

Some Remarks on $\lambda_{p,q}$ -Connectedness

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Abstract

A connected graph is $\lambda_{p,q}$ -connected if there is a set of edges whose deletion leaves two components of order at least p and q , respectively. In this paper we present some sufficient conditions for graphs to be $\lambda_{p,q}$ -connected. Furthermore, we study $\lambda_{2,q}$ -connected graphs in more detail.

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1 Introduction

We consider finite graphs without loops or multiple edges and use standard terminology as in [1] or [3].

It is an easy exercise to show that every connected graph has a vertex whose deletion results in a connected graph. The main problem we consider in this paper concerns similar results for pairs of adjacent vertices. Clearly, a general connected graph need not contain an edge such that the deletion of the two incident vertices results in a connected graph. In fact, deleting the vertices incident with any edge can result in a graph not having any large component.

This motivates the study of so-called $\lambda_{2,q}$ -connected graphs for integers $q \geq 2$. A connected graph is $\lambda_{2,q}$ -connected if it contains an edge such that the deletion of the two incident vertices results in a graph one component of which has order at least q . Equivalently, a connected graph is $\lambda_{2,q}$ -connected if there is an edge cut whose removal results in two components containing at least 2 and at least q vertices, respectively.

This second definition makes it apparent that the problem we consider is closely related to the so-called restricted edge-connectivity first proposed by Harary [10]. In general, for integers $p, q \geq 1$, a connected graph is called $\lambda_{p,q}$ -connected if it contains an edge cut whose removal results in two components containing at least p and at least q vertices, respectively. The smallest size of such an edge cut has been proposed by Esfahanian and Hakimi [6, 7] as a natural measure of fault-tolerance and was studied for various special network topologies [4, 5, 6, 16, 17].

For general graphs, the research mainly focused on the case $p = q$ [2, 11, 13, 14, 15, 18, 19, 20]. Next to explicit characterizations for small values of p and q , sufficient conditions

for $\lambda_{p,q}$ -connectedness and bounds on the sizes of the corresponding edge cuts were studied. We mention just some results and refer to the abundant cited literature for more.

Esfahanian and Hakimi [6] showed that a connected graph of order $n \geq 4$ is $\lambda_{2,2}$ -connected if and only if it is not a star. Later Bonsma, Ueffing and Volkmann [2] characterized all $\lambda_{3,3}$ -connected of order $n \geq 6$. Hellwig, Rautenbach and Volkmann [11] studied sufficient conditions for arbitrary values of p and q and Ou [14] characterized $\lambda_{p,p}$ -connected graphs of order $n \geq 3p - 2$. Recently, Zhao Zhang and Jinjiang Yuan [19] characterized the graphs which are $\lambda_{p,p}$ -connected for some p which is at most the minimum degree of the graph plus one.

In the next section we will first show that the last mentioned result is an immediate consequence of a theorem due to Győri [9] and Lovász [12]. In the third section, we will then consider $\lambda_{2,q}$ -connected graphs in detail.

2 $\lambda_{p,q}$ -connected graphs

The main tool of this section is the following beautiful result which was first conjectured by Frank [8] in 1976. A subgraph of some graph $G = (V, E)$ induced by a set $X \subseteq V$ is denoted by $G[X]$.

Theorem 2.1 (Győri [9] 1978, Lovász [12] 1977) *For every k -connected graph $G = (V, E)$ of order n , k vertices $v_1, v_2, \dots, v_k \in V$, and k positive integers n_1, n_2, \dots, n_k such that $n_1 + n_2 + \dots + n_k = n$ there exists a partition $\{V_1, V_2, \dots, V_k\}$ of V such that $v_i \in V_i$, $|V_i| = n_i$ and $G[V_i]$ is connected for $1 \leq i \leq k$.*

Corollary 2.2 *Let p and q be integers with $q \geq p \geq 1$. A connected graph G of order $n \geq p + q$ and minimum degree δ is $\lambda_{p,q}$ -connected provided one of the following conditions is satisfied.*

- (i) G is 2-connected.
- (ii) G has a block of order at least $p + 1$ containing at most one cut vertex.
- (iii) $p = q \leq \delta + 1$ and G contains a block with at least two cut vertices.
- (iv) $n \geq 2q - 1$ and G contains a cut vertex u such that all components of $G[V \setminus \{u\}]$ are of order at least p .

