Edge Irregular Total Labellings for Graphs of Linear Size

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Abstract Confirming a conjecture by Ivančo and Jendrol' for a large class of graphs we prove that for every graph $G = (V, E)$ of order n, size m and maximum degree Δ with $m > 111000\Delta$ there is a function $f : V \cup E \to \left\{1, 2, ..., \left\lceil \frac{m+2}{3} \right\rceil \right\}$ such that $f(u) + f(uv) + f(v) \neq f(u') + f(u'v') + f(v')$ for every $uv, u'v' \in E$ with $uv \neq u'v'$.

Furthermore, we prove the existence of such a function with values up to $\left\lceil \frac{m}{2} \right\rceil$ for every graph $G = (V, E)$ of order n and size $m \geq 3$ whose edges are not all incident to a single vertex.

Keywords Edge irregular total labelling; total edge irregularity strength; irregular assignment; irregularity strength

1 Introduction

In [5] Bača, Jendrol', Miller and Ryan defined the notion of an *edge irregular total k-labelling* of a graph $G = (V, E)$ to be a labelling of the vertices and edges of G

$$f : V \cup E \to \{1, 2, ..., k\}$$

such that the *weights*

$$F(uv) := f(u) + f(uv) + f(v)$$

are different for all edges, i.e. $F(uv) \neq F(u'v')$ for all edges $uv, u'v' \in E$ with $uv \neq u'v'$. They also defined the *total edge irregularity strength* $\text{tes}(G)$ of G as the minimum k for which G has an edge irregular total k-labelling. As a natural variant of the total edge irregularity strength we consider in [8] the minimum k for which a graph of maximum degree Δ has a total k-labelling whose weights define a proper edge coloring. We prove that this value lies between $\frac{\Delta+1}{2}$ and $\frac{\Delta}{2} + \mathcal{O}\left(\sqrt{\Delta \log(\Delta)}\right)$.

While the original motivation for the definition of the total edge irregularity strength came from *irregular assignments* and the *irregularity strength* of graphs introduced in [10] by Chartrand et al. and studied by numerous authors [1, 2, 6, 9, 11, 16], we are interested

[3] Jozef Miškuf has been supported by the Slovak VEGA grant number 1/3004/06.

in this concept mainly because of the following intriguing conjecture posed by Ivančo and Jendrol'

Conjecture 1 (Ivančo and Jendrol' [13]) *For every graph $G = (V, E)$ with size m and maximum degree Δ that is different from K_5*

$$\mathrm{tes}(G) \quad = \quad \max\left\{\left\lceil\frac{m+2}{3}\right\rceil, \left\lceil\frac{\Delta+1}{2}\right\rceil\right\}. \tag{1}$$

Note that for K_5 the maximum in (1) is 4 while $\mathrm{tes}(K_5) = 5$.

As noted in [5] the two terms in the maximum in (1) are natural lower bounds for the total edge irregularity strength: Let f be an edge irregular total k-labelling of a graph G. Since $3 \leq F(uv) = f(u) + f(uv) + f(v) \leq 3k$ for every edge $uv \in E$, we have $m \leq 3k - 2$ which implies $\mathrm{tes}(G) \geq \left\lceil\frac{m+2}{3}\right\rceil$. Similarly, if $u \in V$ is a vertex of maximum degree Δ, then there is a range of $2k - 1$ possible weights $f(u) + 2 \leq F(uv) \leq f(u) + 2k$ for the Δ edges $uv \in E$ incident with u which implies $\mathrm{tes}(G) \geq \left\lceil\frac{\Delta+1}{2}\right\rceil$. Altogether,

$$\mathrm{tes}(G) \quad \geq \quad \max\left\{\left\lceil\frac{m+2}{3}\right\rceil, \left\lceil\frac{\Delta+1}{2}\right\rceil\right\}. \tag{2}$$

Conjecture 1 has been verified for trees by Ivančo and Jendrol' [13] and for complete graphs and complete bipartite graphs by Jendrol' et al. in [14]. In [7] we proved it for graphs of order n, size m and maximum degree Δ that satisfy $m > 1000\Delta\sqrt{8n}$. As our main result here, we replace the $1000\sqrt{8n}$ factor by a constant. Furthermore, we prove $\mathrm{tes}(G) \leq \left\lceil\frac{m}{2}\right\rceil$ for all graphs G of size $m \geq 3$ whose edges are not all incident to a single vertex.

