Locally Dense Independent Sets in Regular Graphs of Large Girth

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Abstract. For an integer $d \geq 3$ let $\alpha(d)$ be the supremum over all α with the property that for every $\epsilon > 0$ there exists some $g(\epsilon)$ such that every d-regular graph of order n and girth at least $g(\epsilon)$ has an independent set of cardinality at least $(\alpha - \epsilon)n$.

Extending an approach proposed by Lauer and Wormald (Large independent sets in regular graphs of large girth, *J. Comb. Theory, Ser. B* **97** (2007), 999-1009) and improving results due to Shearer (A note on the independence number of triangle-free graphs, II, *J. Comb. Theory, Ser. B* **53** (1991), 300-307) and Lauer and Wormald, we present the best known lower bounds for $\alpha(d)$ for all $d \geq 3$.

Keywords. regular graph; independence; girth; randomized algorithm

1 Introduction

In the present paper we consider the independence number $\alpha(G)$ of finite, simple and undirected graphs $G = (V, E)$ which are d-regular for some $d \geq 3$ and have large girth.

For integers $d \geq 3$ and $g \geq 3$ let $\mathcal{G}(d, g)$ denote the class of all d-regular graphs of girth at least g and let

$$\alpha(d, g) := \sup \left\{ \alpha \mid \alpha(G) \geq \alpha \cdot |V| \text{ for all } G = (V, E) \in \mathcal{G}(d, g) \right\}.$$

Clearly, $\alpha(d, g)$ is monotonic non-decreasing in g and bounded above by 1 and we can consider

$$\alpha(d) := \lim_{g \to \infty} \alpha(d, g).$$

Note that this definition implies that for every $\epsilon > 0$ there exists some $g(\epsilon)$ such that $\alpha(G) \geq (\alpha(d) - \epsilon) \cdot |V|$ for every graph $G = (V, E) \in \mathcal{G}(d, g(\epsilon))$.

Our aim is to prove lower bounds on $\alpha(d)$.

While the first result on the independence number in regular graphs of large girth is due to Hopkins and Staton [5] who proved $\alpha(3) \geq \frac{7}{18} \approx 0.3888$, for quite a long time the best known estimates of $\alpha(d)$ were due to Shearer [8].

Theorem 1 (Shearer 1991) *If*

$$\beta_{\text{Shearer}}(d) := \begin{cases} \frac{125}{302} \approx 0.4139 & \textit{for } d = 3 \textit{ and} \\ \frac{1 + d(d-1)\beta_{\text{Shearer}}(d-1)}{d^2+1} & \textit{for } d \geq 4, \end{cases}$$

then

$$\alpha(d) \geq \beta_{\text{Shearer}}(d)$$

for all $d \geq 3$.

Only very recently Lauer and Wormald [6] improved Shearer's result for $d \geq 7$.

Theorem 2 (Lauer and Wormald 2007) *For all $d \geq 3$*

$$\alpha(d) \geq \beta_{\text{LauWo}}(d) := \frac{1 - (d-1)^{-2/(d-2)}}{2}.$$

From a very abstract viewpoint their approaches are actually similar. On the one hand Shearer constructs an independent set by carefully selecting vertices according to some degree dependent weight function, adding them to the independent set, deleting them together with their neighbours and iterating this process. On the other hand Lauer and Wormald construct an independent set by randomly selecting vertices, adding most of them to the independent set, deleting them together with their neighbours and iterating this process.

In order to get some intuition about how to improve these approaches it is instructive to see that a very simple argument allows to improve Shearer's bound on $\alpha(3)$.

Proposition 3 $\alpha(3) \geq 0.4142 > \beta_{\text{Shearer}}(3) \approx 0.4139$.

Proof: It follows from Theorem 4 in [8] that for every $\epsilon > 0$ there is some $g(\epsilon)$ such that: *If $G = (V, E)$ is a graph of order n and girth at least $g(\epsilon)$, with n_2 vertices of degree 2 no two of which are adjacent and $n_3 = n - n_2$ vertices of degree 3, then*

$$\alpha(G) \geq \left(\frac{79}{151} - \epsilon\right) n_2 + \left(\frac{125}{302} - \epsilon\right) n_3. \tag{1}$$

For a cubic graph G of order n and sufficiently large girth $g(\epsilon)$ the 9-th power G^9 is $(3 + 3 \cdot 2 + 3 \cdot 2^2 + ... + 3 \cdot 2^8) = 1533$-regular. Therefore, G^9 has an independent set I^9 with $|I^9| \geq n/\left(\Delta\left(G^9\right) + 1\right) = n/1534$ where $\Delta\left(G^9\right)$ denotes the maximum degree of G^9 (cf. e.g. [4, 9]).

Let H arise from G by deleting all vertices within distance at most 3 from I^9. We construct an independent set of G by adding all $7|I^9|$ many vertices at distance 0 or 2 from a vertex in I^9 and by applying (1) to H. It follows that

$$\begin{aligned}
\alpha(G) &\geq 7|I^9| + \alpha(H) \\
&\geq 7|I^9| + \left(\frac{79}{151} - \epsilon\right) 24|I^9| + \left(\frac{125}{302} - \epsilon\right)(n - (22 + 24)|I^9|) \\
&\geq (0.4142 - \epsilon)n
\end{aligned}$$

which completes the proof. $\square$

The proof of Proposition 3 suggests that it is worthwhile to consider the iterative deletion not just of a vertex and its neighbours — which would induce a rooted tree of depth 1 — but of rooted trees of larger depths. Locally this should allow us to pack the vertices of the independent set more densely which hopefully yields an overall improvement.

We follow exactly this intuition by generalizing the random procedure and its analysis using differential equations proposed by Lauer and Wormald in [6].

2 The Algorithm $TREE(k, l, p, f)$

In this section we describe a random procedure $TREE(k, l, p, f)$ which depends on two integers $k, l \geq 0$, a real value $p \in [0, 1]$ and a function f which maps rooted trees T with root r to independent subsets $f(T, r)$ of their vertex set. We assume that $|f(T, r)| = |f(T', r')|$ for isomorphic rooted trees, T rooted at r and T' rooted at r'.

The algorithm $TREE(k, l, p, f)$ will be applied to a graph $G = (V, E)$ of girth at least $2(l + 1)$. It executes k rounds and determines disjoint rooted subtrees of G of depth at most l. The value p will serve as a probability.

We need a little more notation to describe the algorithm. Let $u \in V$ be a vertex of the graph G. For an integer $i \geq 0$ let $N_G^i(u)$ and $N_G^{\leq i}(u)$ denote the sets of vertices of G

within distance — measured with respect to G — exactly i from u and at most i from u, respectively, i.e.

$$\begin{aligned} N_G^i(u) &= \{v \in V \mid \mathrm{dist}_G(v,u) = i\} \text{ and} \\ N_G^{\leq i}(u) &= \{v \in V \mid \mathrm{dist}_G(v,u) \leq i\}. \end{aligned}$$

Furthermore, let $B_G^i(u)$ denote the set of vertices $v \in N_G^{\leq i}(u)$ which are not adjacent to a vertex in $V \setminus N_G^{\leq i}(u)$, i.e.

$$B_G^i(u) = \left\{v \in V \mid N_G^{\leq 1}(v) \subseteq N_G^{\leq i}(u)\right\}.$$

For a set $U \subseteq V$ of vertices of G the subgraph of G induced by $V \setminus U$ is denoted by $G - U$ or $G[V \setminus U]$.