Proof: (i) follows immediately from Theorem 2.1.

(ii) In view of (i), we may assume that G is not 2-connected. Let $G' = (V', E')$ denote a block of order at least $p + 1$ containing exactly one cut vertex v_1 . Let $v_2 \in V' \setminus \{v_1\}$. Applying Theorem 2.1 to G' with $n_1 = |V'| - p$ and $n_2 = p$ yields a partition $V_1 \cup V_2$ of V' such that $G[V_2]$ is a connected graph of order (exactly) p and $G[V \setminus V_2]$ is a connected graph of order (exactly) $n - p \geq q$.

(iii) Let G' arise from G by deleting all edges in a block of G with at least two cut vertices v_1 and v_2 . If G_i denotes the component of G' containing v_i , then G_i obviously has order at least $\delta + 1 \geq p = q$ for $i = 1, 2$ which implies the desired result.

(iv) If $G' = (V', E')$ is a smallest component of $G[V \setminus \{u\}]$, then $|V'| \geq p$, $|V \setminus V'| \geq n - \frac{n-1}{2} = \frac{n+1}{2} \geq q$ and $G[V \setminus V']$ is connected. \square

Corollary 2.3 (Zhao Zhang and Jinjiang Yuan [19]) *If $p \in \mathbb{N}$ and G is a connected graph of order at least $2p$ and minimum degree at least $p - 1$, then G is $\lambda_{p,p}$ -connected if and only if G does not arise by identifying one vertex from each of at least three disjoint cliques of order p .*

Proof: This follows immediately from Corollary 2.2 (i)-(iii). \square

Note that Zhao Zhang and Jinjiang Yuan [19] also estimate the size of the corresponding edge cut.

Before we proceed to the next section we phrase the following immediate observation for further reference.

Observation 2.4 *For integers $p, q \geq 1$ a connected graph is $\lambda_{p,q}$ -connected if and only if it has a $\lambda_{p,q}$ -connected spanning tree.*

Proof: The ‘if’-part is immediate. The ‘only if’-part follows for a graph $G = (V, E)$ with $\lambda_{p,q}$ -edge cut S by joining spanning trees of the two components of $(V, E \setminus S)$ by an edge in S . \square

3 $\lambda_{2,q}$ -connected graphs

In the following let a, b, c be non-negative integers with $a, c \geq 2$. Let $S(a, b, c)$ denote the tree of order $a + b + c$ that arises by joining the centers of two stars $K_{1,a-1}$ and $K_{1,c-1}$ by a path containing b internal vertices (cf. Figure 3). Furthermore, let $R(a, b, c)$ denote the graph of order $a + b + c$ that arises by joining the centers of two stars $K_{1,a-1}$ and $K_{1,c-1}$ with a new edge and adding b vertices which are adjacent to the centers of the two stars (cf. Figure 1).

We begin with a result about $\lambda_{2,q}$ -connected trees.

Theorem 3.1 *Let n, q be two integers such that $n \geq q + 2 \geq 4$.*

- (i) *A tree T of order n is $\lambda_{2,q}$ -connected if and only if it contains a non-endvertex u of degree at most $n - q$ which is adjacent to at most one non-endvertex.*
- (ii) *A tree T of order $n \geq \frac{3(q-1)}{2}$ is not $\lambda_{2,q}$ -connected if and only if*

$$T \in \{K_{1,n-1}\} \cup \{S(a, b, c) \mid a, c \geq n - q + 1; a + b + c = n\}.$$

Proof: (i) The ‘if’-part is immediate. For the ‘only if’-part consider a $\lambda_{2,q}$ -cut edge xy in a $\lambda_{2,q}$ -connected tree T . Let the components of $T - xy$ containing x and y have at least 2 and q vertices, respectively. Let u denote a non-endvertex of T in the component of $T - xy$ containing x at maximum possible distance from y . Clearly, u has degree at most $n - q$ in T and is adjacent to at most one non-endvertex.

