex of a cycle of length  $a + 1 \geq 8$ . By the parity conditions, this implies

$$\gamma_0 \leq \frac{a+2}{3} + \frac{b-1}{3} + \left\lceil \frac{c-1}{3} \right\rceil \leq \frac{3(a+b+c+1)}{8} = \frac{3n_0}{8}$$

and we obtain a similar contradiction as in the proof of Claim 1.  $\square$

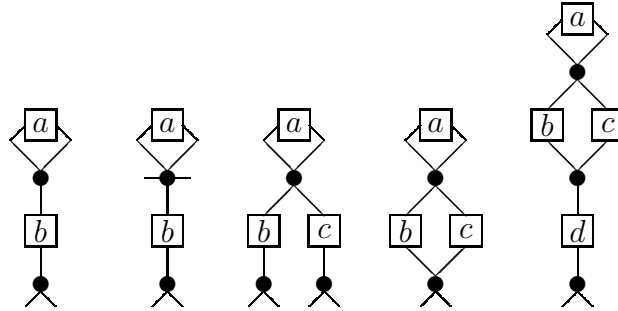


Figure 4

**Claim 4.** *There are no two vertices of degree at least 4 joined by an  $a$ -path and a  $b$ -path with  $a, b \equiv 1 \pmod{3}$ .*

*Proof of Claim 4:* For contradiction, we assume the existence of such vertices  $u$  and  $v$ . If  $V_0$  consists of the internal vertices of the  $a$ -path and the  $b$ -path (cf. Figure 5), then there is a set  $D_0 \subseteq V$  containing  $u$  such that  $|D_0| = \frac{a+b+1}{3}$  and every vertex in  $V_0$  is either in  $D_0$  or adjacent to a vertex in  $D_0$ . Since  $G$  that does not contain cycles of lengths 4 or 7, we have  $a + b \geq 8$  which implies  $|D_0| \leq \frac{3n_0}{8}$ . Clearly,  $\gamma \leq |D_0| + \gamma_1$  and we obtain a similar contradiction as in the proof of Claim 1.  $\square$

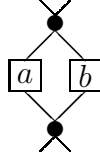


Figure 5

**Claim 5.** *There are no two vertices  $u$  of degree 3 and  $v$  of degree at least 4 that are joined by an  $a$ -path and a  $b$ -path with  $a, b \equiv 1 \pmod{3}$ .*

*Proof of Claim 5:* For contradiction, we assume the existence of such vertices  $u$  and  $v$ . There is a  $c$ -path leading from  $u$  to a vertex  $w$  of degree at least 3. Let  $V_0$  be a minimal set of vertices containing  $u$  for which  $G_1$  has no vertex of degree less than 2.

Either  $v \neq w$  (cf. first graph in Figure 6) or  $v = w$  and  $v$  is of degree at least 5 (cf. second graph in Figure 6) or  $v = w$ ,  $v$  is of degree 4 and there is a  $d$ -path leading from  $v$  to a vertex of degree at least 3 (cf. third graph in Figure 6). In all three cases,  $G_0$  has a spanning subgraph that arises by attaching a path with  $a$  vertices, a path with  $b$  vertices and a path with  $c' \geq 1$  vertices to a single vertex, i.e. this graph is a subdivision of a star. By the parity conditions and since  $a + b \geq 8$ , this implies

$$\gamma_0 \leq 1 + \frac{a-1}{3} + \frac{b-1}{3} + \left\lceil \frac{c'-1}{3} \right\rceil \leq \frac{3(a+b+c'+1)}{8} = \frac{3n_0}{8}$$

and we obtain a similar contradiction as in the proof of Claim 1.  $\square$

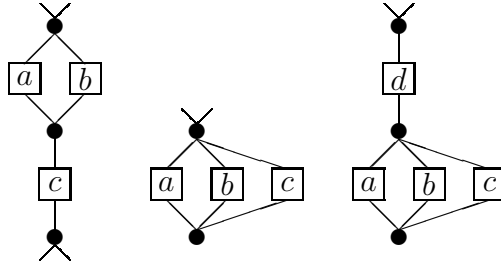


Figure 6

**Claim 6.** *There are no two vertices  $u$  and  $v$  of degree 3 that are joined by an  $a$ -path and a  $b$ -path and such that there is another  $c$ -path starting at  $u$  with  $a, b, c \equiv 1 \pmod{3}$ .*

*Proof of Claim 6:* For contradiction, we assume the existence of such vertices  $u$  and  $v$ . Let the  $c$ -path lead from  $u$  to the vertex  $w$  of degree at least 3. There is a  $d$ -path leading from  $v$  to a vertex  $w'$  of degree at least 3. Let  $V_0$  be a minimal set of vertices containing  $u$  for which  $G_1$  has no vertex of degree less than 2.