$TREE(k,l,p,f)$ proceeds as follows:

(1) Set $G_0 = (V_0, E_0) := G$, $Z_0 := \emptyset$ and $i := 0$.

(2) While $i < k$ select a subset X_i of V by assigning every vertex of G to X_i independently at random with probability p.

Set

$$\begin{aligned} G_{i+1} = (V_{i+1}, E_{i+1}) &:= G_i - \bigcup_{u \in X_i \cap V_i} N_{G_i}^{\leq l}(u), && (2) \\ X_i^* &:= \{v \in X_i \mid \mathrm{dist}_G(v,u) \geq 2l+1 \ \forall u \in X_i \setminus \{v\}\}, && (3) \\ T_i(u) &:= G_i\left[B_{G_i}^l(u)\right], && (4) \\ \Delta Z_i &:= \bigcup_{u \in X_i^* \cap V_i} f\left(T_i(u), u\right), && (5) \\ Z_{i+1} &:= Z_i \cup \Delta Z_i, && (6) \\ i &:= i+1. \end{aligned}$$

(3) Output Z_k.

There are some subtleties we want to stress: The definition of G_{i+1} in (2) and ΔZ_i in (5) use neighbourhoods within the graph G_i while X_i^* is defined in (3) with respect to G. Furthermore, the construction of Z_{i+1} in (4), (5) and (6) does not influence the evolution of the G_i. By the girth condition, $T_i(v)$ is a tree and $f\left(T_i(v), v\right)$ is a well-defined subset of $B_{G_i}^l(v)$.

We first observe that $TREE(k,l,p,f)$ really produces an independent set of G.

Lemma 4 *Z_k is an independent set of G.*

Proof: For contradiction, we assume that $v, w \in Z_k$ with $vw \in E$.

Let $v \in \Delta Z_i$ and $w \in \Delta Z_j$ with, say, $i \leq j$. Let $v \in f\left(T_i(u), u\right)$ for some $u \in X_i^* \cap V_i$. Since, by the definition of X_i^* in (3), the set $N_{G_i}^{\leq l}(u) \cap N_{G_i}^{\leq l}(u')$ is empty for all distinct $u, u' \in X_i^*$ we obtain that $w \in V_i$ is a neighbour of v outside of $N_{G_i}^{\leq l}(u)$ which implies the contradiction $v \notin B_{G_i}^l(u)$. □

3 The Analysis of $TREE(k, l, p, f)$

Throughout this section we will assume that $G = (V, E)$ is a d-regular graph for some $d \geq 3$ and sufficiently large girth. We consider the behaviour of $TREE(k, l, p, f)$ when applied to this graph. We will specify the necessary girth conditions which are all in terms of k and l more exactly whenever they are explicitely needed.

It is one of the key observations made by Lauer and Wormald in [6] that for a sufficiently large girth the probabilities which are suitable to describe the behaviour of their randomized algorithm can be well understood. The next lemma corresponds to Lemma 2 in [6].

Lemma 5 *Let $k \geq 2$ and $0 \leq i \leq k$. Let the girth of G be at least $2(k+1)l + 2$ and let $u \in V$ and $vv' \in E$.*

- *(i) The probabilities $\mathbf{P}[u \in V_i]$, $\mathbf{P}[(v \in V_i) \wedge (v' \in V_i)]$ and $\mathbf{P}[u \in \Delta Z_i]$ as well as the conditional expected value $\mathbf{E}\left[\left|f\left(T_i(u), u\right)\right| \mid u \in X_i^* \cap V_i\right]$ do not depend on the choice of the vertex u or the edge vv'.*
- *(ii) Conditional upon the event $(v \in V_i)$, the event $(v' \in V_i)$ depends only on the intersection of the sets $X_0, X_1, ..., X_{i-1}$ with $N_{G-v}^{\leq il}(v')$.*

Proof: It follows immediately, by induction on i, from the description of $TREE(k, l, p, f)$ that the events $(u \in V_i)$ and $(v \in V_i) \wedge (v' \in V_i)$ depend only on the intersection of the sets $X_0, X_1, ..., X_{i-1}$ with $N_G^{\leq il}(u)$ and $N_G^{\leq il}(v) \cup N_G^{\leq il}(v')$, respectively. Furthermore, the event $(u \in \Delta Z_i)$ depends only on the intersection of the sets $X_0, X_1, ..., X_{i-1}$ with $N_G^{\leq (i+1)l}(u)$ and the intersection of the set X_i with $N_G^{\leq 2l}(u)$. Finally, conditional upon the event $(u \in X_i^* \cap V_i)$, the cardinality of $f\left(T_i(u), u\right)$ depends only on the intersection of the sets $X_0, X_1, ..., X_{i-1}$ with $N_G^{\leq (i+1)l}(u)$.

Since, by the girth condition, the induced subgraphs $G\left[N_G^{\leq 2l}(u)\right]$, $G\left[N_G^{\leq (i+1)l}(u)\right]$ and $G\left[N_G^{\leq il}(v) \cup N_G^{\leq il}(v')\right]$ are isomorphic for all choices of the vertex u or the edge vv', we obtain (i). Similarly, (ii) follows immediately by induction on i. □

By Lemma 5, for $0 \leq i \leq k$ the following quantities

$$\begin{aligned}
r_i &:= \mathbf{P}[u \in V_i], \\
w_i &:= \frac{\mathbf{P}[(v \in V_i) \wedge (v' \in V_i)]}{\mathbf{P}[v \in V_i]} = \mathbf{P}[(v \in V_i) \wedge (v' \in V_i) | v \in V_i], \\
f_l(w_i) &:= \mathbf{E}\left[|f\left(T_i(u), u\right)| \mid u \in X_i^* \cap V_i\right], \\
\Delta z_i &:= \mathbf{P}[u \in \Delta Z_i] \text{ and} \\
z_i &:= \mathbf{P}[u \in Z_k \setminus Z_i]
\end{aligned}$$

are the same for every vertex $u \in V$ and every edge $vv' \in E$.

Using Lemma 5, we can determine the following recursions for these probabilities.

Lemma 6 *Let the girth of G be at least $2(k+1)l+2$.*

(i) $r_0 = w_0 = 1$ and $z_{i+1} = z_i - \Delta z_i$ for $0 \leq i \leq k-1$.

(ii) For $0 \leq i \leq k-1$

$$\begin{aligned}
r_{i+1} &= r_i \left(1 - p \cdot \left(1 + \sum_{j=1}^{l} d(d-1)^{j-1} \cdot w_i^j\right) + O(p^2)\right), \\
w_{i+1} &= w_i \left(1 - p \cdot \left(1 + \sum_{j=1}^{l} (d-2)(d-1)^{j-1} \cdot w_i^j\right) + O(p^2)\right) \text{ and} \\
\Delta z_i &= f_l(w_i) \cdot r_i \cdot p \cdot \prod_{j=1}^{2l} (1-p)^{d(d-1)^{j-1}}
\end{aligned}$$

where the constants implicit in the $O(\cdot)$-terms depend only on d and l.

Proof: (i) is immediate from the definitions and we proceed to the proof of (ii).