(ii) The ‘if’-part is immediate. For the ‘only if’-part consider a tree T of order $n \geq \frac{3(q-1)}{2}$ which is not $\lambda_{2,q}$ -connected. By (i), every non-endvertex which is adjacent to at most one non-endvertex has degree at least $n - q + 1$, i.e. such a vertex together with the adjacent endvertices constitute already at least $n - q + 1$ vertices. Note that every tree with at least three such vertices necessarily also contains a non-endvertex which is adjacent to more than one non-endvertex. Since $3(n - q + 1) + 1 > n$, T has at most two such vertices which immediately implies that T is either a star or $T = S(a, b, c)$ with $a, c \geq n - q + 1$ and $a + b + c = n$. \square

The next result characterizes the $\lambda_{2,q}$ -connected graphs for orders at least $2q - 3$. Note that (i) implies the result due to Esfahanian and Hakimi [6] mentioned in the introduction.

Theorem 3.2 *Let n, q be two integers such that $n \geq q + 2 \geq 4$.*

(i) *A connected graph G of order $n \geq 2q - 1$ is not $\lambda_{2,q}$ -connected if and only if $G = K_{1,n-1}$.*

(ii) *A connected graph G of order $n = 2q - 2$ is not $\lambda_{2,q}$ -connected if and only if*

$$G \in \{K_{1,n-1}, S(q - 1, 0, q - 1)\}.$$

(iii) *A connected graph G of order $n = 2q - 3$ is not $\lambda_{2,q}$ -connected if and only if (cf. Figure 1)*

$$G \in \{K_{1,n-1}, S(q - 1, 0, q - 2), S(q - 2, 1, q - 2), R(q - 2, 1, q - 2)\}.$$

Proof: (i) The ‘if’-part is immediate. For the ‘only if’-part we consider a connected graph G of order $n \geq 2q - 1$ which is not $\lambda_{2,q}$ -connected. By Observation 2.4 and (ii) of Theorem 3.1, the only spanning tree of G is $K_{1,n-1}$. This implies the desired result that $G = K_{1,n-1}$.

(ii) The ‘if’-part is immediate. For the ‘only if’-part we consider a connected graph G of order $n = 2q - 2$ which is not $\lambda_{2,q}$ -connected. By (i) of the present result and (ii) of Theorem 3.1, all spanning trees of G belong to $\{K_{1,n-1}, S(q - 1, 0, q - 1)\}$. Since adding any further edge to one of these trees results in a $\lambda_{2,q}$ -connected graph, it follows that $G \in \{K_{1,n-1}, S(q - 1, 0, q - 1)\}$.

(iii) The ‘if’-part is immediate. For the ‘only if’-part we consider a connected graph G of order $n = 2q - 3$ which is not $\lambda_{2,q}$ -connected. By (i) of the present result and (ii) of Theorem 3.1, all spanning trees of G belong to $\{K_{1,n-1}, S(q - 1, 0, q - 2), S(q - 2, 1, q - 2)\}$. To $S(q - 1, 0, q - 2)$ one can only add one further edge, for example vy_1 (cf. Figure 1), and also to $S(q - 1, 1, q - 2)$ one can only add the edge vw (cf. Figure 1) resulting in a graph, namely $R(q - 2, 1, q - 2)$, which is not $\lambda_{2,q}$ -connected. This leads to the desired result that

$$G \in \{K_{1,n-1}, S(q - 1, 0, q - 2), S(q - 2, 1, q - 2), R(q - 2, 1, q - 2)\}. \quad \square$$

Remark 3.3 Analogously to the proof of Theorem 3.2 (iii), one can show the following supplement to Theorem 3.2.

(iv) A connected graph G of order $n = 2q - 4 \geq q + 2$ is not $\lambda_{2,q}$ -connected if and only if it is isomorphic to a star or to one of the ten graphs in Figure 2. (Note that $n \geq q + 2$ implies $q \geq 6$.)

The next result characterizes $\lambda_{2,q}$ -connected graphs in terms of a specified spanning tree and properties of the additional edges.