2 Results

Before we proceed to our main result we prove a general upper bound.

Theorem 2 *If G is a graph of size $m \geq 3$ whose edges are not all incident to a single vertex, then*

$$\mathrm{tes}(G) \leq \left\lceil\frac{m}{2}\right\rceil.$$

Proof: If $G = (V, E)$ has diameter at least three, there are suitable vertices u and v at distance at least three whose identification results in a graph G' not all edges of which are incident to a single vertex. Clearly, $\mathrm{tes}(G) \leq \mathrm{tes}(G')$. Therefore, we may assume that G has diameter at most two.

It is easy to verify the statement for $m = 3$. Hence we assume $m \geq 4$. Set $k = \left\lceil\frac{m}{2}\right\rceil$. Since for every vertex of G there is an edge not incident to this vertex, for a vertex x of maximum degree there is a partition $V = V_1 \cup V_2$ of the vertex set of G with $x \in V_1$ and two adjacent vertices in V_2. Among all partitions with this property and less than k edges in V_1, choose one that maximizes the number of vertices in V_1.

Let $E(X,Y) = \{uv \in E \mid u \in X, v \in Y\}$ and $m(X,Y) = |E(X,Y)|$ for $X, Y \subseteq V$. If $X = Y$, then we set $E(X) = E(X,X)$ and $m(X) = |E(X)|$.

The choice of the partition immediately implies

$$m(V_1) < k \text{ and} \tag{3}$$
$$m(V_1) + m(V_1, V_2) \leq m - 1 \leq 2k - 1. \tag{4}$$

Our first aim is to show that there is a vertex $y \in V_2$ such that

$$0 < m(V_1, V_2 \setminus \{y\}) < k \text{ and} \tag{5}$$
$$0 < m(V_2) < k. \tag{6}$$

The lower bound of (6) holds by the choice of the partition.

Note that every vertex u different from x satisfies

$$d_G(u) \leq m - d_G(x) - 1 \leq m - d_G(u) + 1 \leq 2k - d_G(u) + 1$$

which implies $d_G(u) \leq k$.

If $m(V_1, V_2 \setminus \{y\}) = 0$, then the diameter condition implies that y is adjacent to all vertices in $V \setminus \{y\}$. By the choice of x, this implies that also x is adjacent to all vertices in $V \setminus \{x\}$ and hence $m(V_1, V_2 \setminus \{y\}) \geq |V_2| - 1 > 0$ which is a contradiction. This shows the lower bound of (5).

If V_2 has at least 3 vertices, then by the choice of the partition we can choose a vertex $y \in V_2$ such that

$$\begin{aligned} m(V_2 \setminus \{y\}) &\geq 1 \text{ and} \\ m(V_1 \cup \{y\}) &\geq k. \end{aligned} \tag{7}$$

By (7) we get the upper bound of (5):

$$m(V_1, V_2 \setminus \{y\}) \leq m - m(V_1 \cup \{y\}) - m(V_2) \leq 2k - k - 1 < k.$$

By (7) and (5), we get

$$m(V_2) \leq m - m(V_1 \cup \{y\}) - m(V_1, V_2 \setminus \{y\}) \leq 2k - k - 1 < k,$$

thus (6) holds as well.

Finally, if V_2 has only two vertices then $V_2 = \{y, z\}$, $yz \in E(G)$, implying (6), and $V_2 \setminus \{y\} = \{z\}$. Thus $m(V_1, V_2 \setminus \{y\}) = d_G(z) - 1 < k$ holds, the upper bound of (5).

We are now ready to define an edge irregular total k-labelling of G

$$f : V \cup E \to \{1, 2, ..., k\}.$$

By (3), $l := m(V_1) + 1$ satisfies $2 \leq l \leq k$.