Let $u \in V$ be fixed. For $v \in N_G^{\leq l}(u)$ let P_v denote the vertex set of the unique path of length at most l from u to v. The event $(u \in V_{i+1})$ holds if and only if

$$(u \in V_i) \wedge (u \not\in X_i) \wedge \bigwedge_{v : 1 \leq \text{dist}_G(u,v) \leq l} \Big((v \not\in X_i) \vee \Big((v \in X_i) \wedge (P_v \not\subseteq V_i)\Big)\Big).$$

Expanding this representation of the event $(u \in V_{i+1})$ to a disjunction of conjunctions, all events corresponding to the conjunctions are disjoint because they differ in

$$X_i \cap \{v : 1 \leq \text{dist}_G(u,v) \leq l\}.$$

Furthermore, all of those events for which two of the independent events $(v \in X_i)$ for some v with $1 \le \mathrm{dist}_G(u,v) \le l$ hold, will contribute together only $\mathbf{P}[u \in V_i] \cdot O(p^2)$ to $\mathbf{P}[u \in V_{i+1}]$ where the constant implicit in the $O(\cdot)$-term depends only on d and l. Therefore,

$$\begin{aligned}
\mathbf{P}[u \in V_{i+1}] = \; & \mathbf{P}\left[(u \in V_i) \wedge (u \notin X_i) \wedge \bigwedge_{v:1 \le \mathrm{dist}_G(u,v) \le l} (v \notin X_i)\right] + \mathbf{P}[u \in V_i] \cdot O(p^2) \\
+ \sum_{v:1 \le \mathrm{dist}_G(u,v) \le l} & \mathbf{P}\left[(u \in V_i) \wedge (v \in X_i) \wedge (P_v \not\subseteq V_i) \wedge \bigwedge_{v':(v' \ne v) \wedge (0 \le \mathrm{dist}_G(u,v') \le l)} (v' \notin X_i)\right] \\
= \; & \mathbf{P}[u \in V_i] \cdot (1-p)^{\left(1+\sum_{j=1}^{l} d(d-1)^{j-1}\right)} \\
& + \sum_{v:1 \le \mathrm{dist}_G(u,v) \le l} \mathbf{P}\left[(u \in V_i) \wedge (P_v \not\subseteq V_i)\right] \cdot p \cdot (1-p)^{\sum_{j=1}^{l} d(d-1)^{j-1}} \\
& + \mathbf{P}[u \in V_i] \cdot O(p^2) \\
= \; & \mathbf{P}[u \in V_i] \cdot \left(1 - \left(1 + \sum_{j=1}^{l} d(d-1)^{j-1}\right) p\right) \\
& + \sum_{v:1 \le \mathrm{dist}_G(u,v) \le l} \mathbf{P}\left[(u \in V_i) \wedge (P_v \not\subseteq V_i)\right] \cdot p \\
& + \mathbf{P}[u \in V_i] \cdot O(p^2).
\end{aligned}$$

In order to evaluate $\mathbf{P}\left[(u \in V_i) \wedge (P_v \not\subseteq V_i)\right]$ let $u = u_0 u_1 u_2 \ldots u_j = v$ be the unique path from u to v for some $1 \le j \le l$.

By Lemma 5 (i) and (ii), we have for $0 \le \nu \le j-1$

$$\begin{aligned}
\mathbf{P}\left[(u_0, u_1, \ldots, u_\nu \in V_i) \wedge (u_{\nu+1} \notin V_i)\right] = \; & \mathbf{P}\left[u_0 \in V_i\right] \\
& \cdot \mathbf{P}\left[u_1 \in V_i \mid u_0 \in V_i\right] \\
& \cdot \mathbf{P}\left[u_2 \in V_i \mid u_0, u_1 \in V_i\right] \\
& \cdot \ldots \cdot \mathbf{P}\left[u_\nu \in V_i \mid u_0, u_1, \ldots, u_{\nu-1} \in V_i\right] \\
& \cdot \mathbf{P}\left[u_{\nu+1} \notin V_i \mid u_0, u_1, \ldots, u_\nu \in V_i\right] \\
= \; & \mathbf{P}\left[u_0 \in V_i\right] \\
& \cdot \mathbf{P}\left[u_1 \in V_i \mid u_0 \in V_i\right] \\
& \cdot \mathbf{P}\left[u_2 \in V_i \mid u_1 \in V_i\right] \\
& \cdot \ldots \cdot \mathbf{P}\left[u_\nu \in V_i \mid u_{\nu-1} \in V_i\right] \\
& \cdot \left(1 - \mathbf{P}\left[u_{\nu+1} \notin V_i \mid u_\nu \in V_i\right]\right) \\
= \; & r_i w_i^\nu (1 - w_i)
\end{aligned}$$

and we obtain

$$\begin{aligned}
\mathbf{P}\left[(u \in V_i) \wedge (P_v \not\subseteq V_i)\right] &= \mathbf{P}\left[(u_0 \in V_i) \wedge ((u_0 \notin V_i) \vee (u_1 \notin V_i) \vee ... \vee (u_j \notin V_i)))\right] \\
&= \mathbf{P}\left[(u_0 \in V_i) \wedge (u_1 \notin V_i)\right] \\
&\quad +\mathbf{P}\left[(u_0, u_1 \in V_i) \wedge (u_2 \notin V_i)\right] \\
&\quad +\mathbf{P}\left[(u_0, u_1, u_2 \in V_i) \wedge (u_3 \notin V_i)\right] \\
&\quad ... \\
&\quad +\mathbf{P}\left[(u_0, u_1, ..., u_{j-1} \in V_i) \wedge (u_j \notin V_i)\right] \\
&= r_i(1-w_i) + r_i w_i (1-w_i) + ... + r_i w_i^{j-1}(1-w_i) \\
&= r_i(1-w_i^j).
\end{aligned}$$

Putting everything together, we obtain

$$\begin{aligned}
r_{i+1} &= \mathbf{P}[u \in V_{i+1}] \\
&= r_i \cdot \left(1 - \left(1 + \sum_{j=1}^{l} d(d-1)^{j-1}\right) p\right) + \sum_{j=1}^{l} d(d-1)^{j-1} r_i \cdot (1-w_i^j) \cdot p + r_i \cdot O(p^2) \\
&= r_i \cdot (1-p) - \sum_{j=1}^{l} d(d-1)^{j-1} r_i \cdot w_i^j \cdot p + r_i \cdot O(p^2) \\
&= r_i \left(1 - p \cdot \left(1 + \sum_{j=1}^{l} d(d-1)^{j-1} \cdot w_i^j\right) + O(p^2)\right).
\end{aligned}$$

By the same type of argument, it follows that for every edge $vv' \in E$

$$\begin{aligned}
&\mathbf{P}[(v \in V_{i+1}) \wedge (v' \in V_{i+1})] \\
= \; &\mathbf{P}[(v \in V_i) \wedge (v' \in V_i)] \left(1 - p \cdot \left(2 + \sum_{j=1}^{l} (2d-2)(d-1)^{j-1} \cdot w_i^j\right) + O(p^2)\right).
\end{aligned}$$

Since $w_i = \frac{\mathbf{P}[(v \in V_i) \wedge (v' \in V_i)]}{r_i}$, the desired equation for w_i follows.

