Some Sobolev Spaces as Pontryagin Spaces

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Abstract

We show that well known Sobolev spaces can quite naturally be treated as Pontryagin spaces. This point of view gives a possibility to obtain new properties for some traditional objects such as simplest differential operators.

Key words: Function spaces, Pontryagin spaces, selfadjoint operators, spectrum

MSC: 47B50, 47E05, 47B25

1 Introduction

Let $\mathcal{H}$ be a separable Hilbert space with a scalar product $(\cdot,\cdot)$. $\mathcal{H}$ is said to be *an indefinite metric space* if it is equipped by a sesquilinear continuous Hermitian form (indefinite inner product) $[\cdot,\cdot]$ such that the corresponding quadratic form has indefinite sign (i.e. $[x,x]$ takes positive, negative and zero values). The indefinite inner product can be represented in the form $[\cdot,\cdot] = (G\cdot,\cdot)$, where G is a so-called Gram operator. The operator G is bounded and self-adjoint. If the Gram operator for an indefinite metric space is boundedly invertible and its invariant subspace corresponding to the negative spectrum of G is finite-dimensional, lets say κ-dimensional, the space is called a Pontryagin space with κ negative squares. There are a lot of problems in different areas of mathematics, mechanics or physics that can be naturally considered as

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problems in terms of Operator Theory in Pontryagin spaces. We have no aim to give here an overview on this theory and its application. We refer only to the standard text books [1,2,10] and to [14] for a brief introduction.

Our scope is a modest illustration of some singular situations that shows an essential difference between Operator Theory in Hilbert spaces and in Pontryagin spaces. For this goal we use Sobolev spaces that represents a new approach.

2 Preliminaries

A Krein space $(\mathcal{K}, [\cdot,\cdot])$ is a linear space $\mathcal{K}$ which is equipped with an (indefinite) inner product (i.e., a hermitian sesquilinear form) $[\cdot,\cdot]$ such that $\mathcal{K}$ can be written as

$$\mathcal{K} = \mathcal{G}_+ [\dot{+}] \mathcal{G}_- \tag{1}$$

where $(\mathcal{G}_\pm, \pm[\cdot,\cdot])$ are Hilbert spaces and $[\dot{+}]$ means that the sum of $\mathcal{G}_+$ and $\mathcal{G}_-$ is direct and $[\mathcal{G}_+, \mathcal{G}_-] = 0$. The norm topology on a Krein space $\mathcal{K}$ is the norm topology of the orthogonal sum of the Hilbert spaces $\mathcal{G}_\pm$ in (1). It can be shown that this norm topology is independent of the particular decomposition (1); all topological notions in $\mathcal{K}$ refer to this norm topology and $\|\cdot\|$ denotes any of the equivalent norms. Krein spaces often arise as follows: In a given Hilbert space $(\mathcal{G}, (\cdot,\cdot))$, every bounded self-adjoint operator G in $\mathcal{G}$ with $0 \in \rho(G)$ induces an inner product

$$[x, y] := (Gx, y), \quad x, y \in \mathcal{G}, \tag{2}$$

such that $(\mathcal{G}, [\cdot,\cdot])$ becomes a Krein space; here, in the decomposition (1), we can choose $\mathcal{G}_+$ as the spectral subspace of $\mathcal{G}$ corresponding to the positive spectrum of G and G_- as the spectral subspace of $\mathcal{G}$ corresponding to the negative spectrum of G. A subspace $\mathcal{L}$ of a linear space $\mathcal{K}$ with inner product $[\cdot,\cdot]$ is called non- degenerated if there exists no $x \in \mathcal{L}$, $x \neq 0$, such that $[x, \mathcal{L}] = 0$, otherwise $\mathcal{L}$ is called degenerated; note that a Krein space $\mathcal{K}$ is always non-degenerated, but it may have degenerated subspaces. An element $x \in \mathcal{K}$ is called positive (non-negative, negative, non-positive, neutral, respectively) if $[x, x] > 0$ ($\geq 0, < 0, \leq 0, = 0$, respectively); a subspace of $\mathcal{K}$ is called positive (non-negative, etc., respectively), if all its nonzero elements are positive (non-negative, etc., respectively). For the definition and simple properties of Krein spaces and linear operators therein we refer to [2], [13] and [1].

If in some decomposition (1) one of the components $\mathcal{G}_\pm$is of finite dimension, it is of the same dimension in all such decompositions, and the Krein space $(\mathcal{K}, [\cdot,\cdot])$ is called a Pontryagin space. For the Pontryagin spaces $\mathcal{K}$ occurring in this paper, the negative component $\mathcal{G}_-$ is of finite dimension, say κ; in this case, $\mathcal{K}$ is called a Pontryagin space with κ negative squares. If $\mathcal{K}$ arises from a Hilbert space $\mathcal{G}$ by means of a self-adjoint operator G with inner product (2), then $\mathcal{K}$ is a Pontryagin space with κ negative squares if and only if the negative spectrum of the invertible operator G consists of exactly κ eigenvalues, counted according to their multiplicities. In a Pontryagin space $\mathcal{K}$ with κ negative squares each non-positive subspace is of dimension $\leq \kappa$, and a non-positive subspace is maximal non-positive (that is, it is not properly contained in another non-positive subspace) if and only if it is of dimension κ. If $\mathcal{L}$ is a non-degenerated linear space with inner product $[\cdot,\cdot]$ such that for a κ-dimensional subspace $\mathcal{L}_-$ we have