Let

$$f(u) := \begin{cases} 1 & , u \in V_1, \\ l & , u = y \text{ and} \\ k & , u \in V_2 \setminus \{y\}. \end{cases}$$

Let

$$\begin{aligned} \{f(e) \mid e \in E(V_1)\} &= \{1, 2, ..., l\} \text{ and} \\ \{f(e) \mid e \in E(\{y\}, V_1)\} &= \{1, 2, ..., m(\{y\}, V_1)\}. \end{aligned}$$

Note that $m(\{y\}, V_1) \leq d_G(y) \leq k$.

Let

$$\{f(e) \mid e \in E(V_1, V_2 \setminus \{y\})\} = \{k - m(V_1, V_2 \setminus \{y\}) + 1, ..., k\}.$$

By (4) and (5), the edges $e \in E(V_1) \cup E(V_1, V_2)$ receive different weights $F(e) \in \{1, 2, \ldots, 2k+1\}$. Now we label the edges in $E(V_2)$ such that they receive different weights $F(e) \in \{2k+2, ..., 3k\}$. If $m(V_2) = 1$, say $E(V_2) = \{e\}$, then let $f(e) = k$.

If $m(V_2) \geq 2$, then (7) implies

$$\begin{aligned} m(\{y\}, V_2 \setminus \{y\}) &\leq d_G(y) - m(\{y\}, V_1) \\ &= d_G(y) - m(V_1 \cup \{y\}) + m(V_1) \\ &\leq k - k + l - 1 = l - 1 \end{aligned}$$

and hence $k - l + 1 + m(\{y\}, V_2 \setminus \{y\}) \leq k$.

Let

$$\{f(e) \mid e \in E(\{y\})\} = \{k - l + 2, ..., k - l + 1 + m(\{y\}, V_2 \setminus \{y\})\}.$$

Finally, let

$$\{f(e) \mid e \in E(V_2 \setminus \{y\})\} = \{k - m(V_2 \setminus \{y\}) + 1, ..., k\}.$$

By (6), the weights of the edges in $E(V_2)$ are as desired which completes the proof. $\square$

We proceed to our main result. As in the previous proof, it relies on a suitable partition of the vertex set whose existence we establish using Azuma's inequality. There is still some space for improving the involved constants. We did not try to optimize them in order to keep the arguments clear and simple.

Theorem 3 (Azuma [3], cf. also [15], p. 92) *If X is a random variable determined by n trials $T_1, T_2, ..., T_n$ such that for each i, and any two possible sequences of outcomes $t_1, ..., t_{i-1}, t_i$ and $t_1, ..., t_{i-1}, t'_i$ we have*

$$|\mathbf{E}(X \mid T_1 = t_1, ..., T_{i-1} = t_{i-1}, T_i = t_i) - \mathbf{E}(X \mid T_1 = t_1, ..., T_{i-1} = t_{i-1}, T_i = t'_i)| \leq d_i,$$

then

$$\mathbf{P}\left(|X - \mathbf{E}(X)| > t\right) \leq 2 \exp\left(-t^2 \Big/ 2\left(\sum_{i=1}^{n} d_i^2\right)\right).$$

for $t > 0$.

In the next lemma we establish the existence of a suitable vertex partition of a graph into 4 sets. Eventually, the vertices in each set will receive the same label.

Lemma 4 *If $0 < \delta < 1$ and $G = (V, E)$ is a graph with order n, size m and degree sequence $(d_1, d_2, ..., d_n)$ such that*

$$\delta^2 m^2 > 2\ln(16)\sum_{i=1}^{n} d_i^2,$$

then there is a partition

$$V = V_1 \cup V_2 \cup V_3 \cup V_4$$

such that

$$\begin{aligned}
\left|m_{1,1} - \frac{m}{9}\right| &\leq \delta m,\\
\left|m_{1,1} + m_{1,2} - \frac{2m}{9}\right| &\leq \delta m,\\
\left|m_{1,1} + m_{1,2} + m_{2,2} - \frac{m}{4}\right| &\leq \delta m,\\
\left|m_{1,1} + m_{1,2} + m_{2,2} + m_{1,3} - \frac{13m}{36}\right| &\leq \delta m,\\
\left|m_{4,4} - \frac{m}{9}\right| &\leq \delta m,\\
\left|m_{4,4} + m_{4,3} - \frac{2m}{9}\right| &\leq \delta m,\\
\left|m_{4,4} + m_{4,3} + m_{3,3} - \frac{m}{4}\right| &\leq \delta m \text{ and}\\
\left|m_{4,4} + m_{4,3} + m_{3,3} + m_{4,2} - \frac{13m}{36}\right| &\leq \delta m.
\end{aligned}$$

where $m_{i,j} = m(V_i, V_j)$ for $1 \leq i \leq j \leq 4$.