Theorem 3.4 Let n, q be two integers such that $n \geq q + 2 \geq 4$. If $G = (V, E)$ is a graph of order n which has a spanning tree that is isomorphic to $S(a, b, c)$ with $a, c \geq n - q + 1$ and $a + b + c = n$, then G is $\lambda_{2,q}$ -connected if and only if one of the following conditions is satisfied. We denote the vertices of G as specified in Figure 3. (Note that the conditions on a, b and c imply $n \leq 2q - 2$.)

(i) $x_i x_j \in E$ for indices i and j with

- either $1 \leq i < j \leq a - 1$,
- or $a + b + 2 \leq i < j \leq n$,
- or $1 \leq i \leq a - 1$ and $a + 3 \leq j \leq n$,
- or $1 \leq i \leq a + b - 2$ and $a + b + 2 \leq j \leq n$,
- or $a \leq i < j \leq a + b + 1$ and $j - i \geq 3$.

(ii) $x_i x_j, x_k x_l \in E$ for indices i, j, k and l with

- either $1 \leq i < k \leq a - 1, j = a + 1 \leq a + b$ and $l = a + 2$,
- or $a + b + 2 \leq i < k \leq n, j = a + b - 1$ and $l = a + b \geq a + 1$,
- or $1 \leq i \leq a - 1, j = a + 1 \leq a + b, k = a$ and $l = a + 2$,
- or $a + b + 2 \leq i \leq n, j = a + b \geq a + 1, k = a + b - 1$ and $l = a + b + 1$,
- or $1 \leq i \leq a - 1, j = a + 2 \leq a + b, k = a + 1$ and $l = a + 3$,
- or $a + b + 2 \leq i \leq n, j = a + b - 1 \geq a + 1, k = a + b - 2$ and $l = a + b$.

(iii) $x_i x_{i+2}, x_{i+1} x_{i+3}, x_{i+2} x_{i+4} \in E$ for $a \leq i \leq a + b - 3$.

(iv) $x_{i_1} x_j, x_{i_2} x_j, \dots, x_{i_r} x_j \in E$ with

- either $2 \leq i_1 < i_2 < \dots < i_r \leq a - 1, r \geq q + a - n$ and $j \in \{a + 1, a + 2\}$,
- or $a + b + 3 \leq i_1 < i_2 < \dots < i_r \leq n, r \geq q + b - n$ and $j \in \{a + b, a + b + 1\}$.

Proof: For the ‘if’-part we have to check that every edge configuration as specified in one of the above conditions would result in a $\lambda_{2,q}$ -connected graph. This is equivalent to showing the existence of an edge e^* such that the deletion of the two vertices incident with e^* results in a graph that has one component with at least q vertices. We will specify

such an edge for the various conditions. Since it is trivial to check the existence of a large component, we leave this task to the reader.

If two endvertices x_i and x_j are adjacent (cf. the first four conditions in (i)), then $e^* = x_i x_j$. If $x_i x_j \in E$ with $1 \leq i \leq a-1$ and $a+3 \leq j \leq a+b+1$, then $e^* = x_{a+1} x_{a+2}$. If $x_i x_j \in E$ with $a \leq i \leq a+b-2$ and $a+b+2 \leq j \leq n$, then $e^* = x_{a+b-1} x_{a+b}$. If $x_i x_j \in E$ with $a \leq i < j \leq a+b+1$ and $j-i \geq 3$, then $e^* = x_{i+1} x_{i+2}$.

If $x_i x_j, x_k x_l \in E$ satisfy any of the conditions given in (ii), then $e^* = x_i x_j$.

If $x_i x_{i+2}, x_{i+1} x_{i+3}, x_{i+2} x_{i+4} \in E$ for $a \leq i \leq a+b-3$, then $e^* = x_{i+1} x_{i+3}$.

If $x_{i_1} x_j, x_{i_2} x_j, \dots, x_{i_r} x_j \in E$ with $2 \leq i_1 < i_2 < \dots < i_r \leq a-1$, $r \geq q+a-n$ and $j \in \{a+1, a+2\}$, then $e^* = x_1 x_a$. If $x_{i_1} x_j, x_{i_2} x_j, \dots, x_{i_r} x_j \in E$ with $a+b+3 \leq i_1 < i_2 < \dots < i_r \leq n$, $r = q+b-n$ and $j \in \{a+b, a+b+1\}$, then $e^* = x_{a+b+1} x_{a+b+2}$.