Finally, we consider Δz_i. By the definitions in (3) and (5) and Lemma 5, we have

$$\begin{aligned}
\Delta z_i &= \mathbf{P}[u \in \Delta Z_i] \\
&= \frac{\mathbf{E}[|\Delta Z_i|]}{|V|} \\
&= f_l(w_i) \cdot \mathbf{P}[u \in X_i^* \cap V_i] = f_l(w_i) \cdot \mathbf{P}[u \in V_i] \cdot \mathbf{P}[u \in X_i^*] \\
&= f_l(w_i) \cdot r_i \cdot p \prod_{j=1}^{2l} (1-p)^{d(d-1)^{j-1}}
\end{aligned}$$

which completes the proof. □

Setting

$$a_i := \frac{z_i w_i}{r_i} = \mathbf{P}\left[u \in Z_k \setminus Z_i \mid u \in V_i\right] \cdot w_i \le w_i$$

for $0 \le i \le k$ and

$$\begin{aligned} \Delta r_i &:= r_{i+1} - r_i, \\ \Delta w_i &:= w_{i+1} - w_i \text{ and} \\ \Delta a_i &:= a_{i+1} - a_i \end{aligned}$$

for $0 \le i \le k-1$, we obtain the following.

Lemma 7 *For* $0 \le i \le k-1$

$$\frac{\Delta a_i}{\Delta w_i} = \frac{f_l(w_i) - 2a_i \frac{((d-1)w_i)^l - 1}{(d-1)w_i - 1}}{1 + (d-2)w_i \frac{((d-1)w_i)^l - 1}{(d-1)w_i - 1}} + O(p)$$

where the constant implicit in the $O(\cdot)$*-term depends only on* d *and* l*.*

Proof: Note that, by definition, $\Delta z_i = z_i - z_{i+1}$. Immediately from the previous definitions it is straightforward to verify that

$$\frac{\Delta a_i}{\Delta w_i} = \frac{w_i}{\Delta w_i}\left(a_i \frac{\frac{\Delta w_i}{w_i}\frac{r_i}{r_{i+1}} - \frac{\Delta r_i}{r_{i+1}}}{w_i} - \frac{\Delta z_i}{r_i}\frac{w_{i+1}}{w_i}\frac{r_i}{r_{i+1}}\right).$$

By Lemma 6,

$$\begin{aligned} \frac{w_{i+1}}{w_i} &= 1 - O(p) \text{ and} \\ \frac{r_i}{r_{i+1}} &= 1 + O(p). \end{aligned}$$

Furthermore,

$$\begin{aligned} \frac{\Delta w_i}{w_i} &= -\left(1 + (d-2)w_i \sum_{j=0}^{l-1} (w_i(d-1))^j\right) p + O(p^2), \\ \frac{\Delta r_i}{r_i} &= -\left(1 + d w_i \sum_{j=0}^{l-1} (w_i(d-1))^j\right) p + O(p^2) \text{ and} \\ \frac{\Delta z_i}{r_i} &= f_l(w_i) p + O(p^2) \end{aligned}$$

which implies that also

$$\frac{\Delta r_i}{r_{i+1}} = -\left(1 + d w_i \sum_{j=0}^{l-1} (w_i(d-1))^j\right) p + O(p^2).$$

Putting everything together, we obtain

$$\frac{\Delta a_i}{\Delta w_i} \quad = \quad \frac{f_l(w_i) - 2a_i \sum_{j=0}^{l-1}((d-1)w_i)^j + O(p)}{1 + (d-2)w_i \sum_{j=0}^{l-1}((d-1)w_i)^j + O(p)}.$$

Note that $f_l(w_i)$ is bounded from above by the order of a d-regular tree of radius l, i.e. it is bounded above in terms of d and l. Clearly, $\left(2a_i \sum_{j=0}^{l-1}((d-1)w_i)^j\right)$ is bounded from above in terms of d and l while $\left(1 + (d-2)w_i \sum_{j=0}^{l-1}((d-1)w_i)^j\right)$ is bounded from below by 1 and bounded from above in terms of d and l. Altogether this implies the stated equation for $\frac{\Delta a_i}{\Delta w_i}$. $\Box$

We proceed to our main result which extends Theorem 1 of [6].

Theorem 8 *Let* $d \geq 3$ *and* $l \geq 0$. *If* $f_l(w)$ *is continuous on* $[0,1]$, *then*

$$\alpha(d) \geq b_{l,f}(1)$$

where $b_{l,f}$ *is the solution of the linear differential equation*

$$b'_{l,f}(w) \quad = \quad c_{l,f,0}(w) + c_{l,f,1}(w) b_{l,f}(w) \quad \text{and} \quad b_{l,f}(0) = 0 \tag{7}$$

with

$$c_{l,f,0}(w) \quad = \quad \frac{f_l(w)}{1 + (d-2)w\frac{((d-1)w)^l - 1}{(d-1)w-1}} \quad \text{and}$$

$$c_{l,f,1}(w) \quad = \quad -\frac{2\frac{((d-1)w)^l - 1}{(d-1)w-1}}{1 + (d-2)w\frac{((d-1)w)^l - 1}{(d-1)w-1}}.$$

Proof: Note that by definition,

$$a_0 = \frac{z_0 w_0}{r_0} = z_0 = \mathbf{P}[u \in Z_k \setminus Z_0 \mid u \in V_0] = \mathbf{P}[u \in Z_k].$$

Therefore, $TREE(k,l,p,f)$ produces an independent set of $G = (V,E)$ of expected cardinality $a_0|V|$ and hence, by the first moment principle, $\alpha(d) \geq a_0$.

Whenever we use the $O(\cdot)$-notation, the implicit constants will be in terms of d and l.

Similarly as in [6], we will prove that for every $\epsilon > 0$ there is some $c_0 = c_0(\epsilon)$ and a function $p_0(c) > 0$ such that for $pk = c > c_0$ and $p < p_0(c)$ we have

$$a_0 \geq b_{l,f}(1) - O(\epsilon)$$

which clearly implies the desired result.

Let some $\epsilon > 0$ be fixed. By Lemma 6, we have

$$w_{i+1} \leq w_i \left(1 - p + O\left(p^2\right)\right)$$

for $0 \leq i < k$. Therefore, for sufficiently small p,

$$w_k \leq (1 - p/2)^k \quad = \quad \left((1 - p/2)^{\frac{1}{p}}\right)^c . \tag{8}$$

Thus for p small enough and $c \to \infty$ we have $w_k \to 0$. Furthermore, by Lemma 6,

$$\Delta w_i \quad = \quad O(p) \tag{9}$$

and hence $\Delta w_i \to 0$ as $p \to 0$ uniformly for every $0 \leq i < k$.

Since $f_l(w)$ is continuous, the function $c_0(w) := c_{l,f,0}(w)$ is continuous and bounded on the compact set $[0, 1]$. (Note that $f_l(w)$ is always bounded in terms of d and l as already noted in the proof of Lemma 7.) Furthermore, the function $c_1(w) := c_{l,f,1}(w)$ is Lipschitz continuous on $[0, 1]$. Hence the solution $b(w) := b_{l,f}(w)$ of (7) is also Lipschitz continuous on $[0, 1]$ where all bounds and Lipschitz constants are in terms of d and l.

Clearly, $a_k = 0$. Let $\tilde{b}(w)$ be the solution of the differential equation with modified initial condition

$$\tilde{b}'(w) = c_0(w) + c_1(w)\tilde{b}(w) \quad \text{and} \quad \tilde{b}(w_k) = 0.$$

By the mentioned continuity/ Lipschitz continuity conditions, it follows from standard results (cf. Corollary 4 and Corollary 6 in §7 of [2]) that the solution of (7) depends continuously on the initial contition. Hence, by (8), for p small enough and c large enough, $\tilde{b}(w_k) = b(w_k) + O(\epsilon)$ which implies

$$\tilde{b}(1) = b(1) + O(\epsilon).$$

By Lemma 7, we have for $0 \leq i < k$ that

$$a_i = a_{i+1} - \Delta a_i = a_{i+1} - (c_0(w_i) + c_1(w_i)a_i + O(p))\Delta w_i$$

which implies

$$\begin{aligned} a_i \quad &= \quad \frac{a_{i+1} - (c_0(w_i) + O(p))\Delta w_i}{1 + c_1(w_i)\Delta w_i} \\ &= \quad \frac{a_{i+1} - c_0(w_i)\Delta w_i}{1 + c_1(w_i)\Delta w_i} + O(p)\Delta w_i \end{aligned} \tag{10}$$

for p small enough.