Proof: Let $p_1 = p_4 = \frac{1}{3}$ and $p_2 = p_3 = \frac{1}{6}$. We consider a random partition $V = V_1 \cup V_2 \cup V_3 \cup V_4$ of V that arises by assigning every vertex in V independently at random to V_i with probability p_i for $1 \leq i \leq 4$.

Clearly, $\mathbf{E}(m_{i,i}) = p_i^2 m$ and $\mathbf{E}(m_{i,j}) = 2p_i p_j m$ for $1 \leq i < j \leq 4$. We consider the following 8 sums of at most 4 different of the random variables $m_{i,j}$ for $1 \leq i \leq j \leq 4$: $m_{1,1}$, $m_{1,1}+m_{1,2}$, $m_{1,1}+m_{1,2}+m_{2,2}$, $m_{1,1}+m_{1,2}+m_{2,2}+m_{1,3}$, $m_{4,4}$, $m_{4,4}+m_{4,3}$, $m_{4,4}+m_{4,3}+m_{3,3}$ and $m_{4,4} + m_{4,3} + m_{3,3} + m_{4,2}$.

Changing the assignment of the ith vertex can change the expected value of any of these 8 random variables conditional on the assignment of the first i vertices by at most the degree d_i of the i-th vertex. This is exactly the kind of condition that we need to apply Azuma's inequality from Theorem 3. Since

$$2\exp\left(-(\delta m)^2 \middle/ \left(2\sum_{i=1}^{n} d_i^2\right)\right) < 2e^{-\ln 16} = \frac{1}{8},$$

with positive probability all 8 of the random variables S considered above satisfy $|S - \mathbf{E}(S)| \leq \delta m$ which implies the existence of the desired partition. $\square$

We proceed to our main result which defines an irregular total labelling based on the partition from the previous lemma.

Theorem 5 *Every graph* $G = (V, E)$ *of order* n*, size* $m \geq 1000$ *and degree sequence* $(d_1, d_2, ..., d_n)$ *with*

$$m^2 > 2 \cdot 100^2 \cdot \ln(16) \sum_{i=1}^{n} d_i^2$$

satisfies

$$\text{tes}(G) = \left\lceil \frac{|E| + 2}{3} \right\rceil.$$

Proof: Let $G = (V, E)$, n, m and $(d_1, d_2, ..., d_n)$ be as in the statement of the Theorem. In view of the lower bound (2) it suffices to prove the existence of a mapping

$$f : V \cup E \to \left\{0, 1, ..., \left\lceil \frac{m-1}{3} \right\rceil \right\}$$

such that

$$f(u) + f(uv) + f(v) \neq f(u') + f(u'v') + f(v')$$

for every $uv, u'v' \in E$ with $uv \neq u'v'$. Note that we allow 0 as the smallest label, in order to make some arguments more symmetric. (Increasing all values of f by 1 increases all weights by 3 and results in an irregular total labelling as defined above.)

Since $m \geq 1000$ the following conditions hold for $\delta = 10^{-2}$:

$$\left(\frac{1}{9} - \delta\right) m > \left\lceil \frac{m-1}{10} \right\rceil \tag{8}$$

$$\left(\frac{2}{9} - \delta\right) m > 2 \left\lceil \frac{m-1}{10} \right\rceil \tag{9}$$

$$\left(\frac{1}{4} - \delta\right) m > \left\lceil \frac{m-1}{3} \right\rceil - \left\lceil \frac{m-1}{10} \right\rceil \tag{10}$$

$$\left(\frac{13}{36} - \delta\right) m > \left\lceil \frac{m-1}{3} \right\rceil \tag{11}$$

$$\left(\frac{1}{4} + \delta\right) m < \left\lceil \frac{m-1}{3} \right\rceil \tag{12}$$

$$\left(\frac{13}{36} + \delta\right) m < 2 \left\lceil \frac{m-1}{3} \right\rceil - \left\lceil \frac{m-1}{10} \right\rceil. \tag{13}$$

By Lemma 4, there is a partition $V = V_1 \cup V_2 \cup V_3 \cup V_4$ such that for $\delta = 10^{-2}$ the conditions from Lemma 4 hold.