$$[x,x] < 0, \quad x \in \mathcal{L}_-, x \neq 0,$$

but there is no $(\kappa+1)$-dimensional subspace with this property, then there exists a Pontryagin space $\mathcal{K}$ with κ negative squares such that $\mathcal{L}$ is a dense subset of $\mathcal{K}$. This means that $\mathcal{L}$ can be completed to a Pontryagin space in a similar way as a pre-Hilbert space can be completed to a Hilbert space. The spectrum of a selfadjoint operator A in a Pontryagin space with κ negative squares is real with the possible exception of at most κ non-real pairs of eigenvalues λ, $\overline{\lambda}$ of finite type. We denote by $\mathcal{L}_\lambda(A)$ the algebraic eigenspace of A at λ. Then $\dim \mathcal{L}_\lambda(A) = \dim \mathcal{L}_{\overline{\lambda}}(A)$ and the Jordan structure of A in $\mathcal{L}_\lambda(A)$ and in $\mathcal{L}_{\overline{\lambda}}(A)$ is the same. Further the relation

$$\kappa = \sum_{\lambda \in \sigma_0 \cap \mathbb{R}} \kappa_\lambda^-(A) + \sum_{\lambda \in \sigma(A) \cap \mathbb{C}^+} \dim \mathcal{L}_\lambda(A)$$

holds, where σ_0 denotes the set of all eigenvalues of A with a nonpositive eigenvector and $\kappa_\lambda^-(A)$ denotes the maximal dimension of a nonpositive subspace of $\mathcal{L}_\lambda(A)$.

Moreover, according to a theorem of Pontryagin, A has a κ-dimensional invariant non-positive subspace $\mathcal{L}_-^{max}$. If q denotes the minimal polynomial of the resriction $A|\mathcal{L}_-^{max}$, then the polynomial q^*q, where $q^*(z) = \overline{q(\overline{z})}$, is independent of the particular choice of $\mathcal{L}_-^{max}$ and one can show that $[q^*(A)q(A)x,x] \geq 0$ for $x \in \mathcal{D}(A^\kappa)$. As a consequence, a selfadjoint operator in a Pontryagin space possesses a spectral function with possible critical points. For details we refer to [11,13].

The linear space of bounded linear operators defined on a Pontryagin or Krein space $\mathcal{K}_1$ with values in a Pontryagin or Krein space $\mathcal{K}_2$ is denoted by $\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)$. If $\mathcal{K} := \mathcal{K}_1 = \mathcal{K}_2$ we write $\mathcal{L}(\mathcal{K})$. We study linear relations in $\mathcal{K}$, that is, linear subspaces of $\mathcal{K}^2$. The set of all closed linear relations in $\mathcal{K}$ is denoted by $\widetilde{\mathcal{C}}(\mathcal{K})$. Linear operators are viewed as linear relations via their graphs. For the usual definitions of the linear operations with relations and the inverse we refer to [7], [8], [9]. We recall only that the multivalued part $\operatorname{mul} S$ of a linear relation S is defined by $\operatorname{mul} S = \left\{ y \,\middle|\, \binom{0}{y} \in S \right\}$.

Let S be a closed linear relation in $\mathcal{K}$. The resolvent set $\rho(S)$ of S is defined as the set of all $\lambda \in \mathbb{C}$ such that $(S-\lambda)^{-1} \in \mathcal{L}(\mathcal{K})$. The spectrum $\sigma(S)$ of S is the complement of $\rho(S)$ in $\mathbb{C}$. The extended spectrum $\widetilde{\sigma}(S)$ of S is defined by $\widetilde{\sigma}(S) = \sigma(S)$ if $S \in \mathcal{L}(\mathcal{K})$ and $\widetilde{\sigma}(S) = \sigma(S) \cup \{\infty\}$ otherwise. We set $\widetilde{\rho}(S) := \overline{\mathbb{C}} \backslash \widetilde{\sigma}(S)$. The *adjoint* S^+ of S is defined as

$$S^+ := \left\{ \begin{pmatrix} h \\ h' \end{pmatrix} \,\middle|\, [f', h] = [f, h'] \text{ for all } \begin{pmatrix} f \\ f' \end{pmatrix} \in S \right\}.$$

S is said to be *symmetric* (*selfadjoint*) if $S \subset S^+$ (resp. $S = S^+$).

For the description of the selfadjoint extensions of closed symmetric relations we use the so-called boundary value spaces (for the first time the corresponding approach was applied in fact by A.V. Strauss [15], [16] without employing the term "boundary value space").

Definition 1 *Let A be a closed symmetric relation in the Krein space $(\mathcal{K}, [\cdot,\cdot])$. We say that $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a* boundary value space *for A^+ if $(\mathcal{G}, (\cdot,\cdot))$ is a Hilbert space and there exist linear mappings $\Gamma_0, \Gamma_1 : A^+ \to \mathcal{G}$ such that $\Gamma := \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} : A^+ \to \mathcal{G} \times \mathcal{G}$ is surjective, and the relation*

$$[f', g] - [f, g'] = (\Gamma_1 \hat{f}, \Gamma_0 \hat{g}) - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g}) \tag{3}$$

holds for all $\hat{f} = \begin{pmatrix} f \\ f' \end{pmatrix}, \hat{g} = \begin{pmatrix} g \\ g' \end{pmatrix} \in A^+$.

If a closed symmetric relation A has a selfadjoint extension $\widehat{A}$ in $\mathcal{K}$ with $\rho(\widehat{A}) \neq \emptyset$, then there exists a boundary value space $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for A^+ such that $\widehat{A}$ coincides with $\ker \Gamma_0$ (see [4]).