For the ‘only if’-part one has to check that a graph G as in the theorem for which none of the above conditions holds is not $\lambda_{2,q}$ -connected. Equivalently, one can argue that the deletion of the endpoints of any edge $\tilde{e} = \tilde{u}\tilde{v}$ from G results in a graph all components of which have order at most $q-1$. Note that $a+b, b+c \leq q-1$. Let $\tilde{G} = G[V \setminus \{\tilde{u}, \tilde{v}\}]$.

First, we consider the case that \tilde{e} is an edge of $S(a, b, c)$. If \tilde{e} is incident to an endvertex of $S(a, b, c)$, then at least $\min\{(a-1)-(q+a-n-1)-1, (b-1)-(q+b-n-1)-1\} = n-q-1$ of the endvertices of $S(a, b, c)$ will be isolated in \tilde{G} , because condition (iv) does not hold. Therefore, there are at most $n-2-(n-q-1) = q-1$ vertices in any component of \tilde{G} . If \tilde{e} is $x_a x_{a+1}$ or $x_{a+b} x_{a+b+1}$, then a very similar argument applies. If \tilde{e} is any other edge of $S(a, b, c)$, then no component of \tilde{G} contains at least q vertices, because the last three conditions in (i) do not hold.

Next, we consider the case that \tilde{e} is not an edge of $S(a, b, c)$. Similarly, as above one can consider the cases that \tilde{e} is or is not incident to an endvertex of $S(a, b, c)$. In each case it is very simple to see which of the conditions not being satisfied implies that there is no large component. We leave the details to the reader. \square

Theorem 3.1 together with the last result allow a characterization of $\lambda_{2,q}$ -connected graphs for orders between $\max\{q+2, \frac{3q-3}{2}\}$ and $2q-2$.

Corollary 3.5 *Let n, q be two positive integers with $\max\{q+2, \frac{3q-3}{2}\} \leq n \leq 2q-2$. If $G = (V, E)$ is a connected graph of order n and T is a spanning tree of G , then G is $\lambda_{2,q}$ -connected if and only if*

- (i) either $T \notin \{K_{1, n-1}\} \cup \{S(a, b, c) \mid a, c \geq n-q+1; a+b+c = n\}$,
- (ii) or $T = K_{1, n-1}$ and G has at least n edges,
- (iii) or $T = S(a, b, c)$ for some positive integers a, b, c with $a, c \geq n-q+1$ and $a+b+c = n$ such that one of the conditions specified in Theorem 3.4 are satisfied.

Proof: The ‘if’-part is immediate. For the ‘only if’-part let G be a connected and $\lambda_{2,q}$ -connected graph of order n with spanning tree T . If $T \notin \{K_{1, n-1}\} \cup \{S(a, b, c) \mid a, c \geq$

$n - q + 1; a + b + c = n\}$, then (i) holds. Hence, we may assume that either $T = K_{1, n-1}$ or $T \in \{S(a, b, c) \mid a, c \geq n - q + 1, a + b + c = n\}$. In the first case, G needs at least one edge more than T in order to be $\lambda_{2,q}$ -connected. In the second case, the result follows directly from Theorem 3.4. Hence (ii) or (iii) hold which completes the proof. \square

From the last result one can deduce the following extremal result about $\lambda_{2,q}$ -connected graphs for orders between $\max\{q + 2, \frac{3q-3}{2}\}$ and $2q - 2$.

Corollary 3.6 *Let n, q be two positive integers with $\max\{q + 2, \frac{3q-3}{2}\} \leq n \leq 2q - 2$. If $G = (V, E)$ is a connected graph of order n and size at least $2q - 2$, then G is $\lambda_{2,q}$ -connected.*

The graph $R(n - q + 1, 2p - n - 2, n - q + 1)$ of size exactly $2q - 3$ shows that the above bound on the size is best possible.