For $v \in V$ let

$$f(v) = \begin{cases} 0, & v \in V_1, \\ \left\lceil \frac{m-1}{10} \right\rceil, & v \in V_2, \\ \left\lceil \frac{m-1}{3} \right\rceil - \left\lceil \frac{m-1}{10} \right\rceil, & v \in V_3 \text{ and} \\ \left\lceil \frac{m-1}{3} \right\rceil, & v \in V_4. \end{cases}$$

For $1 \leq i \leq j \leq 4$ let $E_{i,j} = \{uv \in E \mid u \in V_i, v \in V_j\}$.

We will now describe how to define values

$$f(uv) \in \left\{0, 1, 2, ..., \left\lceil \frac{m-1}{3} \right\rceil \right\}$$

for the edges $uv \in E$ of G such that the weights $F(uv) = f(u) + f(uv) + f(v)$ are different for all edges $uv \in E$. The inequalities (8)-(13) will imply that this is possible.

Step 1 Since

$$m_{1,1} \leq \left(\frac{1}{9} + \delta\right) m < \left\lceil \frac{m-1}{3} \right\rceil$$

by (12), we can assign labels $f(uv) \in \left\{0, 1, 2, ..., \left\lceil \frac{m-1}{3} \right\rceil \right\}$ to the edges $uv \in E_{1,1}$ such that

$$\{F(uv) \mid uv \in E_{1,1}\} = \{0, 1, 2, ..., m_{1,1} - 1\}.$$

Step 2 Since

$$m_{1,1} \geq \left(\frac{1}{9} - \delta\right) m > \left\lceil \frac{m-1}{10} \right\rceil = f(u) + f(v)$$

for $uv \in E_{1,2}$ by (8) and

$$m_{1,1} + m_{1,2} \leq \left(\frac{2}{9} + \delta\right) m < \left\lceil \frac{m-1}{3} \right\rceil$$

by (12), we can assign values $f(uv) \in \left\{0, 1, 2, ..., \left\lceil \frac{m-1}{3} \right\rceil \right\}$ to the edges $uv \in E_{1,2}$ such that

$$\{F(uv) \mid uv \in E_{1,2}\} = \{m_{1,1}, m_{1,1} + 1, ..., m_{1,1} + m_{1,2} - 1\}.$$

Step 3 Since

$$m_{1,1} + m_{1,2} \geq \left(\frac{2}{9} - \delta\right) m > 2 \left\lceil \frac{m-1}{10} \right\rceil = f(u) + f(v)$$

for $uv \in E_{2,2}$ by (9) and

$$m_{1,1} + m_{1,2} + m_{2,2} < \left\lceil \frac{m-1}{3} \right\rceil$$

by (12), we can assign values $f(uv) \in \{0, 1, 2, ..., \lceil \frac{m-1}{3} \rceil\}$ to the edges $uv \in E_{2,2}$ such that

$$\{F(uv) \mid uv \in E_{2,2}\} \quad = \quad \{m_{1,1} + m_{1,2}, m_{1,1} + m_{1,2} + 1, ..., m_{1,1} + m_{1,2} + m_{2,2} - 1\}.$$