For basic facts on boundary value spaces and further references see e.g. [3], [4], [5] and [6]. We recall only a few important consequences. For the rest of this section let A be a closed symmetric relation and assume that there exists a boundary value space $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for A^+. Then

$$A_0 := \ker \Gamma_0 \quad \text{and} \quad A_1 := \ker \Gamma_1 \tag{4}$$

are selfadjoint extensions of A. The mapping $\Gamma = \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix}$ induces, via

$$A_\Theta := \Gamma^{-1}\Theta = \left\{ \hat{f} \in A^+ \,|\, \Gamma \hat{f} \in \Theta \right\}, \quad \Theta \in \widetilde{\mathcal{C}}(\mathcal{G}), \tag{5}$$

a bijective correspondence $\Theta \mapsto A_\Theta$ between $\widetilde{\mathcal{C}}(\mathcal{G})$ and the set of closed extensions $A_\Theta \subset A^+$ of A. In particular (5) gives a one-to-one correspondence between the closed symmetric (selfadjoint) extensions of A and the closed symmetric (resp. selfadjoint) relations in $\mathcal{G}$. Moreover, A_Θ is an operator if and only if

$$\Theta \cap \Gamma \left\{ \begin{pmatrix} 0 \\ h \end{pmatrix} \,\middle|\, h \in \operatorname{mul} A^+ \right\} = \{0\}. \tag{6}$$

If Θ is a closed operator in $\mathcal{G}$, then the corresponding extension A_Θ of A is determined by

$$A_\Theta = \ker\Big(\Gamma_1 - \Theta\Gamma_0\Big). \tag{7}$$

Let $\mathcal{N}_\lambda := \ker(A^+ - \lambda) = \operatorname{ran}(A - \overline{\lambda})^{[\perp]}$ be the defect subspace of A and set

$$\hat{\mathcal{N}_\lambda} := \Big\{ \big(\begin{smallmatrix} f \\ \lambda f \end{smallmatrix} \big) \Big| f \in \mathcal{N}_\lambda \Big\}.$$

Now we assume that the selfadjoint relation A_0 in (4) has a nonempty resolvent set. For each $\lambda \in \rho(A_0)$ the relation A^+ can be written as a direct sum of (the subspaces) A_0 and $\hat{\mathcal{N}_\lambda}$ (see [4]). Denote by π_1 the orthogonal projection onto the first component of $\mathcal{K}^2$. The functions

$$\lambda \mapsto \gamma(\lambda) := \pi_1(\Gamma_0|\hat{\mathcal{N}_\lambda})^{-1} \in \mathcal{L}(\mathcal{G}, \mathcal{K}), \qquad \lambda \in \rho(A_0),$$

and

$$\lambda \mapsto M(\lambda) := \Gamma_1(\Gamma_0|\hat{\mathcal{N}_\lambda})^{-1} \in \mathcal{L}(\mathcal{G}), \qquad \lambda \in \rho(A_0) \tag{8}$$

are defined and holomorphic on $\rho(A_0)$ and are called the *γ-field* and the *Weyl function* corresponding to A and $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$. For $\lambda, \zeta \in \rho(A_0)$ the relation (3) implies $M(\lambda)^* = M(\overline{\lambda})$ and

$$\gamma(\zeta) = \Big(1 + (\zeta - \lambda)(A_0 - \zeta)^{-1}\Big)\gamma(\lambda) \tag{9}$$

and

$$M(\lambda) - M(\zeta)^* = (\lambda - \overline{\zeta})\gamma(\zeta)^+\gamma(\lambda) \tag{10}$$

hold (see [4]). Moreover, by [4], we have the following connection between the spectra of extensions of A and the Weyl function.

Lemma 2 *If $\Theta \in \widetilde{\mathcal{C}}(\mathcal{G})$ and A_Θ is the corresponding extension of A then a point $\lambda \in \rho(A_0)$ belongs to $\rho(A_\Theta)$ if and only if 0 belongs to $\rho(\Theta - M(\lambda))$. A point $\lambda \in \rho(A_0)$ belongs to $\sigma_i(A_\Theta)$ if and only if 0 belongs to $\sigma_i(\Theta - M(\lambda))$, $i = p, c, r$.*

For $\lambda \in \rho(A_\Theta) \cap \rho(A_0)$ the well-known resolvent formula

$$(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\Big(\Theta - M(\lambda)\Big)^{-1}\gamma(\overline{\lambda})^+ \tag{11}$$

holds (for a proof see e.g. [4]).

3 The Underlying Space

Let $H^{1,2}(0,1)$ be the Sobolev space of all absolutely continuous functions f with $f' \in L^2(0,1)$. Let k be a positive real number, $k > 0$. We define for

$f, g \in H^{1,2}(0,1)$ [1]

$$[f, g]_k := k(f', g')_{L^2(0,1)} - (f, g)_{L^2(0,1)}. \tag{12}$$

If $\mathcal{L}$ is an arbitrary subset of $H^{1,2}(0,1)$ we set

$$\mathcal{L}^{[\perp]_k} := \{x \in H^{1,2}(0,1) : [x, y]_k = 0 \text{ for all } y \in \mathcal{L}\}.$$

Then we have the following.