Proof: The proof relies on Theorem 3.4. By considering the maximum possible number of edges to add to $S(a, b, c)$ which do not result in a $\lambda_{2,q}$ -connected graph, one easily sees that the largest such number of edges can be added to $S(\lfloor \frac{n}{2} \rfloor, 0, \lceil \frac{n}{2} \rceil)$. In fact, one can add $2q - n - 2$ edges to this graph resulting in $R(n - q + 1, 2p - n - 2, n - q + 1)$ which implies the desired result. \square

For orders below $\frac{3q-3}{2}$ a number of edges linear in q does no longer suffice in order to ensure $\lambda_{2,q}$ -connectedness. A simple example for this effect is obtained by attaching $l \geq 2$ new vertices to all but one vertex of a clique of order k . Clearly, this graph has order $n = (k - 1)(l + 1) + 1$, is not $\lambda_{2, n-l}$ -connected and has $\binom{k}{2} + (k - 1)l$ edges which is quadratic in n for fixed l .

The next result analyses the effect of long cycles on $\lambda_{2,q}$ -connectedness.

Theorem 3.7 *For an integer $t \geq 1$ and $r \in \{0, 1, 2, 3\}$ let $q = 4t + r$. If G is a connected graph of order n with $n \geq q + 2$ that contains a cycle C of length l with $l \geq 2t + r + 1$ for $r \in \{0, 1, 2\}$ and $l \geq 2t + r$ for $r = 3$, then G is $\lambda_{2,q}$ -connected.*

Proof: Let $C : v_0 v_1 v_2 \dots v_{l-1} v_0$. It is easy to see that G has a spanning tree T which contains all but one edge from C . Deleting from T the remaining $l - 1$ edges of C yields l components with vertex sets $V_0, V_1, V_2, \dots, V_{l-1}$ where $v_i \in V_i$ for $0 \leq i \leq l - 1$. Let $n_i = |V_i|$ for $0 \leq i \leq l - 1$.

Note that for every $0 \leq i \leq l - 1$ and $1 \leq j \leq l - 1$ the two graphs $G[V_i \cup V_{i+1} \cup \dots \cup V_{i+j-1}]$ and $G[V_{i+j} \cup V_{i+j+1} \cup \dots \cup V_{i-1}]$ are connected (all indices are taken modulo l).

Therefore, if $n_i = 2$ or $n_i = n_{i+1} = 1$ for some $0 \leq i \leq l - 1$, then G is clearly $\lambda_{2,q}$ -connected. Hence we may assume that $n_i \neq 2$ and that no two consecutive n_i 's are equal to 1. This implies that

$$n_i + n_{i+1} \geq 1 + 3 = 4 \tag{1}$$

for $0 \leq i \leq l - 1$. This implies

$$n = \sum_{i=0}^{l-1} n_i = (n_0 + n_1) + (n_2 + n_3) + \dots \geq \begin{cases} 2l & , l \text{ even,} \\ 2(l - 1) + 3 = 2l + 1 & , l \text{ odd.} \end{cases} \tag{2}$$

If one of the l connected graphs $G[V_i \cup V_{i+1} \cup \dots \cup V_{i+l-3}]$ for $0 \leq i \leq l-1$ has order at least q , then G is $\lambda_{2,q}$ -connected because $G[V_{i+l-2} \cup V_{i+l-1}]$ has order at least 2. Hence, we may assume that

$$\sum_{j=0}^{l-3} n_{i+j} \leq q-1$$

for $0 \leq i \leq l-1$. This implies

$$(l-2)n = (l-2) \sum_{i=0}^{l-1} n_i = \sum_{i=0}^{l-1} \sum_{j=0}^{l-3} n_{i+j} \leq (q-1)l = (4t+r-1)l. \quad (3)$$

The second equation follows from a double-counting argument - each n_i is counted exactly $l-2$ times in the inner sums of the right term.

First we assume that l is even. Now (2) and (3) imply $(l-2)2l \leq (4t+r-1)l$, i.e. $l \leq 2t + \frac{3}{2} + \frac{r}{2}$. Since l is even, this implies $l \leq 2t$ for $r=0$ and $l \leq 2t+2$ for $r \in \{1, 2, 3\}$. For $r \in \{0, 2, 3\}$ this contradicts the assumption on l .