Step 4 Since

$$m_{1,1} + m_{1,2} + m_{2,2} > \left\lceil \frac{m-1}{3} \right\rceil - \left\lceil \frac{m-1}{10} \right\rceil = f(u) + f(v)$$

for $uv \in E_{1,3}$ by (10) and

$$m_{1,1} + m_{1,2} + m_{2,2} + m_{1,3} < 2\left\lceil \frac{m-1}{3} \right\rceil - \left\lceil \frac{m-1}{10} \right\rceil$$

by (13), we can assign values $f(uv) \in \{0, 1, 2, ..., \lceil \frac{m-1}{3} \rceil\}$ to the edges $uv \in E_{1,3}$ such that

$$\{F(uv) \mid uv \in E_{1,3}\} \quad = \quad \{m_{1,1} + m_{1,2} + m_{2,2}, ..., m_{1,1} + m_{1,2} + m_{2,2} + m_{1,3} - 1\}.$$

Step 5 By symmetry, it is possible to assign values $f(uv) \in \{0, 1, 2, ..., \lceil \frac{m-1}{3} \rceil\}$ to the edges $uv \in E_{2,4} \cup E_{3,3} \cup E_{3,4} \cup E_{4,4}$ such that

$$\{F(uv) \mid uv \in E_{2,4} \cup E_{3,3} \cup E_{3,4} \cup E_{4,4}\} = \{m - (m_{2,4} + m_{3,3} + m_{3,4} + m_{4,4}), ..., m - 1\}.$$

Step 6 By (11), we have

$$m_{1,1} + m_{1,2} + m_{2,2} + m_{1,3} \quad > \quad \left\lceil \frac{m-1}{3} \right\rceil = f(u) + f(v)$$

and also

$$m_{2,4} + m_{3,3} + m_{3,4} + m_{4,4} \quad > \quad \left\lceil \frac{m-1}{3} \right\rceil = f(u) + f(v)$$

for $uv \in E_{1,4} \cup E_{2,3}$. Therefore, by symmetry, it is possible to assign values $f(uv) \in \{0, 1, 2, ..., \lceil \frac{m-1}{3} \rceil\}$ to the edges $uv \in E_{1,4} \cup E_{2,3}$ such that

$$\begin{aligned} & \{F(uv) \mid uv \in E_{1,6} \cup E_{2,5} \cup E_{3,4}\} \\ = \; & \{m_{1,1} + m_{1,2} + m_{2,2} + m_{1,3}, ..., m_{1,1} + m_{1,2} + m_{2,2} + m_{1,3} + m_{1,4} + m_{2,3}\} \\ \subseteq \; & \{m_{1,1} + m_{1,2} + m_{2,2} + m_{1,3}, ..., m - (m_{2,4} + m_{3,3} + m_{3,4} + m_{4,4}) - 1\}. \end{aligned}$$

Altogether, all values of f have been defined appropriately and the proof is complete. □

We close by deriving a corollary from Theorem 5.

Corollary 6 *Every graph $G = (V, E)$ of order n, size m and maximum degree Δ such that $m > 4 \cdot 100^2 \cdot \ln(16)\Delta \approx 110903.55\Delta$ satisfies* $\mathrm{tes}(G) = \left\lceil \frac{|E|+2}{3} \right\rceil$.

Proof: Let $(d_1, d_2, ..., d_n)$ denote the degree sequence of G. The convexity of the function $x \mapsto x^2$ and the fact that all degrees are bounded by Δ imply that $\sum_{i=1}^{n} d_i^2 \leq \frac{\sum_{i=1}^{n} d_i}{\Delta} \Delta^2 = 2m\Delta$. Now

$$m^2 \;>\; 2 \cdot 100^2 \cdot \ln(16) \cdot (2m\Delta) \geq 2 \cdot 100^2 \cdot \ln(16) \sum_{i=1}^{n} d_i^2. \quad (14)$$

Since $m > 0$ implies $\Delta > 0$ and hence $m > 4 \cdot 100^2 \cdot \ln(16) \geq 1000$, the result follows from Theorem 5. □

Note that $0 < \Delta < \frac{10^{-3}m}{\sqrt{8n}}$ implies $n \geq \frac{2m}{\Delta} > \frac{2 \cdot 1000\Delta\sqrt{8n}}{\Delta}$ and hence $m > 16 \cdot 10^6 \Delta$. Therefore, Corollary 6 improves the main result from [7] in every case.

Since the maximum degree of a graph is always bounded by its order minus 1, Corollary 6 implies Conjecture 1 for graphs of size at least 111000 times their order.

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