Proposition 3 *For the space $(H^{1,2}(0,1), [\cdot,\cdot]_k)$ we have the following properties.*

(1) *If k equals $\frac{1}{n^2\pi^2}$ for some $n \in \mathbb{N}$, then the function $g \in H^{1,2}(0,1)$, defined by $g(x) = \cos(n\pi x)$ belongs to the isotropic part of $(H^{1,2}(0,1), [\cdot,\cdot]_k)$, that is*

$$[f, g]_k = 0 \quad \textit{for all} \quad f \in H^{1,2}(0,1).$$

(2) *If $k > \frac{1}{\pi^2}$, then $(H^{1,2}(0,1), [\cdot,\cdot]_k)$ is a Pontryagin space with one negative square.*

(3) *If $k \leq \frac{1}{\pi^2}$ and $k \neq \frac{1}{n^2\pi^2}$ for all $n \in \mathbb{N}$, then $(H^{1,2}(0,1), [\cdot,\cdot]_k)$ is a Pontryagin space with a finite number of negative squares. Set*

$$\mathcal{H}_- := \operatorname{span}\{f_j \,\Big|\, k \leq \frac{1}{j^2\pi^2},\ j \in \mathbb{N}\},$$

where $f_j \in H^{1,2}(0,1)$ is defined by $f_j(x) = \sin(j\pi x)$. Then the number κ_- of negative squares of $(H^{1,2}(0,1), [\cdot,\cdot]_k)$ satisfies

$$\kappa_- = \dim \mathcal{H}_- + 1.$$

Proof: Assertion (1) is an easy calculation. We assume $k \neq \frac{1}{n^2\pi^2}$ for all $n \in \mathbb{N}$. Define the operator C_0 by

$$\mathcal{D}(C_0) := \{g \in H^{1,2}(0,1) \,\Big|\, g' \in H^{1,2}(0,1) \text{ and } g(0) = g(1) = 0\},$$
$$C_0 g := -g'' \quad \text{for} \quad g \in \mathcal{D}(C_0).$$

Let us note that the functions $f_j(x) = \sin(j\pi x)$, $j = 1, 2\ldots$, are eigen functions of C_0. Moreover, each function g in $L^2(0,1)$ can be written as

$$g = \sum_{j=1}^{\infty} \alpha_j f_j,$$

[1] Let us note that the expression $k(y'(t))^2 - y(t)^2$ with t as the time is (up to a constant) the Lagrangian for free small oscillations in one dimension (see [12], p. 58 for details). From this point of view the corresponding integral represents the action.

where α_j, $j = 1, 2 \ldots$, are some constants.

For $g \in \mathcal{D}(C_0) \cap \mathcal{H}_-^{\perp}$, where $\mathcal{H}_-^{\perp}$ denotes the orthogonal complement with respect to the usual scalar product $(\cdot,\cdot)_{L^2(0,1)}$ but within the Hilbert space $H^{1,2}(0,1)$, we have also that $(f,g)_{L^2(0,1)} = 0$ for all $f \in \mathcal{H}_-$. Thus, g has the representation $g = \sum_{j>\frac{1}{\pi\sqrt{k}}}^{\infty} \alpha_j f_j$, where the sum converges in the norm of $L^2(0,1)$. This implies that there exists an $\epsilon > 0$ with $(C_0 g, g)_{L^2(0,1)} > (\frac{1}{k} + \epsilon)(g,g)_{L^2(0,1)}$ for all $g \in \mathcal{D}(C_0) \cap \mathcal{H}_-^{\perp}$. Therefore there exists constants $c, \widetilde{c} > 0$ with

$$[g,g]_k > c(C_0 g, g)_{L^2(0,1)} + \epsilon(g,g)_{L^2(0,1)} \geq \widetilde{c}(g,g)_{H^{1,2}(0,1)}$$

for $g \in \mathcal{D}(C_0) \cap \mathcal{H}_-^{\perp}$, so $\mathcal{D}(C_0) \cap \mathcal{H}_-^{\perp}$ is a uniformly positive linear manifold. It is easy to see that for $f \in \mathcal{H}_-$ we have $[f,f]_k < 0$. Let us note, that $[f_j, f_l]_k = (kj^2\pi^2 - 1)(f_j, f_l)_{L^2(0,1)} = (kj^2\pi^2 - 1)\delta_{jl}$, where δ_{jl} is the symbol of Kronecker. This means that $(\mathcal{D}(C_0) \cap \mathcal{H}_-^{\perp})[\perp]_k \mathcal{H}_-$. Thus, $\mathcal{D}(C_0)$ is a pre-Pontryagin space with respect to the product $[\cdot,\cdot]_k$, where the number of negative squares equals $\dim \mathcal{H}_-$. Let us note also that the subspace $\mathcal{D}(C_0)^{[\perp]_k}$ is two-dimensional. Indeed, we define $h_1, h_2 \in H^{1,2}(0,1)$ by

$$h_1(x) = \sin(\frac{x}{\sqrt{k}}) + \cos(\frac{x}{\sqrt{k}}), \quad h_2(x) = \sin(\frac{x}{\sqrt{k}}) - \cos(\frac{x}{\sqrt{k}})$$

and $z_1, z_2 \in H^{1,2}(0,1)$ by

$$z_1(x) = -\frac{1}{\sqrt{k} \cdot \sin\frac{1}{\sqrt{k}}} \cos(\frac{x}{\sqrt{k}}), \quad z_2(x) = -\frac{1}{\sqrt{k}} \cot(\frac{1}{\sqrt{k}}) \cos(\frac{x}{\sqrt{k}}) - \frac{1}{\sqrt{k}} \sin(\frac{x}{\sqrt{k}}).$$

For every $g \in H^{1,2}(0,1)$ we have $[g, z_1]_k = g(1)$ and $[g, z_2]_k = g(0)$. Thus,

$$(\mathcal{D}(C_0))^{[\perp]_k} = \text{sp}\,\{z_1, z_2\} = \text{sp}\,\{h_1, h_2\}.$$