By a similar reasoning as in the proofs of the previous claims, we obtain that the local structure of  $G$  is as in one of the four graphs in Figure 7. Since  $a + b \geq 8$ , in all these cases  $\gamma_0 \leq \frac{3n_0}{8}$  and we obtain a similar contradiction as in the proof of Claim 1. (For the first three graphs we can argue exactly as in the proof of Claim 5. For the fourth graph in Figure 7 we need to use  $a + b \geq 8$  and  $c \equiv 1 \pmod{3}$ .)  $\square$

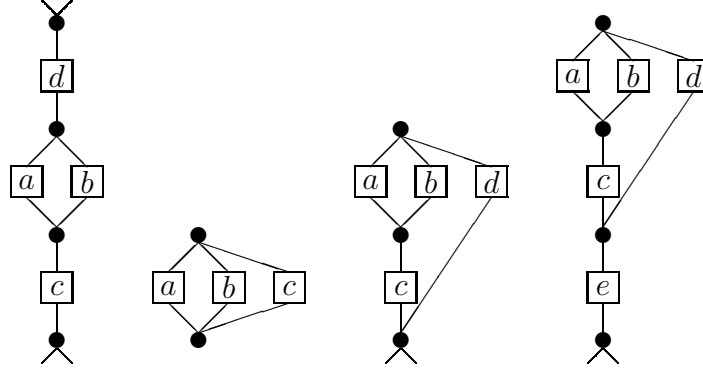


Figure 7

**Claim 7.** *There are no four vertices  $u, v_1, v_2$  and  $v_3$  of degree at least 3 such that  $u$  is joined to  $v_1$  by an  $a$ -path,  $u$  is joined to  $v_2$  by a  $b$ -path and  $u$  is joined to  $v_3$  by a  $c$ -path with  $a, b, c \equiv 1 \pmod{3}$ .*

*Proof of Claim 7:* For contradiction, we assume the existence of such vertices. Let  $V_0$  be a minimal set of vertices containing the internal vertices of the  $a$ -path, the  $b$ -path and the  $c$ -path for which  $G_1$  has no vertex of degree less than 2.

By a similar reasoning as in the proofs of the previous claims, we obtain that the local structure of  $G$  is in one of the nine graphs in Figure 8. In the first case  $V_0$  consists of the internal vertices of the  $a$ -path, the  $b$ -path and the  $c$ -path. There is a set  $D_0 \subseteq V$  containing  $u$  such that  $|D_0| = \frac{a+b+c}{3} = \frac{n_0}{3}$  and every vertex in  $V_0$  is either in  $D_0$  or adjacent to a vertex in  $D_0$ . Again,  $\gamma \leq |D_0| + \gamma_1$  and we obtain a similar contradiction as in the proof of Claim 1. In all of the remaining eight cases,  $G_0$  has a spanning subgraph that arises by attaching a path with  $a$  vertices, a path with  $b$  vertices, a path with  $c$  vertices, and a path with  $d' \geq 0$  vertices to a single vertex; i.e. this graph is a subdivision of a star. By the parity conditions, this implies  $\gamma_0 \leq \frac{n_0}{3}$  and we obtain a similar contradiction as in the proof of Claim 1.  $\square$