Similarly, the differential equation for $\tilde{b}$ together with the mean value theorem imply for $0 \leq i < k$ and some $w_{i+1} \leq \tilde{w}_i \leq w_i$ that

$$\tilde{b}(w_i) = \tilde{b}(w_{i+1}) - \left(c_0(\tilde{w}_i) + c_1(\tilde{w}_i)\tilde{b}(\tilde{w}_i)\right) \Delta w_i.$$

By (9) and the continuity of c_0, c_1 and b, this implies that for every $\delta > 0$ there is some $p_1(\delta)$ such that for $p < p_1(\delta)$

$$\tilde{b}(w_i) = \tilde{b}(w_{i+1}) - \Big(c_0(w_i) + c_1(w_i)\tilde{b}(w_i) + O(\delta)\Big)\,\Delta w_i$$

and thus

$$\begin{aligned} \tilde{b}(w_i) &= \frac{\tilde{b}(w_{i+1}) - (c_0(w_i) + O(\delta))\Delta w_i}{1 + c_1(w_i)\Delta w_i} \\ &= \frac{\tilde{b}(w_{i+1}) - c_0(w_i)\Delta w_i}{1 + c_1(w_i)\Delta w_i} + O(\delta)\Delta w_i \end{aligned} \tag{11}$$

for p small enough.

In view of (10) and (11), we deduce

$$\begin{aligned} \tilde{b}(w_k) - a_k &= 0 \text{ and for } 0 \le i < k \\ \tilde{b}(w_i) - a_i &= \frac{\tilde{b}(w_{i+1}) - a_{i+1}}{1 + c_1(w_i)\Delta w_i} + (O(p) + O(\delta))\Delta w_i. \end{aligned}$$

Since for p small enough

$$\frac{1}{1 + c_1(w_i)\Delta w_i} = 1 + O(\Delta w_i) = 1 + O(p),$$

we obtain, by induction,

$$\tilde{b}(w_0) - a_0 = (O(p) + O(\delta)) \sum_{i=0}^{k-1} \Delta w_i (1 + O(p))^i$$

We have

$$(1 + O(p))^k = \Big((1 + O(p))^{\frac{1}{p}}\Big)^c$$

which is bounded in terms of c. Therefore, choosing δ small enough in terms of c and choosing p small enough in terms of c (and δ), we finally obtain

$$b(1) = a_0 + \Big(b(1) - \tilde{b}(1)\Big) + \Big(\tilde{b}(1) - a_0\Big) = a_0 + O(\epsilon)$$

and the proof is complete. □

3.1 Some Instructive Choices for l

It is very instructive to consider the behaviour of $TREE(k, l, p, f)$ for $l \in \{0, 1\}$ and appropriate choices for f.

For $l = 0$ we have $N^{\leq 0}_{G_i}(v) = \{v\}$ and $X^*_i = X_i$ for all $0 \leq i \leq k-1$. Furthermore, the set $B^l_{G_i}(v)$ contains the vertex v exactly if all neighbours of v in G are not contained in V_i and is empty otherwise.

Choosing $f(T, r) = V_T$ whenever the vertex set V_T of T satisfies $|V_T| \leq 1$ and $f(T, r) = \emptyset$ otherwise, $TREE(k, 0, p, f)$ produces an independent set by Lemma 4. Conditional upon the event $(u \in X^*_i \cap V_i)$, the expected value of $|f(T_i(u), u)|$ equals

$$f_0(w_i) = (1 - w_i)^d,$$

because, by Lemma 5 (ii), each of the d neighbours of u are in V_i independently at random with probability w_i. The differential equation (7) simplifies to

$$b'_{0,f}(w) = f_0(w) \quad \text{and} \quad b_{0,f}(0) = 0$$

which has the solution $b_{0,f}(w) = \frac{1-(1-w)^{d+1}}{1+d}$ and thus

$$b_{0,f}(1) = \frac{1}{1+d}.$$

Therefore, by Theorem 8, asymptotically $TREE(k, 0, p, f)$ produces an independent set of G which contains exactly the same fraction of the vertices, namely $\frac{1}{1+d}$, as guaranteed by the lower bound on the independence number of d-regular graphs proved by Caro [4] and Wei [9].

The reason for this is that for $pk \to \infty$ and $p \to 0$, $TREE(k, 0, p, f)$ essentially processes the vertices of G according to a random linear ordering $v_1, v_2, ..., v_n$ and adds an individual vertex v_i to the constructed independent set exactly if all neighbours of v_i are among $\{v_1 v_2, ..., v_{i-1}\}$. Applying this algorithm to a random linear ordering, the probability that an individual vertex v belongs to the constructed independent set equals exactly $1/(1+d)$, because this is the probability that v is the last vertex from $N^{\leq 1}_G(v)$ with respect to the linear ordering. In fact, this is exactly the argument used in [1] to prove the bound due to Caro and Wei [4, 9].

For $l = 1$ the set $B^1_{G_i}(v)$ always contains the vertex v itself and we can choose $f(T, r) = \{r\}$ in order to obtain an independent set according to Lemma 4.

For this choice $TREE(k, 1, p, f)$ essentially coincides with the randomized **p**-*greedy algorithm* used by Lauer and Wormald in [6] for $\mathbf{p} = (p_1, p_2, ..., p_k) = (p, p, ..., p)$. Solving the differential equation (7) yields the same values for $b_{1,f}(1)$ as obtained by Lauer and Wormald. In the next section, we consider a choice of f which generalizes their **p**-greedy algorithm.

3.2 A Reasonable Choice for f

A reasonable choice for f is to select all vertices within some even distance from the root. Therefore, let $l = 2h + 1$ for some integer $h \geq 0$ and let

$$f_{\text{even}}(T, r) = \{u \in V_T \mid \text{dist}_T(r, u) \text{ is even}\}.$$

Note that $B^{2h+1}_{G_i}(u)$ contains all vertices of $N^{\leq 2h+1}_{G_i}(u)$ within distance at most $2h$ from u.

Conditional upon the event $(u \in X_i^* \cap V_i)$, the probability for the event $(v \in B^{2h+1}_{G_i}(u))$ for some vertex v with $\mathrm{dist}_G(u,v) = 2j$ for some $1 \leq j \leq h$ equals exactly the probability that all vertices of the unique path from u to v within $N^{\leq 2h+1}_{G}(u)$ lie in V_i. Using Lemma 5 in the same way as for the calculation of $\mathbf{P}\left[(u_0, u_1, ..., u_\nu \in V_i) \wedge (u_{\nu+1} \not\in V_i)\right]$ in the proof of Lemma 6, this conditional probability equals w_i^{2j}.