Moreover, we have

$$[h_1, h_1]_k = -\frac{2}{\sqrt{k}}(\sin\frac{1}{\sqrt{k}})^2 \quad \text{and } [h_2, h_2]_k = \frac{2}{\sqrt{k}}(\sin\frac{1}{\sqrt{k}})^2.$$

This proves (3). If $k > \frac{1}{\pi^2}$, then $\mathcal{H}_- = \{0\}$ and the above considerations imply (2). $\square$

4 A Symmetric Operator Associated to the Second Derivative of Defect Four

For the rest of this paper, we assume that k is such, that

$$\sin\frac{1}{\sqrt{k}} \neq 0.$$

Then, according to Proposition 3, the space $(H^{1,2}(0,1),[\cdot,\cdot]_k)$ is a Pontryagin space. We consider the following operator A, defined by

$$\mathcal{D}(A) := \{g \in H^{1,2}(0,1) \,|\, g', g'' \in H^{1,2}(0,1) \text{ with } g(0)=g(1)=g'(0)=g'(1)=g''(0)=g''(1)=0\}$$

and

$$Ag := -g'', \quad g \in \mathcal{D}(A).$$

Let us calculate A^+. By the definition $g \in \mathcal{D}(A^+)$ if and only if there is $g^+ \in H^{1,2}(0,1)$ such that $[Af,g]_k = [f,g^+]_k$ for every $f \in \mathcal{D}(A)$. Let $f \in H^{1,2}(0,1)$ be smooth and $f(x) = 0$ for $x \in [0,\delta) \cup (\epsilon,1]$ for some $\delta,\epsilon > 0$. Then the expression $k\int_0^1 f'''(t)\overline{g'(t)}dt - \int_0^1 f''(t)\overline{g(t)}dt$ can be considered as an action of the generalized function $\overline{-kg^{IV} - g''}$ on the test function f. From the other hand the expression $[f,g^+]_k$ can be consider as an action of the generalized function $\overline{-k(g^+)'' - g^+}$ on the same probe function f. Thus, $kg^{IV}(t) + g''(t) = k(g^+)''(t) + g^+(t)$. The latter yields

$$g^+(t) = g''(t) + \alpha f_1(t) + \beta f_2(t), \tag{13}$$

where f_1, f_2 are defined by

$$f_1(x) = \sin\frac{x}{\sqrt{k}}, \quad f_2(x) = \cos\frac{x}{\sqrt{k}},$$

and

$$g'' \in H^{1,2}(0,1). \tag{14}$$

Now the direct calculation shows that the equality $[Af,g]_k = [f,g^+]_k$ is fulfilled for every $f \in \mathcal{D}(A)$, if g is under Condition (14) and g^+ is defined by (13). So,

$$A^+ = \left\{ \begin{pmatrix} g \\ -g'' \end{pmatrix} + \begin{pmatrix} 0 \\ \alpha f_1 + \beta f_2 \end{pmatrix} \,\middle|\, g', g'' \in H^{1,2}(0,1), \alpha,\beta \in \mathbb{C} \right\}.$$

and

$$\operatorname{mul} A^+ = \operatorname{sp}\{f_1, f_2\}, \tag{15}$$

Lemma 4 *Then A is a closed symmetric operator in $(H^{1,2}(0,1),[\cdot,\cdot]_k)$.*

Proof: Obviously, A is symmetric. The best way to show the closeness is via the calculation of A^{++}. We leave it to the reader. $\square$

Let $\binom{f}{-f''+\alpha_1 f_1+\beta_1 f_2}$ and $\binom{g}{-g''+\alpha_2 f_1+\beta_2 f_2}$ be elements from A^+ with $f, g \in \mathcal{D}(A)$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$. Then we have

$$\begin{aligned}[-f''+&\alpha_1 f_1 + \beta_1 f_2, g]_k - [f, -g'' + \alpha_2 f_1 + \beta_2 f_2]_k = \\ &= -f\overline{g}'\Big|_0^1 + f'\overline{g}\Big|_0^1 - kf''\overline{g}'\Big|_0^1 + kf'\overline{g}''\Big|_0^1 + \\ &+ \sqrt{k}(\alpha_1 f_2 - \beta_1 f_1)\overline{g}\Big|_0^1 - \sqrt{k}\overline{(\alpha_2 f_2 - \beta_2 f_1)}\Big|_0^1 .\end{aligned}$$

We define mappings $\Gamma_0, \Gamma_1 : A^+ \to \mathbb{C}^4$ by

$$\Gamma_0 \binom{f}{-f''+\alpha_1 f_1+\beta_1 f_2} = \begin{pmatrix} f(0)+kf''(0) \\ f(1)+kf''(1) \\ f(0) \\ f(1) \end{pmatrix} \quad \text{and}$$

$$\Gamma_1 \binom{f}{-f''+\alpha_1 f_1+\beta_1 f_2} = \begin{pmatrix} -f'(0) \\ f'(1) \\ -\alpha_1\sqrt{k} \\ \sqrt{k}(\alpha_1 \cos\frac{1}{\sqrt{k}} - \beta_1 \sin\frac{1}{\sqrt{k}}) \end{pmatrix} \quad \text{for } \binom{f}{-f''+\alpha_1 f_1+\beta_1 f_2} \in A^+.$$