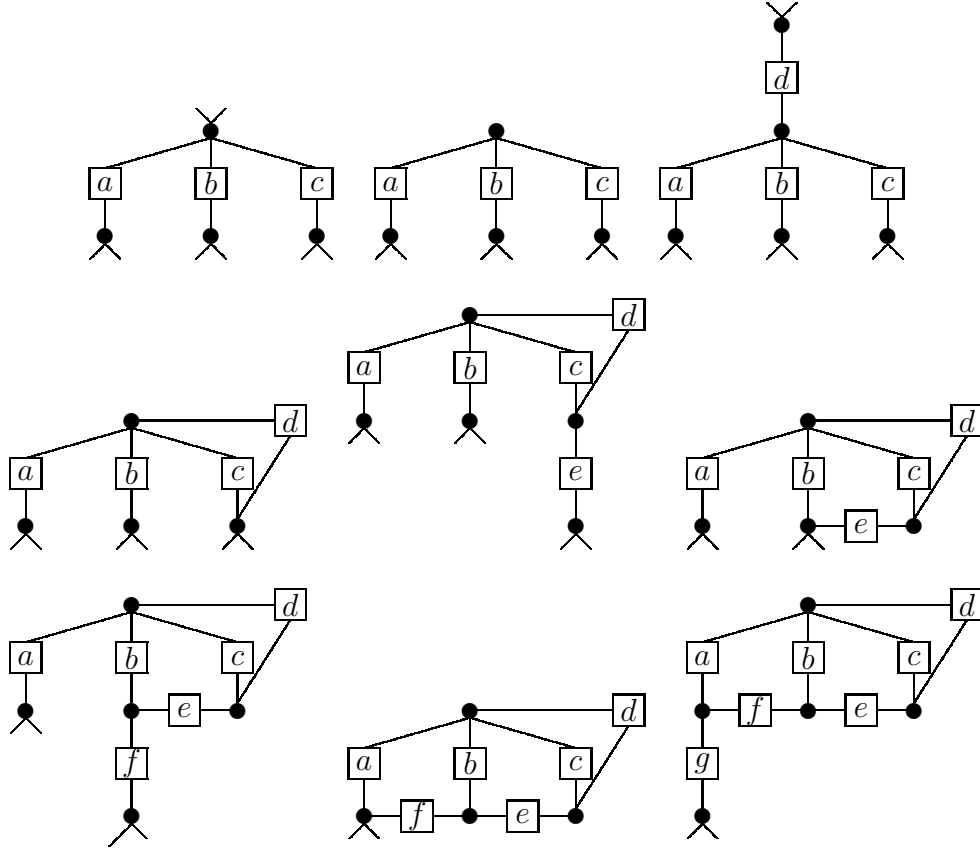


Figure 8

We have by now analyzed the structure of  $G$  far enough in order to describe a sufficiently small dominating set leading to the final contradiction. Let  $V_{\geq 3}$  denote the set of vertices of degree at least 3 and let  $n_{\geq 3} = |V_{\geq 3}|$ . The graph  $G[V \setminus V_{\geq 3}]$  is a collection of paths of order either 1 (mod 3) or 2 (mod 3).

Let  $P_1, P_2, \dots, P_s$  denote the set of vertices of the paths of order 1 (mod 3) and let  $Q_1, Q_2, \dots, Q_t$  denote the set of vertices of the paths of order 2 (mod 3).

By the above claims,

$$s + t \geq \frac{3n_{\geq 3}}{2} \quad \text{and} \quad s \leq n_{\geq 3}$$

which implies

$$t \geq \frac{n_{\geq 3}}{2} \quad \text{and} \quad \left( n_{\geq 3} - \frac{s}{3} - \frac{2t}{3} \right) \leq \frac{n_{\geq 3}}{3}.$$

For  $1 \leq i \leq s$ , the path  $G[P_i]$  without its one or two endvertices has a dominating set  $D_i^P$  of cardinality  $\frac{|P_i|-1}{3}$ . For  $1 \leq j \leq t$ , the path  $G[Q_j]$  without its two endvertices has a dominating set  $D_j^Q$  of cardinality  $\frac{|Q_j|-2}{3}$ .