By linearity of expectation, we deduce

$$\begin{aligned}(f_{\mathrm{even}})_{2h+1}(w) &= 1 + \sum_{j=1}^{h} d(d-1)^{2j-1} w^{2j} \\ &= 1 + d(d-1)w^2 \sum_{j=0}^{h-1} ((d-1)w)^{2j} \\ &= 1 + d(d-1)w^2 \frac{((d-1)w)^{2h} - 1}{((d-1)w)^2 - 1}.\end{aligned}$$

Now the differential equation (7) reads as follows.

$$\begin{aligned} b'_{2h+1,f_{\mathrm{even}}}(w) &= \frac{\left(1 + d(d-1)w^2 \frac{((d-1)w)^{2h}-1}{((d-1)w)^2-1}\right) - 2b_{2h+1,f_{\mathrm{even}}}(w) \frac{((d-1)w)^{2h+1}-1}{(d-1)w-1}}{1 + (d-2)w\frac{((d-1)w)^{2h+1}-1}{(d-1)w-1}} \qquad (12) \\ b_{2h+1,f_{\mathrm{even}}}(0) &= 0. \end{aligned}$$

The algorithm proposed by Lauer and Wormald in [6] corresponds to the choice $h = 0$ in which case (12) simplifies to

$$b'_{1,f_{\mathrm{even}}}(w) = \frac{1 - 2b_{1,f_{\mathrm{even}}}(w)}{1 + (d-2)\,w}.$$

The solution of this differential equation is

$$b_{1,f_{\mathrm{even}}}(w) = \frac{1 - (1 + wd - 2\,w)^{-2/(d-2)}}{2}$$

which together with Theorem 8 immediately implies one of the main results from [6].

Corollary 9 (Lauer and Wormald, cf. Theorem 1 in [6]) *For every $d \geq 3$, we have*

$$\alpha(d) \geq \frac{1 - (d-1)^{-2/(d-2)}}{2}.$$

Next we consider the behaviour of $b_{2h+1,f_{\mathrm{even}}}(w)$ for $h \to \infty$. Our analysis naturally splits into the two cases $(d-1)w < 1$ and $(d-1)w > 1$.

The intuitive reason for this is that for values of w_i with $(d-1)w_i < 1$ the sets $N^{\leq l}_{G_i}(u)$ typically contain no vertices far from u, while for values of w_i with $(d-1)w_i > 1$ the sets

$N^{\leq l}_{G_i}(u)$ may contain vertices up to distance l from u, i.e. the trees induced by the sets $B^l_{G_i}(u)$ "*die out*" quickly for small values of the "*probability of survival*" w_i.

Considering the two intervals $\left[0, \frac{1}{d-1}\right)$ and $\left(\frac{1}{d-1}, 1\right]$ it follows from standard results (cf. Corollary 6 in §7 of [2]) that for $h \to \infty$ the solutions of (12) converge to the solutions of

$$\begin{aligned}
b'_{\infty,f_{\text{even}}}(w) &= \begin{cases} \dfrac{\left(1-\frac{d(d-1)w^2}{((d-1)w)^2-1}\right)+\frac{2b_{\infty,f_{\text{even}}}(w)}{(d-1)w-1}}{1-\frac{(d-2)w}{(d-1)w-1}} & \text{for } 0 \leq w < 1/(d-1) \\[2ex] \dfrac{\left(\frac{d(d-1)w^2}{((d-1)w)^2-1}\right)-2b_{\infty,f_{\text{even}}}(w)\frac{(d-1)w}{(d-1)w-1}}{(d-2)w\frac{(d-1)w}{(d-1)w-1}} & \text{for } 1/(d-1) < w \leq 1 \end{cases} \\[2ex]
&= \begin{cases} \frac{(d-1)w^2+1}{((d-1)w+1)(1-w)} - \frac{2}{1-w}b_{\infty,f_{\text{even}}}(w) & \text{for } 0 \leq w < 1/(d-1) \\[1ex] \frac{d}{((d-1)w+1)(d-2)} - \frac{2}{(d-2)w}b_{\infty,f_{\text{even}}}(w) & \text{for } 1/(d-1) < w \leq 1 \end{cases} \qquad (13) \\
b_{\infty,f_{\text{even}}}(0) &= 0.
\end{aligned}$$

Corollary 10 *For every $d \geq 3$, we have $\alpha(d) \geq b_{\infty,f_{\text{even}}}(1)$.*

Solving (13) for $d = 3$ yields

$$b_{\infty,f_{\text{even}}}(w) = \begin{cases} \frac{w}{3} + \frac{w^2}{6} + \frac{2}{9}\ln\left(\frac{2w+1}{1-w}\right) & \text{for } 0 \leq w < 1/(d-1) \\[1ex] \frac{3}{4}\frac{w-1}{w} + \frac{1}{w^2}\left(\frac{23}{96} + \frac{3}{8}\ln(2w+1) - \frac{25}{72}\ln(2)\right) & \text{for } 1/(d-1) < w \leq 1 \end{cases}$$

and hence in this case

$$b_{\infty,f_{\text{even}}}(1) = \frac{23}{96} + 3/8\ \ln(3) - \frac{25}{72}\ \ln(2) \approx 0.4108.$$

Similarly, we obtain

$$b_{\infty,f_{\text{even}}}(1) \approx \begin{cases} 0.3579 & \text{for } d = 4 \\ 0.3201 & \text{for } d = 5 \\ 0.2911 & \text{for } d = 6. \end{cases}$$

3.3 An Optimal Choice for f

In this section we consider a function f_{opt} for which $f_{\text{opt}}(T, r)$ is a maximum independent set within the rooted tree T. For some tree T with root r the set $f_{\text{opt}}(T, r)$ is obtained by applying the following algorithm $\mathcal{A}_{\text{opt}}$: *Start with $f_{\text{opt}}(T, r) = \emptyset$. Iteratively add to $f_{\text{opt}}(T, r)$ all vertices at maximum distance from the root r within the current tree and delete them together with their parents.*

For this algorithm it follows immediately from the definition of $B^l_{G_i}(v)$ that

$$f_{\text{opt}}(T_i(v), v) = B^l_{G_i}(v) \cap f_{\text{opt}}\left(G_i\left[N^{\leq l+1}_{G_i}(v)\right], v\right)$$

for every $v \in X_i^* \cap V_i$.

By Lemma 5, for $0 \leq i < k$ and $-1 \leq j \leq l+1$ the probability

$$p_l(j,i) = \mathbf{P}\left[v \in f_{\text{opt}}\left(N_{\bar{G}_i}^{\leq l+1}(u), u\right) \mid \left((u \in X_i^* \cap V_i) \wedge \left(v \in N_{G_i}^{l+1-j}(u)\right)\right)\right]$$

does not depend on the choice of $u, v \in V$ with $\text{dist}_G(u,v) = l+1-j$, i.e. $p_l(j,i)$ is well-defined.

Furthermore, by Lemma 5 (ii), the events

$$\left(v \in f_{\text{opt}}\left(N_{\bar{G}_i}^{\leq l+1}(u), u\right)\right) \mid \left((u \in X_i^* \cap V_i) \wedge \left(v \in N_{G_i}^{l+1-j}(u)\right)\right)$$

and

$$\left(v' \in f_{\text{opt}}\left(N_{\bar{G}_i}^{\leq l+1}(u), u\right)\right) \mid \left((u \in X_i^* \cap V_i) \wedge \left(v' \in N_{G_i}^{l+1-j}(u)\right)\right)$$

are independent for different $v, v' \in N_G^{l+1-j}(u)$.