Theorem 5 *The triplet $\{\Gamma_0, \Gamma_1\}$ is a boundary value space for A^+. In particular $A_1 := \ker \Gamma_1$ is an operator and a selfadjoint extension of A, i.e.*

$$\mathcal{D}(A_1) := \{g \in H^{1,2}(0,1) \,\big|\, g', g'' \in H^{1,2}(0,1) \text{ with } g'(0) = g'(1) = 0\}$$

and

$$A_1 g := -g'', \quad g \in \mathcal{D}(A_1).$$

Moreover, for $\lambda \in \rho(A_0)$, the Weyl function is given by $M(\lambda) =$

$$\begin{bmatrix} \frac{1}{1-k\lambda}\left(\frac{\sqrt{\lambda}}{\tan\sqrt{\lambda}} - \frac{\frac{1}{\sqrt{k}}}{\tan\frac{1}{\sqrt{k}}}\right) & \frac{1}{1-k\lambda}\left(-\frac{\sqrt{\lambda}}{\sin\sqrt{\lambda}} + \frac{\frac{1}{\sqrt{k}}}{\sin\frac{1}{\sqrt{k}}}\right) & \frac{1}{\sqrt{k}\tan\frac{1}{\sqrt{k}}} & \frac{-1}{\sqrt{k}\sin\frac{1}{\sqrt{k}}} \\ \frac{1}{1-k\lambda}\left(-\frac{\sqrt{\lambda}}{\sin\sqrt{\lambda}} + \frac{\frac{1}{\sqrt{k}}}{\sin\frac{1}{\sqrt{k}}}\right) & \frac{1}{1-k\lambda}\left(\frac{\sqrt{\lambda}}{\tan\sqrt{\lambda}} - \frac{\frac{1}{\sqrt{k}}}{\tan\frac{1}{\sqrt{k}}}\right) & \frac{-1}{\sqrt{k}\sin\frac{1}{\sqrt{k}}} & \frac{1}{\sqrt{k}\tan\frac{1}{\sqrt{k}}} \\ \frac{1}{\sqrt{k}\tan\frac{1}{\sqrt{k}}} & \frac{-1}{\sqrt{k}\sin\frac{1}{\sqrt{k}}} & -\frac{1-k\lambda}{\sqrt{k}\tan\frac{1}{\sqrt{k}}} & \frac{1-k\lambda}{\sqrt{k}\sin\frac{1}{\sqrt{k}}} \\ \frac{-1}{\sqrt{k}\sin\frac{1}{\sqrt{k}}} & \frac{1}{\sqrt{k}\tan\frac{1}{\sqrt{k}}} & \frac{1-k\lambda}{\sqrt{k}\sin\frac{1}{\sqrt{k}}} & -\frac{1-k\lambda}{\sqrt{k}\tan\frac{1}{\sqrt{k}}} \end{bmatrix}$$

Proof: The above calculations imply that $\{\Gamma_0, \Gamma_1\}$ is a boundary value space for A^+. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Define $g_1, g_2 \in H^{1,2}(0,1)$ by

$$g_1(x) = \cos(\sqrt{\lambda}x) \quad \text{and } g_2(x) = \sin(\sqrt{\lambda}x). \tag{16}$$

Then we have

$$\ker(A^+ - \lambda) = \operatorname{sp}\{g_1, g_2, f_1, f_2\}.$$

Let $f = \alpha g_1 + \beta g_2 + \gamma f_1 + \delta f_2$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Then

$$\Gamma_0\begin{pmatrix} f \\ \lambda f \end{pmatrix} = \Gamma_0\begin{pmatrix} f \\ -f'' + \gamma(\lambda - \frac{1}{k})f_1 + \delta(\lambda - \frac{1}{k})f_2 \end{pmatrix} = \begin{pmatrix} \alpha(1-k\lambda) \\ \alpha(1-k\lambda)\cos\sqrt{\lambda} + \beta(1-k\lambda)\sin\sqrt{\lambda} \\ \alpha+\delta \\ \alpha\cos\sqrt{\lambda} + \beta\sin\sqrt{\lambda} + \gamma\sin\sqrt{k}^{1} + \delta\cos\frac{1}{\sqrt{k}} \end{pmatrix}$$

and

$$\Gamma_1\begin{pmatrix} f \\ \lambda f \end{pmatrix} = \begin{pmatrix} -\beta\sqrt{\lambda} - \gamma\frac{1}{\sqrt{k}} \\ -\alpha\sqrt{\lambda}\sin\sqrt{\lambda} + \beta\sqrt{\lambda}\cos\sqrt{\lambda} + \gamma\sqrt{k}^{1}\cos\frac{1}{\sqrt{k}} - \delta\sqrt{k}^{1}\sin\frac{1}{\sqrt{k}} \\ -\gamma\sqrt{k}(\lambda - \frac{1}{k}) \\ \gamma\sqrt{k}(\lambda - \frac{1}{k})\cos\frac{1}{\sqrt{k}} - \delta\sqrt{k}(\lambda - \frac{1}{k})\sin\frac{1}{\sqrt{k}} \end{pmatrix}$$

Now, by (8), it follows that M is of the above form. □

Now, via (5) we can parameterize all selfadjoint extensions of A via all selfadjoint relations Θ in $\mathbb{C}^4$.