Now the set

$$V_{\geq 3} \cup \bigcup_{i=1}^s D_i^P \cup \bigcup_{j=1}^t D_j^Q$$

is a dominating set of  $G$  and we obtain,

$$\begin{aligned} \gamma &\leq n_{\geq 3} + \sum_{i=1}^s |D_i^P| + \sum_{j=1}^t |D_j^Q| \\ &= n_{\geq 3} + \sum_{i=1}^s \frac{|P_i| - 1}{3} + \sum_{j=1}^t \frac{|Q_j| - 2}{3} \\ &= \left( n_{\geq 3} - \frac{s}{3} - \frac{2t}{3} \right) + \sum_{i=1}^s \frac{|P_i|}{3} + \sum_{j=1}^t \frac{|Q_j|}{3} \\ &\leq \frac{n_{\geq 3}}{3} + \sum_{i=1}^s \frac{|P_i|}{3} + \sum_{j=1}^t \frac{|Q_j|}{3} \\ &= \frac{n}{3}. \end{aligned}$$

This final contradiction completes the proof.  $\square$

If the graph  $G$  arises from  $l \geq 1$  disjoint copies of the cycle  $C_8$  by choosing a set  $L$  of  $l$  vertices intersecting all these cycles and adding the edges of a tree on  $L$ , then  $\gamma(G) = \frac{3}{8}n(G)$ . Furthermore,  $\gamma(C_{16}) = 6 = \frac{3 \cdot 16}{8}$ . These examples show that Theorem 1 is best-possible for infinitely many graphs.

Note that  $\gamma(C_{10}) = 4 > \frac{3n(C_{10})}{8}$ . We believe that the assumption that the graphs in Theorem 1 do not contain cycles of lengths 10 might be replaced by the exclusion of finitely many exceptional graphs. For bipartite graphs we obtain the following.

**Corollary 2** *If  $G$  is a bipartite graph of order  $n$ , minimum degree at least 2 and domination number  $\gamma$  that does not contain cycles of lengths 4 or 10, then  $\gamma \leq \frac{3}{8}n$ .*

We now proceed to bounds for the domination number of bipartite graphs of given minimum degree that are derived using the probabilistic method in a similar way as in the proof of (2) by Alon and Spencer [1]. In order to improve (2) we try to leverage the fact that the graph is bipartite. If for instance one of the partite sets is smaller than the other, then a minimum degree condition for the graph forces the average degree in the smaller partite set to be larger than the minimum degree which can be used to improve the estimate for the domination number.

**Theorem 3** *If  $G$  is a bipartite graph with partite sets of cardinalities  $n_A \leq n_B$ , size  $m$ , minimum degree  $\delta$ , maximum degree  $\Delta$  and domination number  $\gamma$ , then*

$$\gamma \leq g_1 \leq g_2 \leq g_3 \leq g_4$$



where

$$\begin{aligned}
g_1 &= g_1(n_A, n_B, m, \delta, \Delta, a, b) \\
&= an_A + bn_B + (1-a)(1-b)^\delta \frac{\Delta n_A - m}{\Delta - \delta} + (1-a)(1-b)^\Delta \frac{m - \delta n_A}{\Delta - \delta} \\
&\quad + (1-a)^\delta (1-b) \frac{\Delta n_B - m}{\Delta - \delta} + (1-a)^\Delta (1-b) \frac{m - \delta n_B}{\Delta - \delta}, \\
g_2 &= g_2(n_A, n_B, \delta, \Delta, a, b) \\
&= an_A + bn_B + (1-a)(1-b)^\delta \frac{\Delta n_A - \delta n_B}{\Delta - \delta} + (1-a)(1-b)^\Delta \frac{\delta n_B - \delta n_A}{\Delta - \delta} \\
&\quad + (1-a)^\delta (1-b)n_B, \\
g_3 &= g_3(n_A, n_B, \delta, a, b) = an_A + bn_B + (1-a)(1-b)^\delta n_A + (1-b)(1-a)^\delta n_B \text{ and} \\
g_4 &= g_4(n_A, n_B, \delta, a, b) = an_A + bn_B + e^{-a-\delta b} n_A + e^{-b-\delta a} n_B
\end{aligned}$$

for  $0 \leq a, b \leq 1$ .