Hence it remains the case that $r=1$ and $l=2t+2$. If $n=2l=4t+4=q+3$, then we may assume that $n_0=3$ which clearly implies that G is $\lambda_{2,q}$ -connected. If $n \geq 2l+1$, then (3) implies $(l-2)(2l+1) \leq 4tl$, i.e. $l \leq 2t + \frac{3}{2} + \frac{1}{l}$. Since $l \geq 3$ is even, this implies the contradiction $l \leq 2t$.

Next we assume that l is odd. Now (2) and (3) imply $(l-2)(2l+1) \leq (4t+r-1)l$, i.e. $l \leq 2t+1 + \frac{r}{2} + \frac{1}{l}$. Since $l \geq 3$ is odd, this implies $l \leq 2t+1$. For $r \in \{1, 2, 3\}$ this contradicts the assumption on l .

Hence it remains the case that $r=0$ and $l=2t+1$. If $n=2l+1=4t+3=q+3$, then we may assume that $n_0=3$ which clearly implies that G is $\lambda_{2,q}$ -connected. If $n \geq 2l+2$, then (3) implies $(l-2)(2l+2) \leq (4t-1)l$, i.e. $l \leq 2t + \frac{1}{2} + \frac{2}{l}$. If $l \geq 5$, then this implies the contradiction $l \leq 2t-1$. If $l=3$, then $q=4$ and we may assume that $n_0+n_1 \geq 4$ and $n_2 \geq 2$ which clearly implies that G is $\lambda_{2,q}$ -connected. This completes the proof. \square

The next example shows that Theorem 3.7 is best-possible in the case that $r=2$. Similar examples exist for all parities $r \in \{0, 1, 3\}$.

Example 3.8 Let $q=4t+2$ for an integer $t \geq 1$, and let $C = v_1 v_2 \dots v_{q-2t} = v_1 v_2 \dots v_{2t+2}$ be a cycle of length $q-2t=2t+2$. In addition, let $u_1, u_2, \dots, u_{2t+2}$ be $2t+2$ further vertices such that v_{2i-1} is adjacent to u_{2i-1} and u_{2i} for $1 \leq i \leq t+1$. The resulting graph H (cf. Figure 4) is of order $q+2$ with a cycle of length $q-2t$, however, it is a simple matter to verify that H is not $\lambda_{2,q}$ -connected.

We want to close with the following observation which establishes $\lambda_{2,q}$ -connectedness using a structural property.

Observation 3.9 *Let $q \geq 2$ be an integer. If G is a connected and claw-free graph of order $n \geq q+2$, then G is $\lambda_{2,q}$ -connected.*

Proof: Let $u_1u_2 \dots u_t$ be a longest path in G . Since G is claw-free, it is a simple matter to verify that $G - \{u_1, u_2\}$ is connected, and the proof is complete. \square

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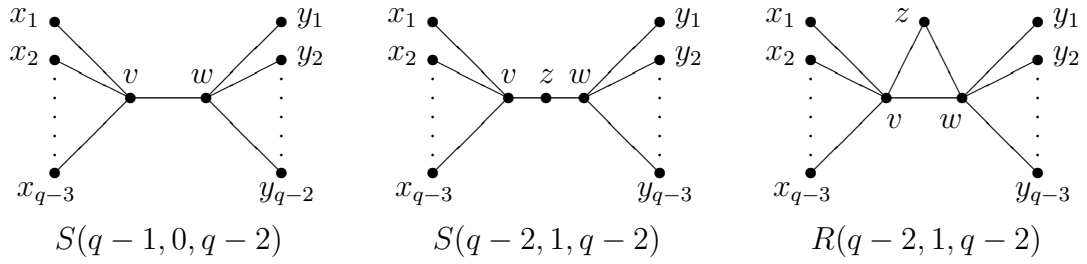


Figure 1: $S(q-1, 0, q-2)$, $S(q-2, 1, q-2)$ and $R(q-2, 1, q-2)$.