Theorem 6 *Let Θ be a selfadjoint relation in $\mathbb{C}^4$. Then A_Θ is a selfadjoint extension of A. If for all $\alpha, \beta \in \mathbb{C}$*

$$\begin{pmatrix} 0 \\ 0 \\ \alpha \\ \beta\sin\frac{1}{\sqrt{k}} - \alpha\cos\frac{1}{\sqrt{k}} \end{pmatrix} \notin \operatorname{mul}\Theta \setminus \{0\} \tag{17}$$

holds, then A_Θ is an operator. If, in particular, Θ is a selfadjoint matrix, then A_Θ is a selfadjoint operator and an extension of A with domain

$$\mathcal{D}(A_\Theta) := \left\{ g \in H^{1,2}(0,1) \,\middle|\, g', g'' \in H^{1,2}(0,1),\ \begin{pmatrix} -g'(0) \\ g'(1) \\ 0 \\ 0 \end{pmatrix} = \Theta \begin{pmatrix} g(0)+kg''(0) \\ g(1)+kg''(1) \\ g(0) \\ g(1) \end{pmatrix} \right\}.$$

Proof: Relation (17) follows from (6), (15) and the definitions of Γ_0 and Γ_1. If Θ is a matrix, (17) is satisfied and the description of $\mathcal{D}(A_\Theta)$ follows from (7). □

5 A Symmetric Operator Associated to the Second Derivative of Defect Two

We start this Section opposite to Section 4. For this we put

$$\mathcal{D}(\tilde{A}) := \{g \in H^{1,2}(0,1) \,|\, g', g'' \in H^{1,2}(0,1)\} \tag{18}$$

and

$$\tilde{A}g := -g'', \quad g \in \mathcal{D}(\tilde{A}).$$

Thus, the operator $\tilde{A}$ corresponds to the same formal differential expression as the operator considered in the previous section, but with a different domain which is in some sense maximal. Let us calculate $\tilde{A}^+$. For $f, g \in \mathcal{D}(\tilde{A})$ we have

$$\begin{aligned}
[\tilde{A}f, g]_k &= -k\int_0^1 f'''(t)\overline{g'(t)}dt + \int_0^1 f''(t)\overline{g(t)}dt = \\
&= -k\Big(f''(t)\overline{g'(t)} - f'(t)\overline{g''(t)}\Big)\Big|_0^1 + \Big(f'(t)\overline{g(t)} - f(t)\overline{g'(t)}\Big)\Big|_0^1 - \\
&- k\int_0^1 f'(t)\overline{g'''(t)}dt + \int_0^1 f(t)\overline{g''(t)}dt = \\
&= -\Big(kf''(1) + f(1)\Big)\overline{g'(1)} + f'(1)\Big(k\overline{g''(1)} + \overline{g(1)}\Big) - \\
&- f'(0)\Big(k\overline{g''(0)} + \overline{g(0)}\Big) + \Big(kf''(0) + f(0)\Big)\overline{g'(0)} + [f, \tilde{A}g]_k
\end{aligned}$$

Note that the maps $f(t) \mapsto \Big(kf''(1) + f(1)\Big)$, $f(t) \mapsto f'(1)$, $f(t) \mapsto \Big(kf''(0) + f(0)\Big)$ and $f(t) \mapsto f'(0)$ represent unbounded linear functionals on $H^{1,2}(0,1)$. Thus, the expression $[\tilde{A}f, g]_k$ gives a continuous linear functional (with respect to f) on $H^{1,2}(0,1)$ if and only if $g'(1) = \Big(kg''(1) + g(1)\Big) = \Big(kg''(0) + g(0)\Big) = g'(0) = 0$ and by the definition of the adjoint operator the latter conditions restrict the domain of $\tilde{A}^+$. For brevity below we set $A := \tilde{A}^+$. Thus, we have the following operator A, defined by

$$\begin{aligned}
\mathcal{D}(A) := \{g \in H^{1,2}(0,1) \,\big|\, & g', g'' \in H^{1,2}(0,1) \text{ with } g'(0) = g'(1) = 0, \\
& g(0) + kg''(0) = 0 \quad \text{and } g(1) + kg''(1) = 0\}
\end{aligned}$$

and

$$Ag := -g'', \quad g \in \mathcal{D}(A).$$

Then A is a closed symmetric operator in $(H^{1,2}(0,1), [\cdot,\cdot]_k)$, which is, in contrast to Section 4, densely defined. In particular

$$A^+ = \tilde{A} = \Big\{\Big(\begin{smallmatrix} g \\ -g'' \end{smallmatrix}\Big) \Big|\, g', g'' \in H^{1,2}(0,1)\Big\}$$

is an operator and therefore all selfadjoint extensions of A are operators.

We define mappings $\Gamma_0, \Gamma_1 : A^+ \to \mathbb{C}^2$ by

$$\Gamma_0\Big(\begin{smallmatrix} f \\ -f'' \end{smallmatrix}\Big) = \Big(\begin{smallmatrix} f(0)+kf''(0) \\ f(1)+kf''(1) \end{smallmatrix}\Big) \quad \text{and} \quad \Gamma_1\Big(\begin{smallmatrix} f \\ -f'' \end{smallmatrix}\Big) = \Big(\begin{smallmatrix} -f'(0) \\ f'(1) \end{smallmatrix}\Big) \quad \text{for } \Big(\begin{smallmatrix} f \\ -f'' \end{smallmatrix}\Big) \in A^+.$$

Theorem 7 *The triplet $\{\Gamma_0, \Gamma_1\}$ is a boundary value space for A^+. The Weyl function is given by*

$$M(\lambda) = \begin{bmatrix} \frac{\sqrt{\lambda}}{(1-k\lambda)\tan\sqrt{\lambda}} & \frac{-\sqrt{\lambda}}{(1-k\lambda)\sin\sqrt{\lambda}} \\ \frac{-\sqrt{\lambda}}{(1-k\lambda)\sin\sqrt{\lambda}} & \frac{\sqrt{\lambda}}{(1-k\lambda)\tan\sqrt{\lambda}} \end{bmatrix}, \quad \lambda \in \rho(A_0).$$