*Proof:* Let the two partite sets be  $A$  and  $B$  with  $n_A = |A|$  and  $n_B = |B|$ . We fix two probabilities  $a \in [0, 1]$  and  $b \in [0, 1]$  and select independently at random vertices from  $A$  with probability  $a$  and vertices from  $B$  with probability  $b$ . If  $A' \subseteq A$  and  $B' \subseteq B$  denote the sets of selected vertices, then  $\mathbf{E}[|A'|] = an_A$  and  $\mathbf{E}[|B'|] = bn_B$ . If

$$A'' = \{u \in A \setminus A' \mid N_G(u) \cap B' = \emptyset\} \text{ and } B'' = \{u \in B \setminus B' \mid N_G(u) \cap A' = \emptyset\},$$

then  $A' \cup A'' \cup B' \cup B''$  is a dominating set of  $G$  whose expected cardinality is an upper bound on  $\gamma$  and equals

$$\mathbf{E}[|A'|] + \mathbf{E}[|A''|] + \mathbf{E}[|B'|] + \mathbf{E}[|B''|] = an_A + bn_B + \mathbf{E}[|A''|] + \mathbf{E}[|B''|].$$

Now

$$\mathbf{E}[|A''|] = (1-a) \sum_{u \in A} (1-b)^{d_G(u)}.$$

Since  $(1-b)^x$  is a convex function of  $x$ ,  $\delta \leq d_G(u) \leq \Delta$  for  $u \in A$  and  $\sum_{u \in A} d_G(u) = m$ , the term  $\sum_{u \in A} (1-b)^{d_G(u)}$  is at most

$$x(1-b)^\delta + (n_A - x)(1-b)^\Delta$$

where  $x$  is chosen as large as possible subject to the condition  $\delta x + \Delta(n_A - x) \geq m$ , i.e.

$$x = \frac{\Delta n_A - m}{\Delta - \delta}.$$

Therefore,

$$\mathbf{E}[|A''|] \leq (1-a)(1-b)^\delta \frac{\Delta n_A - m}{\Delta - \delta} + (1-a)(1-b)^\Delta \frac{m - \delta n_A}{\Delta - \delta}$$

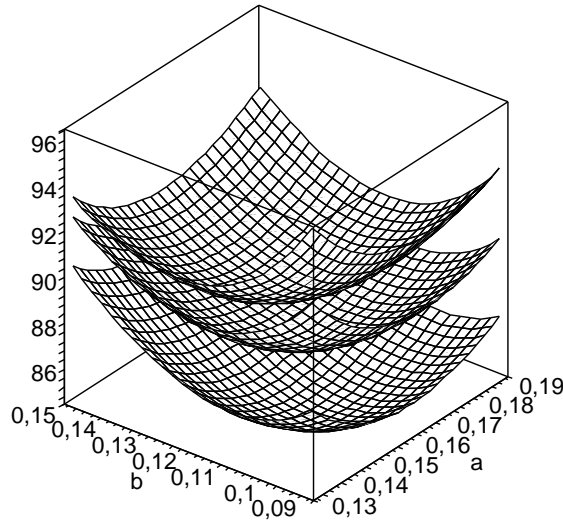
and, by symmetry,

$$\mathbf{E}[|B''|] \leq (1-a)^\delta(1-b) \frac{\Delta n_B - m}{\Delta - \delta} + (1-a)^\Delta(1-b) \frac{m - \delta n_B}{\Delta - \delta}$$

which implies  $\gamma \leq g_1$ .

Since  $g_1$  is decreasing in  $m$  and  $m \geq \delta n_B$ , we have  $g_1 \leq g_2$ . Since  $(1-b)^\delta \geq (1-b)^\Delta$ , we have  $g_2 \leq g_3$  and, finally, since  $1+x \leq e^x$ , we have  $g_3 \leq g_4$  which completes the proof.  $\square$

The problem of the bounds in Theorem 3 is that their evaluation involves the solution of the minimization problem of determining  $a$  and  $b$  such that  $g_i$  is smallest possible. The following observations are immediate:  $g_1$  is smaller than  $g_2$ , if  $m$  is larger than  $\delta n_B$  and  $g_2$  is smaller than  $g_3$ , if  $n_A$  is smaller than  $n_B$ . The problem remains to quantify these differences. The example in Figure 9 shows the functions  $g_1$ ,  $g_2$  and  $g_3$  for  $n_A = 200$ ,  $n_B = 300$ ,  $\delta = 20$ ,  $\Delta = 100$ ,  $m = 8000$ ,  $a \in [0.13, 0.19]$  and  $b \in [0.09, 0.15]$ . In this case  $\lfloor g_1(0.16, 0.11) \rfloor = 84$ ,  $\lfloor g_2(0.16, 0.11) \rfloor = 87$  and  $\lfloor g_3(0.16, 0.11) \rfloor = 89$ .