Let $t_j(w)$ be defined recursively for integers $j \geq -1$ by

$$t_j(w) := \begin{cases} 0 & \text{for } j = -1 \text{ and} \\ (1 - w t_{j-1}(w))^{d-1} & \text{for } j \geq 0. \end{cases}$$

Obviously, by definition and the first step of the algorithm $\mathcal{A}_{\text{opt}}$,

$$\begin{aligned} t_{-1}(w_i) &= p_l(-1,i) = 0 \text{ and} \\ t_0(w_i) &= p_l(0,i) = 1. \end{aligned}$$

For $j \geq 0$ the event

$$\left(v \in f_{\text{opt}}\left(N_{\bar{G}_i}^{\leq l+1}(u), u\right)\right) \mid \left((u \in X_i^* \cap V_i) \wedge \left(v \in N_{G_i}^{l+1-j}(u)\right)\right)$$

is equivalent to the event that none of the vertices $v' \in N_G^1(v) \cap N_G^{l+1-(j-1)}(u)$ is in the set $f_{\text{opt}}\left(N_{\bar{G}_i}^{\leq l+1}(u), u\right)$, i.e. either they are not in V_i or they are in V_i but not in the set $f_{\text{opt}}\left(N_{\bar{G}_i}^{\leq l+1}(u), u\right)$.

By Lemma 5 (ii), conditional upon the event $\left((u \in X_i^* \cap V_i) \wedge \left(v \in N_{G_i}^{l+1-j}(u)\right)\right)$, the probability that $v' \in N_G^1(v) \cap N_G^{l+1-(j-1)}(u)$ is not in V_i equals $(1 - w_i)$, and the probability that such a v' is in V_i but not in $f_{\text{opt}}\left(N_{\bar{G}_i}^{\leq l+1}(u), u\right)$ equals $w_i(1 - p_l(j-1,i))$.

Hence, the probability that such a v' is in not $f_{\text{opt}}\left(N_{\bar{G}_i}^{\leq l+1}(u), u\right)$ equals

$$(1 - w_i) + w_i(1 - p_l(j-1,i)) = 1 - w_i p_l(j-1,i)$$

and, by Lemma 5 (ii),

$$p_l(j,i) = (1 - w_i p_l(j-1,i))^{\left|N_G^1(v) \cap N_G^{l+1-(j-1)}(u)\right|}.$$

For $0 \leq j \leq l$ the values $p_l(j,i)$ satisfy the same recursion as the $t_j(w_i)$ starting with the same value 0 at $j = -1$ and altogether we obtain

$$p_l(j,i) = \begin{cases} t_j(w_i) & \text{for } -1 \leq j \leq l \text{ and} \\ (1 - w_i t_l(w_i))^d & \text{for } j = l+1 \end{cases}$$

(for $j = l+1$ remember that the root u has d possible children while all internal vertices have $d-1$.)

By linearity of expectation,

$$\begin{aligned} (f_{\text{opt}})_l(w_i) &= \mathbf{E}\left[\left|f_{\text{opt}}\left(B^l_{G_i}(u), u\right)\right| \mid u \in X_i^* \cap V_i\right] \\ &= \sum_{v \in N^{\leq l}_G(u)} \mathbf{P}\left[v \in f_{\text{opt}}(B^l_{G_i}(u), u) \mid u \in X_i^* \cap V_i\right] \\ &= p_l(l+1, i) + \sum_{j=1}^{l} d(d-1)^{l-j} w_i^{l+1-j} \cdot p_l(j,i) \\ &= p_l(l+1, i) + \sum_{j=1}^{l} w_i d((d-1)w_i)^{l-j} \cdot p_l(j,i) \\ &= (1 - w_i t_l(w_i))^d + \sum_{j=1}^{l} w_i d((d-1)w_i)^{l-j} \cdot t_j(w_i) \\ &= (1 - w_i t_l(w_i))^d + w_i d \sum_{j=0}^{l-1} ((d-1)w_i)^j \cdot t_{l-j}(w_i). \end{aligned} \tag{14}$$

In the case $w(d-1) < 1$ and $l \to \infty$ the limit behaviour of the recursion for $t_j(w)$ becomes important. The function

$$t \mapsto (1 - wt)^{d-1}$$

maps the unit interval $[0,1]$ into itself and the absolute value of its derivative $(d-1)w(1-wt)^{d-2}$ is strictly smaller than 1 for $t \in [0,1]$ and $w(d-1) < 1$. Hence the recursion for $t_j(w)$ is a contractive map and converges to a unique fixed point $t(w)$ which solves the equation

$$t(w) = (1 - wt(w))^{d-1}.$$

Because for $w(d-1) < 1$ the factors preceeding $t_j(w)$ in (14) decrease exponentially in j, we obtain

$$\begin{aligned} \lim_{l \to \infty} (f_{\text{opt}})_l(w) &= (1 - wt(w))^d + t(w) wd \sum_{j=0}^{\infty} ((d-1)w)^j \\ &= t(w) \left(1 - wt(w) + \frac{wd}{1 - (d-1)w}\right) \end{aligned}$$

and

$$\begin{aligned} c_{\infty,f_{\mathrm{opt}},0}(w) &= \lim_{l\to\infty} c_{l,f_{\mathrm{opt}},0}(w) \\ &= \frac{t(w)}{1-w}\left(dw + (1-wt(w))(1-(d-1)w)\right). \end{aligned}$$

Because in this case

$$\begin{aligned} c_{\infty,f_{\mathrm{opt}},1}(w) &= \lim_{l\to\infty} c_{l,f_{\mathrm{opt}},1} \\ &= \frac{2}{w-1}, \end{aligned}$$

we are able to solve the differential equation for $l \to \infty$ in the interval $[0, \frac{1}{d-1})$ and obtain

$$\begin{aligned} b_{\infty,f_{\mathrm{opt}}}(w) &= (w-1)^2 \int_0^w \frac{c_{\infty,f_{\mathrm{opt}},0}(w)}{(t-1)^2}\delta t \\ &= (w-1)^2 \int_0^w \frac{t(w)\left(dw + (1-wt(w))(1-(d-1)w)\right)}{(1-w)(t-1)^2}\delta t. \end{aligned}$$

The integral is solveable at least for $d = 3$ in which case we obtain

$$b_{\infty,f_{\mathrm{opt}}}(w) = \frac{1+6w-\sqrt{4w+1}}{(1+\sqrt{4w+1})^2}.$$

The most interesting value in this case is at $\frac{1}{d-1} = \frac{1}{2}$

$$b_{\infty,f_{\mathrm{opt}}}\left(\frac{1}{2}\right) = \frac{11}{2} - 6\sqrt{3} \approx 0.3038.$$

Clearly, $(f_{\mathrm{even}})_\infty(w) \leq (f_{\mathrm{opt}})_\infty(w)$ and we can use $(f_{\mathrm{even}})_\infty(w)$ in order to determine a lower bound for $b_{\infty,f_{\mathrm{opt}}}(1)$ in the case $d = 3$ by solving (13) on the interval $\left[\frac{1}{2}, 1\right]$ using as initial condition the value $b_{\infty,f_{\mathrm{opt}}}\left(\frac{1}{2}\right)$ at $w = 1/2$. We obtain for $w \in \left[\frac{1}{2}, 1\right]$ the following lower bound

$$b_{\infty,f_{\mathrm{opt}}}(w) \quad \geq \quad \frac{25-12\sqrt{3}-12w+12w^2+6\ln(\frac{1}{2}+w)}{16w^2}.$$

Corollary 11 $\alpha(3) \geq b_{\infty,f_{\mathrm{opt}}}(1) \geq \frac{25-12\sqrt{3}+6\ln(\frac{3}{2})}{16} \approx 0.4155 > \beta_{\mathrm{Shearer}}(3)$.