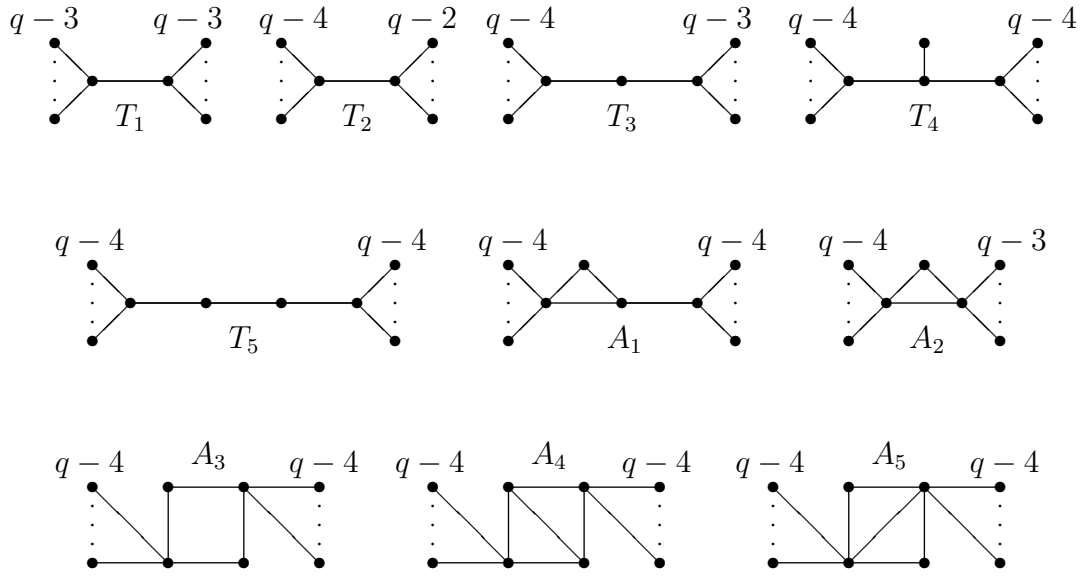


Figure 2: Graphs of order $2q-4$ which are not $\lambda_{2,q}$ -connected.

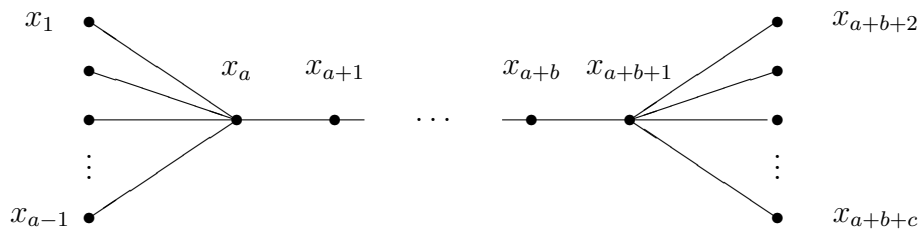


Figure 3: $S(a, b, c)$.

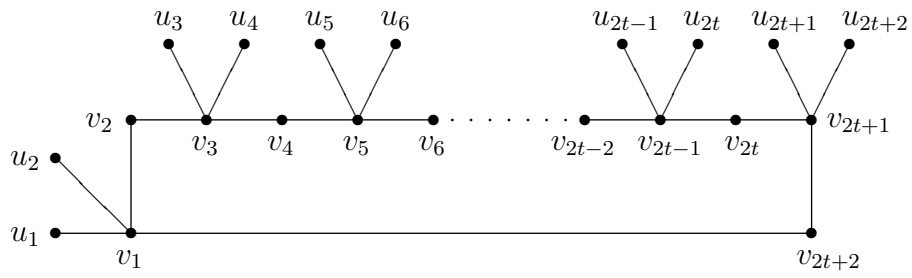


Figure 4: Graph showing that Theorem 3.7 is best-possible.

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Figure 1: $S(q - 1, 0, q - 2)$, $S(q - 2, 1, q - 2)$ and $R(q - 2, 1, q - 2)$.

Figure 2: Graphs of order $2q - 4$ which are not $\lambda_{2,q}$ -connected.

Figure 3: $S(a, b, c)$.

Figure 4: Graph showing that Theorem 3.7 is best-possible.