**Figure 9**

We will now show how to derive an explicit bound from

$$g_4 = g_4(a, b) = an_A + bn_B + e^{-a-\delta b}n_A + e^{-b-\delta a}n_B$$

( $n_A \leq n_B$  and  $\delta \geq 2$  are considered to be fixed). Note that  $g_4(a, b)$  is strictly convex as the sum of two linear and two strictly convex functions.

Let  $n = n_A + n_B$  and  $t = \frac{n_A}{n}$  with  $0 < t \leq \frac{1}{2}$ . We introduce two variables

$$x = e^{-a-\delta b} \text{ and } y = e^{-b-\delta a}.$$

Setting the partial derivatives  $\frac{\partial}{\partial a}g_4(a, b) = n_A - n_Ax - \delta n_By$  and  $\frac{\partial}{\partial b}g_4(a, b) = n_B - \delta n_Ax - n_By$  to zero, yields two simple linear equations for  $x$  and  $y$  which — for  $\delta \geq 2$  — have the unique solution

$$x = x(t) = \frac{\delta n_B - n_A}{(\delta^2 - 1)n_A} = \frac{\delta(1-t) - t}{(\delta^2 - 1)t}$$

and

$$y = y(t) = \frac{\delta n_A - n_B}{(\delta^2 - 1)n_B} = \frac{\delta t - (1-t)}{(\delta^2 - 1)(1-t)}.$$

For the probabilities  $a$  and  $b$  this implies

$$a = a(t) = \frac{1}{\delta^2 - 1}(\ln x - \delta \ln y)$$

and

$$b = b(t) = \frac{1}{\delta^2 - 1}(\ln y - \delta \ln x).$$

In view of the strict convexity of  $g_4$ , the point

$$(a(t), b(t)) \in \mathbb{R}^2$$

is the unique global minimum of  $g_4$ . In order to guarantee that  $a, b \in [0, 1]$ , we need  $1 \leq \frac{x}{y^\delta} \leq e^{\delta^2-1}$  and  $1 \leq \frac{y}{x^\delta} \leq e^{\delta^2-1}$ . This can be ensured for instance by the symmetric conditions

$$e^{-\delta} \leq x \leq e^{-1} \text{ and } e^{-\delta} \leq y \leq e^{-1}.$$

In view of the explicit values given above for  $x$  and  $y$  in terms of  $t$ , this is equivalent to

$$t \geq \max \left\{ \frac{\delta^2 - 1 + e^\delta}{\delta^2 - 1 + e^\delta(\delta + 1)}, \frac{e\delta}{\delta^2 - 1 + e(\delta + 1)} \right\}$$

and

$$t \leq \min \left\{ \frac{\delta^2 - 1 + e}{\delta^2 - 1 + e(\delta + 1)}, \frac{e^\delta \delta}{\delta^2 - 1 + e^\delta(\delta + 1)} \right\}.$$

A simple calculation shows

$$\frac{\delta^2 - 1 + e^\delta}{\delta^2 - 1 + e^\delta(\delta + 1)} \leq \frac{e\delta}{\delta^2 - 1 + e(\delta + 1)}$$

and

$$\frac{1}{2} \leq \frac{\delta^2 - 1 + e}{\delta^2 - 1 + e(\delta + 1)} \leq \frac{e^\delta \delta}{\delta^2 - 1 + e^\delta(\delta + 1)}$$

and the condition on  $t$  simplifies to

$$\frac{e\delta}{\delta^2 - 1 + e(\delta + 1)} \leq t \leq \frac{1}{2}.$$

Note that

$$e^{-a-\delta b}n_A + e^{-b-\delta a}n_B = xn_A + yn_B = \frac{n}{\delta + 1}.$$

Putting all this together we obtain the following.