In general the following observations are useful for estimating $\lim\limits_{l\to\infty} c_{l,f_{\mathrm{opt}},0}(w)$ in the case $w(d-1) > 1$.

- **Observation 1**
  Because the recursion for $t_j(w)$ is based upon a strictly monotonic decreasing function which contracts the unit interval, and starts with $t_{-1}(w) = 0$ we obtain

  $$\begin{aligned} t_{2j}(w) &> t_{2j+1}(w), \\ t_{2(j+1)}(w) &< t_{2j}(w) \text{ and} \\ t_{2(j+1)+1}(w) &> t_{2j+1}(w) \end{aligned}$$

  for $j \geq 0$.

- **Observation 2**
  Consider a modified algorithm $\mathcal{A}^-$ applied to a tree T with root r which behaves like $\mathcal{A}_{\text{opt}}$ up to some distance $j+1$ from r, chooses less vertices at distance j from r than $\mathcal{A}_{\text{opt}}$ and continues like $\mathcal{A}_{\text{opt}}$ for smaller distances to r.

  For distances larger than j to r the output of $\mathcal{A}^-$ coincides with the output of $\mathcal{A}_{\text{opt}}$. For distances at most j to r the output of $\mathcal{A}^-$ coincides with the output of $\mathcal{A}_{\text{opt}}$ when applied to a proper subtree of the tree induced by the vertices at distance at most j to r.

  Therefore, the set produced by $\mathcal{A}^-$ will contain at most as many vertices as the set produced by $\mathcal{A}_{\text{opt}}$.

  Conversely, if $\mathcal{A}^+$ chooses more vertices at distance j from r than $\mathcal{A}_{\text{opt}}$ — possibly neglecting independence — and behaves like $\mathcal{A}_{\text{opt}}$ otherwise, then the set produced by $\mathcal{A}^+$ will contain at least as many vertices as the set produced by $\mathcal{A}_{\text{opt}}$.

Iteratively applying these observations allows to derive lower and upper bounds on $(f_{\text{opt}})_l\,(w)$ for $(d-1)w > 1$:

We choose an integer j^*.

If for all $j \geq j^*$ we replace in (14) t_{2j-1} by t_{2j^*-1} and t_{2j} by t_{2j^*}, then Observation 1 and Observation 2 for $\mathcal{A}^-$ imply that we obtain a lower bound on $(f_{\text{opt}})_l\,(w)$.

If for all $j \geq j^*$ we replace in (14) t_{2j+1} by t_{2j^*+1} and t_{2j} by t_{2j^*}, then Observation 1 and Observation 2 for $\mathcal{A}^+$ imply that we obtain an upper bound on $(f_{\text{opt}})_l\,(w)$.

Therefore, we obtain

$$\begin{aligned} \frac{(f_{\text{opt}})_l(w)}{(w(d-1))^l} &= \frac{(1-wt_l(w))^d}{(w(d-1))^l} + wd\sum_{j=1}^{l} \frac{t_j(w)}{(w(d-1))^j} \\ &\geq wd\left(\sum_{j=1}^{2j^*-2} \frac{t_j(w)}{(w(d-1))^j} + \sum_{j=j^*}^{\lfloor \frac{l}{2} \rfloor} \frac{t_{2j^*}(w)}{(w(d-1))^{2j}} + \sum_{j=j^*}^{\lfloor \frac{l+1}{2} \rfloor} \frac{t_{2j^*-1}(w)}{(w(d-1))^{2j-1}}\right) \end{aligned}$$

and

$$\begin{aligned}
\frac{(f_{\mathrm{opt}})_l(w)}{(w(d-1))^l} &\leq \frac{1}{(w(d-1))^l} \\
&\quad +wd\left(\sum_{j=1}^{2j^*-1}\frac{t_j(w)}{(w(d-1))^j}+\sum_{j=j^*}^{\left\lfloor\frac{l}{2}\right\rfloor}\frac{t_{2j^*}(w)}{(w(d-1))^{2j}}+\sum_{j=j^*}^{\left\lfloor\frac{l-1}{2}\right\rfloor}\frac{t_{2j^*+1}(w)}{(w(d-1))^{2j+1}}\right).
\end{aligned}$$

Using these inequalities it is possible to derive lower and upper bounds for $c_{\infty,f_{\mathrm{opt}},0}(w)$.

$$\begin{aligned}
c_{\infty,f_{\mathrm{opt}},0}(w) &= \frac{w(d-1)-1}{(d-2)w}\lim_{l\to\infty}\frac{(f_{\mathrm{opt}})_l(w)}{(w(d-1))^l} \\
&\geq \frac{d(w(d-1)-1)}{d-2}\left(\sum_{j=1}^{2j^*-2}\frac{t_j(w)}{(w(d-1))^j}+\sum_{j=j^*}^{\infty}\frac{t_{2j^*}(w)+w(d-1)t_{2j^*-1}}{(w(d-1))^{2j}}\right) \\
&= \frac{d}{d-2}\left(\frac{t_{2j^*}(w)+w(d-1)t_{2j^*-1}(w)}{(w(d-1)+1)(w(d-1))^{2j^*-2}}+\sum_{j=1}^{2j^*-2}\frac{(w(d-1)-1)t_j(w)}{(w(d-1))^j}\right)
\end{aligned}$$

and

$$\begin{aligned}
c_{\infty,f_{\mathrm{opt}},0}(w) &\leq \frac{d(w(d-1)-1)}{d-2}\left(\sum_{j=1}^{2j^*-1}\frac{t_j(w)}{(w(d-1))^j}+\sum_{j=j^*}^{\infty}\frac{w(d-1)t_{2j^*+1}(w)+t_{2j^*}(w)}{(w(d-1))^{2j+1}}\right) \\
&= \frac{d}{d-2}\left(\frac{t_{2j^*}(w)+w(d-1)t_{2j^*+1}(w)}{(w(d-1)+1)(w(d-1))^{2j^*-1}}+\sum_{j=1}^{2j^*-1}\frac{(w(d-1)-1)t_j(w)}{(w(d-1))^j}\right).
\end{aligned}$$

Choosing j^* sufficiently large and numerically solving the corresponding two differential equations, we can obtain estimates for $b_{\infty,f_{\mathrm{opt}}}(1)$ with any desired precision.

Corollary 12 *For $d \geq 3$ we have $\alpha(d) \geq b_{\infty,f_{\mathrm{opt}}}(1)$.*

The next table summarizes the numerically obtained values for selected values of d. The entry $\gamma(d)$ is an upper bound on $\alpha(d)$ which is derived from the analysis of random d-regular graphs [3, 7].

d	$\max\{\beta_{\mathrm{Shearer}}(d), \beta_{\mathrm{LauWo}}(d)\}$	$b_{\infty,f_{\mathrm{opt}}}(1)$	$\gamma(d)$
3	0.4139	0.4193	0.4554
4	0.3510	0.3664	0.4136
5	0.3085	0.3279	0.3816
6	0.2771	0.2982	0.3580
7	0.2558	0.2744	0.3357
8	0.2386	0.2548	0.3165
9	0.2240	0.2382	0.2999
10	0.2113	0.2241	0.2852
20	0.1395	0.1455	0.1973
50	0.0748	0.0770	0.1108
100	0.0447	0.0457	0.0679

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