Proof: The above calculations imply that $\{\Gamma_0, \Gamma_1\}$ is a boundary value space for A^+. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $g_1, g_2 \in H^{1,2}(0,1)$ as in (16). Then we have

$$\ker(A^+ - \lambda) = \operatorname{sp}\{g_1, g_2\}.$$

Let $f = \alpha g_1 + \beta g_2$ for some $\alpha, \beta \in \mathbb{C}$. Then

$$\Gamma_0\begin{pmatrix} f \\ \lambda f \end{pmatrix} = \Gamma_0\begin{pmatrix} f \\ -f'' \end{pmatrix} = \begin{pmatrix} \alpha(1-k\lambda) \\ \alpha(1-k\lambda)\cos\sqrt{\lambda}+\beta(1-k\lambda)\sin\sqrt{\lambda} \end{pmatrix}$$

and

$$\Gamma_1\begin{pmatrix} f \\ \lambda f \end{pmatrix} = \Gamma_1\begin{pmatrix} f \\ -f'' \end{pmatrix} = \begin{pmatrix} -\beta\sqrt{\lambda} \\ -\alpha\sqrt{\lambda}\sin\sqrt{\lambda}+\beta\sqrt{\lambda}\cos\sqrt{\lambda} \end{pmatrix}$$

Now, by (8), it is follows that M is of the above form. □

Lemma 8 *The operator $A_0 = \ker \Gamma_0$ is a selfadjoint extension of A with a compact resolvent and*

$$\sigma(A_0) = \sigma_p(A_0) = \{k^{-1}, \pi^2, 4\pi^2, 9\pi^2, 16\pi^2, \ldots\}. \tag{19}$$

Proof: The operator $A_1 = \ker \Gamma_1$ is selfadjoint in the Hilbert space $H^{1,2}(0,1)$. We have for $f \in \mathcal{D}(A_1)$

$$((A_1 + I)f, f)_{H^{1,2}(0,1)} = \|f'\|^2_{L^2(0,1)} + \|f\|^2_{H^{2,2}(0,1)},$$

where $H^{2,2}(0,1)$ is the Sobolev space of all functions $f \in H^{1,2}(0,1)$ with $f' \in H^{1,2}(0,1)$. This gives

$$\|f\|^2_{H^{2,2}(0,1)} \leq \|(A_1 + I)f\|_{H^{1,2}(0,1)} \|f\|_{H^{2,2}(0,1)}.$$

Therefore, as the embedding of $H^{2,2}(0,1)$ into $H^{1,2}(0,1)$ is compact, the selfadjoint operator A_1 has a compact resolvent. By (11) the difference between the resolvents of A_0 and A_1 is of finite rank, hence A_0 has a compact resolvent. We have $\sigma(A_0) = \sigma_p(A_0)$. Now (19) follows from a simple calculation. □

Proposition 9 *Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$ and*

$$|\alpha| < 2\sqrt{k}. \tag{20}$$

Then the operator A_α defined by

$$\mathcal{D}(A_\alpha) := \{g \in H^{1,2}(0,1) \,\big|\, g', g'' \in H^{1,2}(0,1) \text{ with } \alpha g'(0) = g(0) + kg''(0) \quad \text{and} \quad \alpha g'(1) = g(1) + kg''(1)\}$$

and

$$A_\alpha g := -g'', \quad g \in \mathcal{D}(A_\alpha).$$

is a selfadjoint extension of A with non-real eigenvalues.

In the case $\alpha = 2\sqrt{k}$ we have that the selfadjoint extension $A_{2\sqrt{k}}$ of A has a Jordan chain of length two corresponding to the eigenvalue $-\frac{1}{k}$.

Proof: Set

$$\Theta = \begin{bmatrix} -\alpha^{-1} & 0 \\ 0 & \alpha^{-1} \end{bmatrix}.$$

Then $A_\Theta = A_\alpha$, hence, by Lemma 2 and the fact that $\sigma(A_0) \subset \mathbb{R}$ (see Lemma 8), we have for all non-real λ that $\lambda \in \sigma_p(A_\alpha)$ if and only if

$$0 = \det\left(M(\lambda) - \Theta\right) = \frac{k^2}{\alpha^2(1-k\lambda)^2}\left(\lambda^2 + \frac{\lambda\alpha^2}{k^2} - 2\frac{\lambda}{k} + \frac{1}{k^2}\right). \tag{21}$$

Hence,

$$\lambda_{1,2} = \frac{1}{k} - \frac{\alpha^2}{2k^2} \pm \frac{\alpha}{k^2}\sqrt{\frac{\alpha^2}{4} - k}$$

are the solutions of Equation (21). Assertion (20) implies now the existence of two non-real eigenvalues of A_α.

In the case $\alpha = 2\sqrt{k}$ we have that the functions $h_0, h_1 \in \mathcal{D}(A_{2\sqrt{k}})$ given by

$$h_0(x) = e^{x\frac{1}{\sqrt{k}}} \quad \text{and} \quad h_1(x) = -\frac{x}{2}e^{x\frac{1}{\sqrt{k}}}$$

satisfy

$$\left(A_{2\sqrt{k}} + \frac{1}{k}\right)h_1 = h_0 \quad \text{and} \quad \left(A_{2\sqrt{k}} + \frac{1}{k}\right)h_0 = 0,$$

i.e. $\{h_1, h_0\}$ is a Jordan chain of $A_{2\sqrt{k}}$ corresponding to the eigenvalue $-\frac{1}{k}$. □

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