**Corollary 4** *If  $G$  is a bipartite graph of order  $n$  of minimum degree  $\delta \geq 2$  with partite sets of cardinalities  $tn$  and  $(1-t)n$  for some  $t$  with*

$$\frac{e\delta}{\delta^2 - 1 + e(\delta + 1)} \leq t \leq \frac{1}{2},$$

*then*

$$\begin{aligned} \gamma &\leq g_1(tn, (1-t)n, \delta(1-t)n, \delta, (1-t)n, a(t), b(t)) \\ &\leq g_2(tn, (1-t)n, \delta, (1-t)n, a(t), b(t)) \\ &\leq g_3(tn, (1-t)n, \delta, a(t), b(t)) \\ &\leq g_4(tn, (1-t)n, \delta, a(t), b(t)) \\ &= \frac{n}{\delta+1} + \frac{tn}{\delta^2-1} \left( \ln \left( \frac{\delta(1-t)-t}{(\delta^2-1)t} \right) - \delta \ln \left( \frac{\delta t - (1-t)}{(\delta^2-1)(1-t)} \right) \right) \\ &\quad + \frac{(1-t)n}{\delta^2-1} \left( \ln \left( \frac{\delta t - (1-t)}{(\delta^2-1)(1-t)} \right) - \delta \ln \left( \frac{\delta(1-t)-t}{(\delta^2-1)t} \right) \right) \\ &\leq g_4 \left( tn, (1-t)n, \delta, \frac{\ln(\delta+1)}{\delta+1}, \frac{\ln(\delta+1)}{\delta+1} \right) \\ &= \left( \frac{1 + \ln(\delta+1)}{\delta+1} \right) n. \end{aligned}$$

## References

- [1] N. Alon and J. Spencer, *The Probabilistic Method*, John Wiley and Sons, Inc., 1992.
- [2] V.I. Arnautov, Estimation of the exterior stability number of a graph by means of the minimal degree of the vertices, (Russian), *Prikl. Mat. Programm.* **11** (1974), 3-8.
- [3] M. Blank, An estimate of the external stability number of a graph without suspended vertices, *Prikl Math i Programmirovanie Vyp* **10** (1973), 3-11.
- [4] R.C. Brigham and R.D. Dutton, Bounds on the domination number of a graph, *Q. J. Math., Oxf. II. Ser.* **41** (1990), 269-275.
- [5] Y. Caro and Y. Roditty, On the vertex-independence number and star decomposition of graphs, *Ars Combin.* **20** (1985), 167-180.
- [6] Y. Caro and Y. Roditty, A note on the  $k$ -domination number of a graph, *Internat. J. Math. Sci.* **13** (1990), 205-206.
- [7] T. Gerlach and J. Harant, A note on domination in bipartite graphs, *Discuss. Math., Graph Theory* **22** (2002), 229-231.
- [8] J. Harant and A. Pruchnewski, A note on the domination number of a bipartite graph, *Ann. Comb.* **5** (2001), 175-178.

- [9] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of domination in graphs, Marcel Dekker, Inc., New York, 1998.
- [10] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in graphs advanced topics, Marcel Dekker, Inc., New York, 1998.
- [11] K. Kawarabayashi, M.D. Plummer and A. Saito. Domination in a graph with a 2-factor, *J. Graph Theory* **52** (2006), 1-6.
- [12] C. Löwenstein and D. Rautenbach, Domination in Graphs of Minimum Degree at least Two and large Girth, *manuscript* (2006).
- [13] W. McCuaig and B. Shepherd, Domination in graphs with minimum degree two, *J. Graph Theory* **13** (1989), 749-762.
- [14] O. Ore, Theory of graphs, Amer. Math. Soc. Colloq. Publ. **38**, 1962.
- [15] C. Payan, Sur le nombre d'absorption d'un graphe simple, (French), *Cah. Cent. Étud. Rech. Opér.* **17** (1975), 307-317.
- [16] D. Rautenbach, A Note on domination, girth and minimum degree, to appear in *Discrete Math.*
- [17] B. Reed, Paths, stars and the number three, *Combin. Prob. Comput.* **5** (1996), 267-276.
- [18] L. Volkmann, Upper bounds on the domination number of a graph in terms of diameter and girth, *J. Combin. Math. Combin. Comput.* **52** (2005), 131-141.
- [19] L. Volkmann, An upper bound for the domination number of a graph in terms of order and girth, *manuscript* (2004).